Soliton solutions of coupled Maxwell-Bloch equations

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Abstract

In this paper we study the soliton solutions of the coupled Maxwell-Bloch equations which describe pulse propagation in an active optical medium with coherent three-level atomic transitions and inhomogeneous broadening. The soliton solutions and polarization shifts due to soliton interactions are investigated. An infinite set of conservation laws as well as the soliton trace formulae are derived.

Keywords: Maxwell-Bloch equations, Solitons, Conservation laws, Trace formulae

1 Introduction

Nonlinear resonant interaction of radiation with an active optical medium has many important application in optical physics, and has served as one of the basic mechanisms used in laser operation and optical amplifiers [1, 2], as well as self-induced transparency (SIT). The phenomenon of SIT discovered by McCall and Hahn [3] describes the response of a coherent medium to an incident electric field, to which the medium is totally transparent and which undergoes lossless propagation. The complete integrability of the reduced Maxwell-Bloch equations governing the SIT phenomenon in a two-level medium were found by Lamb [4, 5] and others [6, 7]. The general initial value problem for the propagation of a pulse through a resonant two-level optical medium with inhomogeneous broadening was solved by applying the Inverse Scattering Transform (IST) in [8]. The IST was also employed to solve the Maxwell-Bloch equations in a more general setting of two-level unstable optical media to study the superfluorescence phenomenon and related problems in laser optics [9, 10, 11]. The complete integrability of the reduced Maxwell-Bloch equations in the presence of a permanent dipole was investigated in [12, 13].

Optical pulse propagation in a three-level medium under two-photon or double one-photon resonance conditions has also been studied extensively, both theoretically and experimentally, by various authors over the past few decades. In particular, there have been investigations in connection with SIT and simultons [14, 15, 16], lasing without inversion [17, 18], electromagnetically induced transparency [19, 20, 21], and other related topics (see e.g., [21] for a review). In these problems, one typically considers the medium with a three-level atomic configuration to have a Λ -configuration (see Figure 1) or a V-configuration (inverted version of the Λ -configuration). A large class of solutions for the physical problems mentioned above were obtained in earlier works [22, 15, 23], as well as in more recent papers [24, 25]. A comprehensive review of the earlier studies (during the 1980s) on the application of IST to the theory of pulse propagation in multi-level optical media can be found in [26]. Recently, the Maxwell-Bloch equations associated with a three-level medium in the presence of inhomogeneous broadening was studied using the IST technique in [27] where explicit formulas for the n -soliton solution and the corresponding scattering matrix were derived.

In this paper we discuss the special cases of one- and two-soliton solutions of the Maxwell-Bloch equations in a three-level optical medium with inhomogeneous broadening. Compared to many other integrable

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equations, the system studied in this paper has a more complicated soliton dispersion relation, as a result of which the physical behaviors of even the one-soliton solutions vary considerably. In particular, one finds that the soliton envelopes are not in the form of usual traveling waves, rather the soliton pulses travel with non-uniform velocity changing direction as it passes through the optical medium, and only approaches a uniform traveling wave shape asymptotically.

The paper is organized as follows. In section 2 we provide a brief mathematical background of the Maxwell-Bloch equations with inhomogeneous broadening and describe how to construct the solutions from the Lax pair. Then in Section 3 we discuss the varieties of one- and two-soliton solutions and their asymptotic behaviors. Specifically, in Section 3.1 we discuss the different types of possible one-soliton solutions including the real one-solitons. Section 3.2 is devoted to the special case of coherent pulses propagating in a optical medium whose atomic states are resonantly coupled to the electric field envelope. In Section 3.3, we discuss the two-soliton interactions and the associated shifts in the polarization vectors of the component one-solitons due to collision. Finally, in Section 4 we derive the local conservation laws associated with the Maxwell-Bloch equations with inhomogeneous broadening. We find that the integrated densities such as the power, momentum and energy associated with the optical pulses are not generally conserved due to interaction with the optical medium. But for soliton pulses one does have global conserved quantities which can be expressed in terms of soliton parameters (discrete eigenvalues) via the trace formulae.

2 Background

The coupled Maxwell-Bloch (CMB) equations considered in this paper can be expressed as

$$\Omega_z = \frac{1}{4} [\rho_{av}, J] \quad (2.1a)$$

$$\rho_\tau = [\Omega + i\alpha J, \rho] \quad \alpha \in \mathbb{R}, \quad (2.1b)$$

with 3×3 matrices

$$J = \text{diag}(-1, -1, 1), \quad \Omega = \frac{1}{2} \begin{pmatrix} 0 & 0 & \Omega_1 \\ 0 & 0 & \Omega_2 \\ -\Omega_1^* & -\Omega_2^* & 0 \end{pmatrix}, \quad \rho_{av} = \int_{-\infty}^{\infty} \rho(z, \tau, \alpha) g(\alpha) d\alpha, \quad (2.2)$$

and where $[\cdot, \cdot]$ denotes the matrix commutator. In above, $\tau = t - z/c$ where $t \in \mathbb{R}$ is the normalized time, $z \geq 0$ is the normalized propagation distance along the optical medium, and c is the speed of light.

Equation (2.1) describes the propagation of two optical pulses in a medium of three level atoms in the Λ configuration as shown in Figure 1. In this scheme, electronic transitions are allowed between the the excited state $|3\rangle$ and metastable state $|2\rangle$ and the ground state $|1\rangle$ while transition between $|1\rangle$ and $|2\rangle$ is forbidden. The electric fields corresponding to the individual optical pulses $\Omega_1(z, \tau)$ and $\Omega_2(z, \tau)$ are resonantly coupled to the $|3\rangle \leftrightarrow |1\rangle$ and $|3\rangle \leftrightarrow |2\rangle$ atomic transitions, respectively. The material

Figure 1: Schematic of a three-level system, with excited state $|3\rangle$ and lower states $|1\rangle$ and $|2\rangle$.

properties of the optical medium are described by the Bloch matrix ρ which is Hermitian, i.e., $\rho^\dagger = \rho$. The diagonal elements of ρ correspond to the population densities of the atomic levels, and the off-diagonal elements describe the complex valued material polarizability envelopes of the optical medium. In addition to z, τ , the matrix ρ depends also on a real parameter α representing the detuning from the exact quantum transition frequency, and $g(\alpha)$ is the distribution of the detuning resulting from inhomogeneous (Doppler-type) broadening. The term ρ_{av} in (2.2) accounts for the inhomogeneous broadening effect by averaging over the range of detuning. The optical pulses are localized in τ and satisfy the vanishing boundary conditions $\Omega_i \rightarrow 0$ as $\tau \rightarrow \pm\infty$ for all z . For the Bloch matrix one can freely prescribe only one of the boundary values $\rho_- := \rho(\tau \rightarrow -\infty)$, or $\rho_+ := \rho(\tau \rightarrow \infty)$ because (2.1b), regarded as an ordinary differential equation for ρ in τ (for any fixed value of z), uniquely determines the other boundary condition.

Like the 2×2 Maxwell-Bloch equations, the CMB equations (2.1) are also integrable, and admits a Lax representation of the form

$$v_\tau = (ikJ + \Omega)v, \quad v_z = i\langle \rho \rangle v, \quad (2.3)$$

where $v(z, \tau, k)$ is the complex eigenfunction and k the complex spectral parameter. Here and henceforth we define

$$\langle A \rangle(z, \tau, k) := \frac{1}{4} \oint_{-\infty}^{\infty} \frac{g(\alpha)}{\alpha - k} A(z, \tau, \alpha) d\alpha, \quad k \in \mathbb{R} \quad (2.4)$$

where $\oint_{-\infty}^{\infty}$ denoting the Cauchy principal value integral. When $k \notin \mathbb{R}$, the integral above is understood to be over the entire real line instead of principal value. The compatibility condition of the Lax pair (2.3) yields,

$$\Omega_z = i\langle i(k - \alpha)[\rho(\alpha), J] + \rho_\tau(\alpha) + [\rho(\alpha), \Omega + i\alpha J] \rangle,$$

from which one can recover (2.1) after equating terms with the same k -dependence.

