Revival and robustness of Bures distance discord under decoherence channels

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Abstract: In this paper, we demonstrate the revival and robustness of Bures distance discord in comparison with entanglement under local decoherent evolutions. The results show that in depolarizing channel Bures distance discord revives after a dark point of time, while entanglement will damp into death without revival. In addition, in hybrid channel the declining initial condition can enable Bures distance discord to decay more smoothly within a limited time, but speed up the death of entanglement. In this sense, Bures distance discord is typically more robust against decoherence than entanglement. Furthermore, we also provide a geometric interpretation concerning these phenomena.

Key words: revival and robustness, decoherence channels, Bures distance discord and entanglement

1. Introduction

As is well known, the Einstein-Podolsky-Rosen paradox [1] has triggered a long-time debate on the quantum mechanics versus the classical realism. It originated from the fact that quantum mechanics rules the physical world and also advance the physics science. Fundamentally, being different from classical realistic theory, the advantage of quantum theory mainly relies on that there always exists quantum correlation even in the ensembles of separated states [2, 3]. Moreover, how to accurately quantify quantum correlation has been a hot topic in the region of quantum information and computation, and attracted a plenty of attention [4-6]. In this work, we will focus on quantifying quantum correlation by the utilization of Bures distance discord and entanglement, and probing their dynamical behaviors under the noisy

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environment.

As a special quantum correlation, entanglement [7] is quantified by concurrence [8]. It has a remarkable characteristic, but also is capable of fulfilling many tasks of quantum information processing as an important resource [9-12]. Until now, many efforts have been made for the more general witnesses of quantum correlation. Specifically, quantum discord, originally proposed by Olliver and Zurek [13], is a novel quantifier for quantum correlation in bipartite system [14]. Nevertheless, there is a considerable obstacle to pursue its analytical solutions, even for two-qubit systems [15]. Thus, a geometrical version of the quantum discord that built on the true Bures distance has been put forward [16, 17], which is usually called as Bures distance discord.

Note that quantum systems are unavoidably subjected to their surrounding environments that will damage the quantum correlation. Hence, it is extremely intriguing and important to probe the dynamical behaviors of quantum correlation when the open quantum systems interacted with noisy environments [18, 19]. Motivated by this, we here will focus on this theme as well, explicitly, exploration of dynamical behaviors of quantum correlation under depolarizing and hybrid channels. It turns out that depolarizing channel can destroy quantum correlation accompanying a sudden death, when considering the Bell-diagonal states as an input state. We find that there is an interesting result that Bures distance discord revives after a dark point of time, while entanglement damps to death without revival. Additionally, in the hybrid channel the declining initial condition makes the damp of the discord more smoothly within a limited time, whereas it speeds up the entanglement death. In this sense, Bures distance discord is typically more robust against decoherence than entanglement. Besides, the transformations of environments have no effects on concurrence, while these lead to some extreme phenomena for Bures distance discord when bit error and bit-phase error take place with the same probability. Remarkably, a geometric interpretation has been provided for gaining a better understanding for these results.

This paper is organized as follows. In Sec. 2, we briefly review the concepts of

the Bures distance discord and entanglement. In Sec. 3, the dynamical behaviors of two correlation witnesses are given for Bell-diagonal states under the local decoherence channels. We also provide a geometric interpretation of these results in Sec. 4. Finally, conclusions are presented in Sec. 5.

2. Review of quantum correlation quantifiers

To begin with, we first briefly review the definitions as well as the general formalism for Bures distance discord and entanglement. For two quantum states characterized by density operators ρ and σ , the geometric distance between them can be effectively quantified by the true Bures distance, reading [16, 17, 20]

$$d_B(\rho, \sigma) = 2 \left[1 - \sqrt{F(\rho, \sigma)} \right], \quad (1)$$

where $F(\rho, \sigma) = \left[\text{Tr}(\sqrt{\rho\sigma\rho}) \right]^2$ is Uhlmann fidelity [21]. Having this in mind, the

Bures distance discord as a witness of quantum correlation is defined via the Bures distance. In analogy with the Bures measure of entanglement [22], we then have

$$D_B(\rho) = \left\{ (2 + \sqrt{2}) \left[1 - \sqrt{F_{\max}(\rho)} \right] \right\}^{1/2}, \quad (2)$$

with $F_{\max}(\rho) = \max_{\sigma \in \rho_{CQ}} F(\rho, \sigma)$ is the maximum Uhlmann fidelity between ρ and all of the classical-quantum states ρ_{CQ} . Furthermore, for the special case of Bell-diagonal states (the details will be introduced in the next section), the analytical expression of $F_{\max}(\rho)$ is explicitly given as follows [23]

$$F_{\max}(\rho^{BD}) = \frac{1}{2} + \frac{1}{4} \max_{\langle ijk \rangle} \left[\sqrt{(1+c_i)^2 - (c_j - c_k)^2} + \sqrt{(1-c_i)^2 - (c_j + c_k)^2} \right], \quad (3)$$

where the maximum is taken over all the cyclic permutations of $\{1, 2, 3\}$.

