Analytical Solutions for Estimating Tsunami Propagation Speeds

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Abstract

Recent studies suggest that the tsunami speed can be slowed down by around 1% due to Earth elasticity, water compressibility and density stratification. Analytical solutions of wave dispersion relationship, accounting for such effects, were found in previous studies. In this paper, we investigate the additional effects of water viscosity, ocean stratification due to temperature/salinity and numerical dispersion. Theoretical solutions are derived and checked with known solutions. All the formulas are then simplified for long tsunami waves so that the propagation speed can be calculated explicitly. The simplified solutions are evaluated using realistic geophysical parameters. For a typical tsunami wavelength of ~ 200 km, the viscous effect is found to be negligible; ocean stratification due to temperature/salinity causes significant speed reduction because of the high density change rate, which has been ignored before. We also evaluate the numerical dispersion of tsunami simulations, which is shown to be potentially comparable to physical dispersion.

Keywords: Tsunami speed, Earth elasticity, Water viscosity, Water compressibility, Ocean stratification, Numerical dispersion, Analytical solution

1. Introduction

From recent large tsunami events it has been observed that there exists a systematic delay of arrival time compared to the prediction of shallow water wave equations [e.g., Rabinovich et al., 2011, Wei et al., 2008, Hébert et al., 2009, Saito et al., 2010, Kato et al., 2011, Fujii and Satake, 2013, Kimura et al., 2013, Watada et al., 2014]. Watada et al. [2014] summarized the observations and showed that the discrepancy between observation and prediction, which has the order of around 1%, can be explained by the effects of Earth elasticity, water compressibility and geopotential variations. Previous studies have led to the theoretical solutions accounting for one or more of those effects. Mallard et al. [1977] studied the problem of water waves propagating over an elastic bed, using

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a model that consists of a single layer of incompressible potential fluid over half-space homogeneous elastic earth. The dispersion relation of the small amplitude water waves in such a model was given. Dawson [1978] reviewed the problem but took into account the solid inertia and suggested its importance for cases that include thick soft sediment. Dalrymple and Liu [1978] studied the problem of viscous water waves propagating over a mud bottom using a perturbation method, and derived the dispersion relation and the decay rate of wave amplitude. Ward [1980] presented the theory of tsunami generation and propagation on a spherically symmetric, self-gravitating, elastic Earth in terms of normal modes. Okal [1982] studied the asymptotic behavior of the gravity modes of an incompressible spherical oceanic layer surrounding a rigid Earth, as its radius goes to infinity. Okal showed that the flat-layered ocean tsunami solution and its dispersion is an asymptotic limit of the normal modes of a spherical oceanic shell and that only one branch of tsunami modes exists. Comer [1984] considered water waves in an incompressible fluid within a uniform gravitational field overlying an elastic earth in which the gravitational forces are ignored. Comer concluded that elastic forces are far more important than inertia and they also derived the dispersion relation for the water waves when the Poisson's ration of the Earth is 0.25. Panza et al. [2000] derived solutions for a model with multi-layered compressible inviscid fluid on top of a multi-layered solid half-space Earth, with the compressibility of the fluid treated as elastic solid. Tsai et al. [2013] derived theoretical tsunami propagation speeds accounting for Earth elasticity and water compressibility and stratification based on a method of conservation of potential and kinetic energy. Watada [2013] derived the theoretical dispersion relationship accounting for water compressibility and ocean stratification due to compression of gravity. Allgeyer and Cummins [2014] conducted numerical simulations to investigate the effects of elastic Earth. Watada et al. [2014] investigated the data from the 2010 Chilean and 2011 Tohoku tsunamis and showed that the systematic arrival time delay could be explained by the effects of Earth elasticity, water compressibility and geopotential variations.

In this study, we use a method which is slightly different from Watada [2013] to investigate additional secondary effects on tsunami propagation speeds, i.e., the water viscosity and ocean stratification due to temperature/salinity. For completeness the known second order effects, such as the Earth elasticity and water compressibility will be included in the solutions. In addition, we simplify all the theoretical dispersion relationships under the assumption of long waves so that the tsunami propagation speeds can be calculated explicitly. The simplifications are verified for typical tsunami wavelength of $50 \sim 1000$ km (or wave period $260 \sim 5000$ s) using realistic geophysical parameters. The relative importance of the second order processes in affecting the tsunami propagation speeds are assessed. Finally, the numerical dispersions of typical tsunami simulating algorithms are evaluated and compared with the above physical effects.

2. Governing Equations and Boundary Conditions

We consider 2D problems in x and z , with x in the horizontal direction to the right and z pointing upwards (Fig. 1).

Figure 1: **The 2D model:** a tsunami wave $\eta(x, t)$ is propagating over an elastic bed. The z axis origins at the seabed and points upwards; the x axis is in the horizontal direction. The ocean is assumed to be stratified with two layers, with the background density and water depth denoted as (ρ_{01}, h_1) and (ρ_{02}, h_2) , respectively. The total water depth $h = h_1 + h_2$. The fluid is assumed to be viscous and slightly compressible, with constant bulk modulus B and viscosity μ_f ; Earth is elastic with Lamè constants λ_s and μ_s . The constant gravitational acceleration is g .

2.1. Fluid

The linearized governing equations for compressible fluids are derived from the complete Navier-Stokes equations by assuming slight compressibility. They

71 are written as:

$$\left\{ \begin{array}{l} \frac{\partial p_1}{\partial t} + B \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) - \rho_0 g w = 0 \\ \rho_0 \frac{\partial u}{\partial t} = -\frac{\partial p_1}{\partial x} + 2\mu \left[\frac{\partial^2 u}{\partial x^2} - \frac{1}{3} \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) \right] + \frac{\partial \sigma_{xz}}{\partial z} \\ \rho_0 \frac{\partial^2 w}{\partial t^2} = \left(\frac{\rho_0^2 g^2}{B} + g \frac{d\rho_0}{dz} \right) w - \frac{\rho_0 g}{B} \frac{\partial p_1}{\partial t} - \frac{\partial^2 p_1}{\partial t \partial z} + \frac{\partial^2 \sigma_{xz}}{\partial t \partial x} + \frac{\partial^2 \sigma_{zz}}{\partial t \partial z} \\ \sigma_{xz} = \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \\ \sigma_{zz} = 2\mu \left[\frac{\partial w}{\partial z} - \frac{1}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) \right]. \end{array} \right. \quad (1)$$

72 where $u(x, z, t)$ and $w(x, z, t)$ are horizontal and vertical velocities, σ_{xz} and σ_{zz}
73 are components of the stress tensor, μ_f is the fluid viscosity and B is the bulk
74 modulus of the fluid. The subscript f of μ_f is omitted for simplicity. The
75 subscript is used to distinguish between the viscosity in the fluid μ_f and the
76 shear modulus in the solid μ_s only when necessary. The total pressure p is
77 defined as the sum of static pressure p_0 and dynamic pressure p_1 : $p(x, z, t) =$
78 $p_0(z) + p_1(x, z, t)$; and the total density ρ is defined as the sum of the background
79 density ρ_0 and density perturbation ρ_1 due to pressure change: $\rho(x, z, t) =$
80 $\rho_0(z) + \rho_1(x, z, t)$. It is required that $d\rho_0(z)/dz = -\rho_0(z)g$.

81 Note that it has been assumed that the bulk modulus is constant and that
82 the fluid is only slightly compressible, i.e., $\rho_1(x, z, t) \ll \rho_0(z)$ and $B = \rho dp/d\rho$,
83 so we have $d\rho/dt = (\rho/B) dp/dt$, which leads to, by keeping only the leading
84 order terms,

$$\frac{\partial \rho_1}{\partial t} = \frac{\rho_0}{B} \frac{\partial p_1}{\partial t} - \frac{\rho_0^2 g}{B} w - \frac{d\rho_0}{dz} w. \quad (2)$$

85 The total stress tensor is expressed as

$$\begin{aligned} \mathbf{s} &= -(p_0 + p_1) \mathbf{I} + \begin{bmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{bmatrix} \\ &= -(p_0 + p_1) \mathbf{I} + 2\mu \begin{bmatrix} \frac{\partial u}{\partial x} - \frac{1}{3} \left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial z} \right) & \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \\ \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) & \frac{\partial w}{\partial z} - \frac{1}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) \end{bmatrix}, \end{aligned} \quad (3)$$

86 where $\mathbf{I}$ is the identity tensor. The equations are similar to those proposed by
87 Watada [2009], but the shear stress components are included in order to evaluate
88 the viscous effects.