The initial-boundary value problem for (2.1) can be solved via IST for the case where $\Omega(\tau, z)$ decays rapidly as $\tau \rightarrow \pm\infty$ and for all $z > 0$ [27]. In this case, Ω is assigned at $z = 0$ for all $\tau \in \mathbb{R}$ and a boundary condition ρ_- for ρ is assigned as $\tau \rightarrow -\infty$. The solution of the problem via IST involves determining (from the first equation of the Lax pair (2.3)) the basis eigenfunctions: $\Phi(\tau, k) \sim e^{ikJ\tau}$ as $\tau \rightarrow -\infty$ and $\Psi(\tau, k) \sim e^{ikJ\tau}$ as $\tau \rightarrow \infty$, and the corresponding scattering matrix $S(k) = \Phi^{-1}\Psi$ at $z = 0$ in terms of $\Omega(\tau, 0)$. The evolution (with respect to z) of the eigenfunctions and the scattering data are next determined using the second equation of (2.3). The electric fields $\Omega := (\Omega_1, \Omega_2)^T$ are reconstructed via a large k expansion of the basis eigenfunction Φ by setting $\Phi = \mu e^{ikJ\tau}$, where $\mu(k)$ has the asymptotic expansion

$$\mu = \mathbf{I}_3 + \frac{\mu_1}{k} + O(k^{-2}) \quad \text{as } k \rightarrow \infty.$$

From the first equation in (2.3) it follows that μ satisfies $\mu_\tau + ik[\mu, J] = \Omega\mu$. Substituting the large k expansion for μ into this equation, and taking the limit as $k \rightarrow \infty$, yields $\Omega = i[\mu_1, J]$. The Bloch matrix ρ can also be reconstructed from the eigenfunction Φ as follows. Define $F = \Phi C \Phi^{-1}$ where $C = C(z, k)$ is independent of τ . Then it follows from the first equation of (2.3) that F satisfies $F_\tau + [F, ikJ + \Omega] = 0$, which is the same ODE (2.1b) for $\rho(\tau, z, \alpha)$ when $k = \alpha \in \mathbb{R}$. Hence, ρ is uniquely determined by

$$\rho(\tau, z, \alpha) = \Phi(\tau, z, \alpha) C(z, \alpha) \Phi^{-1}(\tau, z, \alpha),$$

where $C(\tau, \alpha)$ is specified via the boundary condition ρ_- according to $\rho(\tau, z, \alpha) \sim e^{i\alpha J\tau} C e^{-i\alpha J\tau} = \rho_-$ as $\tau \rightarrow -\infty$, and satisfies $C^\dagger = C$. Note that the boundary condition ρ_+ as $\tau \rightarrow \infty$ which corresponds to the final state of the optical medium after interaction with the electric field can also be obtained in terms of the scattering matrix S , and is given by

$$\rho = \Phi C \Phi^{-1} = \Psi S^{-1} C S \Psi^{-1} \sim e^{i\alpha J\tau} S^{-1} C S e^{-i\alpha J\tau} = \rho_+ \quad \text{as } \tau \rightarrow \infty.$$

3 Soliton solutions

In this section we consider the multi-soliton solutions of the CMB equation. These solutions are parametrized by a finite number of complex constants $\{k_j\}_{j=1}^n$ with $\text{Im}(k_j) > 0$, and 2-component, complex vector-valued functions $\{\beta_j(k_j, z)\}_{j=1}^n$. From the IST perspective the solitons correspond to the purely reflectionless potentials where the k_j s are the discrete zeros in the upper-half plane of the associated scattering matrix, while the β_j s are the norming constants related to the residues of the complex scattering eigenfunction which have simple poles at k_j , $j = 1, \dots, n$. In this case, the n -soliton solutions can be obtained in closed form in a purely linear algebraic fashion. Consequently, exact explicit expressions for Ω and ρ corresponding to these soliton solutions can be found.

In order to find the explicit formula for the n -soliton solutions, one first defines a $n \times n$ generalized Cauchy matrix K as follows:

$$K_{ij} = \frac{q_i^* q_j + \tilde{\beta}_i^\dagger \tilde{\beta}_j}{k_i^* - k_j}, \quad q_i = \frac{\prod_{j=1}^n (k_i - k_j^*)}{\prod_{j \neq i} (k_i - k_j)}, \quad \tilde{\beta}_j := \beta_j e^{-2ik_j \tau}. \quad (3.1)$$

Then the n -soliton potential Ω in (2.2) is given by [24, 27]

$$\Omega = (\Omega_1, \Omega_2)^T = -4i \sum_{j,m=1}^n (K)_{jm}^{-1} q_m^* \tilde{\beta}_j.$$

Using Cramer's formula, the above expression can be written as the ratio of two determinants,

$$\Omega_r = 4i \frac{\begin{vmatrix} K & \mathbf{q}^* \\ \tilde{\beta}_r & 0 \end{vmatrix}}{|K|}, \quad r = 1, 2, \quad (3.2)$$

where $\mathbf{q} = (q_1, q_2, \dots, q_n)^T$ whose components are given in (3.1), and the row vector $\tilde{\beta}_r$ is defined by $\tilde{\beta}_r = (\tilde{\beta}_{1r}, \tilde{\beta}_{2r}, \dots, \tilde{\beta}_{nr})$ where $\tilde{\beta}_{jr}$ denotes the r^{th} component of each 2-component vector $\tilde{\beta}_j$ given in (3.1). The vector Ω whose components are given by (3.2) corresponds to the electric field envelope for the n -soliton solution for the CMB equation. It is described by the parameters: $\{k_j, \beta_j\}_{j=1}^n$, and in particular, its dependence on τ and z is solely through the quantities $\tilde{\beta}_j(\tau, z) = \beta_j(z) e^{-2ik_j \tau}$. Furthermore, the electric field envelopes $\Omega_r(\tau, z)$ given by (3.2) is regular for all $\tau \in (-\infty, \infty)$ and $z \geq 0$ because it can be shown that the determinant $|K| \neq 0$. The vectors $\beta_j(z)$ prescribing the z -dependence of the potential $\Omega(z, \tau)$, satisfies the linear equation

$$\partial_z \beta_j = i \langle \hat{H} \rangle(k_j) \beta_j, \quad (3.3)$$

where the 2×2 Hermitian matrix $\hat{H} = H - hI_2$ in (3.3) is determined by the boundary condition ρ_- corresponding to the initial atomic configuration of the 3-level medium. For solitons, ρ_- has the block diagonal form [27]

$$\rho_-(z, \alpha) = \begin{pmatrix} H(z, \alpha) & 0 \\ 0 & h(z, \alpha) \end{pmatrix}, \quad (3.4)$$

where $H(z, \alpha)$ is a 2×2 Hermitian matrix, and $h(z, \alpha)$ is a real scalar function. We henceforth assume that $\hat{H} = \hat{H}(\alpha)$ is independent of z , and is diagonalizable for each $j = 1, 2, \dots, n$. Then the solution of (3.3) is

$$\beta_j(z) = c_j^{(1)} e^{i\lambda_j^{(1)} z} v_j^{(1)} + c_j^{(2)} e^{i\lambda_j^{(2)} z} v_j^{(2)}, \quad (3.5)$$

where $v_j^{(\ell)}$ is the eigenvector (with $\|v_j^{(\ell)}\| = 1$) of $\langle \hat{H} \rangle(k_j)$ corresponding to the complex eigenvalue $\lambda_j^{(\ell)}$, $\ell = 1, 2$, and $c_j^{(1)}, c_j^{(2)}$ are complex constants related to the initial value $\beta_j(0)$. Without loss of generality, one can label the eigenvalues such that $\text{Im} \lambda_j^{(1)} \leq \text{Im} \lambda_j^{(2)}$, $j = 1, \dots, n$.

Reality condition: Even though the electric field Ω is in general complex, it is possible to obtain *real* n -soliton solutions from (3.2) by imposing certain constraints on the parameters determining the solutions. This can be accomplished in two ways: (i) choose $n = 2M$, and the parameters $\{k_j, \beta_j\}_{j=1}^{2M}$ with the restrictions $k_{j+M} = -k_j^*$ and $\beta_{j+M}(z) = -\beta_j(z)^*$ for $j = 1, 2, \dots, M$, (ii) $k_j = i\eta_j$, $\eta_j > 0$ and $\beta_j(z) = ib_j(z)$, $b_j \in \mathbb{R}^2$ for $j = 1, 2, \dots, n$. In either case, it follows from (3.2) that the electric field envelope functions satisfy $\Omega_r^* = \Omega_r$, $r = 1, 2$. Note that in order for the restriction $\beta_{j+M}(z) = -\beta_j^*(z)$ to hold consistently for all $z \geq 0$, one must have $\beta_{j+M}(0) = -\beta_j^*(0)$ and $\langle \hat{H} \rangle(-k_j^*) = \langle \hat{H} \rangle^*(k_j)$. The latter condition follows from the evolution equation (3.3) for the β_j . For case (ii), one similarly requires that the matrix $\langle \hat{H} \rangle(i\eta_j)$ be pure imaginary.

