Approximate Analytic Solutions to Coupled Nonlinear Dirac equations

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We consider the coupled nonlinear Dirac equations (NLDE's) in 1+1 dimensions with scalar-scalar self interactions $\frac{g_1^2}{2}(\bar{\psi}\psi)^2 + \frac{g_2^2}{2}(\bar{\phi}\phi)^2 + g_3^2(\bar{\psi}\psi)(\bar{\phi}\phi)$ as well as vector-vector interactions of the form $\frac{g_1^2}{2}(\bar{\psi}\gamma_\mu\psi)(\bar{\psi}\gamma^\mu\psi) + \frac{g_2^2}{2}(\bar{\phi}\gamma_\mu\phi)(\bar{\phi}\gamma^\mu\phi) + g_3^2(\bar{\psi}\gamma_\mu\psi)(\bar{\phi}\gamma^\mu\phi)$. Writing the two components of the assumed rest frame solution of the coupled NLDE equations in the form $\psi = e^{-i\omega_1 t}\{R_1 \cos \theta, R_1 \sin \theta\}$, $\phi = e^{-i\omega_2 t}\{R_2 \cos \eta, R_2 \sin \eta\}$, and assuming that $\theta(x), \eta(x)$ have the *same* functional form they had when $g_3=0$, which is an approximation consistent with the conservation laws, we then find approximate analytic solutions for $R_i(x)$ which are valid for small values of g_3^2/g_2^2 and g_3^2/g_1^2 . In the nonrelativistic limit we show that both of these coupled models go over to the same coupled nonlinear Schrödinger equation for which we obtain two exact pulse solutions vanishing at $x \rightarrow \pm\infty$.

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I. INTRODUCTION

The nonlinear Dirac (NLD) equation in 1 + 1 dimensions [1] has a long history and has emerged as a useful model in many physical systems such as extended particles [2–4], the gap solitons in nonlinear optics [5], light solitons in waveguide arrays and experimental realization of an optical analog for relativistic quantum mechanics [6–8], Bose-Einstein condensates in honeycomb optical lattices [9], phenomenological models of quantum chromodynamics [10], as well as matter influencing the evolution of the universe in cosmology [11]. Further, the multi-component BEC order parameter has an exact spinor structure and serves as the bosonic analog to the relativistic electrons in graphene. To maintain the Lorentz invariance of the NLD equation, the self interaction Lagrangian is built using the bilinear covariants. Of special interest are scalar bilinear covariant and vector bilinear covariant which have particularly attracted a lot of attention.

Classical solutions of nonlinear field equations have a long history as a model of extended particles [12]. In 1970, Soler proposed that the self-interacting 4-Fermi theory was an interesting model for extended fermions. Later, Strauss and Vasquez [13] were able to study the stability of this model under dilatation and found the domain of stability for the Soler solutions. Solitary waves in the 1+1 dimensional nonlinear Dirac equation have been studied [14, 15] in the past in case the nonlinearity parameter $k = 1$, i.e. the massive Gross-Neveu [16] (with $N = 1$, i.e. just one localized fermion) and the massive Thirring [17] models. In those studies it was found that these equations have solitary wave solutions for both scalar-scalar (S-S) and vector-vector (V-V) interactions. The interaction between solitary waves of different initial charge was studied in detail for the S-S case in the work of Alvarez and Carreras [18] by Lorentz boosting the static solutions and allowing them to scatter.

In a previous paper [19] we extended the work of these preceding authors to the case where the nonlinearity was taken to an arbitrary power κ for both the scalar-scalar and vector vector couplings and were able to find solitary wave solutions for an arbitrary nonlinearity parameter κ . In this paper we will extend the previous models in a new direction by looking for solitary wave solutions to the problem of two coupled NLDE's and considering the scalar-scalar coupling as well as the vector-vector coupling between the two fields. That is we assume the interaction Lagrangian has the form $\frac{g_1^2}{2}(\bar{\psi}\psi)^2 + \frac{g_2^2}{2}(\bar{\phi}\phi)^2 + g_3^2(\bar{\psi}\psi)(\bar{\phi}\phi)$ for the scalar-scalar interaction and for the vector-vector interactions of the form $\frac{g_1^2}{2}(\bar{\psi}\gamma_\mu\psi)(\bar{\psi}\gamma^\mu\psi) + \frac{g_2^2}{2}(\bar{\phi}\gamma_\mu\phi)(\bar{\phi}\gamma^\mu\phi) + g_3^2(\bar{\psi}\gamma_\mu\psi)(\bar{\phi}\gamma^\mu\phi)$.

