A Small-world and Scale-free Network Generated by Sierpinski Pentagon

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Abstract

The Sierpinski Pentagon is used to construct evolving networks, whose nodes are all solid regular pentagons in the construction of the Sierpinski Pentagon up to the stage t and any two nodes are neighbors if and only if the intersection of corresponding pentagons is non-empty and non-singleton. We show that such networks have the small-world and scale-free effects, but are not fractal scaling.

Keywords: complex network, Sierpinski Pentagon, scale-free, small-world, non-fractal scaling

1. Introduction

In the past two decades, lots of attention has been paid to the research of complex networks, which are ubiquitous in nature and society such as WWW [1], metabolic networks [2, 3], a network of routers connected by various physical connections [4]. Despite their diversity, most networks appearing in real world obey some organizing principles, which lead to systematic and measurable deviations basing on the random graph theory originating from Erdős and Rényi [5, 6]. In particular, two properties of real networks have aroused considerable scientific and technological interests. First, lots of measurements shows that most networks display the scale-free property [7], which means that the probability that a randomly selected node with exactly k links decays as a power law, following $P(k) \sim k^{-\alpha}$, where α is the degree exponent. Second, lots of networks have the small-world behavior [8], namely, the geodesic distance between two uniform randomly chosen nodes grows proportionally to the logarithm of the number of nodes, simultaneously they display high average clustering coefficient.

At the mean time, various self-similar fractals are used to model and mimic evolving complex networks due to the occurrence of self-similarity in real-world complex networks demonstrated by Song et al. [9]. Zhang et al. [10, 11, 12] use the Sierpinski gasket to construct evolving networks which follow the power-law degree distribution and possess small-world effect. Several complex networks are modeled on self-similar fractals, for example, Apollonian networks [13], Koch networks [14, 15, 16, 17, 18], Platonic solids networks [19], Vicsek networks [20], Sierpinski networks [21, 22], the hierarchical networks [23] and generalized self-similar networks [24].

The fractality of complex networks is introduced in Song et al. [9, 25], see also Gallos et al. [26]. In a series of articles [27, 28, 29, 30, 31, 32], the research group of Makse et al. analyzes the fractality of complex networks and use their techniques of fractal analysis on networks to study the biological networks. An ℓ_B -box is a subset of node set V such that the distance d_{ij} between any pair of nodes i and j in the box has to be $d_{ij} < \ell_B$. Let $N_B = N_B(\ell_B)$ denote the minimal number of ℓ_B -box to cover V , which is the same as ones in [9] and [33]. In [9], a network with node set V has fractal scaling with fractal dimension d_B if

$$N_B \sim (\ell_B)^{-d_B}. \quad (1.1)$$

In this article, we give a self-similar, scale-free and small-world complex network generated from a self-similar fractal, but our networks are non-fractal scaling, which may be a common feature for some self-similar networks as indicated by Kim et al. [33].

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In the Section 2, we give out the construction of the complex network generated by Sierpinski Pentagon. Then In the Section 3 and 4, we show that described network is scale-free and small-world. At last in Section 5, using the technique of fractal geometry, we prove that this network is not fractal scaling.

2. Construction of the network

2.1. Sierpinski Pentagon and its IFS

The Sierpinski Pentagon is generated by the following iteration function system (IFS).

Let $f_i(x) = \frac{3-\sqrt{5}}{2}x + b_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ for $i = 1, 2, 3, 4, 5$, where $b_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

$$b_2 = \begin{pmatrix} \frac{\sqrt{5}-1}{2} \\ 0 \end{pmatrix}, b_3 = \begin{pmatrix} \frac{\sqrt{5}+1}{4} \\ \frac{\sqrt{10-2\sqrt{5}}}{4} \end{pmatrix}, b_4 = \begin{pmatrix} \frac{\sqrt{5}-1}{4} \\ a \end{pmatrix}, b_5 = \begin{pmatrix} \frac{\sqrt{5}-3}{4} \\ \frac{\sqrt{10-2\sqrt{5}}}{4} \end{pmatrix},$$

where $a = \frac{2}{3+\sqrt{5}}(2\sin(2\pi/5) + \sin(\pi/5))$, that means the ratio $\frac{3-\sqrt{5}}{2} \approx 0.382$ and

$$b_2 \approx \begin{pmatrix} 0.618 \\ 0 \end{pmatrix}, b_3 \approx \begin{pmatrix} 0.809 \\ 0.588 \end{pmatrix}, b_4 \approx \begin{pmatrix} 0.309 \\ 0.951 \end{pmatrix}, b_5 \approx \begin{pmatrix} -0.191 \\ 0.588 \end{pmatrix}.$$

