1 Simulation of wave phenomena in the nearshore
2 through application of $O(\mu^2)$ and $O(\mu^4)$
3 pressure–Poisson Boussinesq type models

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10 **Abstract**

The pressure–Poisson Boussinesq models of Donahue *et al.* (2014) are extended to the surf zone and tested at $O(\mu^2)$ and $O(\mu^4)$. This approach resembles a classic Boussinesq–type model with the exception that the dispersive accuracy is obtained through consideration of a Boussinesq–scaled Green–Naghdi type approximation to the nonhydrostatic pressure profile and solved as a solution to the pressure–Poisson equation. The result is a Boussinesq–type model that is similar to the shallow water equations (SWE) with a nonhydrostatic pressure term. This allows for the straightforward modification of established SWE solvers to turn them into fully resolved nearshore models with very little structural change to the underlying SWE model and without the need to solve mixed space/time derivatives. The excellent hyperbolic properties of the discontinuous Galerkin (DG) finite element method are used to solve all terms except pressure and a simple yet accurate finite difference approach is used to solve the elliptic pressure equations. A straightforward numerics based wave breaking algorithm is employed and wave runup is captured using a globally mass conservative wetting/drying algorithm specifically designed for DG finite element methods. The result is a full surf–zone model capable of capturing the propagation of waves offshore, the breaking of waves near shore and full wave runup. The model is verified and validated using analytical and experimental results for both fully

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nonlinear $O(\mu^2)$ and weakly nonlinear $O(\mu^4)$ implementations.

1. Introduction

Boussinesq-type models represent an improvement to the classic shallow water equations in that they are capable of capturing low or high order dispersive effects and thus are capable of resolving more complex wave propagation. Modern Boussinesq-scaled models began in the late 1960's with the work of Peregrine [1]. Nwogu [2] was able to extend the range of applicability of the so-called Boussinesq models by demonstrating that if the equations are scaled about a particular depth, optimum dispersion characteristics can be obtained. Separately, Madsen *et al.* [3, 4, 5, 6] were able to achieve both high-order dispersion and high-order shoaling properties through term rearrangement. Over the last three decades many others have improved upon these results to produce high order Boussinesq models that are capable of capturing nearshore wave physics such as wave breaking and wave runup [7, 8, 9, 10, 11, 12, 13, 14, 15, 16].

Greater accuracy can be accomplished by extending the Boussinesq equations to higher order. Gobbi *et al.* [12] was able to demonstrate that an extension to fourth order provided significant improvement over the second order model due to the increased accuracy in the linear and nonlinear properties. A drawback to this approach was the need to solve higher order derivatives, thus increasing the computational complexity. As a result, higher order Boussinesq models have not found wide spread use in applications.

A separate approach to increasing the accuracy in Boussinesq models is to divide the vertical axis into a number of individual layers and then solve the second order Boussinesq equations on each layer. Lynett *et al.* [16, 17] was able to show that a significant improvement in the model properties could be accomplished with only a few vertical layers.

Recently an effort has been made to produce models that are equivalent in dispersive accuracy, but lack the mixed space/time derivatives that are inherent in the derivation of Boussinesq-type models. Extending the work of Casulli and

39 Stelling [18], Casulli [19] and Stansby *et al.* [20], a new class of dispersive models
 40 have been developed which extend the standard depth integrated hydrostatic
 41 shallow water equations (SWE) to include a dynamic pressure term [21, 22, 23,
 42 24, 25, 26]. The resulting non-hydrostatic shallow water equation based models
 43 consider the dynamic pressure to be a single unknown, which must be found
 44 numerically from the conservation of mass and momentum equations. Bai *et*
 45 *al.* [27] demonstrated using the approach outlined by Yamazaki *et al.* [23] that
 46 with two-layers of equal thickness it is possible to achieve dispersive properties
 47 equivalent to the Boussinesq model of Nwogu [2]. In addition, considering a
 48 similar approach as Nwogu, Bai *et al.* [28] was able to achieve higher order
 49 dispersion in the model through manipulation of the depth at which the two
 50 layers were separated. Similarly, Zhu *et al.* [29] was able to demonstrate that
 51 improvement in the linear and nonlinear properties for the approach outlined by
 52 Zijlema *et al.* [30] could be achieved through an optimum choice of individual
 53 layer thicknesses.

54 The advantage of this approach as compared to Boussinesq models is the lack
 55 of mixed space/time derivatives in the governing equations and the relative sim-
 56 plicity in implementation. Contrary to classic Boussinesq models, which obtain
 57 higher-order accuracy through the inclusion of higher-order terms, higher-order
 58 accuracy in these models is achieved by increasing the number of vertical com-
 59 putational layers. Thus the increase in numerical simplicity is replaced by the
 60 need for more physical layers and thus computational cost.

61 Other methods have examined approaches to manipulate Boussinesq-type
 62 equations (BTE) to include dispersive effects while retaining a relatively simple
 63 structure for numerical implementation. An example would be the work on Dis-
 64 persive Nonlinear Shallow Water Equations (DSNEW's) proposed by Antuono
 65 *et al.* [31, 32], which constructs a fully hyperbolic system of equations that also
 66 include the dispersive effects of standard Boussinesq-type models without the
 67 need to solve higher order spatial derivatives.

68 The present work focuses on a related approach where dispersive terms are
 69 included in the model using a non-hydrostatic pressure approach. Previously,

70 the authors derived a Boussinesq-type approximation to the dynamic pressure
 71 profile that was both simple to implement and shown to be very accurate for
 72 several test cases [33]. The use of polynomial basis functions to represent the
 73 vertical dimension allows for the reduction of the problem to the horizontal
 74 dimension [34]. This approach was employed by Zhang *et al.* [35] to approximate
 75 the vertical variability of the horizontal velocities by a series of polynomial
 76 expansions. In the work discussed here, the Green–Naghdi type basis functions
 77 are used to represent the vertical variability in the pressure profile. In contrast to
 78 standard Boussinesq-type models which consist of a set of coupled equations for
 79 the horizontal velocity vector, the pressure–Poisson based problem only focuses
 80 on the solution to one elliptic problem. The resulting model is consistent with
 81 standard Boussinesq models of second and fourth order dispersive accuracy [33].

82 The present work extends the pressure–Poisson based Boussinesq-type model
 83 presented by Donahue *et al.* [33] to include both wave breaking and wave
 84 runup. The result is a full surf–zone model capable of capturing nearshore wave
 85 physics while remaining simple to implement. The previous implementation of
 86 the model applied a finite difference discretization to the entire system. In the
 87 present work the time rate change of the free–surface and velocity components
 88 of the model will be solved using a DG finite element framework, taking advantage
 89 of the DG method’s excellent hyperbolic properties, especially in regions
 90 where wave breaking occurs. The dynamic pressure component of the model will
 91 be determined through application of a central finite difference discretization to
 92 the pressure–Poisson (PPBOUSS) equation. The PPBOUSS solution is solved
 93 independently of the conservation of momentum and mass, thus producing a dynamic
 94 pressure module which can easily be coupled with any hydrostatic model
 95 in order to include dispersive terms. The result is a coupled finite difference PP-
 96 BOUSS and finite element SWE solution. For the present work only structured
 97 meshes will be applied in order to facilitate a simpler coupling infrastructure
 98 between the DG finite element and finite difference solutions. The resulting
 99 PPBOUSS solver is simple to implement, computationally efficient and can be
 100 applied in conjunction with any SWE solver to transform that solution into a

101 dispersive non-hydrostatic model.