Here we concern ourselves solely with the question of the existence of approximate analytic solutions to these coupled NLDE's in 1 + 1 dimension which reduce to the usual *exact* soliton solutions of the NLDE when the coupling between fields g_3 is set to zero. Of course, there is a huge related literature on solving the related $3+1$ dimensional two-body

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Dirac equations that are used for phenomenologically understanding meson spectroscopy as well as the electromagnetic spectroscopy of the positronium system. A good summary of this approach is found in the work of Crater, Becker, Wong and Van Alstine [20]. Our present paper, however, is only concerned with exploring the existence of bound state solutions of the two coupled 1 + 1 dimensional NLDEs. We note that about four decades ago coupled NLD equations in 1 + 1 dimensions were considered formally [21] but no explicit solutions were obtained.

Our strategy is to write the components of the two coupled nonlinear Dirac equations in rest frame solitary wave form, namely $\psi = e^{-i\omega_1 t} \{R_1 \cos \theta, R_1 \sin \theta\}$, $\phi = e^{-i\omega_2 t} \{R_2 \cos \eta, R_2 \sin \eta\}$ and then assume that the conservation law for linear momentum is satisfied independently for $i = 1, 2$. This assumption is equivalent to saying that $\theta(x), \eta(x)$ have the *same* functional form they had when $g_3=0$. Once one makes that assumption we obtain an analytic expression for $R_i(x)$ which we then show approximately solves the differential equation for $R_i(x)$. The one situation which restricts the validity of this solution occurs in the scalar-scalar interaction case when one of the solitary wave solutions (when $g_3 = 0$) is of a double humped variety. In that case the solution is valid only when the dimensionless coupling constants g_3^2/g_2^2 and g_3^2/g_1^2 are $\leq 1/100$. Otherwise the approximate analytic solutions we have found seem to be numerically accurate in both the scalar-scalar as well as the vector-vector coupled NLD equation as long as the two dimensionless constants are $\leq 1/10$.

II. SCALAR-SCALAR INTERACTIONS

We are interested in solitary wave solutions of the coupled nonlinear Dirac equations (NLDEs) given by

$$(i\gamma^\mu \partial_\mu - m_1)\psi + g_1^2(\bar{\psi}\psi)\psi + g_3^2(\bar{\phi}\phi)\psi = 0, \quad (2.1)$$

$$(i\gamma^\mu \partial_\mu - m_2)\phi + g_2^2(\bar{\phi}\phi)\phi + g_3^2(\bar{\psi}\psi)\phi = 0. \quad (2.2)$$

We can eliminate one of the coupling constants by rescaling the fields, that is if we let $\psi \rightarrow \psi/g_1$, $\phi \rightarrow \phi/g_2$, so that there are two independent dimensionless coupling constants

$$g_{32}^2 = g_3^2/g_2^2, \quad g_{31}^2 = g_3^2/g_1^2. \quad (2.3)$$

as we will discover later. The field equations can be derived from the Lagrangian

$$\begin{aligned} \mathcal{L} &= \bar{\psi}(i\gamma^\mu \partial_\mu - m_1)\psi + \frac{g_1^2}{2}(\bar{\psi}\psi)^2 \\ &+ \bar{\phi}(i\gamma^\mu \partial_\mu - m_2)\phi + \frac{g_2^2}{2}(\bar{\phi}\phi)^2 + g_3^2(\bar{\psi}\psi)(\bar{\phi}\phi) \\ &= \bar{\psi}(i\gamma^\mu \partial_\mu - m_1)\psi + \bar{\phi}(i\gamma^\mu \partial_\mu - m_2)\phi + \mathcal{L}_{int}. \end{aligned} \quad (2.4)$$

We notice the Lagrangian is symmetric under the interchange $\psi \rightarrow \phi, m_1 \rightarrow m_2$ and $g_1 \rightarrow g_2$.

We next choose the following representation of the γ matrices:

$$\gamma^0 = \sigma_3, \quad i\gamma_1 = \sigma_2, \quad (2.5)$$

where the σ_i are the usual Pauli spin matrices.

