Estimating the level of dynamical noise in time series by using fractal dimensions

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Abstract

We present a method for estimating the dynamical noise level of a ‘short’ time series even if the dynamical system is unknown. The proposed method estimates the level of dynamical noise by calculating the fractal dimensions of the time series. Additionally, the method is applied to EEG data to demonstrate its possible effectiveness as an indicator of temporal changes in the level of dynamical noise.

Key words: Dynamical noise level, Fractal dimension, Short time series, EEG data

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1. Introduction

In general, time series data—for example, biological, meteorological, and economic data—are a collection of measurements recording the dynamical behavior of a system. When the underlying dynamics of the system is stochastic, the associated time series will behave like a stochastic process. Note, however,

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that the time series generated from deterministic systems may also be associated with stochastic fluctuations if the measured signals have been influenced by noise.

Typically, a time series contains two basic types of noise: dynamical noise and observational noise. The former refers to the noise affecting the system's dynamics, whereas the latter describes the noise influencing the observed or measured signal of the system. The system including these two distinctive kinds of noise is defined as follows [1]:

$$dx_q = f_q(\vec{x})dt + \rho_q(t)dW_q(t), \text{ for } q = 1, 2, \dots, d, \quad (1)$$

$$X_{\text{obs}}(t) = H(\vec{x}) + \sigma_X \xi_X(t), \quad (2)$$

where x_q denotes the state variable, $\vec{x} = [x_1, x_2, \dots, x_d]^T$ is the state vector, function $f_q(\cdot)$ is either linear or nonlinear, and the dynamical noise is described by $\rho_q(t)dW_q(t)$. Here, $dW_q(t)$ indicates the **Gaussian white noise**, which follows a normal distribution with mean 0 and standard deviation $\sqrt{dt}$. **Note that, in order to define the dynamical noise as colored noise, the process of colored noise generated by the Gaussian white noise should be included in $f_q(\vec{x})$ so that the definition of a system [Eq. (1)] works more generally in terms of modeling the dynamical noise.** We explicitly assume that the temporal change of $\rho_q(t)$ is slow compared with the dynamics of x_q . Note that Eq. (1) denotes a d -dimensional stochastic dynamical system, and the corresponding observational process of this system is described by Eq. (2). Furthermore, from Eq. (1) and Eq. (2), it can be seen that $\vec{x}$ is converted to a one-dimensional variable $X_{\text{obs}}(t)$ through an observational function $H(\cdot)$ with observational noise $\sigma_X \xi_X(t)$, where $\xi_X(t)$ follows a normal distribution with mean 0 and standard deviation 1. Usually, the corresponding time series of $\{X_{\text{obs}}(t)\}$ is defined as follows:

$$\{X(i)\} = \{X_{\text{obs}}(t) | X_{\text{obs}}(t) = X_{\text{obs}}(i\Delta t), i \in \mathbb{N}\}, \quad (3)$$

where $\{X_{\text{obs}}(t)\}$ is converted to a time series $\{X(i)\}$ with sampling time Δt .

In this study, we assume that the amount of dynamical noise can be described as a time-dependent variable, which is denoted by $\rho_q(t)$ in Eq. (1).

32 For example, consider a situation in which the amount of noise in a neural sys-
 33 tem is temporally adjusted by input from other areas of the brain to improve
 34 brain functions. The amount of observational noise, denoted by σ_X in Eq. (2),
 35 is assumed to be constant. This assumption is equivalent to the assumption
 36 that the measurement environment does not change during the observational
 37 process. For a measured signal of the form $X_{\text{obs}}(t) = x_q(t) + \sigma_X \xi_X(t)$, the dy-
 38 namical and observational noise levels have been defined as $\rho_q(t)/\sigma_{\text{dyn}x_q}(t)$ and
 39 $\sigma_X/\sigma_{\text{dyn}x_q}(t)$, respectively, where $\sigma_{\text{dyn}x_q}(t)$ indicates the standard deviation of
 40 the variable x_q . The time dependency of $\sigma_{\text{dyn}x_q}(t)$ can be attributed to the fact
 41 that, in general, the dynamical system [Eq. (1)] behaves as a nonstationary
 42 process.

43 From the system's dynamics of Eq. (1), it is clear that dynamical noise
 44 may influence the behavior of the dynamical system. In fact, a large dynamical
 45 noise level increases the probability of a large trajectory change. The steady
 46 state of the dynamical system may drift into different regions in the state space.
 47 Thus, knowing the dynamical noise level is a fundamental to analyzing and
 48 understanding dynamical systems [2, 3].

49 However, dynamical noise is more difficult to be estimated than observational
 50 noise because the former is directly added to the dynamics of time evolution.
 51 If the dynamical system is known, we can formulate algorithms for estimating
 52 the dynamical noise level [4, 5]. Nevertheless, it should be noted that actual
 53 systems in the real world are usually unknown. For these systems, few meth-
 54 ods have been proposed to estimate the dynamical noise level. For example,
 55 according to the work of Siefert et al. [6], an interesting quantitative method
 56 for analyzing dynamical noise has been proposed. This method can estimate
 57 the amount of noise in systems with unknown dynamics by using the Kramers-
 58 Moyal coefficients [7, 8], but the method requires a 'long' time series to estimate
 59 the mean and variance of a trajectory moving on an attractor. On the other
 60 hand, Urbanowicz and Holyst [9] proposed an important noise-level estimation
 61 method that assumes observational noise. Although they have suggested that
 62 their method is valid for observational noise as well as dynamical noise, we have

found that their algorithm does not necessarily work well for estimation of the dynamical noise level.

In this paper, we propose a method for estimating the level of dynamical noise, which can be applied to even ‘short’ time series generated from unknown dynamics. This paper is organized as follows. First, in Section 2, we present some definitions that are useful for the subsequent results. Next, in Section 3, we recall Urbanowicz and Holyst’s method [9], with the numerical verification whether their method is applicable for even short time series. Then, in Section 4, we introduce the proposed method and verify its effectiveness as a dynamical noise indicator. Furthermore, in Section 5, our method is applied to human electroencephalographic (EEG) data in order to illustrate its applicability to real-world systems. Finally, discussions of the obtained results are presented in Section 6, and some conclusions are provided in Section 7.

2. Preliminaries

We analytically derive some definitions for short and long time series as follows. First, we define

$$I_1(c) = \int_{t_0}^c \frac{|\mathbb{E}_{\tau+h}[\{X_{\text{obs}}(t)\}] - \mathbb{E}_{\tau}[\{X_{\text{obs}}(t)\}]|}{h} d\tau, \quad (4)$$

$$I_2(c) = \int_{t_0}^c \frac{|\text{Var}_{\tau+h}[\{X_{\text{obs}}(t)\}] - \text{Var}_{\tau}[\{X_{\text{obs}}(t)\}]|}{h} d\tau, \quad (5)$$

where $0 < h \ll 1$, and operators $\mathbb{E}_{\tau}[\cdot]$ and $\text{Var}_{\tau}[\cdot]$ denote the mean and variance, respectively, and are defined as follows:

$$\mathbb{E}_{\tau}[\{X_{\text{obs}}(t)\}] = \frac{1}{h'} \int_{\tau}^{\tau+h'} X_{\text{obs}}(t) dt, \quad (6)$$

$$\text{Var}_{\tau}[\{X_{\text{obs}}(t)\}] = \frac{1}{h'} \int_{\tau}^{\tau+h'} (X_{\text{obs}}(t) - \mathbb{E}_{\tau}[\{X_{\text{obs}}(t)\}])^2 dt, \quad (7)$$

with $0 < h' \ll 1$. The two quantities $I_1(c)$ and $I_2(c)$ indicate the sums of discrepancies $|\cdot|$ of ‘partially’ observational signals $\{X_{\text{obs}}(s)\}$ over $\tau \leq s \leq \tau + h + h'$ with $0 \leq t_0 \leq c$ and $t_0 \leq c \leq T$ [see the bidirectional arrows with width h in Fig. 1(a)];

⁸⁴ t_0 and T are a starting point and an ending point of an observation, respec-
⁸⁵ tively. Then, if $\{X_{\text{obs}}(s)\}$ behaves as a stationary process, both quantities $I_1(c)$
⁸⁶ and $I_2(c)$ are almost equal to zero, even if c becomes large, because the mean
⁸⁷ and variance of the stationary process do not change over time. However, in
⁸⁸ this study, we analyze signals observed from a stochastic dynamical system [see
⁸⁹ Eq. (1)], in which variables usually behave as nonstationary processes, so that
⁹⁰ observational signals also stand for nonstationarity. In fact, each of quantities
⁹¹ $I_1(c)$ and $I_2(c)$ represents a margin between stationarity and nonstationarity
⁹² [see Fig. 1]. Note that in Eq. (4) and Eq. (5), if h is set to a very small value,
⁹³ the partial signal $\{X_{\text{obs}}(s)\}$ can be considered as a ‘semi-stationary’ process.

⁹⁴ Depending on whether the target signal is near stationarity, the present
⁹⁵ method works either as a universal indicator among systems or as an intrinsic
⁹⁶ indicator for each system. If the target signal is near stationarity, i.e., it is ob-
⁹⁷ served as a short time series, the method can universally extract the dynamical
⁹⁸ noise level for many systems; if the target signal is far from stationarity, i.e., it
⁹⁹ is observed as a long time series, the method can specifically extract the dynam-
¹⁰⁰ ical noise level for each system so that the extracted levels are not comparable
¹⁰¹ among systems, but involve properties of their respective systems. Thus, it is
¹⁰² essential to flexibly determine a boundary between short and long time series.

