How isotropic are turbulent flows generated by using periodic conditions in a cube?

Z.C. Qin^a, L. Fang^a, J. Fang^b

^aLaboratory of Mathematics and Physics, Ecole Centrale de Pékin, Beihang University, Co-Innovation Center for Advanced Aero-Engine, Beijing 100191, China

^bSchool of Energy and Power Engineering, Beihang University, Co-Innovation Center for Advanced Aero-Engine, Beijing 100191, China

Abstract

In numerical simulations, “isotropic” turbulent flows are always generated by using periodic conditions. We show that these periodic conditions mathematically lead to large-scale anisotropy which can be about 10% of the mean values, and thus prevent existing post-processing results from being accurate. A decomposition method by employing spherical harmonics is then proposed to distinguish this scale-dependent anisotropy effect from others and to minimize the related post-processing error.

Keywords: isotropic turbulence; periodic condition; large-scale anisotropy; energy-containing scale

1. Introduction

In order to numerically generate three-dimensional homogeneous isotropic turbulent (HIT) flows, the commonly-used method is to assume three-dimensional periodic conditions in a cube. There have been enormous literatures following this methodology, for instance Refs. [1, 2, 3, 4, 5, 6, 7, 8]. These generated data are widely used in various topics, including the analysis of small-scale isotropic structures [9, 10, 11, 12]. It is then quite important to know that in which range the turbulent flow can be isotropic. In particular, the upper bound of this isotropic range corresponds to the largest isotropic scale, which is usually considered to be the order of energy-containing scale or integral scale. However, to our knowledge, there is still no systematic discussion on this largest isotropic scale for the numerically generated turbulent flows with a cubic box and the periodic boundary condition applied in all directions.

Practical numerical calculations usually consider a cube with length $R = 2\pi$ in each direction. From existing direct numerical simulations (DNS) the integral scale is usually chosen to be smaller than 1.4 to avoid the influence of cube (see e.g. Refs. [5, 2, 13, 14]). One of the few discussions on this influence can be found from Gotoh *et al.* [14], who remarked that the small number of energy-containing Fourier modes can cause large-scale anisotropy, which is measured by comparing the longitudinal and transverse structure functions. However, as will be analyzed in the present paper, this is a tensor-level anisotropy effect that is presented as redistribution among tensor components, whereas the lower-level anisotropy effect, which stems from periodic conditions, has not been systematically discussed.

In the present paper, we aim at discussing the large-scale influence of the periodic condition in a cube. By theoretical and numerical investigations, it will be shown that there always exists obvious anisotropy among the axis directions, the face diagonal directions and the cube diagonal directions, which even

Fig. 1: Two-dimensional sketch for the anisotropy of longitudinal structure functions in the axis directions and the face diagonal directions.

leads to different summation structure functions along these directions. We will show that this anisotropy effect differs from traditional tensor-level studies and is a congenital defect of numerical simulations. Decomposition by spherical harmonics functions will quantitatively show this large-scale anisotropy influence.

2. Descriptions of problem

In this section, we start from a 2D description and show that there are always anisotropy influence from periodic conditions to smaller scales. Unlike the discussions of Gotoh *et al.* [14], this anisotropy is expressed as a difference between the longitudinal structure functions in the axis directions and the face diagonal directions. It would also be easy to extend the following analysis to a 3D cube and to add cube diagonal directions into comparison.

See Fig. 1 as a sketch. The square $ABCD$ is a periodic domain in both directions. In the following we consider a scalar field and a vector field respectively to express the problem and to clarify the difference between the present problem and traditional studies.

Email address: le.fang@zoho.com (L. Fang)

2.1. Second-order structure function in a scalar field

We consider a scalar field Θ in the periodic domain and expect it to be homogeneous isotropic. From periodic conditions we have

$$\Theta(A) = \Theta(B) = \Theta(C) = \Theta(D). \quad (1)$$

Then we define the ensemble average operation as $\langle \cdot \rangle$. In particular, the second-order structure function at a two-point displacement is defined as

$$S(\mathbf{x}, \mathbf{r}) = \langle (\Theta(\mathbf{x} + \mathbf{r}) - \Theta(\mathbf{x}))^2 \rangle. \quad (2)$$

Because of homogeneity¹, it can also be rewritten as

$$S(\mathbf{r}) = \langle (\Theta(A + \mathbf{r}) - \Theta(A))^2 \rangle. \quad (3)$$

Under ideal isotropy, we can calculate the second-order structure function of a two-point distance r by an average operation on a 1/4 **circumference** (which is also frequently employed in practical post-processing procedures):

$$S(r) = \frac{2}{\pi r} \int_0^{\pi/2} S(\mathbf{r}) r d\alpha. \quad (4)$$

With the isotropy assumption, $S(\mathbf{r})$ does not depend on the angle α and can exit the integer, and this equation will be simplified to the identical equation $S(\mathbf{r}) = S(r)$.