89 Adopting a plane wave solution in x direction, we have

$$\begin{cases} \eta(x, t) = A e^{i(kx - \omega t)} \\ u(x, z, t) = \hat{u}(z) e^{i(kx - \omega t)} \\ w(x, z, t) = \hat{w}(z) e^{i(kx - \omega t)} \\ p_1(x, z, t) = \hat{p}_1(z) e^{i(kx - \omega t)} \\ \sigma_{xz}(x, z, t) = \hat{\sigma}_{xz}(z) e^{i(kx - \omega t)} \\ \sigma_{zz}(x, z, t) = \hat{\sigma}_{zz}(z) e^{i(kx - \omega t)}, \end{cases} \quad (4)$$

90 where A , k and ω are the constant wave amplitude, wave number and angular
 91 frequency respectively. Substituting the assumed solution (4) into the governing
 92 equations (1), we obtain a set of first order differential equations:

$$\frac{d}{dz} \begin{bmatrix} \hat{u}(z) \\ \hat{w}(z) \\ \hat{p}_1(z) \\ \hat{\sigma}_{xz}(z) \end{bmatrix} = \begin{bmatrix} 0 & -ik & 0 & \frac{1}{\mu} \\ -ik & \frac{\rho_0 g}{B} & i \frac{\omega}{B} & 0 \\ m_{31} & m_{32} & m_{33} & m_{34} \\ 2k^2 \mu - i \rho_0 \omega & i \frac{2k \mu \rho_0 g}{3B} & ik - \frac{2k \mu \omega}{3B} & 0 \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{w} \\ \hat{p}_1 \\ \hat{\sigma}_{xz} \end{bmatrix} \quad (5)$$

93 and

$$\hat{\sigma}_{zz}(z) = -i 2k \mu \hat{u}(z) + \frac{4\mu \rho_0 g}{3B} \hat{w}(z) + i \frac{4\mu \omega}{3B} \hat{p}_1(z),$$

94 where the third row of the coefficient matrix is

$$\begin{cases} m_{31} = -i \frac{4k \mu \rho_0 g}{3B - i 4\mu \omega} \\ m_{32} = \frac{1}{1 - i \frac{4\mu \omega}{3B}} \left[i \omega \rho_0 - 2k^2 \mu + \frac{4\mu g}{3B} \frac{d\rho_0}{dz} + \frac{4\mu \rho_0^2 g^2}{3B^2} + i \frac{\rho_0^2 g^2}{\omega B} + i \frac{g}{\omega} \frac{d\rho_0}{dz} \right] \\ m_{33} = \frac{1}{1 - i \frac{4\mu \omega}{3B}} \left[i \frac{4\omega \mu \rho_0 g}{3B^2} - \frac{\rho_0 g}{B} \right] \\ m_{34} = -\frac{ik}{1 - i \frac{4\mu \omega}{3B}}. \end{cases} \quad (6)$$

95 The analytical solutions of Eqns. (5) can be found generally by solving the
 96 eigen-system of the coefficient matrix. However, a full 4×4 matrix does not
 97 have an explicit solution of the eigen-system, so in later sections we decouple the
 98 effects of viscosity, compressibility and background density change and obtain
 99 analytical solutions accounting for those effects separately.

100 *2.2. Solid*

101 In the solid, the governing equations are

$$\rho \ddot{\mathbf{U}} = \nabla \cdot \boldsymbol{\sigma}, \quad (7)$$

102 where ρ is the solid density, $\mathbf{U}(x, z, t)$ is the displacement vector and $\boldsymbol{\sigma}(x, z, t)$
 103 is the stress tensor. The gravitational field is ignored. Note that the density
 104 will be denoted as ρ_f and ρ_s for fluid and solid density, respectively, when
 105 necessary. Assuming homogeneous elastic properties in the solid region and
 106 adopting plane wave solutions similar to (4), the governing equations in the
 107 solid can be converted to a set of ODEs, which are then solved by solving the
 108 eigen-system of coefficient matrix. The solutions in the solid are found to be

$$\begin{cases} U(x, z, t) = (C_a v_{11} e^{\lambda_1 z} + C_c v_{31} e^{\lambda_3 z}) e^{i(kx - \omega t)} \\ W(x, z, t) = (C_a v_{12} e^{\lambda_1 z} + C_c v_{32} e^{\lambda_3 z}) e^{i(kx - \omega t)} \\ \sigma_{xz}(x, z, t) = (C_a v_{13} e^{\lambda_1 z} + C_c v_{33} e^{\lambda_3 z}) e^{i(kx - \omega t)} \\ \sigma_{zz}(x, z, t) = (C_a v_{14} e^{\lambda_1 z} + C_c v_{34} e^{\lambda_3 z}) e^{i(kx - \omega t)}, \end{cases} \quad (8)$$

109 where $U(x, z, t)$ and $W(x, z, t)$ are displacements in x and z direction, C_a, C_c are
 110 constants to be determined from boundary conditions, and $\lambda_{1,2,3,4}$ and $\mathbf{v}_{1,2,3,4}$
 111 are the eigen solution of the coefficient matrix, given by

$$\lambda_{1,2} = \pm k \sqrt{1 - \frac{c^2}{\beta^2}}, \quad \lambda_{3,4} = \pm k \sqrt{1 - \frac{c^2}{\alpha^2}}, \quad (9)$$

112 and

$$\begin{cases} \mathbf{v}_{1,2} = \left(i \sqrt{1 - \frac{c^2}{\beta^2}}, \pm 1, \pm i k \mu \left(2 - \frac{c^2}{\beta^2} \right), 2k\mu \sqrt{1 - \frac{c^2}{\beta^2}} \right) \\ \mathbf{v}_{3,4} = \left(i, \pm \sqrt{1 - \frac{c^2}{\alpha^2}}, \pm 2i k \mu \sqrt{1 - \frac{c^2}{\alpha^2}}, k\mu \left(2 - \frac{c^2}{\beta^2} \right) \right). \end{cases} \quad (10)$$

113 Here c is the tsunami wave phase speed defined as $c = \omega/k$, and α and β are P
 114 and S wave speeds in the solid:

$$c = \frac{\omega}{k}, \quad \alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad \beta = \sqrt{\frac{\mu}{\rho}}. \quad (11)$$

115 λ_s and μ_s , with the subscripts omitted for simplicity, are the Lamè constants.
 116 Note that here we have applied the boundary condition at $z = -\infty$, which
 117 states $U \rightarrow 0$ and $W \rightarrow 0$, and therefore the terms associated with the negative
 118 eigenvalues λ_2 and λ_4 vanish in the solutions (8).