Then the Sierpinski Pentagon F is a self-similar fractal satisfying

$$F = \bigcup_{i=1}^5 f_i(F).$$

Fix the solid pentagon K which is the left one in Figure 1, then $f_i(K) \subset K$. Let $f_6 = f_1$. Notice that $f_i(K) \cap f_{i+1}(K)$ is a singleton.

Figure 1: The first two iterations for the Sierpinski Pentagon

2.2. Structure of Network

Given a word $\sigma = i_1 \cdots i_k$, we consider a small solid pentagon

$$K_\sigma = f_{i_1} \circ \cdots \circ f_{i_k}(K).$$

Then its boundary ∂K_σ is $f_{i_1} \circ \cdots \circ f_{i_k}(M)$, where $M = \cup_{i=1}^5 [A_i, A_{i+1}]$ ($A_6 = A_1$) and $[a, b]$ is the line segment between points a and b in the plane. Let $K_\emptyset = K$ where $\emptyset$ is the empty word.

If $\sigma = i_1 \cdots i_k$ and $\tau = i_1 \cdots i_k i_{k+1} \cdots i_{k+m}$ with $m \geq 1$, we denote $\sigma < \tau$.

Figure 2: G_4

Fix an integer t , we construct a network G_t as follows. Its nodes consist of all regular pentagons in the construction of the Sierpinski pentagons up to the stage t , thus its node set is identified with the set of words of length t

$$V_t = \{\sigma : 0 \leq |\sigma| \leq t\},$$

Since each word of length t encodes such a solid regular pentagon, for two different words σ and τ , they are neighbors, denoted by $\sigma \sim \tau$, if and only if

$$\partial K_\sigma \cap \partial K_\tau \text{ is neither the empty set nor a singleton,}$$

that means the intersection contains at least one line segment.

The total number of nodes for G_t is

$$\#V_t = 1 + 5 + 5^2 + \cdots + 5^t = \frac{1}{4}(5^{t+1} - 1). \quad (2.1)$$

Example 1. $1 \sim \emptyset$, since $\partial K_1 \cap \partial K \subset [A_5, A_1] \cup [A_1, A_2]$ is not a singleton. $11, 12, 21, 22 \sim \emptyset$, but $14, 25 \approx \emptyset$.

To illustrate the complexity of the graphs G_t , we draw G_4 in Figure 2. Now, we have the following claims.

Claim 1. If $\sigma \sim \tau$, then $\sigma < \tau$ or $\tau < \sigma$.

Notice that $\sigma \sim \emptyset$ and $(\partial K_1 \cap \partial K) \cap [A_1, A_2] \neq \emptyset$, then σ is consists of letters 1 and 2. Similarly, for letters $i, (i+1)$, where letter 6 is thought as 1, we have

Claim 2. Suppose $\sigma \sim \tau$ and $\sigma < \tau$. If $\tau = \sigma j_1 \cdots j_k$, then there exists $i \in \{1, \cdots, 5\}$ such that every letter appearing in $j_1 \cdots j_k$ belongs to $\{i, i+1\}$.

Now, we have the following criteria of neighbors:

Proposition 1. $\sigma \sim \tau$ with $|\sigma| \leq |\tau|$ if and only if $\tau = \sigma j_1 \cdots j_k$ with $k \geq 1$ and there exists $i \in \{1, \cdots, 5\}$ such that each letter j_p ($1 \leq p \leq k$) belongs to $\{i, i+1\}$.