102 The paper is organized as follows: section 2 reviews the model theory; sec-
 103 tion 3 discusses the numerical implementation of the model and discusses the
 104 coupling paradigm between the dynamic pressure and the free-surface and ve-
 105 locity solution; section 4 presents the verification and validation of the model
 106 with several test cases; and section 5 draws conclusions and discusses future
 107 work.

108 2. Model Details

We use the model derived by Donahue *et al.* [33]. For brevity the model details are summarized here, for more details the reader is directed to the paper by Donahue *et al.* [33]. The governing equations for the free-surface and the depth-integrated velocity are given by,

$$\frac{\partial \eta}{\partial t} + \nabla \cdot \left((h + \delta \eta) \bar{\mathbf{u}} \right) = 0, \quad (1)$$

$$\begin{aligned} \frac{\partial}{\partial t} \left((h + \delta \eta) \bar{\mathbf{u}} \right) + \delta \nabla \cdot \left((h + \delta \eta) \bar{\mathbf{u}} \bar{\mathbf{u}}^T \right) + \\ \int_{-h}^{\delta \eta} (\nabla P) dz = \int_{-h}^{\delta \eta} \left(\mu^2 \nabla \tau_{\mathbf{x}\mathbf{x}} + \frac{\partial \tau_{z\mathbf{x}}}{\partial z} \right) dz. \end{aligned} \quad (2)$$

109 The depth-averaged horizontal velocity in the x, y direction is given by $\bar{\mathbf{u}} = (\bar{u}, \bar{v})$
 110 respectively. The horizontal and vertical stresses are given by $\tau_{\mathbf{x}\mathbf{x}}$ and $\tau_{z\mathbf{x}}$
 111 respectively. The free-surface displacement is given by η , the pressure by P ,
 112 the bathymetric depth with respect to the still water level by h and time by t .
 113 The nondimensional constants $\delta = a_0/h_0$ and $\mu = k_0 h_0$ represent measures of
 114 the nonlinearity and shallowness of the wave respectively, where the values k_0 ,
 115 h_0 and a_0 represent a reference wavenumber, a reference depth and reference
 116 wave amplitude respectively

117 The main difference between the present approach and the standard Boussinesq-
 118 type approach is in the treatment of the pressure term, P , in equation (2). In
 119 the standard Boussinesq approach the vertical momentum equation is integrated
 120 to find an explicit expression for P in terms of the velocities. The result is a dif-
 121 ferent set of velocity equations which contain mixed space/time derivatives and

higher order spatial derivatives. In the present work we consider an approach similar to that of Stelling *et al.* [21] and others [23, 25, 26] where pressure is broken into a hydrostatic component, P_h , and a dynamic component, P_d ,

$$P = P_h + P_d = g(\delta\eta - z) + P_d, \quad (3)$$

where g is the gravitational constant. Substitution of equation (3) into equation (2) and simplification using the Leibniz integration rule yields an expression for the depth integrated momentum in conservative form,

$$\begin{aligned} \frac{\partial((h + \delta\eta)\bar{\mathbf{u}})}{\partial t} + \delta\nabla \left(\delta u^2(h + \delta\eta) + \frac{1}{2}g\delta\eta^2 + gh\eta + \hat{P}_d \right) = \\ \nabla h(g\eta + P_b) + \int_{-h}^{\delta\eta} \left(\mu^2 \nabla \tau_{\mathbf{xx}} + \frac{\partial \tau_{z\mathbf{x}}}{\partial z} \right) dz, \end{aligned} \quad (4)$$

where,

$$\hat{P}_d = \int_{-h}^{\delta\eta} P_d dz, \quad (5)$$

$$P_b = P_d|_{z=-h}. \quad (6)$$

It is noted that if $P_d = 0$, then the hydrostatic pressure assumption is retained and the standard SWE's are recovered. Once the dynamic pressure is known, implementation of this dispersive system may be conducted with simple modifications to SWE solvers. Donahue *et al.* [33] demonstrated that the pressure profile, P , could be approximated as a perturbation to the still water pressure through the application of a Green–Naghdi type series expansion, as follows,

$$P = gh(1 - q) + \sum_{n=1}^N \mu^{\hat{\beta}_n} \phi_n P_n, \quad (7)$$

where $q \in [0, 1]$ is a sigma coordinate defined as,

$$q = \frac{z + h}{\delta\eta + h}. \quad (8)$$

140 The P_n are unknown expansion coefficients, $\phi_n = \phi_n(q)$ are vertical basis
 141 functions expressed as a Green–Naghdi type series expansion, and $\hat{\beta}_1 = 0$, oth-
 142 erwise $\hat{\beta}_n = n$ when n is even and $\hat{\beta}_n = n + 1$ when n is odd, is a measure
 143 of the order of the expansion. The basis functions are given by the following
 144 approximation,

$$145 \quad \phi_n = \sum_{m=1}^n \hat{\phi}_{mn} (1 - q^m), \quad (9)$$

146 without loss of generality $\hat{\phi}_{nn} = 1$. In this coordinate frame the hydrostatic
 147 pressure is given by $P_h = g(h + \delta\eta)(1 - q)$, thus the dynamic pressure according
 148 to Equation (3) is determined to be,

$$149 \quad P_d = -\delta g\eta(1 - q) + \sum_{n=1}^N \mu^{\hat{\beta}_n} \phi_n P_n. \quad (10)$$

150 Substitution of equation (10) into equations (5) and (6) provides the following
 151 expressions for $\hat{P}_d$ and P_b ,

$$\hat{P}_d = -\frac{1}{2} \delta g\eta(h + \delta\eta) + (h + \delta\eta) \sum_{n=1}^N \mu^{\hat{\beta}_n} G_n P_n, \quad (11)$$

$$P_b = -\delta g\eta + \sum_{n=1}^N \mu^{\hat{\beta}_n} \phi_n|_{q=0} P_n, \quad (12)$$

152 where,

$$153 \quad G_n = \int_0^1 \phi_n dq. \quad (13)$$

154 A solution for the unknown P_n expansion terms is found by simultaneously
 155 solving the pressure–Poisson equations using a weighted residual approach with
 156 arbitrary weighting functions and pressure bottom boundary equation. For more
 157 detail on the form of these equations the reader is directed to Donahue *et al.*
 158 [33].