119 2.3. Boundary Conditions

120 The boundary conditions on the free surface are:

$$z = h + \eta, \quad \text{fluid 1 : } \frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} = w, \quad \mathbf{n} \cdot \boldsymbol{\sigma} = 0, \quad (12)$$

121 where $\mathbf{n}$ is the normal direction of the free surface: $\mathbf{n} = (\partial \eta / \partial x, -1)$. They are
122 linearized to be [e.g., Mei, 1989, p10]

$$z = h, \quad \text{fluid 1 : } \begin{cases} \frac{\partial \eta}{\partial t} = w \\ \sigma_{xz} = 0 \\ \rho_0 g \eta - p_1 + \sigma_{zz} = 0. \end{cases} \quad (13)$$

123 Similarly, the linearized boundary conditions at the fluid-fluid interface are

$$z = h_2 : \begin{cases} u|_{\text{fluid 1}} = u|_{\text{fluid 2}}, & w|_{\text{fluid 1}} = w|_{\text{fluid 2}} \\ \sigma_{xz}|_{\text{fluid 1}} = \sigma_{xz}|_{\text{fluid 2}} \\ -p_1 + \sigma_{zz}|_{\text{fluid 1}} = -p_1 + \sigma_{zz}|_{\text{fluid 2}}. \end{cases} \quad (14)$$

124 The linearized boundary conditions at the fluid-solid interface are

$$z = 0 : \begin{cases} u|_{\text{fluid 2}} = \frac{\partial U}{\partial t}|_{\text{solid}}, & w|_{\text{fluid 2}} = \frac{\partial W}{\partial t}|_{\text{solid}} \\ \sigma_{xz}|_{\text{fluid 2}} = \sigma_{xz}|_{\text{solid}} \\ -p_1 + \sigma_{zz}|_{\text{fluid 2}} = \sigma_{zz}|_{\text{solid}}. \end{cases} \quad (15)$$

125 Note that the static pressure p_0 in the fluid is ignored in the balance of stress
126 due to the cancellation of the equivalent pressure in the solid. The boundary
127 conditions at $z = -\infty$ are

$$z = -\infty, \quad \text{solid : } U = 0, \quad W = 0, \quad (16)$$

128 which have already been applied to obtain the solution (8) in the solid.

129 3. Dispersion Relationships

130 In this section, we first derive the analytical solutions in the fluid that sepa-
131 rately account for the effects of Earth elasticity, water viscosity, compressibility
132 and ocean stratification due to compression and temperature/salinity. The so-
133 lutions are generally found by solving the eigen-system of coefficient matrix of
134 Eqns. (5) in the fluid. They are then applied to the boundary conditions along

135 with the analytical solutions in the solid. For each case, the number of bound-
 136 ary conditions is one more than the number of constants in the solutions, and
 137 the extra boundary condition leads to the dispersion relationship. Thereafter,
 138 the theoretical dispersion relationships are approximated to be an explicit form
 139 under the assumption of long waves. Note that the solutions for Earth elas-
 140 ticity, water compressibility and ocean stratification due to compression have
 141 been given by previous studies [Comer, 1984, Tsai et al., 2013, Watada, 2013].
 142 Here we re-derive them for completeness and show the simplifications of the full
 143 dispersion relationships and associated assumptions.

144 3.1. Earth elasticity

145 By neglecting all other effects except for Earth elasticity, the fluid is inviscous
 146 ($\mu = 0$), incompressible ($B = \infty$) and the density is constant over the entire
 147 region ($\rho_{10} = \rho_{20} = \rho_0 = \text{const}$). The ODEs in the fluid (5) are simplified to

$$\frac{d}{dz} \begin{bmatrix} \hat{w}(z) \\ \hat{p}_1(z) \end{bmatrix} = \begin{bmatrix} 0 & -i \frac{k^2}{\rho_0 \omega} \\ i \rho_0 \omega & 0 \end{bmatrix} \begin{bmatrix} \hat{w} \\ \hat{p}_1 \end{bmatrix}, \quad (17)$$

148 and

$$\begin{cases} \hat{u}(z) = \frac{k}{\rho_0 \omega} \hat{p}_1(z) \\ \hat{\sigma}_{xz} = 0 \\ \hat{\sigma}_{zz} = 0. \end{cases} \quad (18)$$

149 By solving the eigen-system of the coefficient matrix, the solutions can be readily
 150 obtained. They are written in hyperbolic functions as conventionally used in
 151 hydrodynamics:

$$\begin{cases} u(x, z, t) = i \left(C_1 \cosh kz + C_2 \sinh kz \right) e^{i(kx - \omega t)} \\ w(x, z, t) = \left(C_1 \sinh kz + C_2 \cosh kz \right) e^{i(kx - \omega t)} \\ p_1(x, z, t) = i \frac{\rho_0 \omega}{k} \left(C_1 \cosh kz + C_2 \sinh kz \right) e^{i(kx - \omega t)} \\ \sigma_{xz}(x, z, t) = 0 \\ \sigma_{zz}(x, z, t) = 0, \end{cases} \quad (19)$$

152 where C_1 and C_2 are constants to be determined from boundary conditions.
 153 The static pressure in the fluid is simply $p_0(z) = -\rho_0 g(z - h)$, derived from
 154 $dp_0/dz = -\rho_0 g$.

155 By neglecting water stratification, the boundary conditions at the fluid-fluid
 156 interface (14) are ignored; by neglecting water viscosity, the horizontal veloc-
 157 ity at the fluid-solid interface is not necessarily continuous. Substituting the
 158 solution in the fluid (19) and the solution in the solid (8) into the bound-
 159 ary conditions at the free surface (13) and fluid-solid interface (15), given that
 160 $\eta(x, t) = A e^{i(kx - \omega t)}$, we obtain

$$\begin{cases} \sinh(kh) C_1 + \cosh(kh) C_2 = -i\omega A \\ \omega \cosh(kh) C_1 + \omega \sinh(kh) C_2 = -i k g A \\ C_2 + i\omega v_{12} C_a + i\omega v_{32} C_c = 0 \\ v_{13} C_a + v_{33} C_c = 0 \\ i \rho_0 \omega C_1 + k v_{14} C_a + k v_{34} C_c = 0. \end{cases} \quad (20)$$

161 The first two equations are inhomogeneous and can be converted into a homo-
 162 geneous one, resulting in a set of homogeneous equations for C_1 , C_2 , C_a and
 163 C_b . Thereafter, to have non-trivial solutions, the determinant of the coefficient
 164 matrix has to be zero, which leads to the dispersion relationship. After some
 165 manipulation, we obtain

$$c^2 = c_0^2 - F \left(c_0^2 \coth(kh) - c^2 \tanh(kh) \right), \quad (21)$$

166 where c_0 is the standard dispersive wave speed and

$$c_0^2 = \frac{g}{k} \tanh(kh), \quad F = \frac{\rho_f}{\rho_s} \frac{\sqrt{1 - \xi_\alpha} \xi_\beta^2}{4\sqrt{1 - \xi_\alpha} \sqrt{1 - \xi_\beta} - (2 - \xi_\beta)^2}, \quad (22)$$

$$\xi_\alpha = c^2/\alpha^2, \quad \xi_\beta = c^2/\beta^2.$$

167 Again, α and β are the P and S wave speeds in the solid respectively, given by
 168 (11).

169 Eqn. (21) can be solved numerically. Alternatively, it can be simplified under
 170 the assumption that the tsunami wave speed c is much smaller than the seismic
 171 wave speeds in the solid, i.e., $c^2 \ll \alpha^2$ and $c^2 \ll \beta^2$, or, $\xi_\alpha \ll 1$ and $\xi_\beta \ll 1$.
 172 Expanding F near $\xi_\alpha = 0$ and $\xi_\beta = 0$ and only keeping the leading order terms,
 173 we find

$$F \approx \frac{\rho_f}{2\rho_s \left(1 - \frac{\beta^2}{\alpha^2}\right)} \frac{c^2}{\beta^2}, \quad (23)$$

174 where $0 \leq \xi_\alpha/\xi_\beta \leq 0.5$. Thus, F is a small variable with order of the squared
 175 ratio of tsunami wave speed and P or S wave speed, and the correction of the
 176 tsunami wave speed due to Earth elasticity has the same order. Substituting
 177 the simplified F into the full dispersion relationship (21), we obtain a quadratic
 178 equation for c^2 . It is then solved analytically and simplified given that $c_0^2 \ll \beta^2$,
 179 leading to

$$c^2 \approx \left[\frac{1}{1 + \xi_e \coth(kh)} + \frac{\xi_e \tanh(kh)}{(1 + \xi_e \coth(kh))^3} \right] c_0^2, \quad (24)$$

180 where

$$\xi_e = \frac{\rho_f}{2\rho_s \left(1 - \frac{\beta^2}{\alpha^2}\right)} \frac{c_0^2}{\beta^2} \ll 1. \quad (25)$$

181 If we further assume that $\xi_e \coth(kh) \ll 1$, i.e.,

$$\xi_e = \frac{\rho_f}{2\rho_s \left(1 - \frac{\beta^2}{\alpha^2}\right)} \frac{c_0^2}{\beta^2} \ll \tanh(kh), \quad (26)$$

182 the dispersion relationship can be simplified to

$$c \approx \left[1 - \frac{\coth(kh) - \tanh(kh)}{4 \left(1 - \frac{\beta^2}{\alpha^2}\right)} \frac{\rho_f}{\rho_s} \frac{c_0^2}{\beta^2} \right] c_0. \quad (27)$$

183 Again, α and β are the P and S wave speeds in the solid, given by (11). The
 184 solution is the same as Comer [1984]'s analytical result obtained by assuming
 185 $\lambda = \mu$. Note that solution (27) requires that the squared ratio of tsunami and
 186 seismic speeds c_0^2/β^2 should be much smaller than the ratio of water depth and
 187 wavelength (kh) (26), which is stronger than the assumption (25) required by
 188 solution (24). Later we will verify that, for typical tsunami wavelength around
 189 $50 \sim 1000$ km (or wave period around $260 \sim 5000$ s), solution (27) is a good
 190 approximation of the full solution (21).