In the rest frame we assume that the two components of the solutions can be written as

$$\begin{aligned} \psi(x) &= \begin{pmatrix} A(x) \\ B(x) \end{pmatrix} e^{-i\omega_1 t} = R_1(x) \begin{pmatrix} \cos \theta(x) \\ \sin \theta(x) \end{pmatrix} e^{-i\omega_1 t}, \\ \phi(x) &= \begin{pmatrix} C(x) \\ D(x) \end{pmatrix} e^{-i\omega_2 t} = R_2(x) \begin{pmatrix} \cos \eta(x) \\ \sin \eta(x) \end{pmatrix} e^{-i\omega_2 t}. \end{aligned} \quad (2.6)$$

In the absence of interactions ($g_3 = 0$), the solutions are of two types [19]. When $1 > \omega/m > \omega_c/m$ then the solutions are single humped as they are always in the case of vector-vector interactions discussed below. However for the case $1 > \omega_c/m > \omega/m$ the solutions are double humped and in that regime if the solutions when $g_3 = 0$ are of two different types, then we will find the approximate solutions we obtain are only valid for very small $g_{3i}^2 \leq 1/100$. In component form these two coupled NLDEs can be written as

$$\begin{aligned} \partial_x A + (m_1 + \omega_1)B - g_1^2(A^2 - B^2)B - g_3^2(C^2 - D^2)B &= 0, \\ \partial_x B + (m_1 - \omega_1)A - g_1^2(A^2 - B^2)A - g_3^2(C^2 - D^2)A &= 0, \\ \partial_x C + (m_2 + \omega_2)D - g_2^2(C^2 - D^2)D - g_3^2(A^2 - B^2)D &= 0, \\ \partial_x D + (m_2 - \omega_2)C - g_2^2(C^2 - D^2)C - g_3^2(A^2 - B^2)C &= 0. \end{aligned} \quad (2.7)$$

These are symmetric under the interchange $\{A, B\} \rightarrow \{C, D\}$, $m_1 \rightarrow m_2$ and $g_1 \rightarrow g_2$. These four equations can also be written if we let $y_i = R_i^2(x)$ as:

$$\begin{aligned}\frac{dy_1}{dx} &= 2[g_1^2 y_1^2 (\cos 2\theta) + g_3^2 y_1 y_2 (\cos 2\eta) - y_1 m_1] \sin 2\theta, \\ \frac{dy_2}{dx} &= 2[g_2^2 y_2^2 (\cos 2\eta) + 2g_3^2 y_1 y_2 (\cos 2\theta) - y_2 m_2] \sin 2\eta,\end{aligned}\quad (2.8)$$

and

$$\begin{aligned}\frac{d\theta}{dx} &= g_1^2 y_1 \cos^2 2\theta + g_3^2 y_2 \cos 2\theta \cos 2\eta - m_1 \cos 2\theta + \omega_1, \\ \frac{d\eta}{dx} &= g_2^2 y_2 \cos^2 2\eta + g_3^2 y_1 \cos 2\theta \cos 2\eta - m_2 \cos 2\eta + \omega_2.\end{aligned}\quad (2.9)$$

We can rewrite these equations in terms of the two dimensionless coupling constants by scaling $y_1 \rightarrow y_1/g_1^2$, $y_2 \rightarrow y_2/g_2^2$.

A. Conservation Laws

We have that energy and momentum are conserved, namely

$$\partial_\mu T^{\mu\nu} = 0, \quad (2.10)$$

where the energy-momentum tensor is defined as

$$T_{\mu\nu} = i\bar{\psi}\gamma_\mu\partial_\nu\psi + i\bar{\phi}\gamma_\mu\partial_\nu\phi - g_{\mu\nu}\mathcal{L}. \quad (2.11)$$

and $\mathcal{L}$ is given by Eq. (2.4). From total momentum conservation, we find, just like for the single field NLDE, that for a solution that vanishes at $\pm\infty$ we have

$$T_{10} = \omega_1\bar{\psi}\gamma_1\psi + \omega_2\bar{\phi}\gamma_1\phi = 0 \quad (2.12)$$

and also

$$T_{11} = \omega_1\psi^\dagger\psi - m_1\bar{\psi}\psi + \omega_2\phi^\dagger\phi - m_2\bar{\phi}\phi + \mathcal{L}_{int} = 0. \quad (2.13)$$