¹⁰³ To choose a threshold value c^* , by which we can define whether a time series
¹⁰⁴ is short or long, we use Eq. (4) and Eq. (5) with two upper bounds ϵ_1 ($0 < \epsilon_1$)
¹⁰⁵ and ϵ_2 ($0 < \epsilon_2$), and compute c_1^* and c_2^* such that

$$I_1(c_1^*) = \epsilon_1, \quad (8)$$

$$I_2(c_2^*) = \epsilon_2. \quad (9)$$

¹⁰⁶ Then, we define the threshold value c^* as follows:

$$c^* = \min(c_1^*, c_2^*). \quad (10)$$

¹⁰⁷ The computation of c^* is provided in Appendix A. This threshold value c^*
¹⁰⁸ determines whether the length $L_{\text{seg}}(t_0, \Delta t)$ of a time series $\{X(i)\}$ is short or

Figure 1: Concept of a separation between short and long time series by discrepancies introduced between stationarity and nonstationarity arising from the fact that a partially short-term signal $X_{\text{obs}}(s)$ with very short length h is a semi-stationary process. (a) Four time series sorted in accordance with the velocity of the instantaneous mean value of the time series from bottom to top: the harmonic oscillator system, the Rössler system, the coupled Hindmarsh-Rose neuronal system, and the Lorenz system. The time series were obtained with $\rho_q(t)/\sigma_{\text{dyn}x_q}(t) = 0.1$. (b) A schematic example of the accumulation $I_1(t)$ of discrepancies. An upper bound ϵ_1 determines a threshold value c_1^* . If the accumulation $I_1(t) < \epsilon_1$, we call this time series a short time series, and $I_1(t)$ reflects a more universal property among systems. If $I_1(t)$ is large, we call this time series a long time series, and $I_1(t)$ reflects a more specific property of the system because the longer the time series increases, the more its wave form differs from those of other time series.

109 long. If the inequality

$$L_{\text{seg}}(t_0, \Delta t) \leq (c^* - t_0)/\Delta t \quad (11)$$

110 is satisfied, the time series is assumed to be short. Otherwise, the time series is
 111 referred to as a long time series. Note that Eq. (4) and Eq. (5) define $I_1(c_1^*)$
 112 and $I_2(c_2^*)$ like differences, rather than derivatives because $\{X_{\text{obs}}(t)\}$ may be
 113 a non-smooth time series. Equations (4)-(11) can be seen as a set of general
 114 definitions for the length of a time series generated from a continuous-time
 115 dynamical system. Throughout this paper, when we discuss whether a time

series is short or long, we denote the length of the time series as $L_{\text{seg}}(t_0, \Delta t)$,
 otherwise as N , which refers to the data size of the time series.

3. Previous method for noise-level estimation

First, we introduce the previous method for noise-level estimation, which
 uses coarse-grained entropy [9]. Because this algorithm assumes observational
 noise, it may fail when trying to estimate the level of dynamical noise.

3.1. Urbanowicz and Holyst's method

Suppose that a time series $\{X(i)\}$ with $i = 1, 2, \dots, N$ is given. Furthermore,
 assume that it is influenced neither by observational noise nor by dynamical
 noise, i.e. assume a 'clean' time series. Then, we construct M -dimensional
 delay coordinates as $\vec{X}_M(i) = \{X(i), X(i + \Gamma), \dots, X(i + (M - 1)\Gamma)\}$, where M
 and Γ are an embedding dimension and an embedding delay, respectively. We
 define the correlation integral with a threshold value ε for $\vec{X}_M(i)$ as

$$C^M(\varepsilon) = \frac{1}{N^2} \sum_i^N \sum_{p \neq i}^N \theta(\varepsilon - \|\vec{X}_M(i) - \vec{X}_M(p)\|), \quad (12)$$

where $\theta(\cdot)$ is the Heaviside function, defined as

$$\theta(u) = \begin{cases} 0 & (u < 0), \\ 1 & (u \geq 0). \end{cases} \quad (13)$$

For simplicity, we use the maximum norm. In addition, the term $C^M(\varepsilon)$ allows
 to define the coarse-grained correlation entropy, denoted by $K_2(\varepsilon)$, as follows
 [10, 11]:

$$K_2(\varepsilon) = \lim_{\varepsilon \rightarrow 0} \lim_{M \rightarrow \infty} \ln \frac{C^M(\varepsilon)}{C^{M+1}(\varepsilon)} \approx - \frac{d \ln C^M(\varepsilon)}{dM}. \quad (14)$$

The term $K_2(\varepsilon)$ is an ideal entropy, which cannot be obtained in reality. In
 practice, we obtain a noisy version of this entropy, i.e. $K_{\text{noisy}}(\varepsilon)$ [9] instead of
 $K_2(\varepsilon)$, when the Gaussian observational noise is added to $\{X(i)\}$; we call this

136 $K_{\text{noisy}}(\varepsilon)$ as the observed coarse-grained entropy defined as follows:

$$K_{\text{noisy}}(\varepsilon) = -\frac{1}{\Gamma} g\left(\frac{\varepsilon}{2\sigma}\right) \ln \varepsilon + [\kappa + b \ln(1 - \alpha\varepsilon)] \\ \times \left(1 + \sqrt{\pi} \frac{\sqrt{\varepsilon^2/3 + 2\sigma^2} - \varepsilon/\sqrt{3}}{\varepsilon}\right), \quad (15)$$

137 where σ denotes the standard deviation of the Gaussian observational noise.
 138 The term $[\kappa + b \ln(1 - \alpha\varepsilon)]$ represents the approximate form of $K_2(\varepsilon)$; κ is the
 139 correlation entropy; $b \ln(1 - \alpha\varepsilon)$ indicates the effect of the coarse graining. Here,
 140 we define $g(\cdot)$ as follows:

$$g(v) = \frac{2}{\sqrt{\pi}} \frac{v \exp(-v^2)}{\text{Erf}(v)}, \quad (16)$$

141 where $\text{Erf}(v)$ shows the error function defined as

$$\text{Erf}(v) = \frac{2}{\sqrt{\pi}} \int_0^v \exp(-t^2) dt. \quad (17)$$

142 We fit parameters κ , α , b , and σ by using Eq. (15) based on given time series
 143 data, so that a bimodal distribution appears on an $(\varepsilon, K_{\text{noisy}}(\varepsilon)\varepsilon^\mu)$ -plane if
 144 the time series is influenced by the Gaussian observational or dynamical noise,
 145 where μ is a constant of order 1. Consequently, the first local maximum value
 146 corresponding to the smaller ε is equivalent to σ if

$$\mu \approx 3.441717 - \frac{1}{\ln \sigma},$$

147 which has been numerically found by Urbanowicz and Holyst [9].

148 3.2. Examples

149 Here, we discuss some examples of short time series for which Urbanowicz
 150 and Holyst's method does not produce estimates that sufficiently correlate with
 151 the actual dynamical noise levels and universally work among systems [see Figs.
 152 2(a)-(d)]. In these examples, the relationship between the estimated and the
 153 actual levels of dynamical noise has been investigated numerically with the
 154 use of their method [9]. Time series data (sampled observational signals) were
 155 generated with four types of systems, each of which was influenced by dynamical

Figure 2: (a) to (d): Relationship between the noise level estimated with Urbanowicz and Holyst's method and the actual dynamical noise level. (e) to (h): Relationship between the time series dimension (TSD) and the actual dynamical noise level. Time series data are generated for (a, e) the harmonic oscillatory system [Eqs. (18)-(20)], (b, f) the Lorenz system [Eqs. (21)-(24)], (c, g) the Rössler system [Eqs. (25)-(28)], and (d, h) the coupled Hindmarsh-Rose neuronal system [Eqs. (29)-(32)].

156 noise $\rho_w dW_w(t)$ ($w \in \{x, y, z, x_n, y_n, z_n\}$) and converted to observational signals
 157 $\{X_{\text{obs}}(t)\}$:

158 (I) The harmonic oscillator system was simulated with

$$dx = ydt + \rho_x dW_x(t), \quad (18)$$

$$dy = -xdt + \rho_y dW_y(t), \quad (19)$$

$$X_{\text{obs}}(t) = x(t) + \sigma_X \xi_X(t). \quad (20)$$

159 (II) The Lorenz system was simulated with

$$dx = [-10x + 28y] dt + \rho_x dW_x(t), \quad (21)$$

$$dy = [-xz + 28x - y] dt + \rho_y dW_y(t), \quad (22)$$

$$dz = [xy - (8/3)z] dt + \rho_z dW_z(t), \quad (23)$$

$$X_{\text{obs}}(t) = x(t) + \sigma_X \xi_X(t). \quad (24)$$

160 (III) The Rössler system was simulated with

$$dx = [-y - z] dt + \rho_x dW_x(t), \quad (25)$$

$$dy = [x + 0.36y] dt + \rho_y dW_y(t), \quad (26)$$

$$dz = [0.4 + z(x - 4.5)] dt + \rho_z dW_z(t), \quad (27)$$

$$X_{\text{obs}}(t) = x(t) + \sigma_X \xi_X(t). \quad (28)$$

161 (IV) The coupled Hindmarsh-Rose neuronal system [12] with 100 neurons, the
 162 n th neuron of which was influenced by the external input I_n , was simulated
 163 with

$$dx_n = \left[y_n - x_n^3 + 3x_n^2 - z_n + I_n + \sum_{\gamma \neq n}^{100} 0.05\theta(x_\gamma) \right] dt + \rho_{x_n} dW_{x_n}(t), \quad (29)$$

$$dy_n = [1 - 5x_n^2 - y_n] dt + \rho_{y_n} dW_{y_n}(t), \quad (30)$$

$$dz_n = [0.006\{4(x_n + 1.6) - z_n\}] dt + \rho_{z_n} dW_{z_n}(t), \quad (31)$$

$$X_{\text{obs}}(t) = \mathbf{x}_1(t) + \sigma_X \xi_X(t), \quad (32)$$

164 see Eq. (13) for $\theta(\cdot)$, where the amount ρ_w of the dynamical noise was constant
 165 over time. The stochastic variable dW_w follows a normal distribution of mean
 166 0 and standard deviation $\sqrt{dt}$. For systems (I) through (IV), the observational
 167 signal $\{X_{\text{obs}}(t)\}$ has been used to test Urbanowicz and Holyst's method, and
 168 for simplicity, we have assumed that the signal is independent of observational
 169 noise $\sigma_X \xi_X(t)$ by setting $\sigma_X = 0$. Actually, in the network composed of the
 170 Hindmarsh-Rose neurons [Eqs. (29) to (32)], a synchronized chaotic oscillation
 171 appears [13]. Each value of $\{I_n\}$ has been uniformly distributed between 1 and