191 By taking the assumption of long waves, i.e., $kh \ll 1$, $\tanh(kh) \approx kh$ and
 192 $\coth(kh) \approx 1/kh$, we can separate the effects of Earth elasticity and wave dis-
 193 persion, and solution (27) is revised to

$$c \approx \left[1 - \frac{1}{4 \left(1 - \frac{\beta^2}{\alpha^2}\right)} \frac{\rho_f}{\rho_s} \frac{gh}{\beta^2} \right] \sqrt{gh}. \quad (28)$$

194 This result shows that the reduction of tsunami propagation speeds increases
 195 approximately linearly with the wavelength, and the order of the correction is
 196 around $\frac{c_0^2}{\beta^2} \frac{1}{kh}$.

197 3.2. Water viscosity

198 To evaluate the effect of water viscosity, we neglect water compressibility
 199 and density stratification. By taking $B = \infty$ and $\rho_0 = \text{const}$, the ODEs in the
 200 water (5) are simplified to

$$\frac{d}{dz} \begin{bmatrix} \hat{u}(z) \\ \hat{w}(z) \\ \hat{p}_1(z) \\ \hat{\sigma}_{xz}(z) \end{bmatrix} = \begin{bmatrix} 0 & -ik & 0 & \frac{1}{\mu} \\ -ik & 0 & 0 & 0 \\ 0 & -(2k^2\mu - i\rho_0\omega) & 0 & -ik \\ 2k^2\mu - i\rho_0\omega & 0 & ik & 0 \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{w} \\ \hat{p}_1 \\ \hat{\sigma}_{xz} \end{bmatrix},$$

201 and

$$\hat{\sigma}_{zz}(z) = -i2k\mu \hat{u}(z).$$

202 Again, the solutions can be obtained by solving the eigen-system of the coeffi-
203 cient matrix, and they are

$$\left\{ \begin{aligned} u(x, z, t) &= i \left(C_1 \cosh(kz) + C_2 \sinh(kz) + C_3 \frac{\lambda}{k} e^{\lambda(z-h)} - C_4 \frac{\lambda}{k} e^{-\lambda z} \right) e^{i(kx-\omega t)} \\ w(x, z, t) &= \left(C_1 \sinh(kz) + C_2 \cosh(kz) + C_3 e^{\lambda(z-h)} + C_4 e^{-\lambda z} \right) e^{i(kx-\omega t)} \\ p_1(x, z, t) &= i \frac{\rho \omega}{k} \left(C_1 \cosh(kz) + C_2 \sinh(kz) \right) e^{i(kx-\omega t)} \\ \sigma_{xz}(x, z, t) &= i 2k\mu \left(C_1 \sinh(kz) + C_2 \cosh(kz) + C_3 \frac{k^2 + \lambda^2}{2k^2} e^{\lambda(z-h)} + C_4 \frac{k^2 + \lambda^2}{2k^2} e^{-\lambda z} \right) \\ \sigma_{zz}(x, z, t) &= 2k\mu \left(C_1 \cosh(kz) + C_2 \sinh(kz) + C_3 \frac{\lambda}{k} e^{\lambda(z-h)} - C_4 \frac{\lambda}{k} e^{-\lambda z} \right) e^{i(kx-\omega t)}, \end{aligned} \right. \quad (29)$$

204 where

$$\lambda^2 = k^2 - i \frac{\rho_0 \omega}{\mu}, \quad (30)$$

205 and C_1 , C_2 , C_3 and C_4 are constants to be determined from boundary condi-
206 tions. The static pressure is simply $p_0(z) = -\rho_0 g(z - h)$. Note that λ here is
207 a complex wavenumber. The solution is essentially the same as that obtained
208 by Dalrymple and Liu [1978], who used a perturbation method to investigate a
209 two-layer model of viscous fluids.

210 By neglecting Earth elasticity, water compressibility and stratification, the
211 boundary conditions at the fluid-fluid interface (14) are ignored; the boundary
212 conditions at the fluid-solid interface (15) are simplified to that the horizon-
213 tal and vertical fluid velocities vanish. Substituting the solution (29) into the
214 boundary conditions at the free surface (13) and fluid-solid interface (15), we
215 obtain

$$\left\{ \begin{aligned} \sinh(kh)C_1 + \cosh(kh)C_2 + C_3 + e^{-\lambda h}C_4 &= -i\omega A \\ 2k^2 \sinh(kh)C_1 + 2k^2 \cosh(kh)C_2 + (\lambda^2 + k^2)C_3 + (\lambda^2 + k^2)e^{-\lambda h}C_4 &= 0 \\ (2k^2\mu - i\rho_0\omega) \cosh(kh)C_1 + (2k^2\mu - i\rho_0\omega) \sinh(kh)C_2 + 2k\lambda\mu C_3 - 2k\lambda\mu e^{-\lambda h}C_4 &= -k\rho_0 g A \\ kC_1 + \lambda e^{-\lambda h}C_3 - \lambda C_4 &= 0 \\ C_2 + e^{-\lambda h}C_3 + C_4 &= 0. \end{aligned} \right.$$

216 Eliminating the constants on the R.H.S., the equations can be converted into
217 homogeneous ones. Seeking non-trivial solutions, the determinant of the coeffi-
218 cient matrix has to be zero, which leads to

$$\begin{aligned}
& \left[\rho_0 g k^2 (k^2 - \lambda^2) \right] \left(1 - e^{-2\lambda h} \right) \cosh(kh) - \\
& \left[\rho_0 \lambda \omega^2 (k^2 + \lambda^2) + 2ik^2 \lambda \mu \omega (3k^2 + \lambda^2) \right] \left(1 + e^{-2\lambda h} \right) \cosh(kh) + \\
& \left[\rho_0 k \omega^2 (k^2 + \lambda^2) + 2ik^3 \mu \omega (k^2 + 3\lambda^2) \right] \left(1 - e^{-2\lambda h} \right) \sinh(kh) - \\
& \left[\rho_0 g k \lambda (k^2 - \lambda^2) \right] \left(1 + e^{-2\lambda h} \right) \sinh(kh) + \\
& 4k^2 \lambda \omega \left(3ik^2 \mu + i\lambda^2 \mu + \rho_0 \omega \right) e^{-\lambda h} = 0.
\end{aligned} \tag{31}$$

219 We define

$$\xi_v = k \sqrt{\frac{\mu}{\rho_0}} (gk)^{-\frac{1}{4}} \ll 1, \tag{32}$$

220 and further simplify the results by expanding the dispersion relationship near
221 $\xi_v = 0$. First,

$$\begin{aligned}
\lambda^2 &= k^2 - i \frac{\rho_0 \omega}{\mu} = k^2 - i \frac{\omega}{\sqrt{gk} \xi_v^2} k^2, \\
\lambda &= \sqrt{k^2 - i \frac{\rho_0 \omega}{\mu}} = (1 - i) \frac{k \sqrt{\omega}}{\sqrt{2} (gk)^{\frac{1}{4}} \xi_v} \frac{1}{\xi_v} + (1 + i) \frac{k (gk)^{\frac{1}{4}}}{\sqrt{8\omega}} \xi_v + O(\xi_v^3).
\end{aligned} \tag{33}$$