3. Degree formula and power-law distribution

For $k \geq 1$, let

$$E_i(k) = \{\sigma = j_1 \cdots j_k : j_p \in \{i, i+1\} \text{ for all } p \leq k\}.$$

Then $E_i(k) \cap E_j(k) \neq \emptyset$ for $i \neq j$ if and only if $|i - j| = 1$, and in this case, $E_i(k) \cap E_{i+1}(k) \neq \emptyset$, where $[j]^k = \underbrace{j \cdots j}_k$ is a word of length k . Now,

$$\#(\cup_{i=1}^5 E_i(k)) = \sum_{i=1}^5 \#E_i(k) - \sum_{i=1}^5 \#(E_i(k) \cap E_{i+1}(k)) = 5(2^k - 1). \quad (3.1)$$

Let

$$E = \bigcup_{k=1}^{\infty} \bigcup_{i=1}^5 E_i(k). \quad (3.2)$$

Let $\eta(m) = \#\{\sigma : \sigma \sim \emptyset \text{ and } 1 \leq |\sigma| \leq m\}$. Then by Proposition 1 and (3.1), we have

$$\eta(m) = 5(\sum_{k=1}^m 2^k - m) = 10 \cdot 2^m - 5m - 10.$$

Given a word $\sigma = \sigma_1 \sigma_2 \dots \sigma_t \in V_t$, let $T_n(\sigma)$ be the tail of σ with length n , i.e., $T_n(\sigma) = \sigma_{t-n+1} \dots \sigma_t$. Define

$$q_2(\sigma) = \max \left\{ n : T_n(\sigma) \subset \cup_{i=1}^5 E_i(n) \right\}.$$

Let $q_2(\emptyset) = 0$. Therefore, $0 \leq q_2(\sigma) \leq t$.

Using Proposition 1, we have

Proposition 2. For any word $\sigma \in V_t$, we have

$$\deg(\sigma) = q_2(\sigma) + \eta(t - |\sigma|).$$

Since $\eta(t - |\sigma|) \leq \deg(\sigma) \leq \eta(t - |\sigma|) + t$, we have

$$10 \cdot 2^{t-|\sigma|} - 5(t - |\sigma|) - 10 \leq \deg(\sigma) \leq 10 \cdot 2^{t-|\sigma|} - 5(t - |\sigma|) - 10 + t. \quad (3.3)$$

Lemma 1. Suppose $t \geq 4$. We have

(1) If $0 \leq k \leq t - 3$, then $\{\sigma : |\sigma| \leq k\} \subset \{\sigma : \deg(\sigma) \geq 6 \cdot 2^{t-k}\}$.

(2) If $t < 2^{t-k}$, then $\{\sigma : \deg(\sigma) \geq 6 \cdot 2^{t-k}\} \subset \{\sigma : |\sigma| \leq k\}$.

(3) $\{\sigma : \deg(\sigma) \geq 6 \cdot 2^t\} = \{\emptyset\}$.

Proof. (1) When $|\sigma| \leq k \leq t - 3$, since $4 \cdot 2^x - 5x - 10 \geq 0$ for $x \geq 3$, then

$$\begin{aligned} \deg(\sigma) &\geq 10 \cdot 2^{t-|\sigma|} - 5(t - |\sigma|) - 10 \\ &\geq 6 \cdot 2^{t-|\sigma|} \geq 6 \cdot 2^{t-k}. \end{aligned}$$

(2) When $t < 2^{t-k}$, suppose $|\sigma| \geq k + 1$, we have

$$\begin{aligned} \deg(\sigma) &\leq 10 \cdot 2^{t-|\sigma|} - 5(t - |\sigma|) - 10 + t \\ &\leq 10 \cdot 2^{t-k-1} + t < 6 \cdot 2^{t-k}. \end{aligned}$$

(3) It follows from (2) for $k = 0$ and $\deg(\emptyset) \geq 6 \cdot 2^t$ for any $t > 3$. □

Theorem 1. Suppose $t \geq 4$. If $6 \cdot 2^t > n > 12t$, then

$$\left(\frac{6^\alpha}{10}\right) n^{-\alpha} \leq \frac{\#\{\sigma \in V_t : \deg(\sigma) \geq n\}}{\#V_t} \leq \left(\frac{5 \cdot 6^\alpha}{2}\right) n^{-\alpha} \text{ with } \alpha = \frac{\log 5}{\log 2}.$$

If $6 \cdot 2^t \leq n \leq \deg(\emptyset)$, then $\{\sigma : \deg(\sigma) \geq n\} = \{\emptyset\}$.

Figure 3: The log-log graphs of the cumulative degree distributions for $t = 5, 6, 7, 8$. The slope of the solid line is $-\log 5 / \log 2$.