3.1 One-soliton solution

Let us now consider the case $n = 1$, when the soliton solution depends on a single complex parameter $k = \xi + i\eta$ with $\eta > 0$, and a 2-component complex vector $\beta(z)$. In this case, one can readily compute the determinants on the right hand side of (3.2). From (3.5), we have $\beta(z) = c^{(1)}e^{i\lambda^{(1)}z}v^{(1)} + c^{(2)}e^{i\lambda^{(2)}z}v^{(2)}$, where $\lambda^{(1)}, \lambda^{(2)}$ are the eigenvalues of the matrix $\langle \hat{H} \rangle(k)$ and $v^{(1)}, v^{(2)}$ are the corresponding eigenvectors. Next, we express $\tilde{\beta}$ defined in (3.1) as $\tilde{\beta}(\tau, z) = \beta(z)e^{-2ik\tau} = e^{i(\lambda z - 2k\tau)}p(z)$ where λ is either $\lambda^{(1)}$ or $\lambda^{(2)}$ depending on the cases to be discussed below, and the vector $p(z)$ is defined accordingly. Thus we have,

$$\tilde{\beta}(\tau, z) = 2\eta e^{-2i\xi\tau + i\text{Re } \lambda z} \exp[2\eta\tau - \text{Im } \lambda z + \delta(z)]\hat{p}(z), \quad \delta(z) = \ln \frac{||p(z)||}{2\eta}, \quad (3.6)$$

where the unit vector $\hat{p} = p/||p||$. Then from (3.2) with $n = 1$, the one-soliton solution for the optical pulses can be expressed in the form

$$\Omega(\tau, z) = 4i\eta e^{-2i\xi\tau + i\text{Re } \lambda z} \text{sech}[2\eta\tau - \text{Im } \lambda z + \delta(z)]\hat{p}(z), \quad (3.7)$$

where the polarization vector $p(z)$ and the phase shift $\delta(z)$ depend on the propagation distance z and on the constants $c^{(1)}, c^{(2)}$ determined by the initial value $\beta(0)$. Note that due to the nonlinear z -dependence in $\delta(z)$, pulse center does not travel at a constant speed through the optical medium. Therefore, the soliton envelope is not exactly a uniform traveling wave, and the soliton profile “bends” in the (τ, z) -plane (see Figure 2). Below we discuss the one-soliton asymptotics in several situations.

Case (i): For example, if $c^{(2)} = 0$, then $\tilde{\beta}(\tau, z) = e^{i(\lambda_1 z - 2k\tau)}p(z)$ with $p(z) = c^{(1)}v^{(1)}$. In this case the one-soliton solution has a uniform traveling wave of the form

$$\Omega(\tau, z) = 4i\eta e^{-2i\xi\tau + i\text{Re } \lambda^{(1)}z + i\arg c^{(1)}} \text{sech}\left[2\eta\tau - \text{Im } \lambda^{(1)}z + \ln \frac{|c^{(1)}|}{2\eta}\right] v^{(1)},$$

which shows that the soliton polarization vector coincides with the eigenvector $v^{(1)}$ of $\langle \hat{H} \rangle(k)$. Recall that the coordinate τ is related to the normalized time t and the normalized propagation distance z by $\tau = t - z/c$. Hence in the physical coordinates, the soliton travels to the right with a constant velocity

$$v_+ = c \left(1 + \frac{\text{Im } \lambda^{(1)} c}{2\eta} \right)^{-1},$$

which is less than the light speed c in the medium when $\text{Im } \lambda^{(1)} > 0$, and increases with the soliton amplitude parameter η . If instead, $c^{(1)} = 0$, then one obviously has a similar situation as above with the superscripts switched from $1 \rightarrow 2$.

Case (ii): If $c^{(1)} \neq 0$, then we set $\lambda = \lambda^{(1)}$ in (3.6) and (3.7) so that $p(z) = c^{(1)}v^{(1)} + c^{(2)}e^{i(\lambda^{(2)} - \lambda^{(1)})z}v^{(2)}$. In this case, both the soliton velocity and the polarization vector $p(z)$ depend on the propagation distance

Figure 2: One-soliton electric field envelope Ω with Lorentzian inhomogeneous broadening. Left panel: modulus of the first component $|\Omega_1(\tau, z)|$. Center panel: modulus of the second component $|\Omega_2(\tau, z)|$. Right panel: norm $\|\Omega(\tau, z)\|$.

Figure 3: Same plots as in Figure 2 corresponding to periodic solution in Case (iv). The period in z is $2\pi/3$.

z along the optical medium. Note that $e^{i(\lambda^{(2)} - \lambda^{(1)})z}$ decays exponentially for $z > 0$, provided $\text{Im } \lambda^{(1)} < \text{Im } \lambda^{(2)}$. Consequently, as $z \rightarrow \infty$, the one-soliton solution has the asymptotic form which coincides with the case $c^{(2)} = 0$ discussed in Case (i) above. In this case, the asymptotic solution is a single sech profile traveling with a constant speed v_+ and a constant polarization vector $v^{(1)}$.

Case (iii): A more interesting case corresponds to when $0 < |c^{(1)}| \ll |c^{(2)}|$. One can then define a characteristic length along the optical medium by $z_0 = (\text{Im } \lambda^{(2)} - \text{Im } \lambda^{(1)})^{-1} \ln |c^{(2)}/c^{(1)}|$ such that the one-soliton solution behaves differently when $z \ll z_0$ and when $z \gg z_0$. In this case one sets $\lambda = \lambda^{(2)}$ in (3.6) and (3.7) for $z \ll z_0$. Then the polarization vector becomes $p(z) = c^{(1)}v^{(1)}e^{i(\lambda^{(1)} - \lambda^{(2)})z} + c^{(2)}v^{(2)}$, and the one-soliton solution has the asymptotic form

$$\Omega(\tau, z) \sim 4i\eta e^{-2i\xi\tau + i\text{Re } \lambda^{(2)}z + i\arg c^{(2)}} \text{sech}\left[2\eta\tau - \text{Im } \lambda^{(2)}z + \ln \frac{|c^{(2)}|}{2\eta}\right] v^{(2)}.$$

Note that $\Omega(\tau, z)$ in the above expression represents a traveling wave moving with a velocity v_- whose expression is obtained by replacing $\lambda^{(1)}$ by $\lambda^{(2)}$ in the expression for v_+ above. On the other hand, for $z \gg z_0$, the solution has the asymptotic form as given by $\Omega(\tau, z)$ in Case (i). Thus, the soliton velocity and polarization switch from $\{v_-, v^{(2)}\}$ for $z \ll z_0$ to $\{v_+, v^{(1)}\}$ for $z \gg z_0$, and the soliton profile curves in the (τ, z) -plane as in Figure 2.

Case (iv): If the eigenvalues of $\langle \hat{H} \rangle(k)$ are such that $\text{Im } \lambda^{(1)} = \text{Im } \lambda^{(2)} = \omega$, then $\tilde{\beta}$ in (3.6) is expressed as

$$\tilde{\beta}(\tau, z) = 2\eta e^{-2i\xi\tau + i\text{Re } \lambda^{(1)}z} \exp[2\eta\tau - \omega z + \delta(z)] \hat{p}(z), \quad p(z) = c^{(1)}v^{(1)} + c^{(2)}e^{i(\text{Re } \lambda^{(2)} - \text{Re } \lambda^{(1)})z} v^{(2)},$$

and $\delta(z)$ is defined the same way as in (3.6). In this case the one-soliton solution is given by

$$\Omega(\tau, z) = 4i\eta e^{-2i\xi\tau + i \operatorname{Re} \lambda^{(1)} z} \operatorname{sech}[2\eta\tau - \omega z + \delta(z)] \hat{p}(z).$$

Therefore when $c^{(1)} \neq 0$ and $c^{(2)} \neq 0$, the soliton peak oscillates along the line $2\eta\tau - \omega z = 0$ in the (τ, z) -plane (see Figure 3). In this case, the soliton polarization vector $p(z)$ is also a periodic function of z with period $2\pi (\operatorname{Re} \lambda^{(2)} - \operatorname{Re} \lambda^{(1)})^{-1}$, where we take $\operatorname{Re} \lambda^{(1)} < \operatorname{Re} \lambda^{(2)}$ without loss of generality.