222 So we have $e^{-\lambda h} = 0$ and $e^{-2\lambda h} = 0$ if keeping only the leading order small
223 terms. Substituting λ into the dispersion relationship and only keeping the
224 leading order terms, we obtain

$$\omega^2 = gk \tanh(kh) - (1 + i) \frac{\sqrt{2}}{\sinh(2kh) \sqrt[4]{\tanh(kh)}} gk \tanh(kh) \xi_v + O(\xi_v^2), \tag{34}$$

225 or written in phase speed:

$$c^2 \approx \left[1 - (1 + i) \frac{\sqrt{2}}{\sinh(2kh) \sqrt[4]{\tanh(kh)}} k \sqrt{\frac{\mu}{\rho_0}} (gk)^{-\frac{1}{4}} \right] c_0^2. \tag{35}$$

226 The dispersion relationship (35) is complex, i.e., the imaginary part is not trivial
227 compared to the real part, indicating that water viscosity not only affects the
228 wave speed but also damps the wave amplitude. The result is consistent with
229 the simplified result obtained by Mei and Liu [1973] who used a perturbation
230 method to investigate the damping of waves in a channel. Under the assumption
231 of long waves, i.e., $kh \ll 1$, $\tanh(kh) \approx kh$ and $\coth(kh) \approx 1/kh$, the effects of
232 viscosity are separated from wave dispersion and solution (35) is revised to

$$c^2 \approx \left[1 - (1 + i) \frac{1}{\sqrt{2} (kh)^{\frac{5}{4}}} k \sqrt{\frac{\mu}{\rho_0}} (gk)^{-\frac{1}{4}} \right] gh. \tag{36}$$

233 *3.3. Water compressibility*

234 If we only consider the effects of water compressibility, by taking $\mu = 0$ and
 235 $\rho_0 = \text{const}$, the ODEs in the water (5) are simplified to

$$\frac{d}{dz} \begin{bmatrix} \hat{w}(z) \\ \hat{p}_1(z) \end{bmatrix} = \begin{bmatrix} \frac{\rho_0 g}{B} & -i \frac{k^2}{\rho_0 \omega} + i \frac{\omega}{B} \\ i \rho_0 \omega + i \frac{\rho_0^2 g^2}{\omega B} & -\frac{\rho_0 g}{B} \end{bmatrix} \begin{bmatrix} \hat{w} \\ \hat{p}_1 \end{bmatrix}, \quad (37)$$

236 and

$$\begin{cases} \hat{u}(z) = \frac{k}{\rho_0 \omega} \hat{p}_1(z) \\ \hat{\sigma}_{xz}(z) = 0 \\ \hat{\sigma}_{zz}(z) = 0. \end{cases} \quad (38)$$

237 Similarly, we solve the eigen-system of the coefficient matrix and derive the
 238 solutions. They are

$$\begin{cases} u(x, z, t) = k d_1 (C_1 e^{\lambda z} + C_2 e^{-\lambda z}) e^{i(kx - \omega t)} \\ w(x, z, t) = -i \left[C_1 \left(\lambda + \frac{\rho_0 g}{B} \right) e^{\lambda z} + C_2 \left(-\lambda + \frac{\rho_0 g}{B} \right) e^{-\lambda z} \right] e^{i(kx - \omega t)} \\ p_1(x, z, t) = \rho_0 \omega d_1 (C_1 e^{\lambda z} + C_2 e^{-\lambda z}) e^{i(kx - \omega t)} \\ \sigma_{xz}(x, z, t) = 0 \\ \sigma_{zz}(x, z, t) = 0, \end{cases} \quad (39)$$

239 where

$$\lambda = k \sqrt{d_1 - \frac{\rho_0 \omega^2}{B k^2}}, \quad d_1 = 1 + \frac{\rho_0 g^2}{B \omega^2}, \quad (40)$$

240 and C_1 and C_2 are constants to be determined from boundary conditions.

241 By neglecting Earth elasticity and water stratification, the boundary con-
 242 ditions at the fluid-fluid interface (14) are ignored; the boundary conditions at
 243 the fluid-solid interface (15) are simplified to that the vertical fluid velocity van-
 244 ishes. Substituting the solution (39) into the boundary conditions at the free
 245 surface (13) and fluid-solid interface (15), we obtain

$$\begin{cases} C_1 \left(\lambda + \frac{\rho_0 g}{B} \right) e^{\lambda h} + C_2 \left(-\lambda + \frac{\rho_0 g}{B} \right) e^{-\lambda h} = \omega A \\ C_1 e^{\lambda h} + C_2 e^{-\lambda h} = \frac{g A}{\omega d_1} \\ C_1 \left(\lambda + \frac{\rho_0 g}{B} \right) + C_2 \left(-\lambda + \frac{\rho_0 g}{B} \right) = 0. \end{cases} \quad (41)$$

246 Again, by eliminating the constants on the R.H.S. and converting the equations
 247 into homogeneous ones, the determinant of the coefficient matrix must be zero,

248 which leads to

$$c^2 = \frac{\omega^2}{k^2} = g \frac{\tanh(\lambda h)}{\lambda}, \quad (42)$$

249 where λ is given by (40). The result (42) is the same as Eqn. (26) and (10)
 250 derived from a propagator matrix method in Watada [2013].

251 The formula (42) is implicit in terms of phase speed, so the solution is
 252 only available from numerical approaches. Here we simplify it under specific
 253 assumptions. First, we define

$$\xi_c = \frac{\omega^2 \rho_0}{k^2 B} = \frac{c^2}{c_s^2} \ll 1, \quad (43)$$

254 and expand λ near $\xi_c = 0$:

$$\lambda = k\sqrt{d_1 - \xi_c} = k\sqrt{d_1} \left(1 - \frac{\xi_c}{2d_1}\right) + O(\xi_c^2), \quad (44)$$

255 where c_s is the sound speed and $c_s^2 = B/\rho_0$. Substituting the expanded λ into
 256 the full dispersion relationship (42) and keeping only the leading order terms,
 257 we obtain

$$c^2 \approx \frac{g}{k} \frac{\tanh(\sqrt{d_1}kh)}{\sqrt{d_1}} + \frac{g}{k 2d_1} \left[\frac{\tanh(\sqrt{d_1}kh)}{\sqrt{d_1}} - kh + kh(\tanh(\sqrt{d_1}kh))^2 \right] \xi_c. \quad (45)$$

258 For long waves, if we further assume that

$$\xi_d = \sqrt{d_1}kh = \sqrt{1 + \frac{\rho_0 g^2}{B\omega^2}} kh \ll 1, \quad (46)$$

259 by expanding (45) near $\xi_d = 0$ and $kh = 0$ and keeping only the leading order
 260 terms, the dispersion relationship (45) is simplified to

$$c \approx \left[1 - \frac{\rho_0 gh}{6B}\right] c_0, \quad (47)$$

261 where $c_0^2 = g/k \tanh(kh)$. The simplification requires the condition of (46),
 262 which is essentially equivalent to (43) for long waves. It is shown as follows:

$$\xi_d = \sqrt{(kh)^2 + \frac{(gh)^2}{c_s^2 c^2}} \approx \sqrt{(kh)^2 + \frac{c^2}{c_s^2}}, \quad (48)$$

263 where we have used $c^2 \approx gh$ for long waves. Later we will verify that the
 264 simplified solution (47) agrees well with the full solution (42) for typical tsunami
 265 wavelength of $50 \sim 1000$ km (or wave period $260 \sim 5000$ s).