Proof. Suppose $6 \cdot 2^{t-k-1} \leq n < 6 \cdot 2^{t-k}$ with $n > \max(24, 12t)$, which implies

$$t - k \geq 3 \text{ and } t < 2^{t-k-1}.$$

Now, we have

$$\{\sigma : \deg(\sigma) \geq 6 \cdot 2^{t-k}\} \subset \{\sigma : \deg(\sigma) \geq n\} \subset \{\sigma : \deg(\sigma) \geq 6 \cdot 2^{t-k-1}\}.$$

Now, $t - k \geq 3$ and $t < 2^{t-k-1}$, by (1) and (2) in Lemma 1, we have

$$\{\sigma : |\sigma| \leq k\} \subset \{\sigma : \deg(\sigma) \geq 6 \cdot 2^{t-k}\}$$

and

$$\{\sigma : \deg(\sigma) \geq 6 \cdot 2^{t-k-1}\} \subset \{\sigma : |\sigma| \leq k + 1\}.$$

If $6 \cdot 2^t > n > 12t$,

$$\frac{5^{k+1} - 1}{5^{t+1} - 1} \leq \frac{\#\{\sigma \in V_t : \deg(\sigma) \geq n\}}{\#V_t} \leq \frac{5^{k+2} - 1}{5^{t+1} - 1}.$$

Here

$$\begin{aligned} \frac{5^{k+1} - 1}{5^{t+1} - 1} &\geq \frac{1}{2} \cdot \frac{5^{k+1}}{5^{t+1}} = \frac{6^\alpha}{10} (6^{-\alpha} \cdot 5^{-(t-k-1)}) \geq \frac{6^\alpha}{10} (6 \cdot 2^{t-k-1})^{-\alpha} \geq \frac{6^\alpha}{10} n^{-\alpha}. \\ \frac{5^{k+2} - 1}{5^{t+1} - 1} &\leq \frac{1}{2} \cdot \frac{5^{k+2}}{5^{t+1}} = \frac{5 \cdot 6^\alpha}{2} (6^{-\alpha} \cdot 5^{k-t}) \leq \frac{5 \cdot 6^\alpha}{2} (6 \cdot 2^{t-k})^{-\alpha} \leq \frac{5 \cdot 6^\alpha}{2} n^{-\alpha}. \end{aligned}$$

□

4. Small-world Property

4.1. Average shortest path length

For $\sigma, \tau \in V_t$, $\sigma \neq \tau$, we denote $d_t(\sigma, \tau)$ the geodesic distance in G_t between σ and τ , namely the length of the shortest path in G_t from σ to τ . Then the average shortest path length of the network G_t is

$$\overline{D}(t) = \frac{\sum_{\sigma \neq \tau \in V_t} d_t(\sigma, \tau)}{\#V_t(\#V_t - 1)/2},$$

here we only compute $d_t(\sigma, \tau)$ or $d_t(\tau, \sigma)$, not both of them.

Remark 1. If $\sigma, \tau \in V_t$ with $\sigma = i\sigma'$ and $\tau = i\tau'$ for some $i \in \{1, 2, 3, 4, 5\}$, then

$$d_t(\sigma, \tau) = d_{t-1}(\sigma', \tau'),$$

due to the self-similarity of G_t . Moreover, it is not hard to show that for $\sigma, \tau \in V_t$, $d_t(\sigma, \tau) = d_s(\sigma, \tau)$ for any $s \geq t$. That means we can denote $d(\sigma, \tau)$ by ignoring t .

Remark 2. For any $\sigma \neq \emptyset$, as in [34], let $L(\sigma)$ be the minimal number of moves for K_σ to touch the boundary of K , i.e.,

$$L(\sigma) = d(\sigma, \emptyset) - 1. \quad (4.1)$$

For $i \neq j$, we have

$$d(i\tau, j\tau') \geq L(\tau) + L(\tau'). \quad (4.2)$$

As in [34], we also have

$$L(\sigma) + L(\tau) \leq L(\sigma\tau) \leq L(\sigma) + L(\tau) + 1. \quad (4.3)$$

For $t \geq 1$, we denote

$$\overline{\alpha}_t = \frac{\sum_{|\sigma|=t} L(\sigma)}{\#\{\sigma : |\sigma| = t\}},$$

which is the average path moving from K_σ with $|\sigma| = t$ to reach K . Trivially $\overline{\alpha}_1 = 0$. For $|\sigma| = 2$, we can check that there are 10 pentagons K_σ which are not in contact with K on the boundary, which implies $L(\tau) = 1$ for 10 words of length 2. Hence $\overline{\alpha}_2 = \frac{10}{25} = \frac{2}{5}$.