5. Fifty time series of $\{X_{\text{obs}}(t)\}$ were simulated with respect to each noise level, which is defined as $\rho_w/\sigma_{\text{dyn}w}$. Here $\sigma_{\text{dyn}w}$ indicates the standard deviation of the variable w , and the initial condition of w follows the uniform distribution between 0 and 1 for each noise level. Note that $\sigma_{\text{dyn}w}$ is not time-dependent because the simulation was conducted repeatedly over multiple trials. The sampling time Δt was set to 0.001, and the length $L_{\text{seg}}(0, \Delta t)$ of each time series was set to 100 points, in which the transient state following the onset of the simulation was included in the tested time series, i.e., $t_0 = 0$ hereafter.

The time series with length of $L_{\text{seg}}(0, \Delta t) = 100$ were identified as short time series by calculations of the quantities $I_1(c)$ and $I_2(c)$ [Eq. (4) and Eq. (5)], in which $E_\tau[\cdot]$ and $\text{Var}_\tau[\cdot]$ were calculated with an analysis window $h' = 0.005$. Then, h was set to 0.001 so that a partially observational signal $\{X_{\text{obs}}(s)\}$ over $\tau \leq s \leq \tau + h + h'$ can be considered as a semi-stationary process. The upper bounds ϵ_1 and ϵ_2 [see Eq. (8) and Eq. (9)] have been chosen as follows: $\epsilon_1 = 1$ and $\epsilon_2 = 0.05$. For these values, the corresponding threshold values of c^* [Eq. (10)] for the four systems given in Eqs. (18) to (32) were 1878.07 Δt , 103.41 Δt , 229.14 Δt , and 122.33 Δt , respectively. In the calculation of c^* for each system, we have used 100 time series of $\{X_{\text{obs}}(t)\}$ with $L_{\text{seg}}(0, \Delta t) = 50000$. Consequently, for each system we have obtained 100 threshold values for c^* . Finally, the average of these values has been considered as the final threshold value c^* . When we estimated c^* , each system was affected by dynamical noise with the noise level of 0.1 and simulated for different initial conditions (uniformly distributed between 0 and 1) over 100 trials.

A system with any noise level $\rho_w/\sigma_{\text{dyn}w}$ was simulated in two steps: first, a clean system, without any noise, was simulated with an initial condition $\vec{x}_0$, and the resulting standard deviation $\sigma_{\text{dyn}w}$ was stored in memory; next, the system including dynamical noise was simulated using the same initial condition $\vec{x}_0$ with the clean system, and then the noise level was set to the same values over x , y , and z or over $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$, where ρ_w was obtained by multiplication between the arbitrarily set noise level and the stored $\sigma_{\text{dyn}w}$.

In Urbanowicz and Holyst's method, we have defined the estimated noise

level as $\hat{\sigma}/\sigma_{\text{dyn}w}$, where $\hat{\sigma}$ represents the fitted standard deviation of the Gaussian observational noise [see Eq. (15)]. Clearly, the numerical results presented in Fig. 2 reveal that Urbanowicz and Holyst’s method is not necessarily suitable for estimating the level of dynamical noise and for functioning universally among systems.

4. Proposed method for noise-level estimation

In this section, a novel method for detecting changes in the level of dynamical noise is presented. Here consider two types of time series: (1) A series originating from a flow generated from a continuous-time dynamical system; (2) A series originating from a map generated from a discrete-time dynamical system. The proposed method can be applied to the former time series in the real world by calculating the fractal dimension developed by Higuchi [14], even if the dynamical system is unknown. To avoid confusing the fractal dimension with the correlation dimension in the reconstructed attractor, we call the fractal dimension using the Higuchi method as the time series dimension (TSD) because the Higuchi method considers a time series to be a figure represented on a two-dimensional space, in which the horizontal and vertical axes denote time and amplitude, respectively.

Here we assume that an observational signal is sampled with the time interval $\Delta t > 0$. Then, the TSD of the resulting time series reflects both the deterministic and stochastic components of the time series, and ranges from 1 to 1.5. When the dynamical system underlying the time series contains only the deterministic component, the TSD of the time series is 1 because the series locally resembles a line. In contrast, when the dynamical system underlying the time series contains only the stochastic component that represents the Gaussian white noise, the TSD of the time series is 1.5, the fractal dimension of the Wiener process [15]. Therefore, the relative dominance ratio between the deterministic and stochastic components determines the TSD.

We stress the fact that in the present study, our focus is on a system [see

Eq. (1)], in which the deterministic and stochastic components are mixed. Furthermore, since the observed signal [Eq. (2)] is sampled with the certain time interval, the corresponding TSD depends not only on the amount of dynamical noise but also on the sampling time Δt . If the deterministic component does not contribute to the system [Eq. (1)] at all, the TSD is 1.5, independent of the amount of dynamical noise and of the sampling time, as mentioned above. Further, the TSD does not require the power law.

In Subsection 4.1, we explain how to calculate the curve length of a given time series from a stochastic dynamical system of the form [Eq. (1)] by introducing the Higuchi method [14]. In Subsection 4.2, we convert the curve length directly to the TSD. The target time series is the sampled signal $\{X_{\vec{\rho}}(i\Delta t)\}$ ($i = 1, 2, \dots, N$) of the variable x_q perturbed by dynamical noise with $\vec{\rho}(t) = (\rho_1(t), \rho_2(t), \dots, \rho_d(t))$, or more precisely, with $\vec{\rho}(i\Delta t)$; namely, it has been assumed that $\{X_{\vec{\rho}}(i\Delta t)\}$ is influenced by dynamical noise, not by observational noise. Hereinafter, we denote $\vec{\rho}(i\Delta t)$ as $\vec{\rho}$, for simplicity. Finally, in Subsection 4.3, we present a numerical investigation of the proposed method, in which the level of dynamical noise has been estimated.

4.1. Curve length of time series

The Higuchi method [14] is based on the following expression [16] for calculating the curve length,

$$L^{(q)}(\Delta t, \vec{\rho}) = \frac{\sum_{i=1}^{N-1} |X_{\vec{\rho}}((i+1)\Delta t) - X_{\vec{\rho}}(i\Delta t)|}{\Delta t}, \quad (33)$$

where $L^{(q)}(\Delta t, \vec{\rho})$ is an ideal definition of the curve length, but this definition does not work well when we try to get an appropriate TSD. The Higuchi method solves this problem by course-graining the time series and creating some new time series sampled with $\zeta_k = k\Delta t$, in which k is a natural number. A larger k results in a larger sampling time ζ_k .

Consequently, in the Higuchi method, the modified curve length $\tilde{L}^{(q)}$ associated with the time series $\{X_{\vec{\rho}}(i\Delta t)\}$ corresponding to the variable x_q is

259 calculated as follows:

$$\tilde{L}^{(q)}(\zeta_k, \vec{\rho}) = \frac{1}{k} \sum_{m=1}^k L_m^{(q)}(\zeta_k, \vec{\rho}), \quad (34)$$

260 where

$$\begin{aligned} L_m^{(q)}(\zeta_k, \vec{\rho}) &= \left\{ \left(\sum_{j=1}^{\lfloor \frac{N-m}{k} \rfloor} \left| X_{\vec{\rho}}((m+jk)\Delta t) - X_{\vec{\rho}}((m+(j-1)k)\Delta t) \right| \right) \right. \\ &\quad \times \left. \frac{N-1}{\lfloor \frac{N-m}{k} \rfloor k} \right\} / \zeta_k. \end{aligned} \quad (35)$$

261 Here, the notation $\lfloor \cdot \rfloor$ denotes the rounding down; the term $(N-1)/(\lfloor (N-m)/k \rfloor k)$ normalizes the curve length of a segment; and the variable $L_m^{(q)}(\zeta_k, \vec{\rho})$
 262 represents the curve length of the segment with a starting point m . Namely, the
 263 process of Eq. (35) constructs k new segments $\{X_k^m(\vec{\rho})\}$ with $m = 1, 2, \dots, k$,
 264 each segment of which is defined by
 265

$$X_k^m(\vec{\rho}) = \left\{ X_{\vec{\rho}}(m\Delta t), X_{\vec{\rho}}((m+k)\Delta t), X_{\vec{\rho}}((m+2k)\Delta t), \dots, X_{\vec{\rho}}\left(\left(m + \left\lfloor \frac{N-m}{k} \right\rfloor k\right)\Delta t\right) \right\}. \quad (36)$$

266 Note that the Higuchi method targets the time series sampled with ζ_k , so that
 267 the process of Eq. (35) performs division by ζ_k instead of Δt [see Eq. (33)].
 268 Then, any curve length of the time series sampled with ζ_k can be calculated by
 269 coarse-graining of the time series.