Figures 2 and 3 show plots of the magnitudes of the electric field envelopes for the exact one-soliton solutions respectively, for the cases (iii) and (iv) described above. In all the plots, the inhomogeneous broadening function is assumed to be a Lorentzian distribution $g(\alpha) = [\pi(\alpha^2 + 1)]^{-1}$, and the matrix $\hat{H}$ describing the initial preparation of the medium is chosen to be independent of the detuning frequency α . Then one obtains

$$\langle \hat{H} \rangle(k) = -\frac{\hat{H}}{4(k+i)}$$

by explicitly evaluating the integral in (2.4) for $\langle \hat{H} \rangle(k)$. Note that the eigenvalues $\lambda^{(i)}, i = 1, 2$ of $\langle \hat{H} \rangle(k)$ are a (complex) multiple of the real eigenvalues of the constant, Hermitian matrix $\hat{H}$ while the eigenvectors $v^{(i)}, i = 1, 2$ are the same for both $\langle \hat{H} \rangle(k)$ and $\hat{H}$. Moreover, $v^{(1)}$ and $v^{(2)}$ are orthogonal.

In Figure 2 we take the matrix $\hat{H} = \begin{pmatrix} 4 & 1-i \\ 1+i & 5 \end{pmatrix}$ and $k = 1 + i/3$. The constants which determine the initial value $\beta(0)$ are taken to be $c^{(1)} = 1, c^{(2)} = 10$. One can see from the plot of the norm $||\Omega(\tau, z)||$ that the soliton changes its direction in the (τ, z) -plane around $z_0 = 7$ as it switches its asymptotic form from $z < z_0$ to $z > z_0$. In Figure 3 we plot a one-soliton solution corresponding to case (iv) above, where the matrix $\langle \hat{H} \rangle(k)$ has two distinct eigenvalues with the same imaginary parts. Here we have chosen $k = 1 + i/2$, while $\langle \hat{H} \rangle(k) = \begin{pmatrix} 2+i & 1 \\ 2 & 3+i \end{pmatrix}$, and the initial condition $\beta(0)$ is determined by $c^{(1)} = 1, c^{(2)} = 1 + 2i$. The eigenvalues of the matrix $\langle \hat{H} \rangle(k)$ are $\lambda^{(1)} = 4 + i, \lambda^{(2)} = 1 + i$ so that they have the same imaginary part. Here each component of the electric field envelope $\Omega(\tau, z)$ is a periodic function of z with period $2\pi/3$. Finally, in Figure 4 we plot a real one-soliton solution. We take $k = i\eta$ so that $\langle \hat{H} \rangle(i\eta) = i\hat{H}/4(1 + \eta)$ by choosing the same Lorentzian function for $g(\alpha)$ as above. The reality condition discussed earlier implies that $\langle \hat{H} \rangle(i\eta)$ should be pure imaginary, which is accomplished by choosing $\hat{H}$ to be a real matrix. In this case, $\beta(z) = ib(z), b(z) \in \mathbb{R}^2$, and (3.3) reduces to a real linear equation for $b(z)$, namely

$$\partial_z b = -\frac{\hat{H}}{4(1 + \eta)} b.$$

For the plot we choose $\eta = 3/4$, a real, symmetric matrix $\hat{H} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, and $c^{(1)} = -i, c^{(2)} = 5i$. Here we plot the components Ω_1, Ω_2 (left and middle panel) instead of their magnitudes since $\Omega(\tau, z)$ is real. Note that Ω_2 is negative in the (approximate) range $0 < z < 7$.

3.2 Coherent pulse propagation in 3-level media

In this section we consider a special case of the CMB equations (2.1) which concerns matched pulse propagation where the pulses are resonantly coupled to the transition (Rabi) frequencies of the 3-level atomic optical medium [20, 28, 16]. In this situation the medium is initially prepared such that the solitary wave pulse pairs can propagate through the absorbing optical medium while preserving their shape [29, 25]. The Bloch density matrix in this application is expressed as $\rho = \gamma\gamma^\dagger$ where the vector $\gamma(\tau, z, \alpha) = (\gamma_1, \gamma_2, \gamma_3)^T$ denote the complex probability amplitudes for the atomic level occupation. Consequently, ρ is now a *rank one*, 3×3 , Hermitian matrix. It is more natural to write down the equations for the complex probability amplitudes γ_i , instead of ρ as in (2.1b). For the Λ -configuration (cf. Figure 1), these equations are

$$\gamma_{1\tau} = i\Omega_1\gamma_3, \quad \gamma_{2\tau} = i\Omega_2\gamma_3, \quad \gamma_{3\tau} = \alpha\gamma_3 + i(\Omega_1^*\gamma_1 + \Omega_2^*\gamma_2),$$

Figure 4: A real one-soliton solution with $k = 3i/4$. Left panel: $\Omega_1(\tau, z)$, center panel: $\Omega_2(\tau, z)$, right panel: norm $\|\Omega(\tau, z)\|$.

where Ω_1, Ω_2 are the complex electric field envelope functions. One can easily verify that the above equations for γ lead to the equation (2.1b) for the Bloch matrix $\rho = \gamma\gamma^\dagger$. The boundary values of the Bloch matrix $\rho_\pm := \rho(\tau \rightarrow \pm\infty)$ are then also rank one matrices. In particular, ρ_- which is prescribed according to the initial preparation of the 3-level medium, takes the form $\rho_- = bb^\dagger$ for some complex 3-vector $b(\alpha, z)$ denoting the initial probability amplitude of the atomic levels. On the other hand, for soliton solutions ρ_- must also have a block-diagonal form as given by (3.4). Hence, the form of the vector b is restricted to either (i) $b = (0, 0, b_3)^\top$, or (ii) $b = (b_1, b_2, 0)^\top$ for matched soliton pulse propagation in a 3-level medium. For both of these cases, we discuss the form of the electric field $\Omega(\tau, z)$ and the probability amplitude vector $\gamma(\tau, z, \alpha)$ below. Recall from Section 2 that the Bloch density matrix is given as $\rho = \Phi C \Phi^{-1}$ in terms of the basis eigenfunction $\Phi(\tau, z, \alpha)$ and the matrix $C(z, \alpha)$ is related to the boundary value $\rho_- = e^{i\alpha J \tau} C(\alpha, z) e^{-i\alpha J \tau}$ as $\tau \rightarrow -\infty$. Since ρ_- is block-diagonal as in (3.4), it commutes with the diagonal matrix J and we have $C = \rho_- = bb^\dagger$. Moreover, using the form $\Phi = \mu e^{ikJ\tau}$ where $\mu^\dagger(\tau, z, k) = \mu^{-1}(\tau, z, k)$ for real k (cf. [27]), we obtain that $\rho = \gamma\gamma^\dagger = \Phi C \Phi^{-1} = (\mu b)(\mu b)^\dagger$. Finally, using the boundary condition $\mu \rightarrow 1$ as $\tau \rightarrow -\infty$, we get a simple form for the probability amplitude namely, $\gamma = \mu b$ where the one-soliton eigenfunction is given by [27]

$$\mu(\tau, z, \alpha) = \mathbf{I}_3 - \frac{2i\eta}{4\eta^2 + \|\tilde{\beta}\|^2} \begin{pmatrix} \frac{\tilde{\beta}\tilde{\beta}^\dagger}{\alpha - k^*} & \frac{2i\eta\tilde{\beta}}{\alpha - k} \\ -\frac{2i\eta\tilde{\beta}^\dagger}{\alpha - k^*} & -\frac{\|\tilde{\beta}\|^2}{\alpha - k} \end{pmatrix} \quad (3.8)$$

with the one-soliton parameters $k = \xi + i\eta$ and β introduced in Section 3.1.