Lemma 2. For $t \geq 3$, we have

$$\overline{\alpha}_t \geq \frac{1}{5}(t - 1).$$

Proof. We will prove this result by induction. Note that $\overline{\alpha}_2 = \frac{2}{5}$. For any $k > 2$, let $\sigma = \tau\sigma'$ with $|\sigma| = k$ and $|\sigma'| = k - 2$, we obtain that

$$\frac{\sum_{|\sigma|=t} L(\sigma)}{\#\{\sigma : |\sigma| = t\}} = \frac{\sum_{|\tau|=2} \sum_{|\sigma'|=t-2} L(\tau\sigma')}{\sum_{|\tau|=2} \#\{\sigma' : \sigma = \tau\sigma', |\sigma'| = t - 2\}}. \quad (4.4)$$

If $\tau \sim \emptyset$, then $L(\sigma) \geq L(\sigma') + L(\tau) = L(\sigma') + 1$ by (4.2), we have

$$\frac{\sum_{|\sigma'|=t-2} L(\tau\sigma')}{\#\{\sigma' : \sigma = \tau\sigma', |\sigma'| = t - 2\}} \geq \frac{\sum_{|\sigma'|=t-2} (L(\sigma') + 1)}{\#\{\sigma' : \sigma = \tau\sigma', |\sigma'| = t - 2\}}. \quad (4.5)$$

If $\tau \sim \emptyset$, then $L(\sigma) \geq L(\sigma') + L(\tau) = L(\sigma')$, we have

$$\frac{\sum_{|\sigma'|=t-2} L(\tau\sigma')}{\#\{\sigma' : \sigma = \tau\sigma', |\sigma'| = t - 2\}} \geq \frac{\sum_{|\sigma'|=t-2} L(\sigma')}{\#\{\sigma' : \sigma = \tau\sigma', |\sigma'| = t - 2\}}. \quad (4.6)$$

It follows from (4.4)-(4.6) that

$$\begin{aligned}
\bar{\alpha}_t &\geq \frac{\#\{|\tau| = 2: L(\tau) = 1\}}{25}(\bar{\alpha}_{t-2} + 1) + \frac{\#\{|\tau| = 2: L(\tau) = 0\}}{25}\bar{\alpha}_{t-2} \\
&= \frac{10}{25}(\bar{\alpha}_{t-2} + 1) + \frac{15}{25}\bar{\alpha}_{t-2} \\
&\geq \bar{\alpha}_{t-2} + \frac{2}{5}.
\end{aligned} \tag{4.7}$$

Thus $\bar{\alpha}_3 \geq \bar{\alpha}_1 + \frac{2}{5} = \frac{2}{5}$ and $\bar{\alpha}_4 \geq \bar{\alpha}_2 + \frac{2}{5} = \frac{4}{5}$. Therefore,

$$\bar{\alpha}_t \geq \frac{1}{5}(t-1) \text{ for } t = 3, 4.$$

When $t \geq 5$, by the induction hypothesis, we have $\bar{\alpha}_{t-2} \geq \frac{1}{5}(t-3)$, then

$$\bar{\alpha}_t \geq \frac{1}{5}(t-3) + \frac{2}{5} = \frac{1}{5}(t-1),$$

which ends the proof of the lemma. $\square$

Theorem 2. For any $t \geq 6$, we have

$$\frac{128}{625}(t-2) \leq \bar{D}(t) \leq 2t.$$

Proof. By the criterion of neighbors, for any $\sigma \in V_t$, there is a path of length $\leq |\sigma|$ from the empty word $\emptyset$ to σ , i.e., $d(\sigma, \emptyset) \leq |\sigma|$. Hence, for any $\sigma, \tau \in V_t$,

$$d(\sigma, \tau) \leq d(\sigma, \emptyset) + d(\emptyset, \tau) \leq |\sigma| + |\tau| \leq 2t,$$

which implies

$$\bar{D}(t) \leq 2t. \tag{4.8}$$

For $i, j \in \{1, 2, 3, 4, 5\}$ with $i < j$, by (4.2), we have

$$\sum_{|\sigma'|=t-1} \sum_{|\tau'|=t-1} d_t(i\sigma', j\tau') \geq \sum_{|\sigma'|=t-1} \sum_{|\tau'|=t-1} (L(\sigma') + L(\tau')) = 2 \cdot 5^{2(t-1)} \bar{\alpha}_{t-1}.$$