270 4.2. Relationship between time series dimension and noise level

271 The definition of the TSD is based on the box-counting dimension [15], and
 272 is given as follows:

$$D = \lim_{\phi \rightarrow 0} \frac{\log N(\phi)}{\log(1/\phi)}, \quad (37)$$

273 where the variable $N(\phi)$ represents a minimal value of the total number of
 274 hyperspheres or the hypercubes with size ϕ needed to entirely cover a structure
 275 like a coast line. When the size ϕ and the variable $N(\phi)$ are substituted with the
 276 sampling time ζ_k and the curve length $\tilde{L}^{(q)}(\zeta_k, \vec{\rho})$ of the time series associated

Figure 3: Diverse scaling property of time series data with $L_{\text{seg}}(0, 0.001) = 10000$, observed through (a) the harmonic oscillatory system, (b) the Lorenz system, (c) the Rössler system, and (d) the coupled Hindmarsh-Rose neuronal system, each system of which was affected by dynamical noise with the noise level of 0.1. For each panel, both the horizontal and vertical axes are displayed by log scales, and the time series dimension (TSD) is calculated as the slope of the approximate line given by only two points $(1/\zeta_1, \tilde{L}^{(q)}(\zeta_1, \vec{\rho}))$ and $(1/\zeta_2, \tilde{L}^{(q)}(\zeta_2, \vec{\rho}))$, indicated by filled red circles.

with the variable x_q , respectively, we get an idealized version of the TSD instead of Eq. (37), as follows:

$$D^{(q)} = \lim_{\Delta t \rightarrow 0} \frac{\log \tilde{L}^{(q)}(\zeta_k, \vec{\rho})}{\log (1/\zeta_k)}. \quad (38)$$

We show below the relationship between $D^{(q)}$ and $\vec{\rho}$ based on Eq. (38), where the TSD is converted to an approximate version; note that only the approximate TSD depends on $\vec{\rho}$. For simplicity, we investigate the situation in which only one variable x_q of the dynamical system [Eq. (1)] is observed in the observational process and furthermore, we assume that observational noise does not influence the process. The TSD is calculated in the case of $\Delta t \rightarrow 0$ as shown in Eq. (38), so that the TSD is approximated as follows:

$$D^{(q)} \approx \frac{\log \tilde{L}^{(q)}(\zeta_1, \vec{\rho}) - \log \tilde{L}^{(q)}(\zeta_2, \vec{\rho})}{\log 2} \equiv \tilde{D}^{(q)}(\vec{\rho}, \Delta t), \quad (39)$$

where this approximate version is the proposed TSD [see examples in Fig. 3] and becomes a function of $\vec{\rho}$ and of Δt ; hereinafter, we denote this TSD as $\tilde{D}^{(q)}(\vec{\rho})$ because Δt is constant throughout this paper, i.e. $\Delta t = 0.001$ except for Fig. 9.

289 Note that if Δt is small enough so that the time series $\{X_{\vec{\rho}}(i\Delta t)\}$ behaves like
 290 a flow, the difference $\log \tilde{L}^{(q)}(\zeta_1, \vec{\rho}) - \log \tilde{L}^{(q)}(\zeta_2, \vec{\rho})$ reflects almost entirely the
 291 stochastic component, thus the TSD does not depend on $\vec{\rho}$ and always indicates
 292 1.5, except at $\vec{\rho} = \vec{0}$. However, in this study we assume that the observa-
 293 tional signal is sampled with a certain sampling time Δt , and consequently, the
 294 TSD reflects properties of both the deterministic and stochastic components
 295 and ranges between 1 and 1.5 in a certain range of $\vec{\rho}$ [see [Subsection 4.3](#)]. Fur-
 296 thermore, it can be seen that despite the fact that the approximate calculation
 297 [Eq. (39)] uses only two points to form the power law between $\tilde{L}^{(q)}(\zeta_k, \vec{\rho})$ and
 298 ζ_k [see Fig. 3], these two points might be essential to construct the power law;
 299 this view seems to be equivalent to the operation of $\Delta t \rightarrow 0$ included in the ideal
 300 definition [Eq. (39)] of the TSD. Namely, the power law tends to converge to a
 301 slope formed by the two points as $\Delta t \rightarrow 0$ [see Fig. 3]. Hence, the proposed TSD
 302 becomes a more flexible version of the conventional Higuchi's fractal dimension
 303 (which handles only fractal signals) and can be applied to any system.

304 Using Eqs. (34), (35), and (39), we can calculate the TSD. If the right-
 305 hand side of Eq. (39) is smooth with respect to ρ_r ($r = 1, 2, \dots, d$), we can
 306 compute its derivative in order to reveal the monotonic relationship between
 307 the TSD and the noise level. However, when the Higuchi method is applied
 308 to the time series to calculate its curve length, the resulting TSD becomes a
 309 non-smooth function because the absolute function is contained in the curve
 310 length calculation [Eq. (35)] and consequently, the derivative of Eq. (39) does
 311 not exist at certain points. In order to circumvent this problem, we conduct
 312 piecewise differentiation, excluding the non-smooth set defined as follows:

$$\Omega_r^* = \left\{ \rho_r \mid \lim_{\Delta \rho_r \rightarrow +0} \frac{\tilde{D}^{(q)}(\vec{\rho} + \Delta \vec{\rho}_r) - \tilde{D}^{(q)}(\vec{\rho})}{\Delta \rho_r} \neq \lim_{\Delta \rho_r \rightarrow -0} \frac{\tilde{D}^{(q)}(\vec{\rho} + \Delta \vec{\rho}_r) - \tilde{D}^{(q)}(\vec{\rho})}{\Delta \rho_r} \right\} \cup \{-\infty, \infty\}, \quad (40)$$

313 where $\Delta \vec{\rho}_r$ is a small deviation around ρ_r such that $\Delta \vec{\rho}_r = (0, 0, \dots, \Delta \rho_r, \dots, 0, 0)$

314 and implicitly depends on time. Using the following set

$$\Omega_r^{(S,E)} = \left\{ \rho_r \mid \lim_{\Delta\rho_r \rightarrow +0} \frac{\tilde{D}^{(q)}(\vec{\rho} + \Delta\vec{\rho}_r) - \tilde{D}^{(q)}(\vec{\rho})}{\Delta\rho_r} = \lim_{\Delta\rho_r \rightarrow -0} \frac{\tilde{D}^{(q)}(\vec{\rho} + \Delta\vec{\rho}_r) - \tilde{D}^{(q)}(\vec{\rho})}{\Delta\rho_r}, \right. \\ \left. S < \rho_r < E, S < E, S \in \Omega_r^*, E \in \Omega_r^*, (S, E) \cap \Omega_r^* = \emptyset \right\}, \quad (41)$$

315 which sorts the differentiable domains, we can perform piecewise differentiation
316 of Eq. (39) by $\rho_r \in \Omega_r^{(S,E)}$ as follows:

$$\frac{\partial \tilde{D}_{\text{pw}}^{(q)}(\vec{\rho})}{\partial \rho_r} = \frac{1}{\log 2} \left[\frac{1}{\tilde{L}_{\text{pw}}^{(q)}(\zeta_1, \vec{\rho})} \frac{\partial \tilde{L}_{\text{pw}}^{(q)}(\zeta_1, \vec{\rho})}{\partial \rho_r} - \frac{1}{\tilde{L}_{\text{pw}}^{(q)}(\zeta_2, \vec{\rho})} \frac{\partial \tilde{L}_{\text{pw}}^{(q)}(\zeta_2, \vec{\rho})}{\partial \rho_r} \right]. \quad (42)$$

317 Subindex “pw” indicates that the functions $\tilde{D}^{(q)}(\vec{\rho})$ and $\tilde{L}^{(q)}(\zeta_k, \vec{\rho})$ are piece-
318 wisely and sufficiently smooth over interval (S, E) for ρ_r . If $r \neq q$ [see Eq.
319 (1)], the TSD $\tilde{D}^{(q)}(\vec{\rho})$ does not change with ρ_r because the dynamical noises
320 $\{\rho_r(t)dW_r(t)\}_{r \neq q}$ are integrated twice over t when they influence the variable
321 x_q and the integrated Wiener process becomes smooth; namely, the derivative
322 of $\tilde{D}_{\text{pw}}^{(q)}(\vec{\rho})$ with respect to ρ_r is zero. Consequently, if the following inequality

$$\frac{\partial \tilde{D}_{\text{pw}}^{(q)}(\vec{\rho})}{\partial \rho_q} > 0 \quad (43)$$

323 is satisfied over the concatenation of S and E , and if $\tilde{D}^{(q)}(\vec{\rho})$ is continuous with
324 respect to ρ_q , then $\tilde{D}^{(q)}(\vec{\rho})$ increases with ρ_q , so that the noise level can be
325 estimated by the TSD. Here, we can easily prove that the absolute function
326 within $\tilde{L}^{(q)}(\zeta_k, \vec{\rho})$, see Eq. (35), is continuous with respect to ρ_q by using the
327 (ϵ, δ) -definition of limit [17] and consequently, $\tilde{L}^{(q)}(\zeta_k, \vec{\rho})$ is a continuous func-
328 tion. Note that the inner operation, the increment of $X_{\vec{\rho}}(i\Delta t)$, of the absolute
329 function is also continuous [Eq. (35)] because $X_{\vec{\rho}}(i\Delta t)$ continuously changes
330 with the dynamical noise level.