Multiplying Eq. (2.1) on the left by $\bar{\psi}$ and Eq. (2.2) on the left by $\bar{\phi}$ and adding those two equations and then using Eq. (2.13) to eliminate the interaction terms of $\mathcal{L}_{int}$, we then obtain the equation:

$$\omega_1\psi^\dagger\psi - m_1\bar{\psi}\psi + i\bar{\psi}\gamma_1\partial_1\psi + \omega_2\phi^\dagger\phi - m_2\bar{\phi}\phi + i\bar{\phi}\gamma_1\partial_1\phi = 0, \quad (2.14)$$

which becomes using our ansatz

$$R_1^2 \left(\frac{d\theta}{dx} + \omega_1 - m_1 \cos 2\theta \right) + R_2^2 \left(\frac{d\eta}{dx} + \omega_2 - m_2 \cos 2\eta \right) = 0. \quad (2.15)$$

One also has that energy is conserved. The energy density is given by

$$T_{00}(x) = m_1\bar{\psi}\psi + m_2\bar{\phi}\phi = m_1 R_1^2 \cos 2\theta + m_2 R_2^2 \cos 2\eta, \quad (2.16)$$

with the total energy being conserved

$$E = \int_{-\infty}^{\infty} dx (m_1 R_1^2 \cos 2\theta + m_2 R_2^2 \cos 2\eta). \quad (2.17)$$

The other conserved quantities are charges Q_ψ and Q_ϕ defined by

$$Q_\psi = \int_{-\infty}^{+\infty} dx \psi^\dagger\psi, \quad (2.18)$$

$$Q_\phi = \int_{-\infty}^{+\infty} dx \phi^\dagger\phi. \quad (2.19)$$

B. Approximate Solution

We will obtain our approximate analytic solution by assuming that each of the two terms in Eq. (2.15) is identically zero. Then we obtain

$$\frac{d\theta}{dx} = -(\omega_1 - m_1 \cos 2\theta), \quad \frac{d\eta}{dx} = -(\omega_2 - m_2 \cos 2\eta), \quad (2.20)$$

whose solutions are the same as when $g_3 = 0$, namely

$$\theta(x) = \tan^{-1}(\alpha_1 \tanh \beta_1(x)), \quad \eta(x) = \tan^{-1}(\alpha_2 \tanh \beta_2(x)), \quad (2.21)$$

where

$$\alpha_i = \sqrt{\frac{m_i - \omega_i}{m_i + \omega_i}}, \quad \beta_i = \sqrt{m_i^2 - \omega_i^2}. \quad (2.22)$$

Using Eqs. (2.8), (2.20) and (2.21) we can solve for y_1 and y_2 and obtain

$$\begin{aligned} y_1 &= 2 \sec 2\theta \frac{[-g_2^2 \omega_1 \sec 2\theta + g_2^2 m_1 + g_3^2 \omega_2 \sec 2\eta - g_3^2 m_2]}{g_1^2 g_2^2 - g_3^4} \\ &= \frac{1}{(1 - g_{32}^2 g_{31}^2)} \left[y_{10} - g_{31}^2 y_{20} \frac{\sec 2\theta}{\sec 2\eta} \right], \end{aligned} \quad (2.23)$$

$$\begin{aligned} y_2 &= 2 \sec 2\eta \frac{[-g_1^2 \omega_2 \sec 2\eta + g_1^2 m_2 + g_3^2 \omega_1 \sec 2\theta - g_3^2 m_1]}{g_1^2 g_2^2 - g_3^4} \\ &= \frac{1}{(1 - g_{32}^2 g_{31}^2)} \left[y_{20} - g_{32}^2 y_{10} \frac{\sec 2\eta}{\sec 2\theta} \right], \end{aligned} \quad (2.24)$$

where y_{i0} is the value of y_i when $g_3 = 0$, and $g_{3i} = g_3/g_i$ for $i = 1, 2$. Letting $\theta_i = (\theta, \eta)$ we have that the values of various trigonometric functions of θ_i , valid when $g_3 = 0$ are given by:

$$\tan \theta_i = \frac{\sqrt{m_i - \omega_i} \tanh \left(x \sqrt{m_i^2 - \omega_i^2} \right)}{\sqrt{m_i + \omega_i}}, \quad (2.25)$$

$$\sec 2\theta_i = \frac{\omega_i + m_i \cosh 2\beta_i x}{m_i + \omega_i \cosh 2\beta_i x}, \quad (2.26)$$

$$\sin 2\theta_i = \frac{\beta_i \sinh 2\beta_i x}{m_i \cosh 2\beta_i x + \omega_i}. \quad (2.27)$$