When the periodic conditions are involved, additional restrictions are implied at large scales. For example with the periodicity $\Theta(A) = \Theta(B)$, we can rewrite Eq. (3) as

$$S(\mathbf{r}) = \langle (\Theta(A + \mathbf{r}) - \Theta(B))^2 \rangle. \quad (5)$$

Since $A + \mathbf{r} = B + \mathbf{s}$ we have

$$S(\mathbf{r}) = S(\mathbf{s}). \quad (6)$$

When we perform the average operation (4), this can also be regarded as an integration over the arc $E'G'$:

$$S(r) = \frac{2}{\pi} \int_0^{\pi/2} S(\mathbf{r} + (A - B)) d\alpha. \quad (7)$$

Note that this equation is derived by using the periodic condition in the direction of AB . The periodic conditions in other directions can lead to other equations, but the discussions are similar. The assumption of isotropy implies that the statistical variables do not depend on the direction. Hence Eq. (7) can be simplified as

$$S(r) = \frac{2}{\pi} \int_0^{\pi/2} S(s) d\alpha. \quad (8)$$

¹We remark that non-trivial homogeneous scalar fields can exist with periodic boundary conditions. For construction, we suppose that a scalar field Θ is periodic at time 0 and is advected by a non-zero constant velocity, then using time average instead of ensemble average can lead to the independence of $\mathbf{x}$ in Eq. (2).

From the law of cosines $s = (R^2 + r^2 - 2Rr \cos \alpha)^{1/2}$, it leads to

$$S(r) = \frac{2}{\pi} \int_0^{\pi/2} S((R^2 + r^2 - 2Rr \cos \alpha)^{1/2}) d\alpha. \quad (9)$$

With the initial condition $S(0) = 0$, one puts $r = 0$ in Eq. (9) and can immediately obtain $S(R) = 0$. Then, putting $r = R$ in Eq. (9) and by noting that Eq. (9) can be rewritten in the following formula when $r = R$:

$$0 = S(R) = \int_0^{\sqrt{2}R} S(s) w(s) ds, \quad (10)$$

with a weight function $w(s) = 2s/(4R^2s^2 - s^4)^{1/2}$, which satisfies that $\forall 0 < s < 2R, w(s) > 0$. From Eq. (3) we also have $S(s) \geq 0$. Continuity of $S(s)$ then implies that $\forall 0 \leq s \leq \sqrt{2}R, S(s) \equiv 0$ (otherwise, supposing $S(t) > 0$ with $0 < t \leq \sqrt{2}R, \exists \epsilon > 0, \text{s.t. } \forall \tau \in (t - \epsilon, t + \epsilon), S(\tau) > 0$, then $\int_0^{\sqrt{2}R} S(s) w(s) ds \geq \int_{t-\epsilon}^{t+\epsilon} S(s) w(s) ds > 0$ which conflicts with Eq. (10)) which is a non-physical trivial solution. In order to show the influence on a real field, we assume a classical scaling behavior $S(r) = r^{2/3}$, then the right hand side of Eq. (9) can be simplified as

$$S_{sl}(r) = \frac{2}{\pi r} \int_0^{\pi/2} (R^2 + r^2 - 2Rr \cos \alpha)^{1/3} r d\alpha, \quad (11)$$

where S_{sl} means using the scaling law. Clearly, a paradox arises that $S_{sl}(r)$ is not of the order $r^{2/3}$. Fig. 2 shows the scaling behaviors of $S_{sl}(r)$, where $R = 1$. From Fig. 2(a) we find that $r^{-2/3} S_{sl}(r)$ is not constant; from Fig. 2(b) the scaling exponent of $S_{sl}(r)$ is not 2/3 either. Note that this is just a specific example to show that any isotropic scaling cannot be guaranteed, while for any other scalings the conclusions are the same. This obviously illustrates the necessity of investigating the influence of large-scale periodic condition.

