Odd Azimuthal Anisotropy of the Glasma for pA Scattering

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Abstract

In this paper we analytically extract the odd azimuthal anisotropy in the Classical Yang Mills equations for the Glasma for pA collisions. We compute the first non-trivial term in the expansion of the proton sources of color charge. The computation is valid in the limit of a large nucleus when the produced particle momenta are larger than the saturation momentum of the proton.

1. Introduction

Computations using the theory of the Color Glass Condensate can generate even flow harmonics from initial state correlations [1, 2, 3, 4]. These correlations are non-vanishing in the limit of an infinite number of color sources, but suppressed by the number of colors. This is in distinction from fluctuations generated by a finite number of scattering centers which are non-vanishing in the limit of a large number of colors but vanish in the limit of an infinite number of color sources [5, 6, 7, 8, 9, 10, 11, 12]. Four- and higher order-particle elliptical anisotropies also demonstrate a non-trivial behaviour as a function of number of colors and number of sources [10, 13, 14].

The situation for odd harmonics is very interesting. Unlike the case for even harmonics, to obtain odd harmonics at small x requires final state interactions at least in the classical approximation to the CGC ¹. This is a

¹ As demonstrated in Ref. [15], odd azimuthal anisotropy is present in the CGC wave function beyond the classical approximation.

consequence of time reversal invariance. In models of the Glasma, where classical equations are computed numerically, one sees odd harmonics, and they indeed develop after the collision of the nuclei has taken place while final state interactions are in play [17, 18, 19].

It is the purpose of this paper to elaborate somewhat on the generation of flow in the classical equations that describe the evolution of the Glasma [20, 21] and to build a bridge between the analytical calculations in the dilute-dense limit (see e. g. Ref. [25, 26]) and the dense-dense numerical results [16, 19]. We begin by solving the classical Yang Mills equations around the free field equations for a distribution of fluctuating sources. We consider a proton nucleus collision in a momentum range where the field of the proton can be treated as weak. We show explicitly that there are no odd harmonics generated by this lowest order solution. We then iterate the equations around the leading order, treating the color field of the nucleus to all orders, and we find that we generate odd moments of azimuthal anisotropy in the first such iteration of these equations. The non-zero contribution to odd harmonics arises from the interference of the leading and next-to-leading orders. We find remarkable simplification for the result for such odd moments when gluons are put on mass shell, and we integrate over intermediate coordinates associated with iterating the equations.

This exercise is not only academic, in that it provides analytic confirmation of what is already known from numerical simulation, but it is also useful for describing dilute systems such provided by pp and pA collisions, since the analytical form may be somewhat simpler to use than numerical solutions to the full scattering problem.

2. Notation and review of known results

In this section we set up our notation, and will use well known results from the literature concerning the classical equations that describe the Color Glass Condensate and the Glasma [22]. We begin by writing down the color field of an isolated nucleon or nucleus as

$$\alpha_m^i(x_\perp) = -\frac{1}{ig}U_m(\vec{x}_\perp)\partial^i U_m^\dagger(\vec{x}_\perp) = \frac{1}{ig}[\partial^i U_m(\vec{x}_\perp)]U_m^\dagger(\vec{x}_\perp), \quad (1)$$

where the Wilson lines are in the fundamental representation. The field is generated by valence color charges

$$\partial_i \alpha_i(\vec{x}_\perp) = g\rho(\vec{x}_\perp). \quad (2)$$

The label m is 1 for the field of a proton and 2 for the nucleus. This is the field that describes the nucleon or nucleus before the collision, and is the same as for an isolated nucleon or nucleus. We consider the source for the proton ρ_1 to be weak and expand the corresponding Wilson lines into power series to get

$$\alpha_1^i(\vec{x}_\perp) = \partial^i \Phi_1(\vec{x}_\perp) - \frac{ig}{2} \left(\delta_{ij} - \frac{\partial_i \partial_j}{\partial^2} \right) [\partial^j \Phi_1(\vec{x}_\perp), \Phi_1(\vec{x}_\perp)] + \mathcal{O}(\Phi_1^3), \quad (3)$$

where

$$\Phi_1(\vec{x}_\perp) = \frac{g}{\partial^2} \rho(\vec{x}_\perp). \quad (4)$$

The only Wilson lines we will encounter in the text correspond to the strong, nucleus field $m = 2$. In order to simplify the notation we will omit the redundant subscript, that is

$$U(\vec{x}_\perp) \equiv U_2(\vec{x}_\perp). \quad (5)$$

In what follows we will perform the expansion in the weak field; the notation for the expansion coefficients is defined by

$$f(k) = \lim_{N \rightarrow \infty} \sum_{n=1}^N f^{(n)}(k). \quad (6)$$

We warn the reader that there might be a potential confusion with the notation of the Hankel functions, which also involves bracketed integers in the superscript.