Different from that of the Bures distance discord, entanglement can be conveniently quantified by concurrence [7, 24], which is related to the density operator ρ ,

$$C(\rho) = \max \{0, \chi_1 - \chi_2 - \chi_3 - \chi_4\}, \quad (4)$$

where χ_i are the square roots of the eigenvalues in descending order of the matrix $\rho\tilde{\rho}$, $\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y)$ is the spin-flip matrix. Further, for the two-qubit X states ρ^X , a simpler closed expression for concurrence is derived as [25]

$$C(\rho^X) = 2 \max \left\{ 0, |\rho_{14}(t)| - \sqrt{\rho_{22}(t)\rho_{33}(t)}, |\rho_{23}(t)| - \sqrt{\rho_{11}(t)\rho_{44}(t)} \right\}, \quad (5)$$

where $\rho_{ij}(t)$ are elements of the time-evolved density matrix $\rho^X(t)$.

It is noteworthy that both above quantum correlation quantifiers are computable for Bell-diagonal states despite they do not enjoy a closed analytical form in general bipartite system. Hence, in following we will comparatively explore their dynamical behaviors in the presence of decoherence channels.

3. The dynamical behaviors of quantum correlation

With the help of the above preliminaries, we now begin our discussion about the revival and robustness of Bures distance discord in comparison with entanglement under local decoherence channels. In this regard, we restrict our attention to the Bell-diagonal states with maximally mixed marginal [26]. Concretely, its density matrix in Bloch representation can be expressed as

$$\rho^{BD} = \frac{1}{4} \left[I^A \otimes I^B + \sum_{i=1}^3 c_i (\sigma_i^A \otimes \sigma_i^B) \right], \quad (6)$$

where $I^{A(B)}$ is identity matrix, $\{\sigma_i^{A(B)}\}$ denote Pauli matrices, and vector $\vec{c} = (c_1, c_2, c_3)$

composed by correlation functions $c_i = \langle \sigma_i \otimes \sigma_i \rangle$ completely specify the quantum

state. In matrix notation, we explicitly have

$$\rho^{BD} = \frac{1}{4} \begin{pmatrix} 1+c_3 & 0 & 0 & c_1-c_2 \\ 0 & 1-c_3 & c_1+c_2 & 0 \\ 0 & c_1+c_2 & 1-c_3 & 0 \\ c_1-c_2 & 0 & 0 & 1+c_3 \end{pmatrix}. \quad (7)$$

As the name implies, Bell-diagonal states have four maximally entangled Bell states as eigenvectors, with the corresponding eigenvalues $\lambda_{a,b}^{BD}(a, b = 0, 1)$ [11, 27]

$$\lambda_{a,b}^{BD} = \frac{1}{4} \left[1 + (-1)^a c_1 - (-1)^{a+b} c_2 + (-1)^b c_3 \right]. \quad (8)$$

In what follows, we will explore the dynamical behaviors of Bures distance discord and entanglement when Bell-diagonal states coupled with local decoherence channels. In this context, we consider the system-environment interaction through the operator-sum representation formalism [28]. Following the Kraus operators approach, the time-evolved state of a two-qubit system ρ^{AB} under local noisy environment can be described by the trace-preserving quantum operation $\mathcal{E}(\square)$, which is

$$\mathcal{E}(\rho^{AB}) = \sum_{i,j} (K_i^A \otimes K_j^B) \rho^{AB} (K_i^A \otimes K_j^B)^\dagger, \quad (9)$$

where $\{K_i^A, K_j^B\}$ are the sets of Kraus operators of the decoherence associated to a decohering process of the subspaces A and B , satisfying the trace-preserving condition $\sum_{i,j} K_{i,j}^\dagger K_{i,j} = I$.

Moreover, in the actual quantum information processing tasks, the local noisy environment can be modeled as several typical quantum errors, such as bit flip, bit-phase flip and phase flip errors [29]. If we choose $\{|0\rangle, |1\rangle\}$ as the orthonormal basis for the qubit, the three types of errors is characterized as,

$$\begin{aligned} \text{a. Bit flip error:} & \begin{array}{l} |0\rangle \rightarrow |1\rangle \\ |1\rangle \rightarrow |0\rangle \end{array} \quad \text{or } |\varphi\rangle \rightarrow \sigma_x |\varphi\rangle, \\ \text{b. Bit-phase flip error:} & \begin{array}{l} |0\rangle \rightarrow +i|1\rangle \\ |1\rangle \rightarrow -i|0\rangle \end{array} \quad \text{or } |\varphi\rangle \rightarrow \sigma_y |\varphi\rangle, \\ \text{c. Phase flip error:} & \begin{array}{l} |0\rangle \rightarrow |0\rangle \\ |1\rangle \rightarrow -|1\rangle \end{array} \quad \text{or } |\varphi\rangle \rightarrow \sigma_z |\varphi\rangle. \end{aligned} \quad (10)$$

where $\sigma_i (i = x, y, z)$ is the Pauli matrix, and $|\varphi\rangle$ is a quantum state. Then, we will consider the cases that these errors occur with the equal and anti-equal likelihoods, which corresponding to the depolarizing channel and hybrid channel.