159 2.1. Use of optimal basis functions

160 The basis functions for the Green–Naghdi type expansion of the pressure
 161 profile, equation (9), can be optimized for certain properties. In particular,
 162 it was shown that for the $O(\mu^2)$ fully nonlinear solution it was possible to
 163 improve the dispersive accuracy to be formally fourth order with a choice of
 164 basis functions,

$$\phi_1 = (1 - q), \quad (14)$$

$$\phi_2 = -\frac{4}{3}(1 - q) + (1 - q^2). \quad (15)$$

165 Similarly for the $O(\mu^4)$ weakly nonlinear solution it was possible to achieve
 166 formally eighth order dispersion accuracy through the appropriate choice of
 167 basis functions [33]. For brevity the basis for the $O(\mu^4)$ model are not given
 168 here, however for all simulations in this paper, the basis functions and weighting
 169 functions are taken to be the same optimal values used in the Donahue *et al.*
 170 paper for both the $O(\mu^2)$ and weakly nonlinear $O(\mu^4)$ implementations.

171 As previously mentioned, the use of optimal basis functions improves the
 172 linear and nonlinear properties of the model for both the fully nonlinear $O(\mu^2)$
 173 and weakly nonlinear $O(\mu^4)$ formulations. Not only can the linear dispersion
 174 properties of the model be improved to be formally fourth and eighth order
 175 respectively, but it is shown in Donahue *et al.* [33] that the accuracy in shoaling
 176 and Stoke’s second harmonic can also be improved.

177 2.2. Treatment of Wave Breaking

178 The free–surface solution is single valued and as such is incapable of repre-
 179 senting the multi–valued surface that would be expected during a wave breaking
 180 event. As a result a significant amount of research has been conducted on ap-
 181 proaches to handling breaking waves within the framework of Boussinesq–type
 182 models. One approach is to include a viscosity term in the conservation of
 183 momentum equations. The early work of Zelt [36] focused on detection of the
 184 wave breaking front through the gradient of the velocity which then triggered

185 an artificial viscosity term in the momentum equations. Other approaches con-
 186 sidered an eddy viscosity type term which, such as the work of Karambas *et al.*
 187 [37] and Kennedy *et al.* [13]. For example, in the approach of Kennedy *et al.*
 188 the breaking wave front was detected by considering the time rate of change in
 189 the free-surface, which then determines the location and magnitude of the eddy
 190 viscosity term.

191 A separate approach applied by Schäffer *et al.* [7] and extended to two di-
 192 mensions by Madsen *et al.* [38] was built upon the detection of surface rollers
 193 in the solution through analysis of the steepness in the wave. The effect of the
 194 excess momentum due to a non-uniform velocity distribution in the presence of
 195 a breaking wave is represented by the inclusion of extra terms in the momentum
 196 equations.

197 More recently an approach that has become popular is to treat the break-
 198 ing waves as nondispersive bores which are ideally solved by the SWE. This is
 199 achieved through temporarily suspending the dispersive terms in the momen-
 200 tum equations in the presence of a breaking wave. Tonelli *et al.* [39] proposed a
 201 relatively simple mechanism for detecting a breaking wave through an approx-
 202 imation to the Froude number based on the ratio of the wave height to water
 203 depth. Tissier *et al.* [40] extended the detection of the wave breaking criteria to
 204 consider the dissipation of energy across the wave front. Building on the work
 205 of Kennedy *et al.* [13], Smit *et al.* [41] determined the location of a breaking
 206 wave front based on the spatial gradient of the free-surface. Kazolea *et al.* [42]
 207 combined the approaches of Kennedy *et al.* with the approach of Schäffer *et al.*
 208 [7] to construct a hybrid wave breaking detection mechanism. An advantage of
 209 these approaches is the relative simplicity of the implementation as well as the
 210 lack of a requirement to track a travelling wave front.

211 The nondispersive bore breaking model is employed here: thus turbulent
 212 stress terms, τ_{xx} , are ignored. Treatment of the bottom stress term, τ_{zx} , is ap-
 213 proximated by the Manning's n formulation, where n represents the Manning's
 214 roughness coefficient.

215 This work utilizes a relatively simple wave breaking detection method which

216 takes advantage of the DG framework. Wave breaking is measured through
 217 analysis of the discontinuity across the elemental interface of free-surface solu-
 218 tion. This approach is described in detail by Duran *et al.* [43]. A wave breaking
 219 detector is used, T_j , defined as follows,

$$220 \quad T_j = \frac{\Delta^- |H_{j-1/2}^+ - H_{j-1/2}^-| + \Delta^+ |H_{j+1/2}^+ - H_{j+1/2}^-|}{|\Omega_j| |H_j|_\infty}, \quad (16)$$

221 where,

$$222 \quad \Delta^- = \begin{cases} 1, & \text{if } Q_{j-1/2}^+ \geq 0, \\ 0, & \text{otherwise,} \end{cases} \quad \Delta^+ = \begin{cases} 1, & \text{if } Q_{j+1/2}^- \leq 0, \\ 0, & \text{otherwise,} \end{cases} \quad (17)$$

223 and $|H_j|_\infty$ is the absolute maximum water column value over the element j .
 224 An element is considered to contain a breaking wave, if $T_j > 1$. In this case,
 225 the dispersive terms in the element are turned off, i.e. $\hat{P}_d = 0$ and $P_b =$
 226 0. Furthermore, to avoid spurious oscillations due to sharp transitions from
 227 dispersive to non-dispersive elements a stencil of five elements to the left and
 228 to the right of the troubled element is chosen to create an eleven element range
 229 where dispersive terms are turned off.

230 2.3. Moving Shoreline

231 The moving shoreline algorithm follows the work of Bunya *et al.* [44]. This
 232 wetting and drying scheme was specifically designed to work well with DG finite
 233 element schemes and takes advantage of their local mass conservation properties.
 234 As such the scheme is capable of tracking a moving shoreline while preserving
 235 local and global mass conservation properties.

236 A minimum water level, H_0 , is set to ensure that a thin layer of water exists
 237 throughout the element. Elements that do not have a water level higher than
 238 this threshold are considered dry. The minimum water level in a dry element is
 239 set to be parallel to the bathymetry and the gravitational constant is set to zero
 240 to eliminate nonphysical drainage of the element. Furthermore, the dynamic

241 pressure within these elements is set to zero. The reader is directed to Bunya
 242 *et al.* [44] for details.

243 3. Numerical Discretization

244 The governing equations are solved using a mixed finite element/finite dif-
 245 ference scheme in space and an explicit third order Strong–Stability–Preserving
 246 Runge Kutta method in time. The elliptic pressure–Poisson problem is solved
 247 using second order central finite differences in space. The modified SWE’s are
 248 solved using a second order DG finite element discretization with linear nodal
 249 basis functions. The present work focuses on problems in one–horizontal di-
 250 mension. Extension to two horizontal dimensions will be presented in future
 251 work.