In this paper we will consider only first two nontrivial corrections, i.e. we terminate the expansion at $N = 2$. When one computes the cross section for particle production, one evaluates a square of the amplitudes associated with the color field. The first non-trivial correction to particle production that involves a second order iteration of the proton field, is of the order ρ_1^3 and originates from the interference of the leading and next-to-leading order expansion coefficients. We will argue that this is the leading order correction contributing to the odd azimuthal anisotropy of double inclusive gluon production.

Following a widely accepted convention, we define

$$\tau = \sqrt{2x_+ x_-}, \quad (7)$$

where

$$x_{\pm} = \frac{x_0 \pm x_z}{\sqrt{2}}. \quad (8)$$

When it is convenient we will use the Milne metric, or, the τ - η -coordinates. In this case the Minkowski coordinates are parametrized by

$$x = (\tau \cosh \eta, \vec{x}_{\perp}, \tau \sinh \eta).$$

Here $\vec{x}_{\perp}$ is a two-dimensional vector.

We work in the gauge

$$A_{\tau} = x_{-}A_{+} + x_{+}A_{-} = 0.$$

Therefore the η component of the vector potential is

$$A_{\eta} = x_{-}A_{+} - x_{+}A_{-} = \tau^2\alpha, \quad (9)$$

where α is introduced for convenience.

Since the quantum corrections are explicitly ignored in our classical Yang-Mills approach, the field created in collisions is η -independent.

For Bessel (Neumann) functions of n -th order the following notation is used $J_n(x)$ ($Y_n(x)$).

3. Equations of motion

In the upper light cone, assuming independence of rapidity, the Classical Yang Mills (CYM) equations $[D_{\mu}, F^{\mu\nu}] = 0$ can be written as

$$\begin{aligned} & \frac{1}{\tau} \partial_{\tau} \tau \partial_{\tau} A_i - \partial_j (\partial_j \partial A_i - \partial_i \partial A_j) \\ & + ig \left(\partial_i [A_j, A_i] + \frac{1}{\tau^2} [A_{\eta}, F_{\eta i}] + [A_j, F_{ij}] \right) = 0, \\ & \partial_{\tau} \tau^{-1} \partial_{\tau} A_{\eta} - \frac{1}{\tau} \partial_{\perp}^2 A_{\eta} + ig \left(\frac{1}{\tau} \partial_j [A_j, A_{\eta}] + \frac{1}{\tau} [A_j, F_{j\eta}] \right) = 0, \\ & \partial_{\tau} \partial_i A_i - ig \left(\frac{1}{\tau^2} [A_{\eta}, \partial_{\tau} A_{\eta}] + [A_i, \partial_{\tau} A_i] \right) = 0. \end{aligned} \quad (10)$$

Note that the τ derivatives in the second equation can be written in the following equivalent form

$$\partial_{\tau} \tau^{-1} \partial_{\tau} A_{\eta} \equiv \tau \partial_{\tau}^2 \alpha + 3 \partial_{\tau} \alpha \equiv \tau^{-2} \partial_{\tau} \tau^3 \partial_{\tau} \alpha. \quad (11)$$

4. Solutions of CYM to leading order in weak field

In this section we review the result of Ref. [20, 21, 23] using the notation we defined in Sec. 2 and present them in the form that will be useful for what follows.

The gluon field has the following dependence

$$A^\pm(x^+, x^-, \vec{x}_\perp) = \pm x^\pm \alpha(\tau, \vec{x}_\perp) \theta(x^+) \theta(x^-), \quad (12)$$

$$\begin{aligned} A^i(x^+, x^-, \vec{x}_\perp) &= \alpha^i(\tau, \vec{x}_\perp) \theta(x^+) \theta(x^-) \\ &\quad + \alpha_1^i(\vec{x}_\perp) \theta(-x^+) \theta(x^-) + \alpha_2^i(\vec{x}_\perp) \theta(x^+) \theta(-x^-) \end{aligned} \quad (13)$$

with the initial conditions obtained by matching the singularities on the light cone

$$\alpha(\tau \rightarrow 0, \vec{x}_\perp) = \frac{ig}{2} [\alpha_1^i(\vec{x}_\perp), \alpha_2^i(\vec{x}_\perp)], \quad (14)$$

$$\alpha^i(\tau \rightarrow 0, \vec{x}_\perp) = \alpha_1^i(\vec{x}_\perp) + \alpha_2^i(\vec{x}_\perp). \quad (15)$$

The gauge rotation

$$\alpha(\tau, \vec{x}_\perp) = U(\vec{x}_\perp) \beta(\tau, \vec{x}_\perp) U^\dagger(\vec{x}_\perp), \quad (16)$$

$$\alpha_i(\tau, \vec{x}_\perp) = U(\vec{x}_\perp) \left(\beta_i(\tau, \vec{x}_\perp) - \frac{1}{ig} \partial_i \right) U^\dagger(\vec{x}_\perp) \quad (17)$$

enables us to perform a systematic expansion in powers of ρ_1 .