Note that from Ref. [15] we can also find another effect of the domain size: when the domain size is not large enough (e.g. the order of integral scale), some intermittency is observed. By contrast, from the present study, we show that the periodical domain can also lead to (large-scale) anisotropy.

2.2. Second-order structure function in a vector field

Consider the structure functions in a vector field. Similarly to the previous subsection, we can define the summation $D_{ii}(\mathbf{r}) = \langle (u_i(A + \mathbf{r}) - u_i(A))(u_i(A + \mathbf{r}) - u_i(A)) \rangle$ which is a scalar function. It is easy to show that the conclusions in the previous subsection are also satisfied for $D_{ii}(r)$. In particular, the periodic condition prevents $D_{ii}(r)$ from being isotropic.

In practice we are usually interested in the behavior of the longitudinal structure function, defined as $D_{ll}(\mathbf{r}) = \langle (u_l(A + \mathbf{r}) - u_l(A))^2 \rangle$ with l the direction of $\mathbf{r}$, rather than the summation $D_{ii}(\mathbf{r})$. By defining the transverse structure function $D_{nn}(\mathbf{r}) = \langle (u_n(A + \mathbf{r}) - u_n(A))^2 \rangle$ with n the normal direction to $\mathbf{r}$, the tensor-level isotropic condition yields

$$D_{nn}(r) = D_{ll}(r) + \frac{r}{2} \frac{dD_{ll}(r)}{dr}, \quad (12)$$

Fig. 2: $S_{sl}(r)$ using the scaling law. (a) $S_{sl}(r)/r^{2/3}$. (b) $r \cdot S'_{sl}(r)/S_{sl}(r)$.

which implies the relation

$$\frac{dD_{ii}(r)}{dr} = \frac{2(D_{nn}(r) - D_{ll}(r))}{r} + (\mathcal{N} - 1) \frac{dD_{nn}(r)}{dr} \quad (13)$$

with $\mathcal{N}$ the dimension of domain. Similarly to the procedure of the previous subsection, by assuming a scaling, it is easy to show that the periodic condition prevents $D_{ll}(r)$ from being isotropic.

Here we remark that the isotropy property discussed in the present paper is distinct from many numerical isotropy verifications (*c.f.*, Refs. [14, 16]), which use Eq. (12) to estimate the tensor-level isotropy over the subscripts. In this section we have illustrated that a scalar function (*e.g.*, $S(r)$ and $D_{ii}(r)$) would also be anisotropic due to the periodic conditions. The following parts will focus on the numerical verifications of DNS results.

2.3. Numerical evidence

In order to qualitatively investigate the influence of periodic conditions, we start from the open database of the Johns Hopkins University with numerical resolution 1024^3 and Reynolds number $Re_\lambda = 433$. Numerical details about the DNS computation can be found in Ref. [5].

Three typical groups of directions are chosen here to illustrate the problem of anisotropy described in the previous subsections: the three axis, the six face diagonals, and four cube diagonals. In each group we calculate the average of the corresponding directions of the summation structure function D_{ii} . As shown in Fig. 3(a), at large scales ($r/\Delta > 250$ with Δ the grid size, or say $r/R > 1/4$) the axis directions yield different values of D_{ii} from the other directions. The scaling exponent $n(r) = \frac{r}{D_{ii}(r)} \frac{dD_{ii}(r)}{dr}$ is then calculated in Fig. 3(b) to physically locate the scales. Comparing with the classical scaling value $2/3$ [17] or the anomalous scaling value 0.7 [18, 19], the inertial range is about $40 < r/\Delta < 120$, thus the large scales $r/\Delta > 250$ can be considered as the energy-containing range. Obviously, in this range the axis directions behave differently with the other two. **Also, the face diagonal directions and the cube diagonal directions are distinct at larger scales ($r/\Delta > 600$) as shown in Fig. 3(a), showing obviously the large-scale anisotropy.**