3.1. Depolarizing channel

The depolarizing channel is a model of a decohering qubit that has particularly nice symmetry properties. We can describe it by saying that, with probability $1 - p$

the qubit remains intact, while with probability p an error occurs. In addition, when an error occurs, then state $|\varphi\rangle$ evolves to an ensemble of the three states $\sigma_x|\varphi\rangle$, $\sigma_y|\varphi\rangle$ and $\sigma_z|\varphi\rangle$, all occurring with equal likelihood. Furthermore, depolarizing channel can be represented by a unitary operator acting on the Hilbert space, and the corresponding Kraus operators are

$$M_0 = \sqrt{1-p}I, \quad M_1 = \sqrt{p/3}\sigma_x, \quad M_2 = \sqrt{p/3}\sigma_y, \quad M_3 = \sqrt{p/3}\sigma_z. \quad (11)$$

By virtue of the symmetry property, the depolarizing channel processes can preserve the Bell-diagonal form of the time-evolved state $\varepsilon(\rho^{BD})$, with

$$c_{1,2,3}(t) = \left(\frac{4p}{3} - 1\right)^2 c_{1,2,3}(0). \quad (12)$$

It can be readily seen that different initial conditions $c_i(0)$ lead to various dynamics of quantum correlation, without any general agreement between witnesses we consider.

Here, we will choose a specific subset of the initial conditions, which specified as

$$c_1(0) = \pm 1 \quad c_2(0) = \mp c_3(0) \quad \text{with } |c_3(0)| \equiv c. \quad (13)$$

Based on them, we obtain some peculiar dynamics of quantum correlation. In **Fig.1**, we depicted the dynamical behaviors of Bures distance discord and entanglement as a function of the decoherence parameter p . From which one can find that for a certain initial state both of them decreasing with the increase of p . Moreover, the increasing of condition c reduces the initial values of quantum correlation, and stimulates a faster decay only for Bures distance discord. More importantly, the distinct difference between them is that Bures distance discord revives after a dark point of time, while concurrence will damps into death without revival.

Fig.1 (Color online) The dynamical behaviors of quantum correlation for Bell-diagonal state under depolarizing channel with different initial conditions c , subject to $c = 1$ (green line), $c = 0.8$ (red line) and $c = 0.6$ (blue line). Inset: quantum correlations versus p with the range taken as $(0.75, 1)$ and $(0.2, 0.32)$.

3.2. Hybrid channel

Now, we consider the other case that the quantum state simultaneously subjected to the multitier quantum errors occurred with anti-equal probability, i.e., the hybrid channel. Then, the time-evolved states $\varepsilon(\rho)$ is characterized by the superimposed interactions, which is

$$\varepsilon(\rho) = \alpha \varepsilon_{BF}(\rho) + \beta \varepsilon_{BPF}(\rho) + \gamma \varepsilon_{PF}(\rho), \quad (14)$$

where α , β and γ denote the probabilities of the bit flip, bit-phase flip and phase flip errors occurred, respectively. Meanwhile, they also satisfy the condition $\alpha + \beta + \gamma = 1$.

Considering the state ρ^{BD} as inputs, the time-evolved states $\varepsilon(\rho^{BD})$ also remain in the Bell-diagonal form, with

$$\begin{aligned}
c_1(t) &= \left[\alpha + (\beta + \gamma)(1-p)^2 \right] c_1(0), \\
c_2(t) &= \left[\beta + (\alpha + \gamma)(1-p)^2 \right] c_2(0), \\
c_3(t) &= \left[\gamma + (\alpha + \beta)(1-p)^2 \right] c_3(0).
\end{aligned} \tag{15}$$

In the following, we first study the effects of the initial conditions $c_i(0)$ specified as equation (13) on the quantum correlation under certain hybrid channel. In **Fig.2** we plotted the dynamical behaviors of Bures distance discord and concurrence versus decoherence parameter p . From which, we can find that different initial conditions lead to diverse dynamical evolutions. With the decrease of c , Bures distance discord damps more smoothly within a limited time, while, beyond the time there is no longer effect on it. However, for concurrence the declining c not only reduces its initial value and also results in an earlier death.