331 Further, we analytically derive an inequality including Δt , $\vec{\rho}$, and the length
332 $L_{\text{seg}}(0, \Delta t)$ of the time series for the case of linear stochastic dynamical sys-
333 tems [see Appendix B]. Moreover, we summarize our analytic result in the form
334 of a proposition related to the monotonic relationship between the TSD and

Figure 4: Relationship between the time series dimension (TSD) and the actual dynamical noise level, which ranges from 0 to 5. Time series data **are** generated by (a) the harmonic oscillator system, (b) the Lorenz system, (c) the Rössler system, and (d) the coupled Hindmarsh-Rose neuronal system, respectively, over 100 trials for each noise level.

the noise level [see Proposition B.1 in Appendix B]. Additionally, the monotonic relationship between $\tilde{D}^{(q)}(\vec{\rho})$ and ρ_q on the $(L_{\text{seg}}(0, \Delta t), \rho_q / \sigma_{\text{dyn}x_q})$ -plane is numerically illustrated for the case of a harmonic oscillator.

4.3. Numerical investigation

In the proposed method, we can detect changes in the level of dynamical noise by using the TSD. Here, we numerically investigate the relationship between the TSD and the noise level in (I) the harmonic oscillator system [Eqs. (18) to (20)], (II) the Lorenz system [Eqs. (21) to (24)], (III) the Rössler system [Eqs. (25) to (28)], and (IV) the coupled Hindmarsh-Rose neuronal system [Eqs. (29) to (32)], through which we observed short time series of $\{X_{\text{obs}}(t)\}$ with the length $L_{\text{seg}}(0, \Delta t) = 100$, and generated TSDs among 50 trials to create the monotonic properties, with respect to each noise level. As well as in Subsection 3.2, the transient state following the onset of the simulation was included in the tested time series and the initial condition of variables for each trial followed uniform distributions between 0 and 1. The obtained results, depicted in Figs. 2(e) through 2(h), show that the TSD increases with the noise level, which ranges from 0 to 5 [see Fig. 4], and further, the detailed monotonically increasing properties **are** similar among systems, so that there exists a possibility that

Figure 5: An illustrative comparison among systems' monotonic properties of the time series dimension (TSD) with respect to the actual dynamical noise level of short time series, observed through four types of systems: the harmonic oscillator system, the Lorenz system, the Rössler system, and the coupled Hindmarsh-Rose neuronal system. Time series data used here **are** the same with Fig. 4.

the TSD can be a universal value related to the dynamical noise level [see Fig. 5].

However, if the length of the time series increases to include properties of the respective system, the monotonic behavior differs among systems, as shown in Fig. 6. Actually, we displayed the $(L_{\text{seg}}(0, \Delta t), \rho_w / \sigma_{\text{dyn}w})$ diagrams to reveal the effect of the length of time series on the monotonic behavior for the four systems. Each parameter point on the $(L_{\text{seg}}(0, \Delta t), \rho_w / \sigma_{\text{dyn}w})$ diagram indicates the average value of 100 TSDs calculated from time series generated from the corresponding system; namely, for each point in the $(L_{\text{seg}}(0, \Delta t), \rho_w / \sigma_{\text{dyn}w})$ -plane, we generated TSDs among 100 trials as well as Fig. 4, where each simulation **has been** conducted for initial conditions following the uniform distribution between 0 and 1. In particular, the diagrams for the Lorenz system and for the coupled Hindmarsh-Rose neuronal system show specific monotonic

Figure 6: The effect of the length of time series and the dynamical noise level on the monotonic properties among systems: (a) the harmonic oscillatory system, (b) the Lorenz system, (c) the Rössler system, and (d) the coupled Hindmarsh-Rose neuronal system. A larger time series dimension (TSD) is indicated by a lighter color. The effect resulting from the length differs among systems, as shown in areas indicated by open green circles, due to a reflection of systems' own properties to the TSD.

behavior, which is different from the other systems [see Figs. 6(b) and 6(d)]. Note, however, that the TSD can still estimate the noise level for each system. In conclusion, the TSD can estimate the level of dynamical noise for a system on the basis of a monotonic relation.

5. Application to EEG data

We applied the proposed method to human EEG data [see Fig. 7]. Two healthy subjects gave written informed consent prior to participation in the experiment. The EEG study was approved by the ethics committees of RIKEN and the University of Tokyo. EEG signals from 63 electrodes were obtained during eyes-open and eyes-closed resting periods. We ran the experiment in a sound-proof, dimly lit room. A cycle of the measurement consists of the eyes-open and eyes-closed periods, each of length 35 s. The cycle of 70 s was repeated 10 times for two subjects. Sound stimuli (500 Hz or 1 kHz beep) were used to ask subjects to open or close their eyes. Subjects were required to be fixated on a white cross on a black background on a CRT display in the eyes-open condition.

Figure 7: (a) A representative EEG signal obtained from electrode Fp2 during the eyes-open and eyes-closed resting periods. (b) The estimated noise level with Urbanowicz and Holyst’s method. (c) The time series dimension (TSD), and (d) the variance. The arrows indicate the onset of the eyes-open or eyes-closed state. (e) A typical time series of the eyes-open state, the TSD of which was 1.22. (f) A typical time series of the eyes-closed state, the TSD of which was 1.12.

381 The reference electrodes were placed on both the left and right earlobes, and
 382 the sampling frequency was set to 1 kHz. Then, the frequencies of the low- and
 383 high-pass filters were set to 250 Hz and 0.016 Hz, respectively.

384 When we applied Urbanowicz and Holyst’s method and our proposed method
 385 to the EEG signals, we used an analysis window with 1500 points, shifting along
 386 the time axis every 30 points [see Figs. 7(b) and 7(c)]. No preprocessing was
 387 performed.

388 By applying the proposed method, we found that the difference between the
 389 eyes-open and eyes-closed brain states becomes evident [see Fig. 7(c)]. Analy-
 390 sis of the 63 time series of the EEG signals showed that the TSD of electrode

391 Fp2 can be used to robustly discriminate between the eyes-open and eyes-closed
 392 states. With respect to each subject, the Wilcoxon rank sum test was applied
 393 to two datasets: TSDs (11400 points) of the eyes-open period and TSDs (11400
 394 points) of the eyes-closed period, both of which include 10 trials; 1140 TSDs
 395 were calculated in the eyes-open or eyes-closed periods for each trial. The re-
 396 sults showed that the median of the eyes-open dataset was significantly greater
 397 than that of the eyes-closed dataset ($P < 0.0001$ for two subjects); medians
 398 of the eyes-open and eyes-closed datasets were 1.24 and 1.18 for subject 1 and
 399 1.19 and 1.16 for subject 2, respectively. The TSD is less influenced by dis-
 400 turbances such as eye movement [see Fig. 7(c)] so that the TSD of each state
 401 of the eyes-open and eyes-closed is temporally stable, i.e., if we set an appro-
 402 priate threshold value, each state can be definitely distinguished. In contrast,
 403 Urbanowicz and Holyst’s method did not clearly discriminate between the eyes-
 404 open and eyes-closed states [see Fig. 7(b)]. The variance of the time series also
 405 differed between the eyes-open and eyes-closed states [see Fig. 7(d)], where an
 406 analysis window similar to those applied to Figs. 7(b) and 7(c), was used for
 407 dynamically obtaining the variance, but the difference between the states was
 408 smaller than that using the TSD [see Fig. 7(c)]. Moreover, the variance is very
 409 sensitive to disturbances so that the time series of the variance fluctuates over a
 410 wide range of amplitudes. Thus, the proposed method can estimate the level of
 411 the dynamical noise more reliably and distinguish the eyes-open and eyes-closed
 412 states robustly.

413 **6. Discussions**

414 A novel dimension called TSD, for estimating the level of dynamical noise
 415 perturbing in time series, observed through stochastic dynamical systems has
 416 been proposed. We have defined the dynamical noise in the sense of the Wiener
 417 process, and based on this definition, it has been validated whether a monotonic
 418 relationship between the TSD and the dynamical noise level is achieved among
 419 four specific systems [see Subsection 4.3]. The TSD may be a well-defined

dimension because it has been derived directly from the scaling property [see Fig. 3] of the conventional Higuchi’s fractal dimension so that the TSD does not require the power law and is applicable to non-fractal signals. Furthermore, the applicability of the TSD to real time-series data has been demonstrated with the application of the proposed method to human EEG signals. The obtained results have shown that the estimated TSD temporally changes due to visual inputs [see Fig. 7(c)].

Possibly due to the noise-induced order [18], the variance of the eyes-opened period was low compared to that of the eyes-closed period [see Fig. 7(d)]. In particular, the dynamical neural noise caused by the light stimulus may play a role in realizing the noise-induced order. Typical time series [see Figs. 7(e) and 7(f)] of the eyes-open and eyes-closed states have implied that the TSD reflects the relative dominance ratio between the deterministic and stochastic components. In the case of the eyes-open state [see Fig. 7(e)], the TSD was larger corresponding with miniature fluctuations in the EEG data. In the case of the eyes-closed state [see Fig. 7(f)], the TSD was smaller corresponding with large fluctuations in the EEG data.

The proposed TSD may also work as an effective indicator of dynamical noise even for transient time series data (containing transient components). We have supported our claim with the numerical analysis presented in [Subsection 4.3](#). In particular, short time series with the length $L_{\text{seg}}(0, \Delta t) = 100$, obtained from four different systems, have shown a similar monotonically increasing property of the TSD with respect to the dynamical noise level, as depicted in Fig. 4. Thus, the present method may be used to analyze time series containing transient behavior.