At the leading order, the CYM equations are

$$\left[\partial_\tau^2 + \frac{3}{\tau} \partial_\tau - \partial_\perp^2 \right] \beta^{(1)}(\tau, \vec{x}_\perp) = 0, \quad (18)$$

$$\partial_\tau \partial_i \beta_i^{(1)}(\tau, \vec{x}_\perp) = 0, \quad (19)$$

$$\left[\delta^{ij} \left(\partial_\tau^2 + \frac{1}{\tau} \partial_\tau - \partial_\perp^2 \right) + \partial_i \partial_j \right] \beta_j^{(1)}(\tau, \vec{x}_\perp) = 0 \quad (20)$$

with solutions

$$\beta(\tau, \vec{k}_\perp) = b_1(\vec{k}_\perp) \frac{J_1(k_\perp \tau)}{k_\perp \tau}, \quad (21)$$

$$\beta_i(\tau, \vec{k}_\perp) = i \frac{\varepsilon^{ij} k_j}{k_\perp^2} b_2(\vec{k}_\perp) J_0(k_\perp \tau) + i k_i \Lambda(\vec{k}_\perp). \quad (22)$$

The newly introduced functions are defined by the initial conditions

$$b_1(\vec{x}_\perp) = igU^\dagger(\vec{x}_\perp)[\alpha_1^{(1)i}(\vec{x}_\perp), \alpha_2^i(\vec{x}_\perp)]U(\vec{x}_\perp), \quad (23)$$

$$b_2(\vec{x}_\perp) = \epsilon^{ij}\partial^j \left(U^\dagger(\vec{x}_\perp)\alpha_1^{(1)i}(\vec{x}_\perp)U(\vec{x}_\perp) \right), \quad (24)$$

$$\Lambda(\vec{x}_\perp) = \frac{\partial^i}{\partial_\perp^2} \left(U^\dagger(\vec{x}_\perp)\alpha_1^{(1)i}(\vec{x}_\perp)U(\vec{x}_\perp) \right). \quad (25)$$

Note that these functions are manifestly real (to be precise the components), thus the following holds for their Fourier images

$$f(\vec{k}_\perp) = f^*(-\vec{k}_\perp), \quad (26)$$

where $f(\vec{k}_\perp)$ is either of $b_1(\vec{k}_\perp)$, $b_2(\vec{k}_\perp)$ or $\Lambda(\vec{k}_\perp)$.

Equations (23) and (24) can also be rewritten in a similar form. For this, we use the definition of $\alpha_2^i(\vec{x}_\perp)$ and simplify the commutator in Eq. (23): (23):

$$b_1(\vec{x}_\perp) = \delta^{ij}\Omega_{ij}, \quad (27)$$

$$b_2(\vec{x}_\perp) = \epsilon^{ij}\Omega_{ij} \quad (28)$$

with

$$\begin{aligned} \Omega^{ij}(\vec{x}_\perp) &= \left(\alpha_1^{(1)i}(\vec{x}_\perp) \right)_a \partial^j \left(U^\dagger(\vec{x}_\perp)t_a U(\vec{x}_\perp) \right) \\ &= g \left[\frac{\partial_i}{\partial^2} \rho_1^a(\vec{x}_\perp) \right] \partial^j W_{ba}(\vec{x}_\perp) t^b, \end{aligned} \quad (29)$$

where we used the adjoint Wilson line

$$W_{ab}(\vec{x}_\perp) = 2 \operatorname{tr} \left(U^\dagger(\vec{x}_\perp)t_b U(\vec{x}_\perp)t_a \right).$$

To derive these equations we have made explicit use of the form of the solution for α_1^i when expanded to first order in the strength of the proton source, see Eq. (3).

5. Particle production

5.1. LSZ

We start this section from reviewing the Lehmann-Symanzik-Zimmermann (LSZ) reduction formula for a scalar field. The time-dependent creation operator describing one particle at state $\vec{k}$ is defined by

$$a^+(\vec{k}, t) = \frac{1}{i} \int d^3x \exp(-ik \cdot x) \overleftrightarrow{\partial}_0 \phi(x), \quad (30)$$

where k and x are four-dimensional vectors and $k \cdot x = k_\mu x^\mu$.