252 3.1. Computational Grid

253 The spatial domain is partitioned into N elemental domains,

$$254 \quad \Omega = \{\Omega_i : \Omega_i \in [x_{i-1/2}, x_{i+1/2}], i = 1 \dots N\}, \quad (18)$$

255 where the nodal values, x_i , are located at each elemental center and the element
 256 face values, $x_{i\pm 1/2}$, are located at the elemental interfaces. In order to facilitate
 257 a simpler coupling paradigm between the DG finite element solution and the
 258 finite difference pressure–Poisson problem, the finite element mesh is taken to
 259 be structured. The finite difference nodes are defined at the elemental centers,
 260 x_i for $i = 1 \dots N$, such that the computational domain makes up an Arakawa
 261 staggered C grid in space.

262 3.2. Dynamic Pressure

263 Given a particular free–surface displacement and velocity field it is possible
 264 to determine the dynamic pressure profile through the solution of an elliptic
 265 pressure–Poisson equation [33]. For the $O(\mu^2)$ model the system reduces to an
 266 equation of one variable, P_1 . Each of the terms in the expansion of equation
 267 (10), can be found through the following pressure–Poisson problem,

$$\hat{b}_{3,1,1} \frac{\partial^2 P_1}{\partial x^2} + \hat{b}_{2,1,1} \frac{\partial P_1}{\partial x} + \hat{b}_{1,1,1} P_1 = \hat{b}_{0,1}. \quad (19)$$

Considering the optimum basis functions and weighting function as discussed by Donahue *et al.* [33] the coefficients are,

$$\hat{b}_{3,1,1} = -\mu^2 \frac{1}{2} (h + \delta\eta)^2, \quad (20)$$

$$\hat{b}_{2,1,1} = -\mu^2 \frac{1}{12} (h + \delta\eta) \left(3 \frac{\partial h}{\partial x} + 8\delta \frac{\partial \eta}{\partial x} \right), \quad (21)$$

$$\begin{aligned} \hat{b}_{1,1,1} = & -\frac{1}{12} \left(-15 - 3\mu^2 \left(\frac{\partial h}{\partial x} \right)^2 - 6\mu^2 \delta \eta \frac{\partial^2 h}{\partial x^2} + 4\mu^2 \delta \frac{\partial h}{\partial x} \frac{\partial \eta}{\partial x} \right. \\ & \left. - 8\mu^2 \delta^2 \left(\frac{\partial \eta}{\partial x} \right)^2 + 4\mu^2 \delta^2 \eta \frac{\partial^2 \eta}{\partial x^2} + 2\mu^2 h \left(2\delta \frac{\partial^2 \eta}{\partial x^2} - 3 \frac{\partial^2 h}{\partial x^2} \right) \right), \end{aligned} \quad (22)$$

$$\begin{aligned} \hat{b}_{0,1} = & \frac{1}{12} g \left[6\delta\mu^2 \eta^2 \frac{\partial^2 h}{\partial x^2} + 4\mu^2 h \left(3 \frac{\partial h}{\partial x} \frac{\partial \eta}{\partial x} - 2\delta \left(\frac{\partial \eta}{\partial x} \right)^2 + h \frac{\partial^2 \eta}{\partial x^2} \right) \right. \\ & \left. \eta \left(15 + 3\mu^2 \left(\frac{\partial h}{\partial x} \right)^2 + 8\delta\mu^2 \frac{\partial h}{\partial x} \frac{\partial \eta}{\partial x} + 2\mu^2 h \left(3 \frac{\partial^2 h}{\partial x^2} + 2\delta \frac{\partial^2 \eta}{\partial x^2} \right) \right) \right]. \end{aligned} \quad (23)$$

Once P_1 has been determined then P_2 can be found explicitly with the following expression,

$$P_2 = \frac{3}{4} \left[g \left(1 + \left(\frac{\partial h}{\partial x} \right)^2 \right) \eta - (h + \delta\eta) \frac{\partial h}{\partial x} \frac{\partial P_1}{\partial x} + \left(1 + \left(\frac{\partial h}{\partial x} \right) \right) P_1 \right]. \quad (24)$$

Equations (19)–(24) are solved on a regular structured mesh, as described above, using a second order central difference method in space. Once P_1 and P_2 have been obtained they are used to find $\hat{P}_d$ and P_b as follows,

$$\hat{P}_d = \frac{1}{2} H \left(-g\eta + P_1 \right) \quad (25)$$

$$P_b = -g\eta + P_1 - \frac{1}{3} P_2 \quad (26)$$

A similar technique is applied to determine the solution for $\hat{P}_d$ and P_b in the weakly nonlinear $O(\mu^4)$ case, however for brevity, details are not shown here but may be found in Donahue *et al.* [33].

280 3.3. Nonlinear Shallow Water Equations

281 A DG finite element solution is used to discretize the spatial derivatives in
 282 the conservation of mass and momentum equations. A focus of this work is to
 283 generate a simple coupling paradigm between these equations and the pressure-
 284 Poisson based dynamic pressure solution for application in coastal ocean cir-
 285 culation models. Many of the most popular coastal ocean circulation models
 286 employ some discontinuous framework for solving the conservation of mass and
 287 momentum equations. For example the Discontinuous Galerkin Shallow Water
 288 Equation Model (DG-SWEM) [45, 46] and the Finite Volume Coastal Ocean
 289 Model (FVCOM) [47] both solve the SWE's on an unstructured triangle based
 290 grid using the DG finite element method and finite volume method respectively.

291 Assuming a time invariant bottom bathymetry, $\partial h / \partial t = 0$, the conservation
 292 of mass and conservation of momentum with a Manning's n bottom friction,
 293 equations (1) and (2), can be rewritten as,

$$\frac{\partial H}{\partial t} + \frac{\partial Q}{\partial x} = 0, \quad (27)$$

$$\frac{\partial Q}{\partial t} + \delta \frac{\partial}{\partial x} \left(\delta u^2 H + \frac{1}{2} g \delta \eta^2 + g h \eta + \hat{P}_d \right) = \frac{\partial h}{\partial x} (g \eta + P_b) - \mu^2 \frac{g n^2 Q |Q|}{H^{1/3}}, \quad (28)$$

294 where $Q = \bar{u}H$ is the flow discharge and $H = (h + \delta \eta)$ is the total water column
 295 height.