On the other hand, if one has multivariate time series, especially two time series, one may be able to adjust their monotonic properties so that the universal monotonicity holds for both of them, by changing the length of time series [see Fig. 8] or by resampling time series [see Fig. 9]. Let us consider the situation that we have two monotonic properties and try to fit these properties with each other. As a particular example, we try to make a monotonic property

Figure 8: Adjustments of the length of short time series to work the systems universally. A case that the monotonic property of (1) the harmonic oscillator system approaches those of other three systems: (2) the Lorenz system, (3) the Rössler system, and (4) the coupled Hindmarsh-Rose neuronal system. Clearly, the minimal values of the RMS errors correspond to $L_{\text{seg}}(0, \Delta t) = 186$ for system (2), $L_{\text{seg}}(0, \Delta t) = 626$ for system (3), and $L_{\text{seg}}(0, \Delta t) = 437$ for system (4), respectively.

451 generated from the harmonic oscillator system, equal to each of monotonic prop-
 452 erties generated from the Lorenz system, the Rössler system, and the coupled
 453 Hindmarsh-Rose neuronal system. The lengths of the time series associated
 454 with these four systems have been fixed to 4502, 182, 574, and 407 points, re-
 455 spectively so that the beginning of this simulation has been equivalent to Fig.
 456 A1(a) in Appendix A; these lengths correspond to $\lfloor c^*/\Delta t \rfloor$ derived from two up-
 457 per bounds $\epsilon_1 = 2.5$ and $\epsilon_2 = 0.07$ [see Eqs. (8) and (9)], where $\lfloor \cdot \rfloor$ denotes the
 458 rounding down. As the length $L_{\text{seg}}(0, \Delta t) = 4502$ of the time series decreases,
 459 the root mean square (RMS) errors [see Eq. (A1)] between two systems also
 460 decrease, as shown in Fig. 8. Among three pairs of two systems (i.e. (i) the
 461 harmonic oscillator system vs. the Lorenz system, (ii) the harmonic oscillator
 462 system vs. the Rössler system, and (iii) the harmonic oscillator system vs. the

Figure 9: The effect of the sampling time Δt on the monotonic property corresponding to the harmonic oscillator system. As Δt decreases, the time series dimension (TSD) becomes constant along with the noise level larger than 0. As Δt increases, the monotonic property changes, but the property still holds even if $\Delta t = 1$.

coupled Hindmarsh-Rose neuronal system), when we change the length of time series associated with the harmonic oscillator system from 4502 to 50, the RMS errors indicate very small values (less than 0.05) at certain length of time series, see Fig. 8. The lengths corresponding to the minimal values of the RMS errors are 186, 626, and 437 for pairs of systems (i) through (iii), respectively. Thus, we can flexibly control the monotonic properties to work them universally by changing the length of time series.

Finally, we discuss the case of adjusting the monotonic property of a system by resampling the corresponding time series data. We have simulated 100 short time series of $\{X_{\text{obs}}(t)\}$ with the lengths of $L_{\text{seg}}(0, 0.0001k) = \lceil 45020/k \rceil$, as well as Fig. 8 when $k = 10$, where $\lceil \cdot \rceil$ denotes the rounding up. These time series correspond to the harmonic oscillator system affected by dynamical noise. Five monotonic properties, each of which corresponds to the sampling time 0.0001,

0.001, 0.01, 0.1, and 1 of Δt , respectively, have been generated. At this point it is worth to remember that the TSD actually depends on Δt as well as $\vec{\rho}$, although until now we have considered a fixed sampling time $\Delta t = 0.001$. Now, consider the monotonic property obtained by using a very small sampling time, namely $\Delta t = 0.0001$. This property almost does not depend on the noise level except at $\vec{\rho} = \vec{0}$, and then the resulting TSD always indicates around 1.5 so that the property cannot work as an indicator of the dynamical noise level. Next, suppose that we resample the time series, which was sampled at $\Delta t = 0.001$, by new sampling times of 0.01, 0.1, and 1, respectively. As Δt increases, the monotonic property changes; that is, the slope of the monotonically increasing behavior gradually decreases. If Δt is a very large value for example $\Delta t = 1$, the monotonic behavior tends to be unstable, but the increasing property still holds. Thus, along with the resampled multivariate time series data in a plausible way, the resulting TSDs may work universally.

7. Conclusions

A method, based on the TSD, for detecting changes in the level of the dynamical noise in stochastic dynamical systems has been proposed. It has been shown that the TSD has a monotonic relationship with the level of dynamical noise. This novel TSD is an extended or, rather, a flexible version of the conventional Higuchi's fractal dimension because the TSD can handle any stochastic dynamical system, including non-fractal signals. Furthermore, a theory of the TSD has been presented for the case of linear stochastic dynamical systems. This theory should be extended to cover more general systems, namely systems affected by colored dynamical noise and systems influenced not only by dynamical noise but also by observational noise. In order to show the applicability of the TSD to real-world systems, the proposed method has been applied to EEG data, in which the TSD has shown robustness to sudden disturbances. We believe that the proposed method may help us understand the behavior of various complex dynamical systems, including the brain.

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512 Appendix

513 A. How to determine the threshold value c^* in Eq. (10)

514 In this appendix, a procedure for determining the threshold value c^* for a
515 given observational signal is presented. This threshold value provides a bound-
516 ary between short and long time series for the TSD analysis. An optimal length
517 or analysis window for the TSD is assessed, based on the threshold value and
518 accordingly, the time series within the analysis window will be converted to the
519 TSD in turn along the time axis. Actually depending on the length of time
520 series, the function of the TSD can be changed from the viewpoint of the mono-
521 tonic behavior, which can be seen between the TSD and the level of dynamical
522 noise [Fig. 6]. The diagrams, consisting of the two upper bounds ϵ_1 and ϵ_2 [Eq.
523 (8) and Eq. (9)] with the monotonic properties, are described. On the diagrams,
524 the similarity among typical monotonic properties—for which the upper bounds
525 can be selected reasonably for a given observational signal—is plotted.

526 Here we regard a monotonic property consisting of TSDs, as a function of
527 the two upper bounds ϵ_1 and ϵ_2 . Furthermore, assume that ϵ_1 and ϵ_2 vary
528 from 1 to 9.5 and from 0.01 to 0.095, respectively; note that if these bounds are
529 set to very large values, they cannot work as the plausible boundary because
530 short time series defined in the sense of this study mean that the time series is
531 referred to as a process close to a semi-stationary process; i.e., it only includes
532 small discrepancies between stationarity and nonstationarity [Fig. 1]. We have

Figure A1: Effects of the two upper bounds ϵ_1 and ϵ_2 , on the monotonic property of the TSD. (a) The (ϵ_1, ϵ_2) diagrams in accordance with the noise level from left to right: 0.1, 0.2, 0.4, and 0.8. The horizontal axis is ϵ_1 ranged from 1 to 9.5 in steps of 0.5, while the vertical axis is ϵ_2 ranged from 0.01 to 0.095 in steps of 0.005. Each parameter point on the diagrams represents the similarity of the monotonic properties among four systems: (1) the harmonic oscillator system, (2) the Lorenz system, (3) the Rössler system, and (4) the coupled Hindmarsh-Rose neuronal system, where the similarity is evaluated by the RMS error so that a smaller error is indicated by a lighter color. The two magenta curves are boundaries of the RMS error of 0.22. (b) Examples of threshold values for c_1^* and c_2^* on a point $(\epsilon_1, \epsilon_2) = (1, 0.01)$, indicated by open blue rectangles in (a). The vertical axes are displayed by the log scale. The arrows indicate the threshold values of c^* , by which we set the length of short time series and calculated RMS errors for (a); namely, we used $\lfloor c^*/\Delta t \rfloor$ as the length of each short time series.

533 generated the (ϵ_1, ϵ_2) diagrams in terms of the similarity among the following
534 four systems: (I) the harmonic oscillator system, (II) the Lorenz system, (III)
535 the Rössler system, and (IV) the coupled Hindmarsh-Rose neuronal system [Eqs.
536 (18) to (32)], depicted in Fig. A1(a), in which each parameter point indicates
537 the similarity among four monotonic properties generated from the systems.

538 This similarity has been measured by the RMS error defined as follows:

$$RMS = \sqrt{\frac{1}{|\Omega|(V^2 - V)} \sum_{\omega \in \Omega} \sum_{e=1}^V \sum_{f \neq e} \left| \tilde{D}_{\text{mean}}^{(e)}(\omega \vec{\rho}_e) - \tilde{D}_{\text{mean}}^{(f)}(\omega \vec{\rho}_f) \right|^2}, \quad (\text{A1})$$

$$\tilde{D}_{\text{mean}}^{(e)}(\omega \vec{\rho}_e) = \frac{1}{100} \sum_{\lambda=1}^{100} \tilde{D}_{\lambda}^{(e)}(\omega \vec{\rho}_e), \quad (\text{A2})$$

539 where $\tilde{D}_{\lambda}^{(e)}(\omega \vec{\rho}_e)$ denotes the TSD of a time series sampled with $\Delta t = 0.001$
 540 observed from d -dimensional system (e) [Eqs. (18) to (32)], corresponding to
 541 the λ th simulation; e represents the index of systems (I) through (IV) so that
 542 $V = 4$; ω is the level of dynamical noise, which indicates an element of a set
 543 $\Omega = \{0, 0.1, 0.2, \dots, 1\}$; and $\vec{\rho}_e = (\sigma_{\text{dyn}x_1}^{(e)}, \sigma_{\text{dyn}x_2}^{(e)}, \dots, \sigma_{\text{dyn}x_d}^{(e)})$. Then, $\sigma_{\text{dyn}x_q}^{(e)}$ for
 544 $q = 1, 2, \dots, d$ is the standard deviation of a time series associated with the
 545 variable x_q included in system (e) . Here $|\Omega|$ represents the number of elements
 546 in Ω , namely $|\Omega| = 11$. The size $|\vec{\rho}_e|$ of $\vec{\rho}_e$ differs among systems; that is, $|\vec{\rho}_1|$,
 547 $|\vec{\rho}_2|$, $|\vec{\rho}_3|$, and $|\vec{\rho}_4|$ are respectively 2, 3, 3, and 300, each of which corresponds
 548 to the number of variables contained in system (e) [Eqs. (18) to (32)].