Thus we have

$$\sum_{|\sigma|=|\tau|=t, \sigma \neq \tau} d(\sigma, \tau) \geq \sum_{i,j \in \{1,2,3,4,5\}, i < j} \sum_{|\sigma'|=|\tau'|=t-1} d(i\sigma', j\tau') \geq C_5^2 \cdot 2 \cdot 5^{2(t-1)} \bar{\alpha}_{t-1},$$

which give us the lower bound for $\bar{D}(t)$, by Lemma 2,

$$\bar{D}(t) \geq \frac{10 \cdot 2 \cdot 5^{2(t-1)}}{\frac{5}{25}(5^{t+1} - 1)(5^t - 1)} \bar{\alpha}_{t-1} \geq \frac{128}{125} \bar{\alpha}_{t-1} \geq \frac{128}{625}(t-2). \tag{4.9}$$

Then Theorem 2 follows from (4.8) and (4.9). $\square$

4.2. Average clustering coefficient

Given $i, j \in \{1, 2, 3, 4, 5\}$, we say that i, j are dual, if $|i - j| = 1$, where 1 can be thought as 6. For example, 1 and 2, 2 and 3, 3 and 4, 4 and 5, 5 and 1 are dual respectively.

Theorem 3. If $t \geq 4$, we have

$$\bar{C}(t) > \frac{24}{125}.$$

Proof. For the general G_t with $t \geq 4$, we consider the word in the form $\sigma = \sigma'123$, where $|\sigma'| = t - 3$. The total number of such word in G_t is 5^{t-3} . The criteria of neighbors implies that there are two neighbors $\sigma'1, \sigma'12$ for each $\sigma = \sigma'123$, and we know that $\sigma'1 \sim \sigma'12$ in G_t , thus the local clustering coefficient is 1.

In the same way, when considering the word $\sigma'i_1i_2i_3$, where i_2, i_3 are dual, $i_1 \neq i_2$ and $i_1 \neq i_3$, we notice that $\sigma'i_1i_2i_3$ has two neighbors $\sigma'i_1, \sigma'i_1i_2$, and thus its local clustering coefficient is 1.

For fixed σ' with $|\sigma'| = t - 3$, we have

$$\#\{\sigma = \sigma'i_1i_2i_3 : i_2, i_3 \text{ are dual, } i_1 \neq i_2 \text{ and } i_1 \neq i_3\} = 3 \cdot 2 \cdot 5 = 30.$$

Hence we have the following lower bound for average clustering coefficient of G_t ,

$$\bar{C}(t) \geq \frac{5^{t-3} \cdot 30}{\frac{1}{4}(5^{t+1} - 1)} > \frac{24}{125},$$

where $\#V_t = \frac{1}{4}(5^{t+1} - 1)$. □

5. Non fractal scaling

As in [9] and [35], an ℓ_B -box is a subset of V such that any two nodes have distance less than ℓ_B . We let $N_B = N_B(\ell_B)$ denote the minimal number of ℓ_B -box to cover V , which is the same as ones in [9] and [33]. In [9], a network with node set V has fractal scaling with fractal dimension d_B if

$$\frac{\#V}{N_B(\ell_B)} \sim (\ell_B)^{d_B}, \quad (5.1)$$

where $\frac{\#V}{N_B(\ell_B)}$ is the average number of nodes with a box of size ℓ_B .

As shown in [36] by Shanker, the number of vertices in a ball or box is a very important quantity. In general, we call a network a generalized fractal network with gauge function H , if the average number of vertices in the box satisfies

$$\frac{\#V}{N_B(\ell_B)} \sim H(\ell_B).$$

In particular, when $H(r) = r^d$, the network is a fractal network with dimension d as in [9]. In Fractal Geometry [37], there are many cases where some function H other than r^d is more useful and natural, for example almost all trajectories of Brownian motion in $\mathbb{R}^2$ has gauge function $H(r) = r^2 \log \log r^{-1}$, and $H(r) = r^2 \log r^{-1} \log \log \log r^{-1}$ for $\mathbb{R}^3$. In [33], Kim et al. show that

$$\frac{\#V}{N_B(\ell_B)} \sim e^{\ell_B/c}, \quad (5.2)$$

for many real-world networks, where c is a constant.