Besides the comparison among typical directions, it would be more important to quantitatively describe the anisotropy. We introduce the relative standard deviation $\mathcal{A}(r, S)$ to denote the anisotropy rate, which is defined as

$$\mathcal{A}(r, S) = \sqrt{\frac{\frac{1}{4\pi} \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} [S(r, \theta, \varphi) - \mu(r)]^2 \sin \theta d\varphi d\theta}{\mu^2}} \quad (14)$$

with $\mu(r)$ the hemispherical surface average of a certain structure function $S(r, \theta, \varphi)$

$$\mu(r) = \frac{1}{2\pi} \int_{\theta=0}^{\pi/2} \int_{\varphi=0}^{\pi} S(r, \theta, \varphi) \sin \theta d\varphi d\theta. \quad (15)$$

For the summation structure function D_{ii} , the related anisotropy rate can be measured by $\mathcal{A}(r, D_{ii})$, which is presented in Fig. 4.

(a)

(b)

Fig. 3: Summation structure function $D_{ii}(r)$ in different directions. (a) $D_{ii}(r)$; (b) Scaling exponent $n(r)$. Δ is the grid size which is about 2.1η with η Kolmogorov scale, R is the domain size, and u_{rms} is the root-mean square of velocity fluctuation.

Fig. 4: The anisotropy rate of summation structure function $\mathcal{A}(r, D_{ii})$.

It is shown that in the energy-containing range, the anisotropy rate is $\mathcal{A}(r, D_{ii}) \sim 0.1$, indicating a non-negligible fluctuation of about 10% the mean value of D_{ii} . This even notably affects the accuracy of the scaling law calculated from DNS results, which is definitely a key issue in theoretical turbulence. As discussed in the subsections above, this anisotropy is involved from the periodic conditions of the computational domain, which cannot be erased within limited domain size. In this case, we propose a mathematical procedure to minimize this anisotropy rate by employing a decomposition with spherical harmonics.

3. Decomposition with spherical harmonics

By expressing the two-point distance $\mathbf{r}$ in a spherical coordinate system (r, θ, φ) , it would be possible to decompose any structure function $S(r, \theta, \varphi)$ to the following spherical harmonics:

$$S(r, \theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} R_{nm}(r) P_n(\cos \theta) P_m(\cos \varphi), \quad (16)$$

where P_n is the Legendre polynomial and $R_{nm}(r)$ is the coefficient at the (n, m) mode. The inverse transform then writes

$$R_{nm}(r) = \frac{2n+1}{2} \int_{\theta=0}^{\pi} \left(\frac{2m+1}{2} \times \int_{\varphi=0}^{\pi} S(r, \theta, \varphi) P_m(\cos \varphi) \sin \varphi d\varphi \right) P_n(\cos \theta) \sin \theta d\theta. \quad (17)$$

This decomposition differs from the studies of Casciola *et al.*[20] and Biferale and Vergassola[21] due to different research objectives. Those existing studies focus on the physical influences by global shear; whereas, in the present paper, as discussed in the previous section, we are interested in the

differences among the axis directions, the face diagonal directions and the cube diagonal directions. The present decomposition method would be therefore more appropriate for expressing these differences by observing the 2-nd order modes. Evidently, the $(0,0)$ mode coefficient $R_{00}(r)$ denotes the isotropic contribution, and the isotropic statistical result can be obtained by removing all other modes. However, this rude operation will also remove other anisotropy contributions, such as the lack of statistical samples and the asymmetry of the initializing method introduced by Rogallo [1], rather than those which are caused by periodic conditions. It would then be important to investigate the real contribution from periodic conditions. Indeed, from the analysis in the previous section, we may conclude that the influence of periodic conditions can only exist in the even modes. To illustrate, we define the contribution of even modes as

$$C_e(r) = \frac{\sum_{2|n \text{ and } 2|m \text{ and } (n,m) \neq (0,0)} R_{nm}^2(r)}{R_{00}^2(r)}, \quad (18)$$

while the contribution of the rest terms is defined as

$$C_o(r) = \frac{\sum_{2 \nmid n \text{ or } 2 \nmid m} R_{nm}^2(r)}{R_{00}^2(r)}, \quad (19)$$

where the modes $R_{nm}(r)$ are calculated by using Eq. (17). In order to further focus on the dominant modes, according to observation we define another group of contributions by limiting the even modes upto 4-th order:

$$C_e^*(r) = \frac{\sum_{(n,m) \in \mathcal{F}} R_{nm}^2(r)}{R_{00}^2(r)}, \quad (20)$$

while the contribution of the rest terms are defined as

$$C_o^*(r) = \frac{\sum_{(n,m) \notin \mathcal{F} \cup \{(0,0)\}} R_{nm}^2(r)}{R_{00}^2(r)}, \quad (21)$$

with the limited modes

$$\mathcal{F} = \{(0,2), (0,4), (2,0), (2,2), (2,4), (4,0), (4,2), (4,4)\}. \quad (22)$$