We see from Eqs. (2.23) and (2.24) that the solution for y_i after rescaling depends on the two dimensionless coupling constants g_{3i}^2 . Our solutions for y_i were found using the differential equation for θ_i . So to see the accuracy of this solution we need to check how well the Eqs. (2.8) are satisfied. We will find that this depends on whether one of the solutions y_{i0} is double humped, since then its derivative near $x = 0$ will be opposite that of the single humped one and then the left hand side can have behavior different from the right hand side in various scenarios near $x = 0$. First, let us look at the case when both y_{i0} are single humped. We can always after rescaling take $g_1 = 1$. For simplicity we will also choose $g_2 = 1$, and are then left with $g_{32} = g_{31} = g_3$ which for illustrative purposes we will usually choose $= \sqrt{1/10}$. For the choice $m_1 = 1$, $m_2 = 9/10$, $\omega_1 = 9/10$, $\omega_2 = 7/10$, $g_3^2 = 1/10$ the left hand side of Eq. (2.8) is shown in blue and the right hand side in yellow in Fig. 1. Here if we take the ratio $(lhs - rhs)/(lhs + rhs)$ we would find that this was always less than 2% over the entire x range. For this choice of parameters y_1 is much more modified by the interaction than y_2 . This is shown in Figs. 2 and 3.

The problematic case is when one of the y_{i0} is double humped. In that case the derivatives of y_i determined from our approximation can be positive, whereas the right hand side can still be negative in our approximation for a range of small x and $g_3^2 = 1/10$. An example of this is given for the values: $m_1 = 1$, $m_2 = 9/10$, $\omega_1 = 9/10$, $\omega_2 = 3/10$, $g_3^2 = 1/10$ in Fig. 4

In this case if we reduce g_3^2 to be $1/100$, then we again get good agreement between the left and right hand sides of the equation for dy_1/dx . This is shown in Fig. 5. When $g_3^2 = 1/100$, y_1 is slightly modified from its value when $g_3 = 0$. However, the double humped y_2 solution is barely modified by the interaction as seen in Figs. 6 and 7.

FIG. 1: lhs (blue curve) vs. rhs of Eq. (2.8) for dy_1/dx when $g_3^2 = 1/10$.

FIG. 2: y_1 (upper curve) and y_{10} when $g_3^2 = 1/10$.

III. VECTOR-VECTOR INTERACTIONS

The coupled nonlinear Dirac equations (NLDEs) with vector-vector interactions are given by

$$(i\gamma^\mu \partial_\mu - m_1)\psi + g_1^2 \gamma^\mu \psi (\bar{\psi} \gamma_\mu \psi) + g_3^2 \gamma^\mu \psi (\bar{\phi} \gamma_\mu \phi) = 0, \quad (3.1)$$

$$(i\gamma^\mu \partial_\mu - m_2)\phi + g_2^2 \gamma^\mu \phi (\bar{\phi} \gamma_\mu \phi) + g_3^2 \gamma^\mu \phi (\bar{\psi} \gamma_\mu \psi) = 0. \quad (3.2)$$

FIG. 3: y_2 (upper curve) and y_{20} when $g_3^2 = 1/10$.

FIG. 4: lhs (blue) vs. rhs of Eq. (2.8) for dy_1/dx when $g_3^2 = 1/10$.

FIG. 5: lhs vs. rhs of Eq. (2.8) for dy_1/dx when $g_3^2 = 1/100$.

FIG. 6: y_1 when $g_3 = 0$ (upper curve) vs. y_1 when $g_3^2 = 1/100$.

FIG. 7: y_2 when $g_3 = 0$ vs. y_2 when $g_3^2 = 1/100$ (no visible difference at this scale.)

Again by scaling $\psi \rightarrow \psi/g_1$, $\phi \rightarrow \phi/g_2$, we have only two independent dimensionless coupling constants g_3^2/g_2^2 and g_3^2/g_1^2 . Equations (3.1) and (3.2) can be derived from the Lagrangian

$$\begin{aligned}\mathcal{L} &= \bar{\psi}(i\gamma^\mu\partial_\mu - m_1)\psi + \frac{g_1^2}{2}(\bar{\psi}\gamma_\mu\psi)(\bar{\psi}\gamma^\mu\psi) \\ &\quad + \bar{\phi}(i\gamma^\mu\partial_\mu - m_2)\phi + \frac{g_2^2}{2}(\bar{\phi}\gamma_\mu\phi)(\bar{\phi}\gamma^\mu\phi) + g_3^2(\bar{\psi}\gamma_\mu\psi)(\bar{\phi}\gamma^\mu\phi) \\ &= \bar{\psi}(i\gamma^\mu\partial_\mu - m_1)\psi + \bar{\phi}(i\gamma^\mu\partial_\mu - m_2)\phi + \mathcal{L}_{int}.\end{aligned}\tag{3.3}$$