In the computation for real-world networks, ℓ_B is usually relatively small due to the capacity of the networks and the complexity of computation, but algorithms for estimating N_B is very sensitive to the parameter ℓ_B , this makes the graphic fitting not very accurate. On the other hand, $H(\ell_B)$ is close to a polynomial according to the Taylor expansion when ℓ_B is relatively small. Thus the experimental results may not detect the exponential $H(\ell_B)$ very well. Because of this, we have to use precise mathematical analysis to estimate the minimum number N_B of covering to obtain right conclusion for $H(\ell_B)$.

For our evolving network G_t with node set V_t , we have

Theorem 4. $\frac{\#V_t}{N_B(\ell_B)} \sim e^{\frac{\log 5}{2} \ell_B}.$

Given a word σ in V_t , $L(\sigma)$ is the defined as (4.1). For example, $L((23)(115)) = 1$, $L((45)(43)(2111)) = 2$, $L((4)(2)(5)(3)(1)) = 4$. Here

$$L(\sigma) = d(\sigma, \emptyset) - 1.$$

Notice that $d(\sigma, \emptyset) = k$, if σ has a decomposition $\sigma = \sigma_1 \sigma_2 \cdots \sigma_k$ such that for every i , the word σ_i is maximal suffix of $\sigma_1 \sigma_2 \cdots \sigma_i$ satisfying $\sigma_i \in E$. Here E is defined in (3.2). Therefore,

$$L(\sigma) \leq |\sigma| - 1, \quad (5.3)$$

and

$$L(\sigma) \geq L(\sigma') \text{ if } \sigma' \text{ is a suffix of } \sigma. \quad (5.4)$$

In particular, for $\sigma^* = \cdots 4253142531 = i_{-|\sigma^*|} \cdots i_{-1}$, where

$$i_{-p} = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{5}, \\ 3 & \text{if } p \equiv 2 \pmod{5}, \\ 5 & \text{if } p \equiv 3 \pmod{5}, \\ 2 & \text{if } p \equiv 4 \pmod{5}, \\ 4 & \text{if } p \equiv 0 \pmod{5}. \end{cases}$$

In the above expression, every pair i_{-k} and $i_{-(k-1)}$ are not dual, hence we have

$$L(\sigma^*) = |\sigma^*| - 1. \quad (5.5)$$

As in [34], if $i \neq j$, then

$$L(\sigma) + L(\tau) \leq d(i\sigma, j\tau) \leq L(\sigma) + L(\tau) + 2. \quad (5.6)$$

and the self-similarity

$$d(i\sigma, i\tau) = d(\sigma, \tau). \quad (5.7)$$

Proof of Theorem 4. It suffice to show that

$$N_B \sim 5^{t - \frac{\ell_B}{2}}.$$

(1) Firstly we give the lower bound for N_B . Let $\sigma^* = \cdots 4253142531$ with the length $|\sigma^*| = \left\lfloor \frac{\ell_B + 1}{2} \right\rfloor + 1$. By (5.5), $L(\sigma^*) = \left\lfloor \frac{\ell_B + 1}{2} \right\rfloor$. For any two different words τ, τ' of length $t - \left\lfloor \frac{\ell_B + 1}{2} \right\rfloor$, we claim that

$$d(\tau\sigma^*, \tau'\sigma^*) \geq \ell_B. \quad (5.8)$$

In fact, suppose the common prefix of τ, τ' is u , and $\tau = uiv, \tau' = u jv'$ with $i \neq j \in \{1, 2, 3, 4, 5\}$. It follows from (5.7), (5.6) and (5.4) that

$$d(\tau\sigma^*, \tau'\sigma^*) = d(iv\sigma^*, jv'\sigma^*) \geq 2L(\sigma^*) \geq \ell_B.$$