549 Toward an application of the (ϵ_1, ϵ_2) diagrams to the real-data analysis, in
 550 fact, we have used the lengths of $\lfloor c^*/\Delta t \rfloor$ among the four systems, where $\lfloor \cdot \rfloor$
 551 denotes the rounding down; the length $\lfloor c^*/\Delta t \rfloor$ corresponds to the upper limit
 552 value of the length of short time series. For the sake of clarity, the above sim-
 553 ulation has reflected transient behavior, where the initial condition of variables
 554 follows the uniform distribution between 0 and 1. We have applied time se-
 555 ries data with the following four kinds of noise levels: 0.1, 0.2, 0.4, and 0.8,
 556 respectively, to a set of ϵ_1 and ϵ_2 ; i.e., we accumulated discrepancies between
 557 stationarity and nonstationarity until their accumulation reached ϵ_1 or ϵ_2 [Eqs.
 558 (4) to (9)], so that only the two (ϵ_1, ϵ_2) diagrams associated with the noise lev-
 559 els of 0.1 and 0.8 show the relatively small RMS errors less than 0.22 [see Fig.
 560 A1(a)].

561 Here we show the two typical points on one (ϵ_1, ϵ_2) diagram realized with
 562 the noise level of 0.1 [Fig. A2(a)] and on the other (ϵ_1, ϵ_2) diagram realized

Figure A2: Two examples of monotonic properties among four systems: (1) the harmonic oscillator system, (2) the Lorenz system, (3) the Rössler system, and (4) the coupled Hindmarsh-Rose neuronal system, where for both panels (a) and (b), the RMS errors over monotonic properties are less than 0.22. The lengths of time series used here are obtained as the upper limit values $\lfloor c^*/\Delta t \rfloor$ of short time series, through the uses of ϵ_1 and ϵ_2 as in Fig. A1. (a) Monotonic behavior. The time series are observed through systems with the noise level of 0.1, and applied to $(\epsilon_1, \epsilon_2) = (2.5, 0.07)$. Each length of time series for systems (1) through (4) is 4502, 183, 574, and 407, respectively. (b) When the noise level is fixed to 0.8 and $(\epsilon_1, \epsilon_2) = (1, 0.03)$, monotonic behavior does not appear for all systems. Only the harmonic oscillator has shown the monotonic property. In this case, each length of time series for systems (1) through (4) is 262, 4, 12, and 76, respectively.

563 with the noise level of 0.8 [Fig. A2(b)]; namely, point $(\epsilon_1, \epsilon_2) = (2.5, 0.07)$ for
564 noise level 0.1 and point $(\epsilon_1, \epsilon_2) = (1, 0.03)$ for noise level 0.8. For the noise
565 level of 0.1, the monotonic behavior can be clearly seen among the four systems
566 [see Fig. A2(a)]. In this case, $\lfloor c^*/\Delta t \rfloor$ of short time series are 4502, 183, 574,
567 and 407 for systems (I) through (IV), respectively. On the other hand, for the
568 noise level of 0.8, the monotonic behavior has not been observed among three
569 systems except for the harmonic oscillator system [see Fig. A2(b)], because in
570 this case, the lengths of short time series are 262, 4, 12, and 76 for systems
571 (I) through (IV), respectively. This irregularity tends to happen when the time

series applied to ϵ_1 and ϵ_2 has been affected by the large dynamical noise level so that the length $\lfloor c^*/\Delta t \rfloor$ of short time series, especially generated from the Lorenz system, the Rössler system, and the coupled Hindmarsh-Rose neuronal system, has been very short [see Fig. A1(b)]. Therefore, in order to select the upper bounds ϵ_1 and ϵ_2 —which in turn set the upper limit value of the length of short time series—we should use nearly clean data, i.e. the noise level should be less than 0.1, especially when we have multivariate time series data and try to work their monotonic properties universally [Fig. 8].

B. Theory of time series dimension

In this appendix, the monotonic relationship between the TSD $\tilde{D}^{(q)}(\vec{\rho})$ and the standard deviation $\vec{\rho}(t)$ of the time-dependent Gaussian dynamical noise is shown. First, the analytic solution of a linear stochastic dynamical system, obtained via the Laplace transform [19], is introduced. Next, an inequality for the monotonic relationship between the TSD and the noise level, occurring in linear stochastic dynamical systems, is derived. Finally, the success rate of the derived inequality—i.e. the ratio between the number of successful trials satisfying the aforementioned inequality, and the number of total trials—is numerically illustrated in the case of a harmonic oscillator.

B.1. Linear stochastic dynamical system

The linear stochastic dynamical system with state vector $\vec{x} = (x_1, x_2, \dots, x_d)$ is given by

$$dx_q = \left(\sum_{l=1}^d a_{ql} x_l \right) dt + \rho_q(t) dW_q(t), \text{ for } q = 1, 2, \dots, d, \quad (\text{B1})$$

where $\rho_q(t)$ is an element of $\vec{\rho}(t) = (\rho_1(t), \rho_2(t), \dots, \rho_d(t))$. This system is the linear version of Eq. (1), where $f_q(\vec{x})$ is substituted with the linear combination of $\vec{x}$; a_{ql} represents the connection from the variable x_l to the variable x_q . Note that Eq. (B1) denotes a d -dimensional linear stochastic dynamical system.

597 Under any initial condition, if the temporal change of $\rho_q(t)$ is negligible com-
 598 pared to that of the deterministic component, the solution $x_q(t)$ of Eq. (B1) is
 599 represented by

$$\begin{aligned} x_q(t) &= F_q(t) + \sum_{l=1}^d \rho_l(t) \{G_{ql}(t) * \eta_l(t)\} \\ &= F_q(t) + \sum_{l=1}^d \rho_l(t) \xi_{ql}(t), \end{aligned} \quad (\text{B2})$$

600 where $\eta_q(t) = dW_q(t)/dt$ follows a normal distribution of mean 0 and standard
 601 deviation 1. The functions $F_q(t)$ and $G_{ql}(t)$ are time-dependent and contribute
 602 to the deterministic component and stochastic components, respectively; $G_{ql}(t) *$
 603 $\eta_l(t)$ denotes the convolution between the terms $G_{ql}(t)$ and $\eta_l(t)$. We cannot give
 604 an explicit form of the functions $F_q(t)$ and $G_{ql}(t)$, but let us show an example
 605 of them by considering the following two-dimensional dynamics:

$$dx_1 = x_2 dt + \rho_1 dW_1(t), \quad (\text{B3})$$

$$dx_2 = -x_1 dt + \rho_2 dW_2(t), \quad (\text{B4})$$

606 where we have set $(a_{11}, a_{12}, a_{21}, a_{22}) = (0, 1, -1, 0)$ in Eq. (B1). Additionally,
 607 we have assumed that ρ_1 and ρ_2 are constant over time, for simplicity. By
 608 applying the Laplace transform to the above system and considering initial
 609 condition $x_1(0) = 1$ and $x_2(0) = 0$, we obtain

$$\begin{aligned} x_1(t) &= \cos(t) + \rho_1 \{\cos(t) * \eta_1(t)\} + \rho_2 \{\sin(t) * \eta_2(t)\} \\ &= \cos(t) + \sum_{l=1}^2 \rho_l \xi_{1l}(t), \end{aligned} \quad (\text{B5})$$

610 and

$$\begin{aligned} x_2(t) &= -\sin(t) - \rho_1 \{\sin(t) * \eta_1(t)\} + \rho_2 \{\cos(t) * \eta_2(t)\} \\ &= -\sin(t) + \sum_{l=1}^2 \rho_l \xi_{2l}(t), \end{aligned} \quad (\text{B6})$$

611 for $t \geq 0$.

612 *B.2. Monotonic relationship between time series dimension and noise level*

613 We derive the inequality for the monotonic relationship between $\tilde{D}^{(q)}(\vec{\rho})$ and
 614 ρ_q , holding in a linear stochastic dynamical system [Eq. (B1)]. First, the curve
 615 length [Eq. (34) and Eq. (35)] is written in terms of the time series associated
 616 with the variable x_q as given in Eq. (B2). This yields

$$\tilde{L}^{(q)}(\zeta_k, \vec{\rho}) = \frac{(N-1)}{k^2 \zeta_k} \sum_{m=1}^k \left[\frac{1}{\lfloor \frac{N-m}{k} \rfloor} \sum_{j=1}^{\lfloor \frac{N-m}{k} \rfloor} \left| J_{\vec{\rho}}^{(q)}(m, j, k) \right| \right], \quad (\text{B7})$$

617 where

$$\begin{aligned} J_{\vec{\rho}}^{(q)}(m, j, k) = & \left\{ F_q((m+jk)\Delta t) + \sum_{l=1}^d \rho_l \xi_{ql}((m+jk)\Delta t) \right\} \\ & - \left\{ F_q((m+(j-1)k)\Delta t) + \sum_{l=1}^d \rho_l \xi_{ql}((m+(j-1)k)\Delta t) \right\}. \end{aligned} \quad (\text{B8})$$