Case (i) When $b = (0, 0, b_3)^\top$, it follows from (3.4) that $H(\alpha, z) = 0$ and $h(\alpha, z) = |b_3|^2(\alpha, z)$. Then $\hat{H}(\alpha, z) = -h(\alpha, z)I_2$ where I_2 is the 2×2 identity matrix. Consequently, the evolution equation (3.3) for the one-soliton parameter $\beta(z)$ takes the simple form

$$\beta_z = -i\lambda(k, z)\beta, \quad \lambda(k, z) = \frac{1}{4} \int_{-\infty}^{\infty} \frac{|b_3|^2(\alpha, z)g(\alpha)}{\alpha - k} d\alpha, \quad (3.9)$$

where $k = \xi + i\eta$. In addition, if we assume that b_3 is independent of z then $\lambda(k, z) = \lambda(k)$ so that $\beta(z) = e^{-i\lambda z}\beta(0)$. Then using (3.6), we find that the one-soliton potential $\Omega(\tau, z)$ has the same form as in (3.7) with $\lambda(k)$ defined above in (3.9), and the polarization vector $p = \beta(0)$. Therefore, the matched electric pulses propagate along the medium preserving their secant hyperbolic shape. Note that in this case the soliton polarization vector p and the phase shift δ are constant so that the pulses are traveling wave moving with a constant velocity

$$v_0 = c \left(1 + \frac{\text{Im } \lambda(k)c}{2\eta} \right)^{-1},$$

which is less than the light speed c in the medium since it follows from (3.9) that $\text{Im } \lambda(k) > 0$.

The complex probability amplitude γ in this case is obtained by using the one-soliton eigenfunction μ in (3.8) and the decomposition of $\hat{\beta}$ in (3.6). Setting $\zeta(\tau, z) = 2\eta\tau - \text{Im } \lambda z + \delta(z)$ and $\theta(\tau, z) = -2\xi\tau + \text{Re } \lambda z$, we get

$$\gamma(\tau, z, \alpha) = \mu b = \begin{pmatrix} 0 \\ 0 \\ b_3 \end{pmatrix} + \frac{\eta b_3}{\alpha - k} \begin{pmatrix} e^{i\theta} \text{sech } \zeta \hat{p} \\ i(1 + \tanh \zeta) \end{pmatrix},$$

where $\hat{p}$ is the two-component soliton polarization (unit) vector. In this case, the medium is initially prepared so that the atoms are all in the excited state $|3\rangle$. The one-soliton pulse initially triggers transitions to the ground states $|j\rangle$, $j = 1, 2$ as seen above, and induces a material dipole polarization along the direction of the soliton polarization vector $\hat{p}$. But as $\zeta \rightarrow \infty$, $\gamma \rightarrow (0, 0, \tilde{b}_3)$ where $\tilde{b}_3 = b_3(\alpha - k)/(\alpha - k^*)$, which implies that the atoms tend to return to the excited state $|3\rangle$ long after the pulse has passed through a given location in the medium, where the dipole polarization also tends to zero. Also, note that since the quantity $\alpha - k/\alpha - k^*$ is unimodular for $\alpha \in \mathbb{R}$, one has $|\tilde{b}_3| = |b_3|$ so that the complex probability amplitude of state $|3\rangle$ changes only by a phase factor. However, the population *density* $|\gamma_i|^2$ of each atomic state $|i\rangle$, $i = 1, \dots, 3$ approaches its initial value as $\zeta \rightarrow \infty$, so that the soliton propagates without any energy loss to the medium.

Case (ii) In this case the initial probability amplitude is of the form $b = (b_1, b_2, 0)^T$. Then from (3.4) we obtain $h = 0$ and $\hat{H} = H = uu^\dagger$ where $u := (b_1, b_2)^T \in \mathbb{C}^2$. Thus H is a 2×2 , rank one matrix with eigenvalues $0, \|u\|^2$ corresponding to eigenvectors $u^\perp$ and u respectively, where $u^\perp$ is orthogonal to u . Note that in general, $\langle \hat{H} \rangle(k)$ need not be rank one if the initial probability amplitude b depends on the detuning frequency α . However, if we assume that the medium is prepared initially such that the vector u is a constant independent of α then $\langle \hat{H} \rangle(k)$ is indeed rank one with eigenvalues

$$\lambda^{(1)} = 0, \quad \lambda^{(2)} = \frac{1}{4} \|u\|^2 \int_{-\infty}^{\infty} \frac{g(\alpha)}{\alpha - k} d\alpha = \frac{\|u\|^2}{4} \int_{-\infty}^{\infty} \frac{[(\alpha - \xi) + i\eta]g(\alpha)}{(\alpha - \xi)^2 + \eta^2} d\alpha$$

and (unit) eigenvectors $v^{(1)} = \hat{u}^\perp, v^{(2)} = \hat{u}$. It is clear from the above expression of $\lambda^{(2)}$ that $\text{Im } \lambda^{(2)} > 0$. Then from (3.5) we have $\beta(z) = c^{(1)}\hat{u}^\perp + c^{(2)}\hat{u}e^{i\lambda^{(2)}z}$, and after setting $\lambda = \lambda^{(1)} = 0$ in equations (3.6) and (3.7) we obtain the one-soliton potential

$$\Omega(\tau, z) = 4i\eta e^{-2i\xi\tau} \text{sech}[2\eta\tau + \delta(z)] \hat{\beta}(z), \quad \delta(z) = \ln \frac{\|\beta(z)\|}{2\eta}.$$

Since $e^{i\lambda^{(2)}z} \rightarrow 0$ as $z \rightarrow \infty$, it follows that $\beta(z) \rightarrow c^{(1)}\hat{u}^\perp$ and $\delta(z) \rightarrow \ln \frac{|c^{(1)}|}{2\eta}$. Thus the electric field envelope approaches a stationary state (i.e. z -independent) in the moving frame as $z \rightarrow \infty$, and the soliton moves with the light speed. The soliton polarization vector $\hat{u}^\perp$ is orthogonal to the vector $u = (b_1, b_2)^T$ associated with the initial probability amplitude of the 3-level medium.

In order to derive the complex probability amplitude $\gamma(\tau, z, \alpha)$, we use $\gamma = \mu b$ where μ is again given by (3.8). After taking into account the form of b and $\beta(z)$ in this case, we find that

$$\gamma = b - \eta \frac{\hat{\beta}^\dagger u}{\alpha - k^*} \begin{pmatrix} i(1 + \tanh \zeta) \hat{\beta} \\ e^{-i\theta} \text{sech } \zeta \end{pmatrix}.$$

In this case, $\gamma \rightarrow b$ as $\zeta \rightarrow -\infty$, i.e., the ground states are populated initially. Then the electric field of the soliton pulse induces transition to the excited state, and as $\zeta \rightarrow \infty$, the atoms tend to return to the ground states, i.e., $\gamma \rightarrow \tilde{b}$ where

$$\tilde{b} = b - \frac{2i\eta}{\alpha - k^*} \begin{pmatrix} (\hat{\beta}^\dagger u) \hat{\beta} \\ 0 \end{pmatrix}.$$

The transition from $b \rightarrow \tilde{b}$ as the pulse propagates through the medium implies that the probability amplitudes of the ground states undergo a unitary transformation since $||\tilde{b}|| = ||b||$. In fact, when $z \rightarrow \infty$, the soliton polarization vector $\hat{\beta} \rightarrow \hat{u}^\perp$ so that $\hat{\beta}^\dagger u \rightarrow 0$, and $\tilde{b} \rightarrow b$, which implies $\gamma \rightarrow b$. That is, the medium returns to its initial ground state configuration. This fact ensures that the soliton pulse propagation is again lossless as in case (i).