Figure 5(a) shows the contributions of all or limited even modes as well as the corresponding rest modes, which are normalized by the $(0,0)$ mode $R_{00}^2(r)$. It is observed that the limited modes $C_e^*(r)$ and $C_o^*(r)$ behave quite similarly to the unlimited ones $C_e(r)$ and $C_o(r)$ respectively. The even modes $C_e(r)$ and $C_e^*(r)$ increase faster than the rest modes $C_o(r)$ and $C_o^*(r)$ at large scales, which agrees with the fact that the periodic conditions affect the large-scale isotropy more than the small-scale isotropy. Figure 5(b) shows the ratios between $C_e(r)$ and $C_o(r)$, and $C_e^*(r)$ and $C_o^*(r)$ respectively. It is shown that at large scales the ratios are quite similar; however, at small scales $C_e^*(r)/C_o^*(r)$ tends to zero, which differs from the large value of $C_e(r)/C_o(r)$ in this range. To explain all these observations, we should emphasize that not all anisotropy contributions are caused by periodic conditions. Instead, as discussed above there may be other

Fig. 5: Contributions of different terms by using the decomposition with spherical harmonics. (a) Contributions of all or limited even modes as well as the corresponding rest modes, normalized by the $(0,0)$ mode $R_{00}^2(r)$; (b) The ratio of all or limited even modes in the anisotropic terms.

anisotropy sources, such as the lack of statistical samples and the asymmetry of the initializing method introduced by Rogallo, which could be considered as a global noise of anisotropy without any scale property. Therefore, if we consider $C_e^*(r)$ as the anisotropy contributions mainly caused by periodic conditions, and $C_o^*(r)$ caused by other anisotropy sources, all the curves in Fig. 5 could be reasonably explained. In particular, from Fig. 5(b) the trend of $C_e^*(r)/C_o^*(r)$ is similar to the anisotropy rate shown in Fig. 4, which illustrates the rationality of choosing the set of modes $\mathcal{F}$ to investigate the effect of periodic conditions.

By removing the modes in $\mathcal{F}$ and employing Eq. (16), we reconstruct a corrected structure function field from the original field D_{ii} , which leads to the statistical results as shown in Fig. 6. Comparing to Fig. 3, the corrected results in Figs. 6(a-b) evidently reduce the differences among typical directions at large scales, which is, as analyzed in Sec. 2, caused by ex-

ternal periodic conditions. Values of the anisotropy rate $\mathcal{A}$ are then calculated to compare with the original values, shown in Fig. 6(c). Clearly, the corrected $\mathcal{A}(r)$ is almost a constant 0.05 against different two-point distances r . As discussed above, this remaining anisotropic part probably stems from the global-noise-type sources, which differs from the periodic conditions that are scale-dependent. These statistical results, in *a posteriori* sense, illustrate the rationality and correctness of choosing the set of modes $\mathcal{F}$ to investigate the effect of periodic conditions. Besides, it is also interesting to observe that at small scales ($r/\eta < 100$) the anisotropy rate $\mathcal{A}$ increases after the correction, which may imply the fact that in this range the anisotropy effect of periodic conditions is partly compensated by the other anisotropy effects. This fact of isotropization can be explained that at small scales the flow is governed by the non-linear physics of turbulence, which always leads to a quasi-isotropic small-scale field and enforces the anisotropy effects to limit and compensate one another [22, 23, 24, 25]. This also indicates the rationality of the local-isotropy assumption of large-eddy simulation at dissipative and inertial scales [26].