We can construct the combination

$$\begin{aligned} a^+(\vec{k}, t \rightarrow \infty) - a^+(\vec{k}, t \rightarrow 0) &= \frac{1}{i} \int_0^\infty dt \partial_0 \left(\int d^3x \exp(-ik \cdot x) \overleftrightarrow{\partial}_0 \phi(x) \right) \\ &= \frac{1}{i} \int_0^\infty dt d^3x \exp(-ik \cdot x) (\square + m^2) \phi(x), \end{aligned} \quad (31)$$

where usually instead of $t \rightarrow 0$ the limit $t \rightarrow -\infty$ is used for the second term. We chose the limit $t \rightarrow 0$ to mimic our problem where the initial conditions are formulated on the light cone. From the equality (31), we can express the creation operator in the final state by

$$\begin{aligned} a^+(\vec{k}, \infty) &= \left[\frac{1}{i} \int d^3x \exp(-ik \cdot x) \overleftrightarrow{\partial}_0 \phi(x) \right]_{t=0} \\ &\quad + \frac{1}{i} \int_0^\infty dt \int d^3x \exp(-ik \cdot x) (\square + m^2) \phi(x). \end{aligned} \quad (32)$$

Therefore under the classical approximation, we deduce that that number of produced particles is given by

$$\begin{aligned} E_k \frac{dN}{d^3k} &= \frac{1}{2(2\pi)^3} \left| \left[\int d^3x \exp(-ik \cdot x) \overleftrightarrow{\partial}_0 \phi(x) \right]_{t=0} \right. \\ &\quad \left. + \int_0^\infty dt \int d^3x \exp(-ik \cdot x) (\square + m^2) \phi(x) \right|^2. \end{aligned} \quad (33)$$

Here we have two distinct contributions. One is from the initial time $t = 0$ “surface” and the other involving the time integration from the “bulk”. Anticipating the results, we want to comment that the surface contribution is manifestly T -even and thus is not expected to produce non-zero odd azimuthal anisotropy.

5.2. Milne metric

A straightforward generalization of Eq. (33) for β_i and β in the Milne metric reads

$$E_k \frac{dN}{d^3k} = \frac{1}{16\pi} \left[\left| \mathfrak{S}_\perp(\vec{k}_\perp) + \mathfrak{B}_\perp(\vec{k}_\perp) \right|^2 + \left| \mathfrak{S}_\eta(\vec{k}_\perp) + \mathfrak{B}_\eta(\vec{k}_\perp) \right|^2 \right], \quad (34)$$

where the surface contributions at $\tau \rightarrow 0+$ are given by

$$\mathfrak{S}_\perp(\vec{k}_\perp) = \lim_{\tau \rightarrow 0+} \left(\tau H_0^{(1)}(k_\perp \tau) \overset{\leftrightarrow}{\partial}_0 \beta_\perp(\tau, \vec{k}_\perp) \right), \quad (35)$$

$$\mathfrak{S}_\eta(\vec{k}_\perp) = \lim_{\tau \rightarrow 0+} \left(\tau^3 \left\{ \frac{H_1^{(1)}(k_\perp \tau)}{\tau} \overset{\leftrightarrow}{\partial}_0 \beta(\tau, \vec{k}_\perp) \right\} \right). \quad (36)$$

The bulk contributions from the upper light cone are

$$\mathfrak{B}_\perp(\vec{k}_\perp) = \int_0^\infty d\tau \tau H_0^{(1)}(k_\perp \tau) \left\{ \frac{1}{\tau} \partial_\tau \tau \partial_\tau \beta_\perp(\tau, \vec{k}_\perp) - \partial_\perp^2 \beta_\perp(\tau, \vec{k}_\perp) \right\}, \quad (37)$$

$$\mathfrak{B}_\eta(\vec{k}_\perp) = \int_0^\infty d\tau \tau^2 H_1^{(1)}(k_\perp \tau) \left\{ \frac{1}{\tau^3} \partial_\tau \tau^3 \partial_\tau \beta(\tau, \vec{k}_\perp) - \partial_\perp^2 \beta(\tau, \vec{k}_\perp) \right\}, \quad (38)$$

where the transverse part of the field β_i is defined as

$$\beta_\perp(\tau, \vec{k}_\perp) = i \frac{\epsilon^{ij} k_j}{k_\perp} \beta_i(\tau, \vec{k}_\perp). \quad (39)$$

The imaginary unit is included to guarantee that the function $\beta_\perp(\tau, \vec{x}_\perp)$ is real.

In order to simplify the notations we will introduce the following combinations

$$j_\perp(\tau, \vec{k}_\perp) = \frac{1}{\tau} \partial_\tau \tau \partial_\tau \beta_\perp(\tau, \vec{k}_\perp) - \partial_\perp^2 \beta_\perp(\tau, \vec{k}_\perp), \quad (40)$$

$$j_i(\tau, \vec{k}_\perp) = \frac{1}{\tau} \partial_\tau \tau \partial_\tau \beta_i(\tau, \vec{k}_\perp) - \partial_\perp^2 \beta_i(\tau, \vec{k}_\perp), \quad (41)$$

$$j(\tau, \vec{k}_\perp) = \frac{1}{\tau^3} \partial_\tau \tau^3 \partial_\tau \beta(\tau, \vec{k}_\perp) - \partial_\perp^2 \beta(\tau, \vec{k}_\perp). \quad (42)$$

which will be referred to as “currents” because they vanish in the absence of non-trivial interaction in the bulk.