618 In the above expressions and the latter arguments, we have denoted $\vec{\rho}(i\Delta t)$ as
 619 $\vec{\rho}$ for simplicity. As a next step, we define the following sets:

$$\Omega_{pq}(\vec{\rho}, m, k) := \left\{ j \mid J_{\vec{\rho}}^{(q)}(m, j, k) > 0 \right\}, \quad (\text{B9})$$

$$\Omega_{zq}(\vec{\rho}, m, k) := \left\{ j \mid J_{\vec{\rho}}^{(q)}(m, j, k) = 0 \right\}, \quad (\text{B10})$$

$$\Omega_{nq}(\vec{\rho}, m, k) := \left\{ j \mid J_{\vec{\rho}}^{(q)}(m, j, k) < 0 \right\}. \quad (\text{B11})$$

620 Then, the summation over j in Eq. (B7) can be written in the following form:

$$\sum_{j=1}^{\lfloor \frac{N-m}{k} \rfloor} \left| J_{\vec{\rho}}^{(q)}(m, j, k) \right| = \sum_{j \in \Omega_{pq}(\vec{\rho}, m, k)} J_{\vec{\rho}}^{(q)}(m, j, k) - \sum_{j \in \Omega_{nq}(\vec{\rho}, m, k)} J_{\vec{\rho}}^{(q)}(m, j, k). \quad (\text{B12})$$

621 Next, in order to perform piecewise differentiation of $\tilde{L}^{(q)}(\zeta_k, \vec{\rho})$ with respect to
 622 ρ_q , we have removed the non-smooth points by using the set defined as

$$\Psi^{(q)}(m, k) := \left\{ \vec{\rho} \mid \Omega_{zq}(\vec{\rho}, m, k) = \emptyset \right\}, \quad (\text{B13})$$

623 which sorts the differentiable domains. Consequently, with $\vec{\rho} \in \Psi^{(q)}(m, k)$, Eq.
 624 (B9) can be converted to a piecewise smooth function as follows:

$$\tilde{L}_{\text{pw}}^{(q)}(\zeta_k, \vec{\rho}) = \frac{(N-1)}{k^2 \zeta_k} \sum_{m=1}^k \left[\frac{1}{\lfloor \frac{N-m}{k} \rfloor} \left\{ \sum_{j \in \Omega_{\text{pq}}(\vec{\rho}, m, k)} J_{\vec{\rho}}^{(q)}(m, j, k) - \sum_{j \in \Omega_{\text{nq}}(\vec{\rho}, m, k)} J_{\vec{\rho}}^{(q)}(m, j, k) \right\} \right], \quad (\text{B14})$$

625 where subindex “pw” in the left-hand side has been used to indicate that the
 626 function is piecewisely and sufficiently smooth with respect to $\vec{\rho}$. Therefore, its
 627 derivative with respect to ρ_q exists and is given by

$$\begin{aligned} \frac{\partial \tilde{L}_{\text{pw}}^{(q)}(\zeta_k, \vec{\rho})}{\partial \rho_q} = & \frac{(N-1)}{k^2 \zeta_k} \sum_{m=1}^k \left[\frac{1}{\lfloor \frac{N-m}{k} \rfloor} \left\{ \sum_{j \in \Omega_{\text{pq}}(\vec{\rho}, m, k)} \{ \xi_{qq}((m+jk)\Delta t) - \xi_{qq}((m+(j-1)k)\Delta t) \} \right. \right. \\ & \left. \left. - \sum_{j \in \Omega_{\text{nq}}(\vec{\rho}, m, k)} \{ \xi_{qq}((m+jk)\Delta t) - \xi_{qq}((m+(j-1)k)\Delta t) \} \right\} \right], \end{aligned} \quad (\text{B15})$$

628 where we have used the fact that elements and sizes of sets $\Omega_{\text{pq}}(\vec{\rho}, m, k)$ and
 629 $\Omega_{\text{nq}}(\vec{\rho}, m, k)$ are not changed due to differentiation with respect to ρ_q ; namely,
 630 $\Omega_{\text{pq}}(\vec{\rho}, m, k) = \Omega_{\text{pq}}(\vec{\rho} + d\vec{\rho}_q, m, k)$ and $\Omega_{\text{nq}}(\vec{\rho}, m, k) = \Omega_{\text{nq}}(\vec{\rho} + d\vec{\rho}_q, m, k)$ where
 631 $d\vec{\rho}_q$ is a small deviation around ρ_q such that $d\vec{\rho}_q = (0, 0, \dots, d\rho_q, \dots, 0, 0)$, and
 632 $d\vec{\rho}_q$ implicitly depends on time. Finally, by substituting Eqs. (B14) and (B15)
 633 in Eq. (42), the inequality

$$\frac{1}{\tilde{L}_{\text{pw}}^{(q)}(\zeta_1, \vec{\rho})} \frac{\partial \tilde{L}_{\text{pw}}^{(q)}(\zeta_1, \vec{\rho})}{\partial \rho_q} > \frac{1}{\tilde{L}_{\text{pw}}^{(q)}(\zeta_2, \vec{\rho})} \frac{\partial \tilde{L}_{\text{pw}}^{(q)}(\zeta_2, \vec{\rho})}{\partial \rho_q} \quad (\text{B16})$$

634 is obtained. If the above inequality is satisfied, and if $\tilde{L}^{(q)}(\zeta_k, \vec{\rho})$ [Eq. (B7)] is
 635 a continuous function, then a monotonic relationship between the TSD $\tilde{D}^{(q)}(\vec{\rho})$
 636 and ρ_q is achieved as a whole. We can easily prove that the absolute function
 637 in Eq. (B7) is continuous by using the (ϵ, δ) -definition of limit [17], and then
 638 $\tilde{L}^{(q)}(\zeta_k, \vec{\rho})$ is also continuous. Note that the inner operation, the increment of
 639 $X_{\vec{\rho}}(t)$, of the absolute function is continuous with respect to $\vec{\rho}$ because $X_{\vec{\rho}}(t)$
 640 changes with $\vec{\rho}$. This conditional equation is built on the appropriate values of
 641 Δt , N , and $\vec{\rho}$. Note that $\tilde{L}_{\text{pw}}^{(q)}(\zeta_1, \vec{\rho})$ and $\tilde{L}_{\text{pw}}^{(q)}(\zeta_2, \vec{\rho})$ can easily be obtained from
 642 Eq. (B14) by using $k = 1$ and $k = 2$, respectively. Likewise, $\partial \tilde{L}_{\text{pw}}^{(q)}(\zeta_1, \vec{\rho}) / \partial \rho_q$

643 and $\partial \tilde{L}_{\text{pw}}^{(q)}(\zeta_2, \vec{\rho})/\partial \rho_q$ can be obtained from Eq. (B15) by considering $k = 1$ and
 644 $k = 2$, respectively.

645 The aforementioned analysis can be summarized in the following proposition.

646 **Proposition B.1** (Monotonic relationship between the TSD and the noise
 647 level). *Consider the d -dimensional linear stochastic system [Eq. (B1)]. Assume
 648 that only the variable x_q is observed during the observational process and that
 649 observational noise does not influence the process. Then, if condition Eq. (B16)
 650 is satisfied, there exists a monotonic relationship between the TSD $\tilde{D}^{(q)}(\vec{\rho})$ and
 651 the noise level $\rho_q(t)/\sigma_{\text{dyn}x_q}(t)$.*

652 The $(L_{\text{seg}}(0, \Delta t), \rho_q/\sigma_{\text{dyn}x_q})$ diagram [see Fig. B1] shows the success rate in
 653 which the derived inequality [Eq. (B16)] holds given a noise level (uniformly dis-
 654 tributed between 0.01 and 1) and length (uniformly distributed between 10 and
 655 500) of a time series taken from the harmonic oscillator system [Eqs. (B3) and
 656 (B4)] simulated with the initial condition $x_1(0) = 1$ and $x_2(0) = 0$. The variable
 657 $x_1(t)$ was used as a target time series; namely, its curve length $\tilde{L}^{(1)}(\zeta_k, \vec{\rho})$ was
 658 calculated. We calculated the inequality [Eq. (B16)] 100 times on each param-
 659 eter point in the diagram to obtain the success rate. We define the success rate as
 660 the ratio between the number of successful simulations satisfying the inequality
 661 [Eq. (B16)] and the number of total simulations. In this simulation, ρ_1 and
 662 $\sigma_{\text{dyn}x_1}$ were constant for each trial. Then, the dynamical noise level of $x_2(t)$
 663 was fixed at 0.1 for each trial. The sampling time Δt was set to 0.001. As a
 664 result, a larger size of data satisfies the inequality in a wider range of the noise
 665 level. Note that when the inequality is simulated, $\text{Prob}[J_{\vec{\rho}}^{(q)}(m, j, k) = 0] = 0$
 666 because $\vec{\rho}$ is optionally given, and thereby any used $\hat{\rho}_q$ is within (S, E) , that is
 667 $\hat{\rho}_q \in \Omega_q^{(S, E)}$ [see Eq. (40) and Eq. (41)]. Moreover, we have revealed that the
 668 original inequality [Eq. (43)] also shows the same diagram with Fig. B1, where
 669 $\tilde{D}_{\text{pw}}^{(1)}(\rho_1 + 0.0001, \rho_2) - \tilde{D}_{\text{pw}}^{(1)}(\rho_1, \rho_2)$ was used for calculation of the success rate.
 670 Then, the correlation coefficient between success rates of the original and the
 671 derived inequalities is more than 0.999. Thus, the derived inequality is valid for
 672 linear stochastic dynamical systems.

Figure B1: The $(L_{\text{seg}}(0, \Delta t), \rho_1 / \sigma_{\text{dyn}x_1})$ diagram in which the harmonic oscillator system was simulated. A larger success rate for the inequality is indicated by a lighter color.

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