In the paragraphs above we focus on the summation structure function D_{ii} , which is a scalar field that is affected directly by periodic conditions. As analyzed in Sec. 2.2, the longitudinal structure function D_{ll} , which is frequently used in theoretical analysis, differs from D_{ii} and involves a tensor-level relation (13). Following the same procedure as in the previous sections and choosing the field function S as D_{ll} , we show the corresponding anisotropy behavior in typical directions in Fig. 7. Comparing to Fig. 3, differences of D_{ll} among these directions are observed to be stronger than that of D_{ii} . In particular, the axis directions obviously differ from the other directions even in the inertial range (about $40 < r/\Delta < 120$), while for D_{ii} they are similar. Clearly, this stems from the anisotropic redistribution (13) among different tensor-level directions, which has already been investigated in many existing studies [14, 16]. Here we remark that, the anisotropic effect of periodic conditions, which is the principal concept of the present paper, is a mathematical effect that does not depend on any physical restriction.

4. Conclusion

In the present paper we analyze the large-scale source of anisotropy arising from periodic conditions. As discussed in Sec. 2, this anisotropy effect depends on neither numerical methods nor physical restrictions, and it may be considered as a congenital defect of numerical simulations for generating isotropic turbulent flows. This effect differs from traditional studies of the tensor-level anisotropy effect [14, 16]: taking the second-order structure functions as example, periodic conditions can lead to the anisotropy of scalar fields such as D_{ii} . DNS results show that periodic conditions cause the anisotropy at energy-containing scales, while the tensor-level anisotropy effect mainly occurs at inertial scales. By defining the anisotropy rate, we then show that the anisotropy contribution by periodic conditions is scale-dependent, while those from other sources are not. It is shown that periodic conditions lead to non-negligible fluctuations of about 10% of the mean value

Fig. 6: Statistical results of the corrected field of D_{ii} . (a) Structure functions in different directions; (b) scaling exponents in different directions; (c) anisotropy rates.

(a)

(b)

(a)

(b)

(c)

Fig. 7: Longitudinal structure function $D_{||}(r)$ in different directions. (a) $D_{||}(r)$; (b) Scaling exponent $n(r)$. Δ is the grid size which is about 2.1η with η Kolmogorov scale, R is the domain size, and u_{rms} is the root-mean square of velocity fluctuation.

Fig. 8: Statistical results of the corrected field of $D_{||}$. (a) Structure functions in different directions; (b) scaling exponents in different directions; (c) anisotropy rates.

of D_{ii} at large scales. A decomposition method based on spherical harmonics is then introduced to correct this anisotropy effect. By removing the even modes upto 4-th order, we show that the anisotropy effect of periodic conditions can be minimized for post-processing. In addition, the conclusion also illustrates the isotropy of existing DNS data [5, 2, 13, 14] at inertial and dissipative scales, since the integral scales are always smaller than $R/4$ in these cases.

Finally, instead of post-processing methods, we remark that changing the periodic condition might be an alternative solution for obtaining isotropic turbulence. As analyzed in the present paper, periodic conditions on instantaneous variables certainly lead to the large-scale anisotropy. By contrast, if we relax this constraint and employ periodic conditions on time-averaged variables instead, it might be possible to guarantee isotropy until large scales. The spirit of this idea would be constructing spherical-symmetric conditions rather than traditional conditions on statistical average (e.g., on a Cartesian grid, straightforwardly. These boundary conditions would introduce anisotropy because of the lack of spherical symmetry.), which restricts the mean values but allows high-order moments to be isotropic. Two main issues are therefore needed to clarify this possibility: *i)* the new mathematical constraint should be derived to satisfy the requirement of isotropy; *ii)* numerically, new algorithms should be introduced to implement this new periodic condition. These interesting problems will be further investigated in future works.