We notice that as in the scalar-scalar case, the Lagrangian in this case is also symmetric under the interchange $\psi \rightarrow \phi$, $m_1 \rightarrow m_2$ and $g_1 \rightarrow g_2$. Again using the representation as given by Eq. (2.6), we have the equations for the components of the two coupled NLDEs which can be written as

$$\begin{aligned}\partial_x A + (m_1 + \omega_1)B + g_1^2(A^2 + B^2)B + g_3^2(C^2 + D^2)B &= 0, \\ \partial_x B + (m_1 - \omega_1)A - g_1^2(A^2 + B^2)A - g_3^2(C^2 + D^2)A &= 0, \\ \partial_x C + (m_2 + \omega_2)D + g_2^2(C^2 + D^2)D + g_3^2(A^2 + B^2)D &= 0, \\ \partial_x D + (m_2 - \omega_2)C - g_2^2(C^2 + D^2)C - g_3^2(A^2 + B^2)C &= 0.\end{aligned}\tag{3.4}$$

These are symmetric under the interchange $\{A, B\} \rightarrow \{C, D\}$, $m_1 \rightarrow m_2$ and $g_1 \rightarrow g_2$. These four equations can also be written if we let $y_i = R_i^2(x)$ as:

$$\begin{aligned}\frac{dy_1}{dx} &= -2y_1m_1\sin 2\theta, \\ \frac{dy_2}{dx} &= -2y_2m_2\sin 2\eta,\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}\frac{d\theta}{dx} &= \omega_1 + g_1^2y_1 + g_3^2y_2 - m_1\cos 2\theta, \\ \frac{d\eta}{dx} &= \omega_2 + g_2^2y_2 + g_3^2y_1 - m_2\cos 2\eta.\end{aligned}\tag{3.6}$$

This reduces, when $g_3 = 0$ to the Eqs. (14) in Chang et al. [22].

A. Conservation Laws

We again have energy-momentum conservation governed by Eqs. (2.10) and (2.11) but where $\mathcal{L}$ is now given by Eq. (3.3). From total momentum conservation, we find, just like for the scalar case, that for a solution that vanishes at $\pm\infty$, T_{10} and T_{11} are again given by Eqs. (2.12) and (2.13) respectively, but where $\mathcal{L}_{int}$ is as given by Eq. (3.3). Multiplying Eq. (3.1) on the left by $\bar{\psi}$ and Eq. (3.2) on the left by $\bar{\phi}$ and adding those two equations and then using

Eq. (2.13) to eliminate the interaction terms of $\mathcal{L}_{int}$, we as in the scalar case, again obtain Eq. (2.14). On using the ansatz (2.6) we then again obtain

$$R_1^2 \left(\frac{d\theta}{dx} + \omega_1 - m_1 \cos 2\theta \right) + R_2^2 \left(\frac{d\eta}{dx} + \omega_2 - m_2 \cos 2\eta \right) = 0. \quad (3.7)$$

As in the scalar case, in the vector case too the energy and the charges Q_ψ and Q_ϕ are conserved and are again given by Eqs. (2.17) to (2.19), respectively.

B. Approximate Solution

We will obtain our approximate analytic solution by assuming that each of the two terms in Eq. (3.7) is identically zero. Then we obtain

$$\frac{d\theta}{dx} = (m_1 \cos 2\theta - \omega_1), \quad \frac{d\eta}{dx} = (m_2 \cos 2\eta - \omega_2), \quad (3.8)$$

whose solutions are given by Eq. (2.20). We can again solve for y_1 and y_2 and obtain

$$\begin{aligned} y_1 &= \frac{2(g_2^2 m_1 \cos(2\theta) - g_2^2 \omega_1 - g_3^2 m_2 \cos(2\eta) + g_3^2 \omega_2)}{g_1^2, g_2^2 - g_3^4} \\ y_2 &= \frac{2(g_1^2 m_2 \cos(2\eta) - g_1^2 \omega_2 - g_3^2 m_1 \cos(2\theta) + g_3^2 \omega_1)}{g_1^2, g_2^2 - g_3^4} \end{aligned} \quad (3.9)$$