266 By taking $c_0 = \sqrt{gh}$ we can separate the effects of compressibility and wave
 267 dispersion, leading to the dispersion relationship accounting for water compress-
 268 ibility only:

$$c \approx \left[1 - \frac{\rho_0 gh}{6B}\right] \sqrt{gh}. \quad (49)$$

269 3.4. Water Stratification

270 In this section two kinds of ocean density stratification are evaluated: the
 271 increase of density due to compression of gravity and the ocean stratification
 272 due to temperature/salinity. With water being compressible, the ocean density
 273 increases slightly and gradually with depth under gravitational compression,
 274 and it is larger at the seabed than surface. In addition, the vertical ocean
 275 column is formed to two layers due to temperature/salinity, with the less dense
 276 layer on top. The first kind has a larger absolute variation of density, while
 277 the second has a smaller density change but a larger changing rate over a small
 278 depth. Previous studies have ignored the second kind. We will show that the
 279 effects of the second kind on tsunami propagation speeds can be the same order
 280 as the first kind.

281 3.4.1. Density stratification due to compression of gravity

282 Considering a background water density $\rho_0(z)$ but neglecting water viscos-
 283 ity ($\mu = 0$) and compressibility ($B = \infty$), the equations in the water (5) are
 284 simplified to

$$\frac{d}{dz} \begin{bmatrix} \hat{w}(z) \\ \hat{p}_1(z) \end{bmatrix} = \begin{bmatrix} 0 & -i \frac{k^2}{\rho_0 \omega} \\ i \rho_0 \omega + i \frac{g}{\omega} \frac{d\rho_0}{dz} & 0 \end{bmatrix} \begin{bmatrix} \hat{w} \\ \hat{p}_1 \end{bmatrix}, \quad (50)$$

285 and

$$\begin{cases} \hat{u}(z) = \frac{k}{\rho_0 \omega} \hat{p}_1(z) \\ \hat{\sigma}_{xz}(z) = 0 \\ \hat{\sigma}_{zz}(z) = 0. \end{cases} \quad (51)$$

286 In this situation, the coefficient matrix depends on z because ρ_0 is a function of
 287 z . Hence we are not able to find the solution by simply solving the eigen-system
 288 of the matrix as in the previous sections. By eliminating $\hat{p}_1$, we obtain

$$\frac{d^2 \hat{w}}{dz^2} + \frac{1}{\rho_0} \frac{d\rho_0}{dz} \frac{d\hat{w}}{dz} - k^2 \left(1 + \frac{g}{\omega^2} \frac{1}{\rho_0} \frac{d\rho_0}{dz} \right) \hat{w} = 0. \quad (52)$$

289 This equation has simple analytical solutions when $1/\rho_0 d\rho_0/dz = \text{const}$. Since
 290 the density variation in the ocean is very small in depth, the following three
 291 density profiles are similar: (1) $1/\rho_0 d\rho_0/dz = -1/H = \text{const}$; (2) density profile
 292 under compression of gravity; (3) density profile that increases linearly in depth.

293 For those three profiles, $\rho_0(z)$ is respectively expressed as

$$\left\{ \begin{array}{l} \rho_0(z) = \rho_b \left(\frac{\rho_t}{\rho_b} \right)^{\frac{z}{h}} = \bar{\rho} \left[1 + \left(\frac{1}{2} - \frac{z}{h} \right) \frac{\Delta\rho}{\bar{\rho}} + O \left(\frac{\Delta\rho}{\bar{\rho}} \right)^2 \right] \\ \rho_0(z) = \frac{\rho_b}{\left(\frac{\rho_b}{\rho_t} - 1 \right) \frac{z}{h} + 1} = \bar{\rho} \left[1 + \left(\frac{1}{2} - \frac{z}{h} \right) \frac{\Delta\rho}{\bar{\rho}} + O \left(\frac{\Delta\rho}{\bar{\rho}} \right)^2 \right] \\ \rho_0(z) = -\frac{\rho_b - \rho_t}{h} z + \rho_b = \bar{\rho} \left[1 + \left(\frac{1}{2} - \frac{z}{h} \right) \frac{\Delta\rho}{\bar{\rho}} + O \left(\frac{\Delta\rho}{\bar{\rho}} \right)^2 \right], \end{array} \right. \quad (53)$$

294 where ρ_b and ρ_t are the fluid density at the bottom and top of the ocean, and

$$\begin{aligned} H &= \frac{h}{\log(\rho_b/\rho_t)} = h \frac{\bar{\rho}}{\Delta\rho} \left[1 + O \left(\frac{\Delta\rho}{\bar{\rho}} \right)^2 \right], \\ B &= gh \frac{\rho_b \rho_t}{\rho_b - \rho_t} = \bar{\rho} gh \frac{\bar{\rho}}{\Delta\rho} \left[1 + O \left(\frac{\Delta\rho}{\bar{\rho}} \right)^2 \right], \\ \bar{\rho} &= \frac{\rho_b + \rho_t}{2}, \quad \Delta\rho = \rho_b - \rho_t \ll \bar{\rho}. \end{aligned} \quad (54)$$

295 Eqn. (53) suggests that the three profiles are the same to the order of $\Delta\rho/\bar{\rho}$,
 296 and the solutions for these three profiles are therefore expected to be the same
 297 to the order of $\Delta\rho/\bar{\rho}$. Taking the first density profile, i.e., $1/\rho_0 d\rho_0/dz = -1/H$,
 298 the solutions in the fluid are found to be

$$\left\{ \begin{array}{l} u(x, z, t) = i \frac{1}{k} \left(C_1 \lambda_1 e^{\lambda_1 z} + C_2 \lambda_2 e^{\lambda_2 z} \right) e^{i(kx - \omega t)} \\ w(x, z, t) = (C_1 e^{\lambda_1 z} + C_2 e^{\lambda_2 z}) e^{i(kx - \omega t)} \\ p_1(x, z, t) = i \frac{\rho_0 \omega}{k^2} \left(C_1 \lambda_1 e^{\lambda_1 z} + C_2 \lambda_2 e^{\lambda_2 z} \right) e^{i(kx - \omega t)} \\ \sigma_{xz}(x, z, t) = 0 \\ \sigma_{zz}(x, z, t) = 0, \end{array} \right. \quad (55)$$

299 where

$$\lambda_1 = \frac{1}{2H} + \sqrt{k^2 + \frac{1}{4H^2} - \frac{gk^2}{\omega^2 H}}, \quad \lambda_2 = \frac{1}{2H} - \sqrt{k^2 + \frac{1}{4H^2} - \frac{gk^2}{\omega^2 H}}, \quad (56)$$

300 and C_1 and C_2 are constants to be determined from boundary conditions.

301 By neglecting Earth elasticity, water viscosity and multiple layers in the
 302 fluid, the boundary conditions at the fluid-fluid interface (14) are ignored; the
 303 boundary conditions at the fluid-solid interface (15) are simplified to that the

vertical fluid velocity vanishes. Substituting the solution (55) into the boundary conditions at the free surface (13) and fluid-solid interface (15), we obtain

$$\begin{cases} C_1 e^{\lambda_1 h} + C_2 e^{\lambda_2 h} = -i\omega A \\ C_1 \lambda_1 e^{\lambda_1 h} + C_2 \lambda_2 e^{\lambda_2 h} = -i \frac{g A k^2}{\omega} \\ C_1 + C_2 = 0, \end{cases} \quad (57)$$

Eliminating the constants in the first two equations and converting the system into homogeneous equations, the determinant of the coefficient matrix has to be zero, leading to:

$$c^2 = \frac{\omega^2}{k^2} = g \frac{\tanh(\lambda h)}{\lambda + \frac{1}{2H} \tanh(\lambda h)}, \quad (58)$$

where

$$\lambda = \sqrt{k^2 + \frac{1}{4H^2} - \frac{gk^2}{\omega^2 H}}. \quad (59)$$

The result (58) is the same as Eqn. (25) and (10) by Watada [2013]. Note that there are c^2 terms on the R.H.S. of Eqn. (25) of Watada [2013].