Fig.2 (Color online) The dynamical behaviors of quantum correlation for Bell-diagonal state under the certain hybrid channel with different initial conditions c , subject to $c = 1$ (green line), $c = 0.8$ (red line), $c = 0.6$ (blue line).

Fig.3 (Color online) The dynamical behaviors of quantum correlation for Bell-diagonal state under various hybrid channels with the specific initial condition $c = 1$, subject to $\alpha = \beta = \gamma/3$

(green line), $2\alpha = \beta = \gamma$ (red line), $\alpha = \gamma = \beta/3$ (blue line).

Then, we study how the hybrid channels affect the quantum correlation for the specific initial state, and the details have been plotted in **Fig. 3**. The results show that the transformations of environments have no effects on concurrence, while it causes several extreme phenomena for Bures distance discord when bit error and bit-phase error occurred with same probability.

4. Geometric interpretation

For the sake of better insight into these revival and robustness of Bures distance discord in comparison with entanglement under local decoherence channels, we should focus on the geometric representation of Bell-diagonal states [29, 30]. Due to that ρ^{BD} corresponds to a physical state, we must comply with the conditions $\lambda_{a,b}^{BD} \geq 0$ and $\sum_{a,b} \lambda_{a,b}^{BD} = 1$, which impose constraints on the entries of vector $\vec{c}$, so that physically Bell-diagonal states turns out to be a restricted tetrahedron $\mathfrak{R}$ of vertices $(-1, -1, -1), (-1, 1, 1), (1, -1, 1)$ and $(1, 1, -1)$ shown as the orange region in **Fig. 4**, and this geometric picture is very appealing to visualize dynamical trajectories. Furthermore, a Bell-diagonal state is separable if and only if its partial transpose is positive, and we can find that the partial transpose just flips the sign of c_2 , thus operators with positive partial transpose occupy the reflection of tetrahedron $\mathfrak{R}$ through the plane $c_2 = 0$. Hence, the region of the set of separable Bell-diagonal states is the intersection of the two tetrahedral, which belongs to the green octahedron $\mathfrak{S}$.

Fig.4 (Color online) Geometrical representation of Bell-diagonal states: the tetrahedron $\mathfrak{R}$ represents valid Bell-diagonal states, which sit at the four vertices. The green octahedron $\mathfrak{S}$ is the set of separable Bell-diagonal states. Classical states, i.e., those diagonal in a product basis, lie on the Cartesian axes.

From the Ref. [31], we derive that concurrence is a convex function in the tetrahedron $\mathfrak{R}$, and maintains zero for the separable states in the octahedron $\mathfrak{S}$. In addition, this geometrical picture also clearly displays that almost all states have nonzero discord. Now, let us focus, without loss of generality, on the depolarizing channel. To explain the revival of Bures distance discord, we should pay close attention to the evolution of vector $\vec{c}_i(t)$. According to Eq. (12) shown as

$$c_{1,2,3}(t) = \left(\frac{4p}{3} - 1 \right)^2 c_{1,2,3}(0), \text{ for the constant initial conditions } c_i(0), \text{ the increasing } p$$

makes the vector coordinates $\vec{c}_i(t)$ close and across the peripheral boundaries of the green octahedron $\mathfrak{S}$, which leads concurrence to undergo a sudden death without revival, while Bures distance discord revives after a dark point of time. In particular, for the classical states at the center $\vec{c}(t) = (0,0,0)$ of the tetrahedron $\mathfrak{R}$ (octahedron $\mathfrak{S}$)

corresponding to the critical point $p = 3/4 \left(p = \frac{3}{4} \left(1 + \sqrt{\frac{c_i(t)}{c_i(0)}} \right) \right)$ as shown in **Fig. 1**,

both Bures distance discord and concurrence are disappeared.

5. Conclusions

In conclusion, we have explored the revival and robustness of Bures distance

discord in comparison with entanglement under local nondissipative decoherence. Considering Bell-diagonal state suffering from depolarizing, its quantum correlation will be destroyed with increase of the decoherent to a large extent. Particularly, the distinct difference between them is that Bures distance discord revives after a dark point of time, while concurrence undergoes the sudden death without resurrection. Furthermore, in hybrid decoherence case we first studied the influences of initial conditions $\bar{c}_k(0)$ on the quantum correlation under the certain noisy environment. With the decrease of c , Bures distance discord damps more smoothly within a limited time, while, beyond the time there is no longer effect on it. However, for concurrence the parameter c not only reduces its initial value and also results in an earlier death subjected. In terms of the above results, we derived a fact that Bures distance discord is typically more robust against decoherence than the entanglement. Moreover, when the specific initial state is subjected to the different hybrid channel, the transformations of environments have no effects on concurrence, while it causes several extreme phenomena for Bures distance discord when bit error and bit-phase error occurred with the same probability. For the sake of better insight into these phenomena, we provide a geometric interpretation for Bell-diagonal states.

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