3.3 Two-soliton solutions

The two-soliton solutions constitute the simplest model of optical pulse interactions described by the CMB equations. The two-soliton interaction manifests itself via phase shifts as well as polarization shifts of the component solitons. Here we will briefly describe these effects by analyzing the two-soliton expression obtained from (3.2) with $n = 2$. A two-soliton solution is determined by two distinct complex parameters $k_j = \xi_j + i\eta_j$, and complex-valued vectors β_j , $j = 1, 2$. Then (3.2) with $n = 2$ yields

$$\Omega(\tau, z) = -4i \frac{|K_1|\tilde{\beta}_1 + |K_2|\tilde{\beta}_2}{|K|}, \quad (3.10)$$

where K is the 2×2 generalized Cauchy matrix

$$K = \begin{pmatrix} \frac{|q_1|^2 + ||\tilde{\beta}_1||^2}{-2i\eta_1} & \frac{q_1^* q_2 + \tilde{\beta}_1^\dagger \tilde{\beta}_2}{k_1^* - k_2} \\ \frac{q_2^* q_1 + \tilde{\beta}_2^\dagger \tilde{\beta}_1}{k_2^* - k_1} & \frac{|q_2|^2 + ||\tilde{\beta}_2||^2}{-2i\eta_2} \end{pmatrix}, \quad q_1 = 2i\eta_1 \frac{k_1 - k_2^*}{k_1 - k_2}, \quad q_2 = 2i\eta_2 \frac{k_1^* - k_2}{k_1 - k_2},$$

K_1, K_2 are 2×2 matrices obtained by replacing respectively, the first and the second column of K by the column $(q_1^*, q_2^*)^T$. Equation (3.10) represents two interacting solitons in the sense that the asymptotic form of the solution before or after the interaction can be represented as a sum of two one-soliton solutions. Furthermore, the polarization vectors associated with the asymptotic one-solitons before the interaction are distinct from those after the interaction. Thus, in addition to the usual phase shift, the solitons also experience a polarization shift after collision.

To analyze the two-soliton interaction dynamics, we first introduce the one-soliton phases defined by

$$\zeta_j = 2\eta_j \tau + \ln \frac{||\beta_j(z)||}{2\eta_j}, \quad j = 1, 2.$$

During a two-soliton interaction in the optical medium, the phases ζ_j completely overlap at the collision point $z = z_0$ which is characterized by

$$\Delta(z_0) = 0, \quad \Delta(z) := \zeta_2/2\eta_2 - \zeta_1/2\eta_1.$$

We assume here that the initial conditions $\beta_j(0)$ and $\langle \hat{H} \rangle(k_j)$ are such that $\Delta(z_0) = 0$ holds for some $z_0 > 0$. In addition, $\Delta(z)$ is assumed to be a strictly monotonic function of z in a sufficiently large neighborhood of z_0 so that the solitons are well separated before and after the interaction. Then it is possible to derive the asymptotic form of the two-soliton solution under these assumptions for both $z \gg z_0$ and $z \ll z_0$. In what follows we assume that $\Delta(z)$ is a strictly increasing function of z , the strictly decreasing case can be handled in a similar fashion.

If $z \ll z_0$, then $\Delta(z) \ll 0$ which implies that $\zeta_2 \ll \zeta_1$ since $\eta_j = O(1)$, $j = 1, 2$. Furthermore, since $||\tilde{\beta}_j|| = ||e^{-2ik_j \tau} \beta_j|| = 2\eta_j e^{\zeta_j}$ we conclude that $||\tilde{\beta}_2||/||\tilde{\beta}_1|| \ll 1$ as $z \ll z_0$. Then there are two cases: (i) $||\tilde{\beta}_1|| = O(1)$, $||\tilde{\beta}_2|| \ll ||\tilde{\beta}_1||$ and (ii) $||\tilde{\beta}_2|| = O(1)$, $||\tilde{\beta}_1|| \gg ||\tilde{\beta}_2||$, for $z \ll z_0$. In the first case it follows from the expression of two-soliton generalized Cauchy matrix K and K_1, K_2 given above, that

$$|K| \sim \begin{vmatrix} \frac{|a_1|^2 + ||\tilde{\beta}_1||^2}{-2i\eta_1} & \frac{a_1^* a_2}{k_1^* - k_2} \\ \frac{a_2^* a_1}{k_2^* - k_1} & \frac{|a_2|^2}{-2i\eta_2} \end{vmatrix}, \quad |K_1| \sim \begin{vmatrix} a_1^* & \frac{a_1^* a_2}{k_1^* - k_2} \\ a_2^* & \frac{|a_2|^2}{-2i\eta_2} \end{vmatrix}, \quad \frac{|K_2|}{|K|} ||\tilde{\beta}_2|| \ll 1,$$

Figure 5: Two-soliton electric field $\Omega(x, \tau)$ with Lorentzian inhomogeneous broadening. Left: modulus of the first component $|\Omega_1(\tau, z)|$. Center: modulus of the second component $|\Omega_2(\tau, z)|$. Right: norm $||\Omega(\tau, z)||$.

so that $\Omega \sim -4i(|K_1|\tilde{\beta}_1/|K|)$. From the above expressions one recovers after some calculation, the one-soliton formula

$$\Omega(\tau, z) \sim 4i\eta_1 e^{-2i\xi_1\tau} \text{sech}\zeta_1 \hat{\beta}_1.$$

For the second case, a similar calculation as in case (i) yields a one-soliton solution as above but with parameters ξ_2, η_2 and one-soliton phase ζ_2 . However, this solution also has an additional phase χ , and the soliton polarization vector depends on both β_1 and β_2 . These expressions are given below. Thus, combining cases (i) and (ii), the asymptotic form of the two-soliton solution as $z \ll z_0$ is as follows

$$\Omega(\tau, z) \sim 4i\eta_1 e^{-2i\xi_1\tau} \text{sech}\zeta_1 p_1^- + 4i\eta_2 e^{-2i\xi_2\tau} \text{sech}(\zeta_2 + \chi) p_2^-,$$

with $p_1^- = \hat{\beta}_1$ and

$$p_2^- = e^{-\chi} \det(\ell) \ell(\hat{\beta}_2), \quad \ell = I_2 - \frac{2i\eta_1}{k_2 - k_1^*} (\hat{\beta}_1^\dagger \hat{\beta}_2) \hat{\beta}_1, \quad e^{2\chi} = |\det(\ell)|^2 ||\ell(\hat{\beta}_2)||^2.$$

Thus, asymptotically as $z \ll z_0$ the two-soliton optical pulse can be viewed as a superposition of two well-separated one-soliton pulses traveling along constant $\zeta_j(\tau, z)$ curves, and with polarizations p_j^- , $j = 1, 2$.

Similarly when $z \gg z_0$, the soliton phase difference $\Delta(z) \gg 0$, and the asymptotic behavior of the solution in (3.10) arise by considering the two cases: (i) $||\tilde{\beta}_2|| = O(1)$, $||\tilde{\beta}_1|| \ll ||\tilde{\beta}_2||$, and (ii) $||\tilde{\beta}_1|| = O(1)$, $||\tilde{\beta}_2|| \gg ||\tilde{\beta}_1||$. The computations are similar to above, and the asymptotic form as $z \gg z_0$ of the two-soliton solution can also be viewed as a sum of two well-separated one-soliton pulses

$$\Omega(\tau, z) \sim 4i\eta_1 e^{-2i\xi_1\tau} \text{sech}(\zeta_1 + \chi) p_1^+ + 4i\eta_2 e^{-2i\xi_2\tau} \text{sech}\zeta_2 p_2^+,$$

with unit polarization vectors $p_2^+ = \hat{\beta}_2$ and

$$p_1^+ = e^{-\chi} \det(\tilde{\ell}) \tilde{\ell}(\hat{\beta}_1), \quad \tilde{\ell} = I_2 + \frac{2i\eta_2}{k_2^* - k_1} (\hat{\beta}_2^\dagger \hat{\beta}_1) \hat{\beta}_2.$$

Note that the matrices ℓ and $\tilde{\ell}$ are related to each other by switching the indices $1 \leftrightarrow 2$ in their expressions. The phase χ here is exactly the same as above in the $z \ll z_0$ case. Comparing the two asymptotic forms for $z \ll z_0$ and $z \gg z_0$, one finds that the component one-soliton pulses suffer an equal and opposite amount of phase shift χ , while the soliton polarization vectors undergo a transition: $\{p_1^-, p_2^-\} \rightarrow \{p_1^+, p_2^+\}$ due to

the interaction process. Both the phase shift and the polarization shift depend *nonlinearly* on β_j , $j = 1, 2$. These results for the CMB equation are similar to those found by Manakov [30] for the coupled nonlinear Schrödinger (NLS) equation. The main difference is that the vectors $\beta_j(z)$ in the coupled NLS have a simpler z -dependence than that of the CMB equation. As a result, the asymptotic polarization vectors $p_j^\pm$ and the phase shift χ are constants in the coupled NLS case.