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- [1] R.S. Rogallo. *Numerical experiments in homogeneous turbulence*. NASA TM 81315, 1981.
- [2] T. Watanabe and T. Gotoh. Statistics of a passive scalar in homogeneous turbulence. *New Journal of Physics*, 6:40, 2004.
- [3] L. Fang, G. X. Cui, C. X. Xu, and Z. S. Zhang. Multi-scale analysis of energy transfer in scalar turbulence. *Chinese Physics Letters*, 22(11):2877–2880, 2005.
- [4] T. Ishihara, Y. Kaneda, M. Yokokawa, K. Itakura, and A. Uno. Small-scale statistics in high-resolution direct numerical simulation of turbulence: Reynolds number dependence of one-point velocity gradient statistics. *Journal of Fluid Mechanics*, 592:335, 2007.
- [5] Y. Li, E. Perlman, M. Wan, Y. Yang, R. Burns, C. Meneveau, S. Chen, A. Szalay, and G. Eyink. A public turbulence database cluster and applications to study lagrangian evolution of velocity increments in turbulence. *Journal of Turbulence*, 9(31), 2008.
- [6] T. Gotoh, T. Watanabe, and Y. Suzuki. Universality and anisotropy in passive scalar fluctuations in turbulence with uniform mean gradient. *Journal of Turbulence*, 12:N48, 2011.
- [7] L. Fang, W.J.T. Bos., L. Shao, and J.-P. Bertoglio. Time-reversibility of Navier-Stokes turbulence and its implication for subgrid scale models. *Journal of Turbulence*, 13:1–14, 2012.
- [8] J. Fang, Y.F. Yao, Z.R. Li, and L.P. Lu. Investigation of low-dissipation monotonicity-preserving scheme for direct numerical simulation of compressible turbulent flows. *Computers and Fluids*, 11:104, 2014.
- [9] H. Chen, J. Herring, R. Kerr, and R. Kraichnan. Non-Gaussian statistics in isotropic turbulence. *Physics of Fluids A*, 1:1844, 1989.
- [10] C.G. Speziale and T.B. Gatski. Analysis and modelling of anisotropies in the dissipation rate of turbulence. *Journal of Fluid Mechanics*, 344:155, 1997.
- [11] W. Bos, R. Rubinstein, and L. Fang. Reduction of mean-square advection in turbulent passive scalar mixing. *Physics of Fluids*, 24(7):075104, 2012.
- [12] L. Fang, W.J.T. Bos, and G.D. Jin. Short-time evolution of lagrangian velocity gradient correlations in isotropic turbulence. *Physics of Fluids*, 27(12):125102, 2015.
- [13] T. Gotoh and D. Fukayama. Pressure spectrum in homogeneous turbulence. *Physical Review Letters*, 86:3775, 2001.
- [14] T. Gotoh, D. Fukayama, and T. Nakano. Velocity field statistics in homogeneous steady turbulence obtained using a high-resolution direct numerical simulation. *Physics of Fluids*, 14:1065, 2002.
- [15] C. Brun and A. Pumir. Statistics of fourier modes in a turbulent flow. *Physical Review E*, 63:056313, 2001.
- [16] G.X. Cui, H.B. Zhou, Z.S. Zhang, and L. Shao. A new dynamic subgrid eddy viscosity model with application to turbulent channel flow. *Physics of Fluids*, 16(8):2835–2842, 2004.
- [17] A. N. Kolmogorov. The local structure of turbulence in incompressible viscous fluid for very large reynolds number. *Proceedings: Mathematical and Physical Sciences*, 30:301, 1941.
- [18] A. N. Kolmogorov. A refinement of previous hypotheses concerning the local structure of turbulence. *Journal of Fluid Mechanics*, 13:82:85, 1962.
- [19] Z. S. She and E. Leveque. Universal scaling law in fully developed turbulence. *Physics Review Letters*, 72:336, 1994.
- [20] C. M. Casciola, P. Gualtieri, B. Jacob, and R. Piva. The residual anisotropy at small scales in high shear turbulence. *Physics of Fluids*, 19:101704, 2007.
- [21] L. Biferale and M. Vergassola. Isotropy vs anisotropy in small-scale turbulence. *Physics of Fluids*, 13:2139, 2001.
- [22] X. Shen and Z. Warhaft. Longitudinal and transverse structure functions in sheared and unsheared wind-tunnel turbulence. *Physics of Fluids*, 14:370, 2002.
- [23] G.X. Cui, C.X. Xu, L. Fang, L. Shao, and Z.S. Zhang. A new subgrid eddy-viscosity model for large-eddy simulation of anisotropic turbulence. *Journal of Fluid Mechanics*, 582:377–397, 2007.
- [24] L. Fang, L. Shao, J.P. Bertoglio, G. Cui, C. Xu, and Z. Zhang. An improved velocity increment model based on Kolmogorov equation of filtered velocity. *Physics of Fluids*, 21(6):065108, 2009.
- [25] L. Fang, Y. Zhu, Y.W. Liu, and L.P. Lu. Spectral non-equilibrium property in homogeneous isotropic turbulence and its implication in subgrid-scale modelling. *Physics Letters A*, 379:2331 – 2336, 2015.
- [26] P. Sagaut. *Large eddy simulation for incompressible flows*. Springer, 2006.