To simplify the full dispersion relationship (58), we assume

$$\xi_s = \frac{h}{H} \ll 1, \quad (60)$$

which can be justified by $H \approx B/\bar{\rho}g \approx 200 \text{ km} \gg h$ (Eqn. 54, taking $B \approx 2 \times 10^9 \text{ Pa}$, $\bar{\rho} \approx 1 \times 10^3 \text{ kg/m}^3$ and $g \approx 10 \text{ m/s}^2$). According to Eqn. (54), ξ_s is also determined to be the same as $\Delta\rho/\bar{\rho}$ to the leading order. By substituting $\omega^2 \approx ghk^2$ for long waves, it is found that

$$(\lambda h)^2 = k^2 h^2 + \frac{h^2}{4H^2} - \frac{gk^2 h^2}{\omega^2 H} \approx k^2 h^2 + \frac{\xi_s^2}{4} - \xi_s \ll 1. \quad (61)$$

Therefore the full dispersion relationship (58) is expanded near $\lambda h = 0$ and then near $\xi_s = 0$ to have

$$c^2 \approx \left[gh - \frac{1}{3}(gh)(kh)^2 \right] - \left(\frac{1}{2} - \frac{ghk^2}{3\omega^2} \right) (gh)\xi_s, \quad (62)$$

where the crossing terms of kh and ξ_s are neglected. Substituting $\omega^2 = ghk^2$ to the right hand side by neglecting small order crossing terms, we obtain

$$c \approx \left[1 - \frac{1}{12} \frac{h}{H} \right] c_0, \quad (63)$$

where $c_0^2 = g/k \tanh(kh)$. Replacing H in (63) with $\bar{\rho}$ and $\Delta\rho$ from Eqn. (54), (63) can be written in terms of bulk modulus or fluid density as follows:

$$c \approx \left[1 - \frac{1}{12} \frac{h}{H} \right] c_0 \approx \left[1 - \frac{1}{12} \frac{\bar{\rho}gh}{B} \right] c_0 \approx \left[1 - \frac{1}{12} \frac{\Delta\rho}{\bar{\rho}} \right] c_0. \quad (64)$$

By taking $c_0 = \sqrt{gh}$ we can separate the effects of stratification and wave dispersion, leading to the dispersion relationship accounting for water stratification only:

$$c \approx \left[1 - \frac{1}{12} \frac{h}{H}\right] \sqrt{gh} \approx \left[1 - \frac{1}{12} \frac{\bar{\rho}gh}{B}\right] \sqrt{gh} \approx \left[1 - \frac{1}{12} \frac{\Delta\rho}{\bar{\rho}}\right] \sqrt{gh}. \quad (65)$$

3.4.2. Density stratification due to temperature/salinity

For this kind of density stratification, the model consists of two layers of fluid, but water viscosity, compressibility and density stratification are ignored in each layer. The solutions (19) are applied to the boundary conditions (13, 14, 15) to derive the dispersion relationship. By neglecting Earth elasticity, water viscosity, compressibility and background density change, the boundary conditions at the fluid-fluid interface (14) do not include the continuity of horizontal velocity; the boundary conditions at the fluid-solid interface (15) are simplified to that the vertical fluid velocity vanishes. Substituting the solution (19) into the boundary conditions (13, 14, 15), we obtain

$$\begin{cases} \sinh(kh)C_1 + \cosh(kh)C_2 = -i\omega A \\ \omega \cosh(kh)C_1 + \omega \sinh(kh)C_2 = -ikgA \\ \rho_1 \cosh(kh_2)C_1 + \rho_1 \sinh(kh_2)C_2 - \rho_2 \cosh(kh_2)C'_1 - \rho_2 \sinh(kh_2)C'_2 = 0 \\ \sinh(kh_2)C_1 + \cosh(kh_2)C_2 - \sinh(kh_2)C'_1 - \cosh(kh_2)C'_2 = 0 \\ C'_2 = 0, \end{cases} \quad (66)$$

where $h = h_1 + h_2$, C_1 and C_2 are coefficients for fluid 1, C'_1 and C'_2 are coefficients for fluid 2. Eliminating the constants in the first two equations and converting the system into homogeneous equations, the determinant of the coefficient matrix has to be zero, leading to:

$$c^2 = \frac{\omega^2}{k^2} = \frac{g}{k} \frac{\tanh(kh_1) + \frac{\rho_1}{\rho_2} \tanh(kh_2)}{1 + \frac{\rho_1}{\rho_2} \tanh(kh_1) \tanh(kh_2)}. \quad (67)$$

The result can be verified by letting $\rho_1 = \rho_2$, which leads to $\omega^2 = gk \tanh(k(h_1 + h_2))$.

Assuming that the density difference is very small, i.e.,

$$\frac{\Delta\rho}{\bar{\rho}} = 2 \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} \ll 1, \quad (68)$$

by expanding the dispersion relationship near $\Delta\rho/\bar{\rho} = 0$ and only keeping the leading order terms, the dispersion relationship is simplified to

$$c \approx \left[1 - \frac{\Delta\rho}{2\bar{\rho}} \frac{\sinh(2kh_2)}{\sinh(2kh)}\right] c_0, \quad (69)$$

where $c_0^2 = g/k \tanh(k(h_1 + h_2)) = g/k \tanh(kh)$.

Taking $c_0 = \sqrt{gh}$, we obtain the dispersion relationship accounting for density stratification due to temperature/salinity:

$$c \approx \left[1 - \frac{\Delta\rho h_2}{2\bar{\rho} h} \right] \sqrt{gh}. \quad (70)$$

The density change $\Delta\rho$ of such stratifications is normally smaller than that of stratifications due to compression of gravity, but the coefficient of the correction term is larger (1/2 in Eqn. 70 and 1/12 in Eqn. 65). Therefore it is expected that such stratifications can slow down the tsunami waves by the same order.

4. Verification of the Simplifications

In this section we verify the simplifications of the dispersion relationships, that is, (a) the full dispersion relationship accounting for Earth elasticity (21) and the simplified formula (27); (b) the full dispersion relationship accounting for water compressibility (42) and the simplified formula (47); (c) the full dispersion relationship accounting for density stratification due to compression (58) and the simplified formula (64); (d) the full dispersion relationship accounting for density stratification due to temperature/salinity (67) and the simplified formula (69). The viscous effects are very small for tsunami waves and the verification for its simplified dispersion relationship (35) is omitted.

The geophysical parameters adopted are as follows: water depth $h = 4000$ m, water density $\rho_0 = \rho_t = 1025$ kg/m³, water viscosity $\mu_f = 1.0 \times 10^{-3}$ Pa s, water bulk modulus $B = 2.2$ GPa, solid density $\rho_s = 3500$ kg/m³, Earth Poisson's ratio $\nu_s = 0.25$, Earth shear modulus $\mu_s = 50$ GPa, water stratification $h_1 = 1000$ m, $h_2 = 3000$ m, $\Delta\rho = 3.0$ kg/m³, $\rho_1 = \rho_0$, $\rho_2 = \rho_2 + \Delta\rho$ and constant gravity acceleration of $g = 9.8$ m/s².

The full dispersion relationships are solved using the Newton's iteration method, and we vary the initial guess in the range of $c_0 \pm 20$ m/s to ensure that the solution is independent of the initial guess. The results are plotted along with the simplified results in Fig. 2. It shows that, for typical tsunami wavelength of $50 \sim 1000$ km (or wave period $260 \sim 5000$ s, the error between the full and simplified solutions is negligible compared to the correction itself. Thus, the simplified solution is a very good approximate of the full.

Figure 2: **Verification of the simplified dispersion relationships:** (a) Earth elasticity; (b) water compressibility; (c) density stratification due to compression; (d) density stratification due to temperature/salinity. Note that all the speeds include the effects of dispersion, so the difference between the blue or red curves and the black curves is the correction and the difference between the blue and red curves is the error caused by simplification. For instance, the black curve in (a) gives the propagation speeds of dispersive tsunami waves, the blue curve shows the speeds of such waves under the effects of Earth elasticity, and the red curve denotes the simplified formula to calculate the blue. The error of simplification is so small that the red curves overlap with the blue ones.