5.3. Absence of odd azimuthal anisotropy at leading order

Lets consider the solutions of the CYM to the leading order in the weak field. Owing to the equations of motions we get

$$j^{(1)} = 0, \quad (43)$$

$$j_\perp^{(1)} = 0. \quad (44)$$

Because of the absence of the currents, there are no non-zero contributions from the upper light-cone. The surface term for the transverse component is defined by the initial conditions

$$\begin{aligned}\mathfrak{S}_{\perp}^{(1)}(\tau, \vec{k}_{\perp}) &= - \lim_{\tau \rightarrow 0+} \left(\tau \partial_{\tau} H_0^{(1)}(k_{\perp} \tau) \beta_{\perp}^{(1)}(\tau, \vec{k}_{\perp}) \right) \\ &= -\frac{2}{\pi} i \beta_{\perp}^{(1)}(\tau = 0, \vec{k}_{\perp}) = \frac{2i}{\pi k_{\perp}} b_2(\vec{k}_{\perp}).\end{aligned}\quad (45)$$

Correspondingly, the contribution from the η component is given by

$$\mathfrak{S}_{\eta}^{(1)}(\tau, \vec{k}_{\perp}) = -\frac{4}{\pi} i \frac{\beta(\tau = 0, \vec{k}_{\perp})}{k_{\perp}} = -\frac{2i}{\pi k_{\perp}} b_1(\vec{k}_{\perp}). \quad (46)$$

Combining these equations together, we conclude that the single inclusive gluon distribution to this order is given by

$$\begin{aligned}E_k \frac{dN}{d^3k} &= \frac{1}{4\pi^3} \left[\beta_i(\tau = 0, \vec{k}_{\perp}) \mathfrak{t}_{ij}(\vec{k}_{\perp}) \beta_j(\tau = 0, -\vec{k}_{\perp}) \right. \\ &\quad \left. + \frac{4}{k_{\perp}^2} \beta(\tau = 0, \vec{k}_{\perp}) \beta(\tau = 0, -\vec{k}_{\perp}) \right],\end{aligned}\quad (47)$$

where $\mathfrak{t}_{ij}(\vec{k}_{\perp})$ is the two-dimensional transverse projector

$$\mathfrak{t}_{ij}(\vec{k}_{\perp}) = \delta_{ij} - \frac{k_i k_j}{k_{\perp}^2}. \quad (48)$$

This expression is manifestly symmetric under $\vec{k}_{\perp} \rightarrow -\vec{k}_{\perp}$. To match this result to the one derived previously [23], we rewrite Eq. (47) in the form

$$E_k \frac{dN}{d^3k} = \frac{1}{4\pi^2} (|a_1|^2 + |a_2|^2) \quad (49)$$

where

$$a_1 = \frac{g}{\sqrt{\pi} k_{\perp}} \int d^2 x_{\perp} e^{-i\vec{k}_{\perp} \cdot \vec{x}_{\perp}} U^{\dagger}(\vec{x}_{\perp}) [\alpha_1^{(1)i}(\vec{x}_{\perp}), \alpha_2^i(\vec{x}_{\perp})] U(\vec{x}_{\perp}), \quad (50)$$

$$a_2 = \frac{1}{\sqrt{\pi} k_{\perp}} \int d^2 x_{\perp} e^{-i\vec{k}_{\perp} \cdot \vec{x}_{\perp}} \epsilon_{ij} \partial_j U^{\dagger}(\vec{x}_{\perp}) \alpha_1^{(1)i}(\vec{x}_{\perp}) U(\vec{x}_{\perp}). \quad (51)$$

This coincides with Dumitru-McLerran result [23] modulo an irrelevant complex phase in the definition of $a_{1,2}$.

Further simplifications are possible if explicitly expand the weak field into a power series in ρ_1 ²

$$\alpha_1^{(1)i}(\vec{x}_\perp) = \partial^i \Phi_1(\vec{x}_\perp) = g \frac{\partial^i}{\partial_\perp^2} \rho_1(\vec{x}_\perp). \quad (52)$$

Substituting to Eq. (49) we obtain

$$\begin{aligned} E_k \frac{dN}{d^3k} &= \frac{1}{8\pi^3 k_\perp^2} (\delta_{ij} \delta_{lm} + \epsilon_{ij} \epsilon_{lm}) \Omega_{ij}^b(\vec{k}_\perp) \left[\Omega_{lm}^b(\vec{k}_\perp) \right]^* \\ &= \frac{g^2}{8\pi^3 k_\perp^2} (\delta_{ij} \delta_{lm} + \epsilon_{ij} \epsilon_{lm}) \\ &\quad \times \int \frac{d^2 p_\perp}{(2\pi)^2} \frac{d^2 q_\perp}{(2\pi)^2} \frac{p_{\perp,i} (k-p)_{\perp,j}}{p_\perp^2} \frac{q_{\perp,l} (k-q)_{\perp,m}}{q_\perp^2} \\ &\quad \times \rho_a^*(\vec{q}_\perp) \left[W^\dagger(\vec{k}_\perp - \vec{q}_\perp) W(\vec{k}_\perp - \vec{p}_\perp) \right]_{ab} \rho_b(\vec{p}_\perp), \end{aligned} \quad (53)$$