296 A finite element space $V_h^k = \{v : v|_{\Omega_i} \in P^k(\Omega_i), i = 1 \dots N\}$ is defined, where
 297 $P^k(\Omega_i)$ is the space of piecewise polynomials of degree of at most k defined over
 298 the element Ω_i . In this work, we only consider the case where $k = 1$. The finite
 299 element representation of the variables on a given element, η , Q and h follows,

$$300 \quad (\eta_h, Q_h, h_h) = \sum_{j=0}^k \psi_j(x) (\hat{\eta}, \hat{Q}, \hat{h})_j, \quad (29)$$

301 where $\psi_j \in V_h^k$ is the j^{th} nodal basis function and the $\hat{\eta}_j$, $\hat{Q}_j$ and $\hat{h}_j$ are
 302 respective weights of the basis functions. A weak formulation of the equations is
 303 found through multiplication of equations (27) and (28) by a test function $w_h|_{\Omega_i}$

and integrating over the local elemental domain. Application of the integration
by parts rule leads to the following local problem,

$$\left(\frac{\partial H}{\partial t}, w_h\right)_{\Omega_i} - \left(Q, \frac{dw_h}{dx}\right)_{\Omega_i} + \langle Q, w_h \rangle_{\partial\Omega_i} = 0, \quad (30)$$

$$\left(\frac{\partial Q}{\partial t}, w_h\right)_{\Omega_i} - \left(\delta r, \frac{dw_h}{dx}\right)_{\Omega_i} - (s, w_h)_{\Omega_i} + \langle r, w_h \rangle_{\partial\Omega_i} = 0, \quad (31)$$

where $r = \delta u^2 H + \frac{1}{2} g \delta \eta^2 + g h \eta + \hat{P}_d$ and $s = \frac{\partial h}{\partial x} (g \eta + P_b) - \mu^2 \frac{g n^2 Q |Q|}{H^{1/3}}$ and
the operators $(a, b)_{\Omega_i}$ and $\langle a, b \rangle_{\partial\Omega_i}$ represent the inner products of a and
 b over the element and along the element boundaries respectively. The exact
solutions for H and Q are approximated by $H_h = (h_h + \delta \eta_h) \in V_h^k$ and $Q_h \in V_h^k$
respectively. Furthermore, the solutions are allowed to be discontinuous across
the element edges located at $x_{i-1/2}$ and $x_{i+1/2}$. Thus the flux terms in each of
the equations, $\langle Q, w_h \rangle_{\partial\Omega_i}$ and $\langle r, w_h \rangle_{\partial\Omega_i}$ can be multi-valued at the boundary.
This problem is addressed through the introduction of single valued, monotone,
numerical fluxes, $\hat{Q} = \hat{Q}(Q_h^+, Q_h^-)$ and $\hat{r} = \hat{r}(r_h^+, r_h^-)$, where,

$$(*)^+ = \lim_{x \rightarrow x_{i-1/2}^+} (*), \quad (32)$$

$$(*)^- = \lim_{x \rightarrow x_{i-1/2}^-} (*), \quad (*) = \eta, u. \quad (33)$$

In this work the local Lax–Friedrichs (LLF) flux was chosen,

$$\hat{f}(a, b) = \frac{1}{2} (f(a) + f(b) - C(b - a)), \quad (34)$$

where C is chosen as the maximum eigenvalue of the flux Jacobian at the element
interface. Considering this definition of the value at each flux interface equations
(30) and (31) can be rewritten as,

$$\left(\frac{\partial H_h}{\partial t}, w_h\right)_{\Omega_i} - \left(Q_h, \frac{dw_h}{dx}\right)_{\Omega_i} + \langle \hat{Q}, w_h \rangle_{\partial\Omega_i} = 0, \quad (35)$$

$$\left(\frac{\partial Q_h}{\partial t}, w_h\right)_{\Omega_i} - \left(\delta r_h, \frac{dw_h}{dx}\right)_{\Omega_i} - (s_h, w_h)_{\Omega_i} + \langle \hat{r}, w_h \rangle_{\partial\Omega_i} = 0. \quad (36)$$

For the work here the DG basis functions and weighting functions were chosen to be nodal Lagrange polynomials. All integrations were performed using Gauss–Legendre quadrature methods.

3.4. Finite Difference (PPBOUSS)/Finite Element (SWE) Coupling

The finite difference mesh is constructed through the center points of each element in the finite element mesh. Coupling from the DG finite element solution to the finite difference pressure–Poisson solution is achieved through a projection of the free surface and velocity solutions onto the center point of each element. Evaluation of the local free–surface and velocity at the element center, x_i , provides,

$$(\eta, Q)_i = \sum_{j=0}^k \psi_j(x_i)(\hat{\eta}, \hat{Q})_j, \quad (37)$$

which can then be used in the finite difference solution to the PPBOUSS problem.

In the reverse direction, the finite difference solution to the PPBOUSS equation is projected onto a nodal finite element framework through interpolation of the cell centered solutions onto the element edges and then treating these values as the prospective weights of a linear nodal finite element solution.

Here we note that extension to higher–order basis functions in the DG solution is possible, however, the convergence properties of the coupled model are limited to the second–order accuracy of the finite difference method. It is for this reason that the present work focuses on linear nodal basis functions in the DG solution.

3.5. Discretization in time

Following the spatial discretization of the non–pressure terms, a set of ordinary differential equations in time must be solved. In this work we use a third order Strong–Stability–Preserving Runge Kutta method [48, 49]. The dynamic pressure profile is found for each stage of the Runge Kutta method and used to inform the solution at that stage.

348 4. Numerical Test Cases

349 Previous validation cases tested with this model focused on the propagation
 350 of solitary and monochromatic waves without wave breaking or wetting and
 351 drying [33]. In order to demonstrate the accuracy of the model and to test
 352 the wetting/drying and breaking algorithm, the model was verified against the
 353 analytical Carrier–Greenspan solution [50] and validated for three experimental
 354 test cases: runup of solitary waves [51], plunging and spilling breakers [52, 53]
 355 and solitary wave propagation over a breakwater [54].

356 In all cases with solitary waves an initial solitary wave profile is imposed on
 357 the free-surface and velocity at $t = 0s$, such that,

$$\eta_{t=0} = a_0 \text{sech}^2 \left(\kappa (x - x_0)^2 \right), \quad (38)$$

$$\bar{u}_{t=0} = c \left(1 - \frac{h_b}{h_b + \eta_{t=0}} \right), \quad (39)$$

$$Q_{t=0} = c a_0 \text{sech}^2 \left(\kappa (x - x_0)^2 \right) \quad (40)$$

358 where,

$$\kappa = \frac{\sqrt{3a_0}}{2h_b\sqrt{h_b + a_0}}, \quad (41)$$

$$c = \sqrt{g(h_b + a_0)}, \quad (42)$$

359 and a_0 is the solitary wave amplitude, h_b is the deep water depth, x_0 is the
 360 location of the solitary wave peak and $g = 9.81$ is the gravitational constant.

361 4.1. Solitary wave runup on a simple beach

362 One of the simplest test problems for the verification of wetting and drying
 363 is the simulation of the interaction between a solitary wave and a simple beach
 364 with a slope angle of α (Figure 1a).

365 The analytical solution of a non-breaking solitary wave on a sloping beach
 366 was explored by Synolakis *et al.* [51, 55] who built upon the theory of Carrier

367 and Greenspan [50] for analytical solutions to the SWE's for non-breaking waves
 368 on a sloping beach.

369 The Carrier-Greenspan solution provides an excellent test case to assess the
 370 accuracy of a model's implementation of wetting and drying. The model is
 371 compared with the analytical Carrier-Greenspan solution for a solitary wave of
 372 initial amplitude to depth ratio of $A/h = 0.0185$ propagating towards a beach
 373 with a slope of $\cot(\alpha) = 1/19.85$.