375 5. Numerical Dispersion and Relative Importance of Secondary Ef- 376 fects

377 When simulating tsunami waves, numerical dispersion is inevitably intro-
378 duced. The shallow water wave equations are written as [Cho, 1995]

$$\begin{cases} \frac{\partial \eta}{\partial t} + \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0 \\ \frac{\partial P}{\partial t} + gh \frac{\partial \eta}{\partial x} = 0 \\ \frac{\partial Q}{\partial t} + gh \frac{\partial \eta}{\partial y} = 0, \end{cases} \quad (71)$$

379 where $\eta(x, y, t)$ is the water elevation, $P(x, y, t)$ and $Q(x, y, t)$ are the horizontal
380 fluxes. As an example, the leap-frog numerical method can be used to solve the
381 equations using the following scheme:

$$\begin{cases} \frac{\eta_{i,j}^{n+1/2} - \eta_{i,j}^{n-1/2}}{\Delta t} + \frac{P_{i+1/2,j}^n - P_{i-1/2,j}^n}{\Delta x} + \frac{Q_{i,j+1/2}^n - Q_{i,j-1/2}^n}{\Delta y} = 0 \\ \frac{P_{i+1/2,j}^{n+1} - P_{i+1/2,j}^n}{\Delta t} + gh \frac{\eta_{i+1,j}^{n+1/2} - \eta_{i,j}^{n+1/2}}{\Delta x} \\ \quad + \frac{\gamma gh}{12 \Delta x} \left[(\eta_{i+1,j+1}^{n+1/2} - 2\eta_{i+1,j}^{n+1/2} + \eta_{i+1,j-1}^{n+1/2}) - (\eta_{i,j+1}^{n+1/2} - 2\eta_{i,j}^{n+1/2} + \eta_{i,j-1}^{n+1/2}) \right] = 0 \\ \frac{Q_{i,j+1/2}^{n+1} - Q_{i,j+1/2}^n}{\Delta t} + gh \frac{\eta_{i,j+1}^{n+1/2} - \eta_{i,j}^{n+1/2}}{\Delta y} \\ \quad + \frac{\gamma gh}{12 \Delta y} \left[(\eta_{i+1,j+1}^{n+1/2} - 2\eta_{i,j+1}^{n+1/2} + \eta_{i-1,j+1}^{n+1/2}) - (\eta_{i+1,j}^{n+1/2} - 2\eta_{i,j}^{n+1/2} + \eta_{i-1,j}^{n+1/2}) \right] = 0, \end{cases} \quad (72)$$

382 where the superscripts denote the time step, the subscripts show the spatial
383 grids, Δt is the time step, Δx and Δy are the spatial grid size and γ is a
384 constant parameter. The scheme is the same as used by Cho [1995] if $\gamma = 1$ and
385 the same as Imamura and Goto [1988] if $\gamma = 0$. Taking $\gamma = 1$ and $\Delta x = \Delta y$,
386 the modified equation associated with the numerical scheme can be obtained
387 after lengthy but straightforward algebra [Cho, 1995]:

$$\frac{\partial^2 \eta}{\partial t^2} - gh \left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} \right) - gh \frac{\Delta x^2}{12} \left(1 - \frac{gh \Delta t^2}{\Delta x^2} \right) \left(\frac{\partial^4 \eta}{\partial x^4} + 2 \frac{\partial^4 \eta}{\partial x^2 \partial y^2} + \frac{\partial^4 \eta}{\partial y^4} \right) = O(\Delta x^3 \dots). \quad (73)$$

388 The first two terms in the equation are the exact shallow water wave equations,
389 and the third term is the dispersive term induced by the numerical scheme.
390 Adopting a plane wave solution, i.e., assuming $\eta(x, y, t) = A e^{i(k_x x + k_y y - \omega t)}$,
391 where A is constant, $k = \sqrt{k_x^2 + k_y^2}$ is the wave number and ω is the angular
392 frequency, and substituting $\eta(x, y, t)$ into (73), we find that the wave speed for
393 such a governing equation is

$$c \approx \left[1 - \frac{\Delta x^2}{24} \left(1 - \frac{gh \Delta t^2}{\Delta x^2} \right) k^2 \right] \sqrt{gh}. \quad (74)$$

Thus, the wave speed is no longer $\sqrt{gh}$ and the modification depends the grid size and time step. Using C.F.L. number of 0.5, i.e., $\Delta t\sqrt{gh}/\Delta x = 0.5$, the numerical dispersion can be estimated purely according to the grid size:

$$c \approx \left[1 - \frac{k^2 \Delta x^2}{32} \right] \sqrt{gh}. \quad (75)$$

In Fig. 3, we plot the speed reduction and percentage due to physical dispersion, Earth elasticity, water viscosity, compressibility, density stratification due to compression of gravity and temperature/salinity and numerical dispersion. The range of wavelength is from 50 ~ 1000 km, corresponding to wave period of 260 ~ 5000 s. The dispersion relationships used are the simplified formulas excluding dispersion, i.e., Eqns. (28), (36), (49), (65), (70) and (74). Fig. 3 shows that, the correction due to Earth elasticity increases roughly linearly with wavelength; the correction due to water viscosity is negligible; the correction due to water compressibility and density stratification is also constant regardless the wavelength. For a typical tsunami wavelength of ~ 200 km, the correction of dispersion for shallow water approximations is around 0.26%, the correction of Earth elasticity is around 0.24%, the correction of water viscosity is around 0.01%, the correction of water compressibility is around 0.30%, the correction of density stratification due to compression is around 0.15%, and the correction of density stratification due to temperature/salinity is around 0.14%. The numerical correction is 0.01% for $\Delta x = 1$ arc min, 0.04% for $\Delta x = 2$ arc min and 0.27% for $\Delta x = 5$ arc min.

6. Conclusions

In this study we derive the theoretical solutions of dispersion relationships accounting for the effects of water viscosity, ocean stratification due to temperature/salinity and numerical dispersion. We find that the water viscosity hardly affects the speed of tsunami waves, but the density stratification due to temperature/salinity is capable of slowing down the tsunami waves by the same order as the stratification due to gravitational compression, because such a stratification has a sudden change of density over a small depth. The numerical dispersion is comparable to the physical dispersion if the spatial grid size is around 5 arc min in the numerical simulation. It is noted that 5 arc min has been commonly used for global tsunami warning purposes. Therefore, we suggest that if the correction of Earth elasticity, water compressibility and density stratification due to compression [e.g., Watada, 2013, Allgeyer and Cummins, 2014, Watada et al., 2014] were deemed to be significant, the effects of stratification due to temperature/salinity and numerical dispersion should also be taken into account. In this study we also simplify the theoretical dispersion relationships so that the simplified formulas are explicit and easy to adopt. They are equations (28), (36), (49), (65), (70) and (74). The simplifications are verified to be good approximations of the full dispersion relationships for typical tsunami wavelength of 50 ~ 1000 km (or wave period 260 ~ 5000 s).

Figure 3: **Speed reduction and percentage due to secondary effects and numerical dispersion.** In the top panel, the tsunami speeds in shallow water ($\sqrt{gh}$) and under effects of wave dispersion ($\sqrt{g/k \tanh(kh)}$), Earth elasticity (Eqn. 28), water viscosity (Eqn. 36), water compressibility (Eqn. 49), water density change due to gravitational compression (Eqn. 65), ocean stratification due to temperature/salinity (Eqn. 70) and numerical dispersion (Eqn. 74) are plotted. In the bottom panel, the percentage of speed reduction under such effects with respect to the shallow water wave speed $\sqrt{gh}$ is plotted. The effect of water viscosity on speed reduction is minor and it is hardly seen from the plot. The numerical dispersion is close to the physical dispersion when the spatial grid size is 5 arc min.

Since all the effects slow down the tsunami speeds by a small order ($< 1\%$), it is expected that the combination of them in a linear system is simply the linear summation, and the crossing terms of the coupling effects are one order even smaller. Using the typical physical parameters given in Section 4, for tsunami waves with wavelength of ~ 200 km, the effects that cause significant speed reduction are wave dispersion (0.26%), Earth elasticity (0.24%), water compressibility (0.30%), density stratification due to compression (0.15%) and density stratification due to temperature/salinity (0.14%). The total reduction is around 1.1%.

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