In Figure 5 we plot a two-soliton solution, with discrete eigenvalues $k_1 = 1 + i/2$ and $k_2 = 5 + i$. The matrix $\hat{H} = \begin{pmatrix} \pi & -8i \\ 8i & \pi \end{pmatrix}$ is chosen to be independent of α , and $g(\alpha)$ is the same as chosen for the one-soliton plots in Section 3.1. The constants $c_j^{(1)}$ and $c_j^{(2)}$ in (3.5) are $c_1^{(1)} = -2i$, $c_1^{(2)} = i$, $c_2^{(1)} = -i$, $c_2^{(2)} = i/2$. Note that after interaction, one of the one-soliton components (shown in the left of each panel) associated with parameter k_1 changes direction in the (τ, z) -plane similar to the one-soliton case (iii) in Section 3.1. That is, $\zeta_1 \sim 2\eta\tau - \text{Im } \lambda^{(2)}z + \ln(|c^{(2)}|/2\eta_1)$ for $z \ll z_0$, whereas $\zeta_1 \sim 2\eta\tau - \text{Im } \lambda^{(1)}z + \ln(|c^{(1)}|/2\eta_1)$ for $z \gg z_0$.

4 Conservation laws

The CMB equations being completely integrable, admit an infinite set of conservation laws of the form

$$\frac{\partial f_n}{\partial z} = \frac{\partial j_n}{\partial \tau}, \quad n = 1, 2, \dots, \quad (4.1)$$

where f_n is the local density while j_n is the corresponding flux. An important feature of the CMB system which distinguishes it from other nonlinear integrable systems is the fact that in presence of inhomogeneous broadening, the integrated density $\int f_n d\tau$ is not *necessarily* a conserved quantity. Thus, although one has local conservation laws, these do not lead to global conserved quantities, including the conservation of energy. The latter is a consequence of irreversible loss of energy from the electric field to the active optical medium. However, for purely soliton pulses the global conservation laws are indeed recovered, and quantities such as the total power, momentum and energy are conserved. This feature was also observed in the case of Maxwell-Bloch equations in a two-level medium with inhomogeneous broadening in [8]. Below we derive these conservation laws from the 3×3 Lax pair (2.3) of the CMB equations.

In the following it will be convenient to use the following decomposition of 3×3 matrices

$$A = A_d + A_m, \quad A_d = \left\{ \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \right\}, \quad A_m = \left\{ \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix} \right\},$$

which satisfy the matrix multiplication rules represented symbolically as follows

$$A_d A_d \subset A_d, \quad A_d A_m \subset A_m, \quad A_m A_d \subset A_m, \quad A_m A_m \subset A_d.$$

Note that the matrix $J = \text{diag}(-1, -1, 1)$ commutes with any element in A_d . Next we consider a solution of the Lax pair (2.3) and express it as $G = (I_3 + M)e^{(D+ikJ\tau)}$ where $M(\tau, z, k) \in A_m$, $Z(\tau, z, k) \in A_d$, and I_3 is the 3×3 identity matrix. Substituting this expression into the first equation of (2.3) and decomposing into the A_d and A_m components leads to the equations

$$M_\tau + M\Omega M - \Omega = ik[J, M], \quad D_\tau = \Omega M. \quad (4.2)$$

The matrix Riccati equation for M can be solved if we consider its asymptotic expansion

$$M(\tau, z, k) = \sum_{n=1}^{\infty} \frac{M_n(\tau, z)}{(2ik)^n},$$

as $k \rightarrow \infty$, and solve for the coefficients M_n recursively. We obtain

$$M_1 = -J\Omega, \quad M_2 = -\Omega_\tau, \quad M_{n+1} = J \left(M_{n\tau} + \sum_{j=1}^{n-1} M_j \Omega M_{n-j} \right), \quad n \geq 2, \quad (4.3)$$

where we have used the relations $[J, M] = 2JM$, $M \in A_m$ and $J^2 = 1$, which can be easily verified. The coefficients $M_n(\tau, z)$ are local expressions which are polynomials in $\Omega(\tau, z)$ and its τ -derivatives. Similarly, from the second equation of (2.3) one obtains the z -evolution equation for $D(\tau, z, k)$ as

$$D_z = i(\langle \rho \rangle_d + \langle \rho \rangle_m M), \quad \langle \rho \rangle(z, \tau, k) := \frac{1}{4} \int_{-\infty}^{\infty} \frac{g(\alpha)}{\alpha - k} \rho(z, \tau, \alpha) d\alpha, \quad (4.4)$$

with $\rho = \rho_d + \rho_m$, $\langle \rho \rangle_d = \langle \rho_d \rangle$, $\langle \rho \rangle_m = \langle \rho_m \rangle$. The corresponding equation for M is omitted as it does not play a role here. Note that the quantity $\langle \rho \rangle(\tau, z, k)$ admits the large k expansion

$$\langle \rho \rangle(\tau, z, k) = \sum_{n=1}^{\infty} \frac{\rho_n(\tau, z)}{2ik^n}, \quad \rho_n = \frac{-i}{2} \int_{-\infty}^{\infty} (2i\alpha)^{n-1} \rho(\tau, z, \alpha) g(\alpha) d\alpha,$$

as $k \rightarrow \infty$, and where the coefficient matrices can be also decomposed as $\rho_n = (\rho_n)_d + (\rho_n)_m$.

The conservation laws for the CMB equation can now be derived from the simple integrability condition $\text{Tr}(JD_\tau)_z = \text{Tr}(JD_z)_\tau$ which from (4.2) and (4.4) translates to

$$\frac{\partial}{\partial z} \text{Tr}(J\Omega M) = i \frac{\partial}{\partial \tau} \text{Tr}(\langle \rho \rangle_d + \langle \rho \rangle_m M).$$

Inserting above the asymptotic expansions of $M(\tau, z, k)$ and $\langle \rho \rangle(\tau, z, k)$ as $k \rightarrow \infty$, and using (4.3) for the coefficients $M_n(\tau, z)$, we obtain an infinite set of local conservation laws given in the form of (4.1). The first few densities (power, momentum and energy) and the corresponding fluxes are given as follows:

$$\begin{aligned} f_1 &= \frac{1}{2}(\Omega^\dagger \Omega), & f_2 &= \frac{1}{4}(\Omega_\tau^\dagger \Omega - \Omega^\dagger \Omega_\tau) & f_3 &= \frac{1}{2}(\Omega^\dagger \Omega_{\tau\tau}) + \frac{1}{8}(\Omega^\dagger \Omega)^2 \\ j_1 &= \frac{1}{2} \int_{-\infty}^{\infty} N(\alpha) g(\alpha) d\alpha, & j_2 &= i \int_{-\infty}^{\infty} \alpha N(\alpha) g(\alpha) d\alpha - \frac{1}{4} \int_{-\infty}^{\infty} \left(\nu^\dagger(\alpha) \Omega - \Omega^\dagger \nu(\alpha) \right) g(\alpha) d\alpha, \\ j_3 &= -2 \int_{-\infty}^{\infty} \alpha^2 N(\alpha) g(\alpha) d\alpha - \frac{i}{2} \int_{-\infty}^{\infty} \alpha \left(\nu^\dagger(\alpha) \Omega - \Omega^\dagger \nu(\alpha) \right) g(\alpha) d\alpha + \frac{1}{4} \int_{-\infty}^{\infty} \left(\Omega_\tau^\dagger \nu(\alpha) + \nu^\dagger(\alpha) \Omega_\tau \right) g(\alpha) d\alpha, \end{aligned}$$

where $\Omega = (\Omega_1, \Omega_2)^T$, $N(\tau, z, \alpha) = \rho_{11} + \rho_{22} - \rho_{33}$, and $\nu(\tau, z, \alpha) = (\rho_{13}, \rho_{23})^T$. Integrating (4.1) over τ , we find the evolution of the integrated densities,

$$\frac{dF_n}{dz} = j_n(\tau, z) \Big|_{\tau \rightarrow -\infty}^{\tau \rightarrow \infty}, \quad F_n(z) := \int_{-\infty}^{\infty} f_n d\tau, \quad n = 1, 2, \dots$$

For localized optical pulses, when Ω_i vanishes rapidly as $\tau \pm \infty$, and for all z , the above equation becomes

$$\frac{dF_n}{dz} = K_n \int_{-\infty}^{\infty} \alpha^{n-1} (N_+(z, \alpha) - N_-(z, \alpha)) g(\alpha) d\alpha, \quad n = 1, 2, \dots,$$

where $\{K_n\}$ is a set of numerical coefficients and $N_\pm(z, \alpha) = (\rho_\pm)_{11} + (\rho_\pm)_{22} - (\rho_\pm)_{33}$. The last quantities $N_\pm$ depend on the boundary values $\rho_\pm$ of the Bloch matrix before and after the passage of the optical pulses through a given point in the three-level medium, and in general $N_-(z, \alpha) \neq N_+(z, \alpha)$. Consequently, F_n is *not* a global conserved quantity for the CMB equations (2.1) for an arbitrary optical pulse Ω .