where we introduced the Fourier transforms of the components of Eq. (29)

$$\Omega_{ij}^b(\vec{k}_\perp) = g \int \frac{d^2 p_\perp}{(2\pi)^2} \frac{p_{\perp,i} (k-p)_{\perp,j}}{p_\perp^2} \rho_a(\vec{p}_\perp) W_{ba}(\vec{k}_\perp - \vec{p}_\perp) \quad (54)$$

to simplify the notation in the coming section. In Appendix A, we provide yet another alternative form of Eq. (49).

6. Second order

At second order we expect some non-trivial modification of the particle production owing to the presence of non-trivial currents

$$j^{(2)}(\tau, \vec{x}_\perp) = -ig \left(\partial_i [\beta_i^{(1)}(\tau, \vec{x}_\perp), \beta^{(1)}(\tau, \vec{x}_\perp)] \right. \quad (55)$$

$$\left. + [\beta_i^{(1)}(\tau, \vec{x}_\perp), \partial_i \beta^{(1)}(\tau, \vec{x}_\perp)] \right), \quad (56)$$

$$j_i^{(2)}(\tau, \vec{x}_\perp) = -\partial_i \partial_j \beta_j^{(2)}(\tau, \vec{x}_\perp) - ig \left(\partial_j [\beta_j^{(1)}(\tau, \vec{x}_\perp), \beta_i^{(1)}(\tau, \vec{x}_\perp)] \right. \quad (57)$$

$$\left. + [\beta_j^{(1)}(\tau, \vec{x}_\perp), \partial_j \beta_i^{(1)}(\tau, \vec{x}_\perp) - \partial_i \beta_j^{(1)}(\tau, \vec{x}_\perp)] \right. \\ \left. - \tau^2 [\beta^{(1)}(\tau, \vec{x}_\perp), \partial_i \beta^{(1)}(\tau, \vec{x}_\perp)] \right).$$

²See Eq. (3).

Note that to this order, we do not have to solve the equations of motion for $\beta^{(2)}$. The contribution of the currents to particle production is solely defined by combinations of $\beta^{(1)}$ except for the term proportional to the gradient of the divergence of $\beta^{(2)}$ (the first term in Eq. (57)). Fortunately this term does not contribute to the transverse current $j_\perp$ and thus drops out from the particle production equations, see Eq. (37) and Eq. (40). This becomes obvious in momentum space

$$\begin{aligned}
j^{(2)}(\tau, \vec{k}_\perp) &= g \int \frac{d^2 q}{(2\pi)^2} [(2\vec{k}_\perp - \vec{q}_\perp) \cdot \vec{\beta}^{(1)}(\tau, \vec{q}_\perp), \beta^{(1)}(\tau, \vec{k}_\perp - \vec{q}_\perp)], \\
j_\perp^{(2)}(\tau, \vec{k}_\perp) &= g \int \frac{d^2 q}{(2\pi)^2} \left(i \left[(2\vec{k}_\perp - \vec{q}_\perp) \cdot \vec{\beta}^{(1)}(\tau, \vec{q}_\perp), \frac{\vec{\beta}^{(1)}(\tau, \vec{k}_\perp - \vec{q}_\perp) \times \vec{k}_\perp}{k_\perp} \right] + \right. \\
&\quad \left. i \frac{\vec{q}_\perp \times \vec{k}_\perp}{k_\perp} \left(\tau^2 [\beta^{(1)}(\tau, \vec{q}_\perp), \beta^{(1)}(\tau, \vec{k}_\perp - \vec{q}_\perp)] \right. \right. \\
&\quad \left. \left. + [\vec{\beta}^{(1)}(\tau, \vec{q}_\perp), \vec{\beta}^{(1)}(\tau, \vec{k}_\perp - \vec{q}_\perp)] \right) \right).
\end{aligned}$$

6.1. η -component of bulk contribution

The goal of this subsection is to compute

$$\mathfrak{B}_\eta^{(2)}(\vec{k}_\perp) = \int d\tau \tau^2 H_1^{(1)}(k_\perp \tau) j^{(2)}(\tau, \vec{k}_\perp). \quad (58)$$