374 The computational domain is broken into elements with a length of $\Delta x =$
 375 $0.05m$ and the simulation was run with a CFL condition of 0.1 to ensure that
 376 spatial errors dominated. The analytical solution is derived without considering
 377 bottom friction, thus for this simulation the Manning's n value is set to zero,
 378 the wetting/drying criteria was set to $H_0 = 0.00001m$.

379 Figure 2 shows the results of the simulation for the entire domain at seven
 380 times. The figure demonstrates that the model performs very well at capturing
 381 the propagation of the solitary wave and the wetting and drying front. Figure 3
 382 shows the time history of the model and the analytical solution for two locations
 383 in the domain, $x/h = 9.95$ and $x/h = 0.25$. Note that the location at $x/h = 0.25$
 384 is initially wet and then undergoes a transition from wet to dry as the solitary
 385 wave propagates up the beach and then recedes. This is accurately represented
 386 in the model results supporting the argument that the model captures both the
 387 propagation of the solitary wave and the wetting/drying front. The comparison
 388 of the analytical solution and model results is nearly perfect.

389 4.2. Laboratory comparison for runup of solitary wave on a simple beach

390 A set of two experiments conducted by Synolakis [51, 55] in a basin of di-
 391 mension (31.73 m long, 0.6096 m deep and 0.3937 m wide) provide the basis for
 392 validation of the model for the propagation of both breaking and nonbreaking
 393 waves. Two solitary waves propagate towards a sloping beach with a slope of
 394 $\cot(\alpha) = 1/19.85$. The first experiment considered a solitary wave with an am-
 395 plitude of $a_0/h = 0.0185$, which is not expected to break on the approach to
 396 the beach. The second experiment considered a solitary wave with a consider-

ably higher amplitude of $a_0/h = 0.30$, which would be expected to break at the beach. This test case further validates the ability of the model to simulate the propagation of solitary waves in the nearshore and capture both wave breaking and wave runup.

The computational domain is discretized into elements with a length of $\Delta x = 0.05m$ and the simulation was conducted using a CFL condition of 0.1 to ensure that spatial errors dominate. A Manning's n value of 0.004 was used for each simulation to approximate the bottom friction of the experimental setup, with a wetting/drying criteria of $H_0 = 0.0005m$.

Figures 4 and 5 show a comparison between the model and the data for the nonbreaking and breaking waves respectively. In both cases the model performs very well, capturing both the breaking and wetting/drying dynamics. A minor discrepancy can be seen in Figure 4 at $t = 70s$ during the drying process. The model appears to dry more quickly than the measurements show, however the difference is relatively minor and the overall dynamics of the system are accurately captured.

Similarly the breaking solitary wave case is well represented by the model. There is slight phase lag in the initial wetting front, see Figure 5 at $t = 25s$. However, in the next panel, for $t = 30s$, it is observed that wetting front represents the data very well during the wave run-up. Once again a lag is observed for the run-down of the wave, see the solution at $t = 50s$ however the overall dynamics of the data are captured very well. This lag in the propagation of the wave run-up and run-down may be due to a combination of the choice of bottom friction and wetting/drying criteria, however a rigorous analysis of this is not presented here.

Importantly, both the $O(\mu^2)$ and $O(\mu^4)$ cases produce fairly similar results. Outside of the breaking zone this demonstrates how accurate the $O(\mu^2)$ model is at capturing the propagation of solitary waves. Within the breaking zone nearshore the dispersive terms have been turned off as per the breaking wave criteria in the model and both simulations are solving the same shallow water equations.

428 In addition to the three cases discussed so far, Synolakis *et al.* [51, 55]
 429 tested the theory behind the maximum run-up of solitary waves on a sloping
 430 beach for numerous breaking and non-breaking cases. Synolakis experimentally
 431 tested 77 cases for varying ratios of wave height to water depth (a_0/h). Each
 432 experiment was run for still water levels of varying depths. For each case the
 433 ratio of maximum level of run-up to depth was recorded (R/h). In order to
 434 test the ability of the model to simulate the run-up of solitary waves for both
 435 breaking and non-breaking cases, all 77 cases were simulated for a beach of
 436 slope $\cot(\alpha) = 1/19.85$. The computational domain divided into elements of
 437 size $\Delta x/h = 0.1$ with a CFL condition of 0.2 to ensure that spatial errors
 438 dominate. A Manning's n value of 0.004 was chosen to approximate the bottom
 439 friction associated with the experimental setup. Considering that each of the
 440 77 cases represented solitary waves of varying amplitude, the wetting/drying
 441 criteria was set to $H_0 = 0.0001m$.

442 Figure 6 shows a comparison of the recorded run-up level versus the sim-
 443 ulated run-up level. According to the analysis conducted by Synolakis *et al.*
 444 [55], solitary waves with an initial amplitude to depth ratio of $a_0/h > 0.055$
 445 became breaking waves upon approaching the shore. The comparison of the
 446 maximum run-up for the 77 cases shows good agreement between the model
 447 and the recorded data, with a good correlation coefficient of $R^2 = 0.8495$ for the
 448 $O(\mu^2)$ case and a significant improvement for the $O(\mu^4)$ case with $R^2 = 0.9006$
 449 determined by calculating the correlation between the measured and modelled
 450 data. The modelled run-up is accurate for both the breaking and non-breaking
 451 regime, however it is clearly less accurate when the waves are expected to break.
 452 This is illustrated in greater detail by the observation that for just the waves in
 453 the non-breaking zone the correlations for the $O(\mu^2)$ and $O(\mu^4)$ are $R^2 = 0.9724$
 454 and $R^2 = 0.9764$ respectively. In the breaking regime the correlations drop to
 455 $R^2 = 0.5613$ and $R^2 = 0.7107$ for the $O(\mu^2)$ and $O(\mu^4)$ models respectively.
 456 This can be attributed to the approximate nature of the wave breaking criteria.
 457 Although a travelling bore representation of a breaking wave is fairly accurate,
 458 it does not capture all of the three-dimensional dynamics in the breaking zone.

459 Furthermore, in the breaking regime the run-up is consistently over-predicted.
 460 One explanation for this might be due to mesh resolution, however the similar
 461 results were observed at finer resolution with only a minor improvement. An ad-
 462 justment of wetting/drying threshold did improve results in the breaking regime
 463 at the cost of accuracy in the non-breaking regime.

464 4.3. *Plunging and Spilling Breakers*

465 To further test the model's ability to capture the dynamics of breaking wave
 466 in the nearshore, we compare model results to experiments conducted by Ting
 467 and Kirby [52, 53]. In the previous cases only a single solitary wave is present,
 468 in this test case a train of regular nonlinear waves is generated on the left hand
 469 side of the domain which propagate towards the shore. This test case allows us
 470 to test not only the ability for the model to resolve complex nonlinear waves, but
 471 to also simulate the train of breaking waves as well as the interaction between
 472 the wave train and wave energy reflected off the beach.