Recall that $\#\{\tau\sigma^* : |\tau| = t - \left\lfloor \frac{\ell_B + 1}{2} \right\rfloor - 1\} = 5^{t - \left\lfloor \frac{\ell_B + 1}{2} \right\rfloor - 1}$, using (5.8), we get the lower bound

$$N_B \geq 5^{t - \left\lfloor \frac{\ell_B + 1}{2} \right\rfloor - 1}.$$

(2) Now we estimate the upper bound for N_B . Given $\ell_B \geq 3$ and a word τ of length $\left(t - k \left\lfloor \frac{\ell_B - 1}{2} \right\rfloor\right)$ with some integer $k \geq 1$, consider the box

$$B_\tau = \{\tau u : |u| = \left\lfloor \frac{\ell_B - 1}{2} \right\rfloor\}.$$

In fact, by (5.7), (5.6) and (5.3), for any two different words $\tau\sigma, \tau\sigma' \in B_\tau$, we have

$$d(\tau\sigma, \tau\sigma') = d(\sigma, \sigma') \leq L(\sigma) + L(\sigma') + 2 \leq 2(|\sigma| - 1) + 2 = 2 \left\lfloor \frac{\ell_B - 1}{2} \right\rfloor < \ell_B,$$

which implies

$$\text{diam}(B_\tau) < \ell_B.$$

Let $\Gamma = \{\tau = \emptyset \text{ or } |\tau| = t - k \left\lfloor \frac{\ell_B - 1}{2} \right\rfloor \text{ for } k \in \mathbb{N}^+\}$. It is easy to know that $\{B_\tau\}_{\tau \in \Gamma}$ is a covering of G_t . Hence we can give the upper bound

$$\begin{aligned} N_B &\leq \#\Gamma \leq 1 + 5^{t - \left\lfloor \frac{\ell_B - 1}{2} \right\rfloor} (1 + 5^{-\left\lfloor \frac{\ell_B - 1}{2} \right\rfloor} + 5^{-2\left\lfloor \frac{\ell_B - 1}{2} \right\rfloor} + \dots) \\ &\leq 1 + \frac{5^{t - \left\lfloor \frac{\ell_B - 1}{2} \right\rfloor}}{1 - 5^{-\left\lfloor \frac{\ell_B - 1}{2} \right\rfloor}} \\ &\leq 1 + \frac{5}{4} \cdot 5^{t - \left\lfloor \frac{\ell_B - 1}{2} \right\rfloor}. \end{aligned}$$

Therefore, $5^{t - \left\lfloor \frac{\ell_B - 1}{2} \right\rfloor} \leq N_B \leq 1 + \frac{5}{4} \cdot 5^{t - \left\lfloor \frac{\ell_B - 1}{2} \right\rfloor}$, which implies $N_B \sim 5^{t - \frac{\ell_B}{2}}$. $\square$

We think the exponential gauge function $H(r) = e^{cr}$ also appears frequently in complex networks. Because the real networks have more complexity than Euclidean spaces of finite dimensions, i.e., we can not embed many real networks into Euclidean spaces, the complex network of real world is not restricted by 3-dimensional space and 1-dimensional time. It may not be appropriate to describe them by the classical gauge function r^d which is used to investigate the compact fractals in Euclidean spaces.

6. Conclusions

We use the Sierpinski Pentagon to construct evolving networks G_t , ($t \geq 1$), whose nodes is the solid regular pentagons in the construction of the Sierpinski Pentagon up to the stage t and any two nodes are neighbors if and only if the intersection of corresponding pentagons is non-empty and non-singleton.

In order to show that our network is scale-free (power-law degree distribution) and have the small-world effect (small average path length and high clustering coefficient), we use the encoding method, which is commonly used in the fractal geometry, to give a criteria of neighbors and then to get the accurate degree formula for each node in G_t by Proposition 2. Firstly, we give the degree formula to show the scale-free property of prescribed network, in fact, the power law exponent of cumulative distribution is $-\frac{\log 5}{\log 2}$ by Theorem 1. Secondly we prove the small-world effect of G_t by Theorems 2 and 3. At last, by Theorem 4 we show G_t is not fractal scaling in the sense of references [9] and [25]. **It should be noted that all the methods in this article are also applicable to other self-similar networks.** It is interesting that the network G_t generated from a fractal set Sierpinski Pentagon is not fractal scaling. We expect our work can provide a new view into the scale-free and small-world networks with non fractal scaling.

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