For this we note two useful identities obtained based on the equations from Appendix B:

$$\begin{aligned}
\int d\tau \tau H_1^{(1)}(k_\perp \tau) J_0(q_\perp \tau) J_1(|\vec{k}_\perp - \vec{q}_\perp| \tau) &= \frac{1}{\pi} \frac{1}{|\vec{k}_\perp - \vec{q}_\perp| k_\perp} \times \\
&\left(\frac{(\vec{k}_\perp - \vec{q}_\perp) \cdot \vec{k}_\perp}{|\vec{q}_\perp \times \vec{k}_\perp|} - i \right), \quad (59)
\end{aligned}$$

$$\int d\tau \tau H_1^{(1)}(k_\perp \tau) J_1(|\vec{k}_\perp - \vec{q}_\perp| \tau) = i \frac{2}{\pi} \frac{|\vec{k}_\perp - \vec{q}_\perp|}{k_\perp} \frac{1}{k_\perp^2 - |\vec{k}_\perp - \vec{q}_\perp|^2}. \quad (60)$$

Thus the bulk contribution for the η -component at second order reads

$$\begin{aligned}
\mathfrak{B}_\eta^{(2)}(\vec{k}_\perp) &= \frac{2ig}{\pi k_\perp} \int \frac{d^2 q}{(2\pi)^2} \left(\frac{\vec{k}_\perp \times \vec{q}_\perp}{q^2 |\vec{k}_\perp - \vec{q}_\perp|^2} \left(\frac{(\vec{k}_\perp - \vec{q}_\perp) \cdot \vec{k}_\perp}{|\vec{q}_\perp \times \vec{k}_\perp|} - i \right) \times \right. \\
&\quad \left. [b_2(\vec{q}_\perp), b_1(\vec{k}_\perp - \vec{q}_\perp)] + i[\Lambda(\vec{q}_\perp), b_1(\vec{k}_\perp - \vec{q}_\perp)] \right).
\end{aligned}$$

6.2. Transverse vector-component of current

Analogously, using the integrals from the Appendix B we get

$$\begin{aligned}
\mathfrak{B}_\perp^{(2)}(\vec{k}_\perp) &= \int d\tau \tau H_0^{(1)}(k_\perp \tau) j_\perp^{(2)}(\tau, \vec{k}_\perp) \\
&= g \frac{2}{\pi k_\perp} \int \frac{d^2 q}{(2\pi)^2} \left(\frac{1}{2} \vec{k}_\perp \times \vec{q}_\perp [\Lambda(\vec{q}_\perp), \Lambda(\vec{k}_\perp - \vec{q}_\perp)] \right. \\
&\quad - \frac{\vec{k}_\perp \cdot \vec{q}_\perp}{q_\perp^2} [b_2(\vec{q}_\perp), \Lambda(\vec{k}_\perp - \vec{q}_\perp)] \\
&\quad - i \frac{1}{2} \frac{\vec{k}_\perp \times \vec{q}_\perp}{|\vec{k}_\perp \times \vec{q}_\perp|} \frac{k_\perp^2 + \vec{q}_\perp \cdot (\vec{q}_\perp - \vec{k}_\perp)}{q_\perp^2 |\vec{k}_\perp - \vec{q}_\perp|^2} [b_2(\vec{q}_\perp), b_2(\vec{k}_\perp - \vec{q}_\perp)] \\
&\quad \left. + \frac{1}{2} \frac{\vec{k}_\perp \times \vec{q}_\perp}{q_\perp^2 |\vec{k}_\perp - \vec{q}_\perp|^2} \left(1 + i \frac{\vec{q}_\perp \cdot (\vec{k}_\perp - \vec{q}_\perp)}{|\vec{k}_\perp \times \vec{q}_\perp|} \right) [b_1(\vec{q}_\perp), b_1(\vec{k}_\perp - \vec{q}_\perp)] \right) .
\end{aligned} \tag{61}$$

Although the last equation is complicated, we expect significant simplifications for the asymmetric part, as we will demonstrate below.

6.3. Surface contributions

To obtain the final equation for particle production we have to derive the surface contributions as well. They are

$$\mathfrak{S}_\eta^{(2)}(\vec{k}) = -i \frac{4}{\pi} \frac{\beta^{(2)}(\tau = 0, \vec{k}_\perp)}{k_\perp} \tag{62}$$

and

$$\mathfrak{S}_\perp^{(2)}(\vec{k}) = -i \frac{2}{\pi} \beta_\perp^{(2)}(\tau = 0, \vec{k}_\perp), \tag{63}$$

where the functions are defined by the second-order expansion coefficient in the weak field of the initial conditions

$$\beta^{(2)}(\tau \rightarrow 0, \vec{x}_\perp) = \frac{ig}{2} U^\dagger(\vec{x}_\perp) [\alpha_1^{(2)i}(\vec{x}_\perp), \alpha_2^i(\vec{x}_\perp)] U(\vec{x}_\perp), \tag{64}$$

$$\beta_i^{(2)}(\tau \rightarrow 0, \vec{x}_\perp) = U^\dagger(\vec{x}_\perp) \alpha_1^{(2)i}(\vec{x}_\perp) U(\vec{x}_\perp). \tag{65}$$

Figure 1: Illustration of the surface and bulk contributions to the single inclusive gluon production.