473 Ting and Kirby ran a series of experiments in a basin with dimensions of
 474 (40m long, 0.6m wide and 1.0m deep) to examine regular waves breaking on
 475 plane slopes. Both plunging [52] and spilling [53] breakers were considered and
 476 data was collected at a series of stations throughout the domain. Regular nonlin-
 477 ear waves were generated in a water depth of 0.4m and propagated downstream
 478 to a plane beach with slope $\cot(\alpha) = 1/35$, see figure 1b. In the present model,
 479 the wave train is generated using the sponge layer generation/absorption tech-
 480 nique of Zhang *et al.* [56]. A Fourier decomposition of the regular nonlinear
 481 waves is calculated using the techniques of Fenton [57]. In the present work an
 482 approximation of the regular nonlinear wave train uses 32 Fourier modes.

483 The details for each case are as follows,

- 484 1. Plunging Breaker, $H/h = 0.32$, $T/\sqrt{gh} = 2.524$
- 485 2. Spilling Breaker, $H/h = 0.3125$, $T/\sqrt{gh} = 1.01$

486 where T is the period of the wave in seconds, h is the offshore still water depth
 487 and H is the overall wave amplitude. The computational domain was broken

into elements of length $\Delta x = 0.05m$ to ensure that higher frequency modes in the Fourier decomposition were accurately represented. A CFL condition of 0.2 was chosen to ensure that spatial errors dominate. Considering the amplitudes of the incident waves, the wetting/drying threshold was set to $H_0 = 0.00001m$ and a Manning's n value of 0.01 was chosen to represent the bottom friction of the experiment.

At each measurement location, see Figure 1b, Ting and Kirby performed the experiments several times in order to measure the velocity at varying depths. In addition, for each experiment, a free-surface displacement was similarly recorded. Due to this, the experimental data at measurement location has a set of 20–30 distinct time histories for the free-surface. Here, we compare the displacement of the free-surface about the mean water level of the numerical simulations with the displacement of the free-surface about the mean water level of each dataset. Given the volume of datasets for each measurement location the numerical simulation is compared with the minimum and maximum mean water levels at each data point. The model is compared over all experiments for the $O(\mu^2)$ and $O(\mu^4)$ simulations.

Figures 7 and 8 show the model timeseries comparison for the spilling and plunging type breaker respectively. Figures 9a and 9b show a comparison at each station for the average wave crest, mean water level and average wave trough elevations for the $O(\mu^2)$ and $O(\mu^4)$ simulations respectively.

As can be seen from the figures, the model is capable of capturing the propagation of the regular nonlinear waves very well, and in each case is able to represent the breaking dynamics, including the high frequency, low amplitude waves that are present due to reflection of the wave off the sloped beach. For the direct timeseries comparison of the spilling breaker case as shown in figures 7 and 8, the model results are nearly identical to the mean water level over all experiments. There are minor discrepancies at wave crests and troughs which can be explained by the numerical technique to capture wave breaking. A more diffusive breaking model would provide a smoother breaking front. This reasoning can also be used to explain the discrepancies observed in the wave peaks

519 between $x = 6m$ and $x = 8m$, see figures 9a and 9b.

520 4.4. Solitary waves travelling over a breakwater

521 Hsiao *et al.* [54] performed a series of experiments to simulate the interaction
522 of tsunami type waves with an impermeable seawall. Hsiao *et al.* considered
523 solitary waves to be representative of tsunami type waves. The experiments
524 took place at the Tainan Hydraulics Laboratory (THL) at the National Cheng
525 Kung University, Taiwan. Solitary waves were generated in a two-dimensional
526 wave flume with dimensions of ($22m$ long, $0.5m$ wide and $0.75m$ deep). A
527 simple sloping beach made of aluminum, with a Plexiglas surface and a slope
528 of $\cot(\alpha) = 1/20$ was situated at a distance of $10m$ from the wave generation
529 paddle. A separate Plexiglas trapezoidal section with a seaward slope of $1 : 4$
530 and a landward slope of $1 : 1.8$ was placed on the beach to represent a seawall.
531 See Figure 1c for a schematic of the initial setup.

532 This test case is well suited for determining how well this model can simulate
533 the overtopping of nearshore structures. The simulation was run using a CFL
534 condition of 0.1, a Manning's n value of 0.01 and a wetting/drying criteria
535 of $H_0 = 0.00001m$. The computational domain was divided into elements of
536 two resolutions, the first was of length $\Delta x = 0.025m$, and a finer resolution of
537 $\Delta x = 0.01m$ was also examined during this study. The higher resolution $O(\mu^2)$
538 simulation exhibited high frequency oscillations which limited the accuracy of
539 the solution. This illustrated a well known limitation in Boussinesq modelling
540 that at very fine resolutions higher frequency waves are able to be resolved
541 which can lead to instabilities in the solution, the $O(\mu^4)$ model with it's higher
542 dispersive accuracy did not exhibit the same oscillations at this resolution. In
543 order to provide a more fair comparison between the two models the results of
544 the $\Delta x = 0.025m$ simulations are presented here.

545 Three solitary waves were tested, as shown in Table 1. Figures 10 through
546 12 show the results of the laboratory measurements compared with model pre-
547 dictions at stations with locations given in Table 2. For stations on the seaward
548 side of the seawall and on top of the seawall the model performs very well,

549 predicting the advancing breaking wave as well as the appropriate wetting and
 550 drying front, stations s_1 – s_7 . The solution fidelity breaks down on the landward
 551 side of the seawall, stations s_8 – s_{11} . The model accurately predicts overtopping
 552 and inundation of the landward side. Similarly it does a good job of capturing
 553 the overall landward side flooding state after some time has passed. The model
 554 is unable, however, to capture the full dynamics of the wave overtopping. This
 555 can be attributed to the fact that the model is not fully three-dimensional, and
 556 thus not capable of capturing the multi-valued surface that occurs when the
 557 wave crashes into the seawall, causing water to spray forward.

558 Although the model is incapable of capturing all of the fine scale processes
 559 that occur when a wave interacts with a barrier, it is noted that the model does
 560 accurately capture the overall dynamics of wave breaking and inundation of the
 561 beach on the landward side of the seawall.

562 **5. Conclusions and Discussion**

563 In the surf zone, dispersive and shoaling accuracy play a major role in captur-
 564 ing the nearshore wave dynamics. The nondispersive properties of the shallow
 565 water equations (SWE) for higher frequency waves makes them inapplicable in
 566 this regime. In order to capture these higher order physics in the nearshore a
 567 separate model is often required. The present work has developed a Boussinesq-
 568 type system that is capable of seamlessly extending SWE models, thereby intro-
 569 ducing a framework for extending the applicability of standard SWE's instead
 570 of replacing them in the nearshore.

571 A full surf-zone phase resolving wave model has been developed and vali-
 572 dated against one analytical and several experimental results. The dispersive
 573 terms in the model are captured through the integration of the dynamic pressure
 574 term into the standard conservation of mass and momentum equations. The re-
 575 sult is a computationally efficient model that is relatively simple to implement.