Since in the absence of interactions ($g_3 = 0$) we have

$$y_{10} = \frac{2}{g_1^2} [m_1 \cos(2\theta) - \omega_1], \quad y_{20} = \frac{2}{g_2^2} [m_2 \cos(2\eta) - \omega_2], \quad (3.10)$$

we can rewrite Eq. (3.9) as

$$\begin{aligned} y_1 &= \frac{1}{1 - g_{31}^2 g_{32}^2} [y_{10} - g_{31}^2 y_{20}], \\ y_2 &= \frac{1}{1 - g_{31}^2 g_{32}^2} [y_{20} - g_{32}^2 y_{10}]. \end{aligned} \quad (3.11)$$

So we see that we need both $g_{31}^2 \ll 1$ and $g_{32}^2 \ll 1$ for this approximation to make sense. Now let us see to what extent we violate Eq. (3.5). We have, letting $\theta_i = (\theta, \eta)$, the approximate expression for θ_i , valid when $g_3 = 0$, given by Eq. (2.20).

Now unlike the scalar-scalar case, the solutions y_{i0} are single humped and so typical values of the parameters give generic results. Setting $g_1 = g_2 = 1$ and $g_3^2 = 1/10$ and $m_1 = 1$, $m_2 = 1/2$, $\omega_1 = 1/2$, $\omega_2 = 1/4$ we find that the relative error on comparing the lhs and rhs of Eq. (3.5), i.e. $(lhs - rhs)/(lhs + rhs)$, is less than 3%, (see Fig. 8) for $|x| < 1$. At the same time, y_2 is changed quite a bit from its uncoupled value when we choose these values of the parameters as seen in Fig. 9. The effect is not as dramatic for y_1 for these values as seen in Fig. 10.

IV. NONRELATIVISTIC LIMIT

In our previous paper [19], we had started with NLD equations and using Moore's decoupling method [23] we had obtained the nonrelativistic limit of our NLD equations in both the scalar and the vector coupling cases. In this section we essentially follow the same decoupling method to obtain the nonrelativistic limit of coupled NLD equations in both scalar and vector coupling cases.

Let us start from the coupled Eqs. (2.1) and (2.2) or (3.1) and (3.2). They can be reexpressed in the form

$$i\sigma_3 \partial_t \psi + \sigma_x \partial_x \psi - m_1 \psi - V_I^1 \psi = 0, \quad (4.1)$$

$$i\sigma_3 \partial_t \phi + \sigma_x \partial_x \phi - m_2 \phi - V_I^2 \phi = 0, \quad (4.2)$$

FIG. 8: $(\text{lhs-rhs})/(\text{lhs+rhs})$ of Eq. (3.5) for dy_1/dx when $g_3^2 = 1/10$.

FIG. 9: y_2 (upper curve) and y_{20} when $g_3^2 = 1/10$.

where $V_I^1 = -\frac{\partial L_I}{\partial \psi}$ while $V_I^2 = -\frac{\partial L_I}{\partial \phi}$ and L_I is as given by Eq. (2.4) or (3.3). On using

$$\psi = \begin{pmatrix} u_0 \\ w_0 \end{pmatrix} e^{-i\omega_1 t}, \quad (4.3)$$

$$\phi = \begin{pmatrix} v_0 \\ z_0 \end{pmatrix} e^{-i\omega_2 t}, \quad (4.4)$$

and the Moore's decoupling method as well as essentially following the steps given in our previous paper [19], we find that in both the scalar-scalar and the vector-vector cases we get the coupled NLS equations

$$-(u_0)_{xx} + (m_1^2 - \omega_1^2)u_0 - (m_1 + \omega_1)[g_1^2|u_0|^2 + g_3^2|v_0|^2]u_0 = 0, \quad (4.5)$$

FIG. 10: y_1 (upper curve) and y_{10} when $g_3^2 = 1/10$.

$$-(v_0)_{xx} + (m_2^2 - \omega_2^2)v_0 - (m_2 + \omega_2)[g_2^2|v_0|^2 + g_3^2|u_0|^2]v_0 = 0, \quad (4.6)$$

under the assumption that $(m_1 - \omega_1)/2m_1 \ll 1$ and $(m_2 - \omega_2)/2m_2 \ll 1$.