Trace formulae: If the envelope of the electric field propagating through the optical medium is in the form of pure soliton pulse, then it is possible to show that the right hand side vanishes in the above equation for dF_n/dz . This is because the quantities $N_{\pm}$ are the *same* although the initial and final states of the medium represented by the boundary values $\rho_{\pm}$ can be different. Consequently, $\{F_n\}$ forms an infinite set of global conserved quantities in the instance of interactions between soliton pulses with a three-level active optical medium. In order to establish this interesting fact we will need some results from IST briefly mentioned at the end of Section 2. A detailed discussion can be found in [27].

Recall that in the case of pure soliton solutions, the boundary value ρ_- has a block diagonal form as given in (3.4). Then from our discussion in (the end of) Section 2, we deduce that the matrix $C(z, \alpha) = e^{-i\alpha J\tau} \rho_- e^{i\alpha J\tau} = \rho_-$ since the diagonal matrix J commutes with any block diagonal matrix. Furthermore, it turns out that the scattering matrix $S(k)$ for pure solitons is a block diagonal matrix of the form

$$S(k) = \begin{pmatrix} \mathbf{a}(k) & 0 \\ 0 & a(k) \end{pmatrix},$$

where $\mathbf{a}(k)$ is a 2×2 matrix, invertible when $k \in \mathbb{R}$, and $a(k)$ is a scalar. Then from the expression of the boundary value ρ_+ given in Section 2 and equation (3.4), one can write

$$\rho_+(z, \alpha) = S^{-1}(\alpha) \rho_-(z, \alpha) S(\alpha) = \begin{pmatrix} (\mathbf{a}^{-1} H \mathbf{a})(z, \alpha) & 0 \\ 0 & h(z, \alpha) \end{pmatrix},$$

which shows that ρ_+ is also a block diagonal matrix. Since the quantity $N(\tau, z, \alpha)$ can be also expressed as $N = \text{Tr } \rho - 2\rho_{33}$, it follows from (3.4) and the expression of ρ_+ given above that $N_-(z, \alpha) = N_+(z, \alpha) = \text{Tr } H(z, \alpha) - h(z, \alpha)$. Hence the quantities F_j , $j = 1, 2, \dots$ are indeed constants for soliton solutions. We will now find an explicit expression for F_j in terms of the n -soliton parameters k_j , $j = 1, 2, \dots, n$ introduced in Section 3.

As shown above, the densities f_n for the local conservation laws are generated using the asymptotic expansion of $M(\tau, z, k)$ as $k \rightarrow \infty$. In particular, we have

$$\text{Tr}(JD_{\tau}) = \text{Tr}(J\Omega M) = \sum_{n=1}^{\infty} \frac{-f_n}{(2ik)^n}, \quad f_n(\tau, z) = -\text{Tr}(J\Omega M_n),$$

where the coefficients M_n are obtained recursively from (4.3). Integrating over τ , we obtain the relation

$$\int_{-\infty}^{\infty} \text{Tr}(JD_{\tau}) d\tau = \text{Tr}(JD) \Big|_{\tau \rightarrow -\infty}^{\tau \rightarrow \infty} = \sum_{n=1}^{\infty} \frac{-F_n}{(2ik)^n} \quad (4.5)$$

generating the integrated densities. Next, we show that the limiting value of the matrix $D(\tau, z, k)$ as $\tau \rightarrow \infty$ is related to the scattering matrix $S(k)$. We choose $G = \Phi$ for the solution of the Lax pair (2.3) where $\Phi(\tau, z, k)$ is the basis eigenfunction satisfying the boundary condition $\Phi \sim e^{ikJ\tau}$ as $\tau \rightarrow -\infty$. Then the decomposition $\Phi = (I_3 + M)e^{(D+ikJ\tau)}$ implies that both M and D vanish as $\tau \rightarrow -\infty$. To find the limiting behavior of D as $\tau \rightarrow \infty$, we introduce the basis eigenfunction Ψ satisfying the boundary condition $\Psi \sim e^{ikJ\tau}$ as $\tau \rightarrow \infty$, and use the relation $\Phi = \Psi S^{-1}$ (see Section 2). Then as $\tau \rightarrow \infty$, we get

$$\Phi \sim (I_3 + M_{\infty})e^{D_{\infty}+ikJ\tau} = e^{ikJ\tau} S^{-1}, \quad M_{\infty} := \lim_{\tau \rightarrow \infty} M(\tau, z, k), \quad D_{\infty} := \lim_{\tau \rightarrow \infty} D(\tau, z, k).$$

Decomposing the above relation into A_d and A_m components and using the fact that $[J, D_{\infty}] = 0$, one then recovers the relation $(S^{-1})_d = e^{D_{\infty}}$. But since the scattering matrix S (and its inverse) is block diagonal in the case of pure soliton solutions, we have $(S^{-1})_d = S^{-1}$ leading to the simple formula

$$S(k) = \begin{pmatrix} \mathbf{a}(k) & 0 \\ 0 & a(k) \end{pmatrix} = e^{-D_{\infty}}, \quad (4.6)$$

which relates the scattering data to the conserved quantities F_n as shown below.

It can be shown that the scattering matrix satisfies $\det S = 1$. Hence, from (4.6), $\text{Tr}(D_\infty) = 0$. Then $\text{Tr}(JD_\infty) = 2(D_\infty)_{33} - \text{Tr}(D_\infty) = 2(D_\infty)_{33}$. Since $D_\infty \in A_d$, its (33)-component can be read off from (4.6) as $(D_\infty)_{33} = -\ln a(k)$. Returning to (4.5), and using the fact that $D \rightarrow 0$ as $\tau \rightarrow -\infty$, we then obtain

$$\sum_{j=1}^{\infty} \frac{F_m}{(2ik)^m} = -\text{Tr}(JD_\infty) = 2 \ln a(k) = 2 \ln \left(\prod_{j=1}^n \frac{k - k_j}{k - k_j^*} \right),$$

where we have introduced the n -soliton formula for $a(k)$ obtained in [27] in the last equality above. Finally, expanding the expression for $\ln a(k)$ as $k \rightarrow \infty$ yields the following relations for the global conserved quantities F_j in terms of the n -soliton parameters

$$F_m = 2 \frac{(2i)^m}{m} \sum_{j=1}^n k_j^{*m} - k_j^m, \quad m = 1, 2, \dots \quad (4.7)$$

The expressions such as above for the conserved quantities are known as trace formulae for soliton equations integrable via IST.

5 Conclusion

In this article we have investigated the one- and two-soliton solutions of the CMB equations in a three-level optical medium. In particular, we find that in the presence of inhomogeneous broadening, the one-soliton waveform evolve in a non-trivial fashion unlike many well-known $(1+1)$ -dimensional integrable equations such as the NLS equation. The asymptotics of the two-soliton solution have been analyzed to obtain the polarization shifts of the vector solitons before and after the collision. These results are similar to that found by Manakov [30] for the coupled NLS equation. We have also considered the soliton solutions for special case of matched pulse propagation when the Bloch matrix is rank one, and is expressed in terms of the transition probability amplitude of the the three-level atoms. In this case the soliton pulses propagate preserving their shape, and the medium returns to its initial state after the passage of the optical pulses. Finally, we have derived an infinite set of local conservation laws for the CMB equation, and have found that the integrated densities do not in general form conserved quantities except in the case of soliton pulses propagating through the medium. In the latter case, we also give explicit trace formulae for the conserved quantities in terms of the soliton parameters.

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