576 Use of a Green–Naghdi type expansion of the pressure profile leads to a stand
 577 alone set of equations for the pressure built on the pressure–Poisson equation

578 and the bottom boundary equation for pressure. The pressure–Poisson approxi-
579 mation allows for the implementation of both $O(\mu^2)$ and $O(\mu^4)$ models without
580 the need to solve any mixed space/time derivatives or to resolve third order or
581 higher spatial derivatives. Given the prohibitive nature of solving the $O(\mu^4)$
582 equations in mixed space–time Boussinesq systems the present work offers an
583 advantage for solving problems where higher–order dispersive accuracy is re-
584 quired.

585 In the present work a finite difference solution for the pressure is coupled
586 with a DG finite element solution for the depth integrated SWE’s to include
587 nonhydrostatic terms, however the theory supports coupling of the pressure–
588 Poisson problem to any SWE solver. The coupling paradigm is relatively simple,
589 amounting to a simple modification in the convective term and the source term of
590 the conservation of momentum equation in the SWE’s. Thus the present model
591 represents a straightforward means to incorporate higher order dispersion into
592 already established SWE solvers with very little structural change to the SWE
593 model.

594 Verification of the model with the well known Carrier–Greenspan solution
595 showed excellent agreement, demonstrating accuracy in the wetting/drying al-
596 gorithm. Validation of the model for both breaking and nonbreaking wave prop-
597 agation over both a simple sloping beach topography as well as an impermeable
598 seawall demonstrated the models ability to capture the nearshore breaking and
599 shoaling wave dynamics and inundation on the landward side of a barrier. Al-
600 though the model is limited to single–valued free–surfaces and thus is unable
601 to accurately capture the small scale vertical structures that form during fluid–
602 structure interactions or wave overtopping, the overall dynamics of the system
603 are accurately represented.

604 In comparison with equivalent order Boussinesq–type models the accuracy
605 of the present model is similar, however the present work is better suited for
606 coupling with current SWE based coastal ocean circulation models. In addition,
607 the set of equations solved in order to resolve the dispersive terms is a scalar
608 equation and, contrary to standard Boussinesq approaches, does not involve

609 solutions to mixed space/time derivatives.

610 Future work will focus on extension of the model to two-horizontal dimen-
611 sions. In order to take full advantage of the DG finite element approach, future
612 work will also build upon the work of Panda *et al.* [58] towards the development
613 of a Local Discontinuous Galerkin (LDG) approximation of the pressure-Poisson
614 problem.

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628 **7. Figures**

(a) Solitary wave on a simple beach [51, 55]. Station locations are at a) $x/h = 9.95$ and b) $x/h = 0.25$.

(b) Ting and Kirby [52, 53] breaking waves experiments. Measurement locations are shown by stations s_1 – s_7 .

(c) Hsiao *et al.* [54] experimental setup, stations s_1 – s_{11} are shown with dashed lines.

Figure 1: Computational and Experimental Domain Setups

Figure 2: Comparison of model solution with the Carrier–Greenspan analytical solution for solitary wave run-up on a simple beach [51, 55]. Each panel shows the spatial comparison of the two solutions at a different timesnap. Gray circles: analytical, $\circ$; Solid red line: $O(\mu^2)$ model, $-$; Blue dashed line: $O(\mu^4)$ model, $--$.

Figure 3: Comparison of model solution with Carrier–Greenspan analytical solution for solitary wave run-up on a simple beach [51, 55]. The top panel shows the time history for a location near the beginning of the beach, $x/h = 9.95$. The bottom panel shows the time history for a location near the still water beach front, $x/h = 0.25$. The flat region in the bottom panel between $t\sqrt{g/h} = 60$ and $t\sqrt{g/h} = 80$ represents a period when the location was dry. Colors are the same as for Figure 2

Figure 4: Comparison of model and experimental results for run-up of a non-breaking solitary wave on a simple beach [51, 55]. Gray circles: data, $\circ$; Solid red line: $O(\mu^2)$ model, $-$; Blue dashed line: $O(\mu^4)$ model, $--$.

Figure 5: Comparison of model and experimental results for run-up of a breaking solitary wave on a simple beach [51, 55]. Gray circles: data, $\circ$; Solid red line: $O(\mu^2)$ model, $-$; Blue dashed line: $O(\mu^4)$ model, $--$.

Figure 6: Comparison of depth normalized wave run-up for seventy-seven solitary wave run-up experiments conducted by Synolakis *et al.* [51, 55]. The breaking threshold is set to $A/h = 0.055$. Model predicted run-up (dashed black or dash-dot magenta line) for $O(\mu^2)$ and $O(\mu^4)$ respectively, experimental measurements (red circles).

Figure 7: Comparison of $O(\mu^2)$ (red line) and $O(\mu^4)$ (dashed blue line) modeled free-surface displacement to experimental measurements for a spilling breaker regular nonlinear wave train on a simple beach at several measurement station locations [53]. The shaded area represents the data range for all experimental data.

Figure 8: Comparison of $O(\mu^2)$ (red line) and $O(\mu^4)$ (dashed blue line) modeled free-surface displacement to experimental measurements for a plunging breaker regular nonlinear wave train on a simple beach at several measurement station locations [52]. The shaded area represents the data range for all experimental data.

(a) Spilling Breaker

(b) Plunging Breaker

Figure 9: Comparison of mean water level (magenta circles), mean peak wave height (blue triangles) and mean trough depth (red inverted triangle) for the spilling and plunging breaker data and model simulations. Model simulations are shown using a solid line and dashed line for the $O(\mu^2)$ and $O(\mu^4)$ models respectively.

Figure 10: Comparison of modeled free-surface displacement and experimental measurements for case 1 of solitary waves interacting with a impermeable seawall, see Table 2. Experimental data (gray circles), $O(\mu^2)$ (red line) modeled data and $O(\mu^4)$ (blue dashed line) modeled data[54].

Figure 11: Comparison of modeled free-surface displacement and experimental measurements for case 2 of solitary waves interacting with a impermeable seawall, see Table 2. Experimental data (gray circles), $O(\mu^2)$ (red line) modeled data and $O(\mu^4)$ (blue dashed line) modeled data[54].

Figure 12: Comparison of modeled free-surface displacement and experimental measurements for case 3 of solitary waves interacting with a impermeable seawall, see Table 2. Experimental data (gray circles), $O(\mu^2)$ (red line) modeled data and $O(\mu^4)$ (blue dashed line) modeled data[54].

Table 1: Experimental setup for three solitary waves in Hsiao *et al.* [54] experiment.

Case	h_0 (m)	H_0 (m)	$\epsilon = H_0/h_0$
1	0.2	0.07	0.35
2	0.22	0.0638	0.29
3	0.256	0.0589	0.23

Table 2: Station locations for Hsiao *et al.* [54] experiment.

Location	Station #	x (m)
Deep Water	1	5.9
	2	7.6
	3	9.644
Seaward of seawall	4	10
	5	10.462
	6	0.732
On the seawall	7	11.005
	8	11.024
	9	11.045
Landward of seawall	10	11.12
	11	11.57

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