Let us now look for exact solutions of the coupled Eqs. (4.5) and (4.6) under the assumption that both u_0 and v_0 vanish in the limit $x \rightarrow \pm\infty$. It turns out that there are two such solutions and we discuss these one by one.

A. Solution I

It is easy to check that

$$u_0 = A \operatorname{sech} \beta x, \quad v_0 = B \operatorname{sech} \beta x, \quad (4.7)$$

is an exact solution to the coupled Eqs. (4.5) and (4.6) provided

$$m_1^2 - \omega_1^2 = m_2^2 - \omega_2^2 = \beta^2, \quad (4.8)$$

$$g_1^2 A^2 + g_3^2 B^2 = 2(m_1 - \omega_1), \quad (4.9)$$

$$g_2^2 B^2 + g_3^2 A^2 = 2(m_2 - \omega_2). \quad (4.10)$$

On solving Eqs. (4.9) and (4.10) we find that

$$A^2 = \frac{2[(m_1 - \omega_1)g_2^2 - (m_2 - \omega_2)g_3^2]}{g_1^2 g_2^2 - g_3^4}, \quad B^2 = \frac{2[(m_2 - \omega_2)g_1^2 - (m_1 - \omega_1)g_3^2]}{g_1^2 g_2^2 - g_3^4}, \quad (4.11)$$

provided $g_1^2 g_2^2 \neq g_3^4$. In case $g_1 g_2 = g_3^2$ then A and B remain undetermined and we only have the constraint

$$g_1 A^2 + g_2 B^2 = \frac{2(m_1 - \omega_1)}{g_1} = \frac{2(m_2 - \omega_2)}{g_2}. \quad (4.12)$$

B. Solution II

Another solution to the coupled Eqs. (4.5) and (4.6) satisfying the boundary condition $u_0, v_0 \rightarrow 0$ as $x \rightarrow \pm\infty$ is

$$u_0 = A \operatorname{sech}^2 \beta x, \quad v_0 = B \operatorname{sech} \beta x \tanh \beta x, \quad (4.13)$$

provided

$$m_1^2 - \omega_1^2 = 4\beta^2, \quad m_2^2 - \omega_2^2 = \beta^2, \quad g_1 g_2 = g_3^2, \quad g_1 A^2 = g_2 B^2, \quad (4.14)$$

$$g_1(m_1 + \omega_1) = g_2(m_2 + \omega_2), \quad (m_1 + \omega_1)g_1^2 A^2 = 6\beta^2. \quad (4.15)$$

It is thus worth noting that while the first solution is valid for any values of g_1, g_2, g_3 , the second solution is only valid when $g_1 g_2 = g_3^2$.

V. CONCLUSIONS

In this paper we have found approximate bound state solutions and discussed them in the context of coupled NLD equations in $1+1$ dimensions with both scalar-scalar and vector-vector interactions, generalizing the solutions found in the absence of coupling. In particular, we have given the approximate analytic solitary wave solution to two coupled NLDEs for both scalar-scalar interactions and vector-vector interactions in $1+1$ dimensions. These solutions are relevant in nonlinear optics [5] as well as for light solitons in waveguide arrays [6–8] among other applications in BECs and cosmology. Further, we have shown using the Moore's decoupling method that in the non-relativistic limit, NLDEs with both scalar-scalar and vector-vector interactions reduce to the same coupled nonlinear Schrödinger equation (NLSE). We have obtained two exact pulse solutions to these coupled NLSE. Using the results found in [19],

one can extend these solutions to the case where the scalar-scalar as well as vector-vector interactions are taken to an arbitrary (nonlinearity) power κ . We hope to address this issue as well as the question of stability of the solutions found here in the near future.

We emphasize that the two-body Dirac equations in $3+1$ dimensions have been considered extensively in literature [20] but the present discussion about the coupled NLD in $1+1$ dimensions is totally different. Also, coupled $1+1$ dimensional NLD equations have been considered formally in the literature [21] but no explicit or approximate bound state solutions were obtained. We found our solutions under the approximation of small coupling (g_3). A detailed error analysis is necessary to calibrate how far our solutions get distorted when g_3 is large. Indeed, it would be desirable to numerically find exact solutions to the coupled NLD equations for various g_3 values but this is beyond the scope of the present paper. We hope to address these issues in detail in the near future.

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