Here the weak proton field at second order is given by ³

$$\alpha_1^{(2)i} = \left(\delta_{ij} - \frac{\partial_i \partial_j}{\partial^2} \right) [\partial^j \Phi_1(\vec{x}_\perp), \Phi_1(\vec{x}_\perp)] . \quad (66)$$

6.4. Odd azimuthal anisotropy on event-by-event basis

The goal of this section is to show that the single inclusive particle production configuration-by-configuration (before performing the average with respect to ρ_1 and ρ_2) has odd azimuthal harmonics at second order. This can be straight-forwardly shown using the results obtained in the previous sections. We however prefer to use the following line of argumentation. Lets consider the single inclusive cross-section

$$E_k \frac{dN}{d^3k} = \sum_{\gamma=\eta,\perp} (a_\gamma^{(1)}(\vec{k}_\perp) + a_\gamma^{(2)}(\vec{k}_\perp)) (a_\gamma^{(1)}(\vec{k}_\perp) + a_\gamma^{(2)}(\vec{k}_\perp))^* . \quad (67)$$

As we discussed previously the first order is entirely defined by the surface contribution, i.e. $a_\gamma^{(1)}(\vec{k}_\perp) = \mathfrak{S}_\gamma^{(1)}(\vec{k}_\perp)$ with the following property for $\mathfrak{S}_\gamma^{(1)}(\vec{k}_\perp)$

$$\mathfrak{S}_\gamma^{(1)}(\vec{k}_\perp) = -(\mathfrak{S}_\gamma^{(1)}(-\vec{k}_\perp))^* . \quad (68)$$

³See Eq. (3).

An analogous relation holds also for second order

$$\mathfrak{S}_\gamma^{(2)}(\vec{k}_\perp) = -(\mathfrak{S}_\gamma^{(2)}(-\vec{k}_\perp))^* . \quad (69)$$

The asymmetric part of the single inclusive production is

$$\begin{aligned} & \frac{E_k}{2} \left(\frac{dN(\vec{k}_\perp)}{d^3k} - \frac{dN(-\vec{k}_\perp)}{d^3k} \right) \\ &= \frac{1}{2} \sum_{\gamma=\eta,\perp} (a_\gamma^{(1)}(\vec{k}_\perp) + a_\gamma^{(2)}(\vec{k}_\perp))(a_\gamma^{(1)}(\vec{k}_\perp) + a_\gamma^{(2)}(\vec{k}_\perp))^* \\ & - \frac{1}{2} \sum_{\gamma=\eta,\perp} (a_\gamma^{(1)}(-\vec{k}_\perp) + a_\gamma^{(2)}(-\vec{k}_\perp))(a_\gamma^{(1)}(-\vec{k}_\perp) + a_\gamma^{(2)}(-\vec{k}_\perp))^* \\ &= \Re \left((\mathfrak{S}_\gamma^{(1)}(\vec{k}_\perp))^* \left[\mathfrak{B}_\gamma^{(2)}(\vec{k}_\perp) + (\mathfrak{B}_\gamma^{(2)}(-\vec{k}_\perp))^* + \mathfrak{S}_\gamma^{(2)}(\vec{k}_\perp) + (\mathfrak{S}_\gamma^{(2)}(\vec{k}_\perp))^* \right] \right) + \mathcal{O}(\rho_1^4) \\ &= \Re \left((\mathfrak{S}_\gamma^{(1)}(\vec{k}_\perp))^* \left[\mathfrak{B}_\gamma^{(2)}(\vec{k}_\perp) + (\mathfrak{B}_\gamma^{(2)}(-\vec{k}_\perp))^* \right] \right) + \mathcal{O}(\rho_1^4). \end{aligned} \quad (70)$$

The surface contribution in the square bracket cancels, as we alluded to before. In order to compute the bulk contribution in the square brackets, lets go back and consider Eq. (38). Since the functions $\beta(\vec{x}_\perp)$ and $\beta_\perp(\vec{x}_\perp)$ are real we have

$$\mathfrak{B}_\eta^{(2)}(\vec{k}_\perp) + (\mathfrak{B}_\eta^{(2)}(-\vec{k}_\perp))^* = 2 \int_0^\infty d\tau \tau^2 J_1(k_\perp \tau) j^{(2)}(\tau, \vec{k}_\perp) \quad (71)$$

and

$$\mathfrak{B}_\perp^{(2)}(\vec{k}_\perp) + (\mathfrak{B}_\perp^{(2)}(-\vec{k}_\perp))^* = 2 \int_0^\infty d\tau \tau J_0(k_\perp \tau) j_\perp^{(2)}(\tau, \vec{k}_\perp) . \quad (72)$$

The cancellation of the Neumann functions simplifies the computation of the right-hand side

$$\begin{aligned} & \mathfrak{B}_\eta^{(2)}(\vec{k}_\perp) + (\mathfrak{B}_\eta^{(2)}(-\vec{k}_\perp))^* \\ &= \frac{4g}{k_\perp \pi} \int \frac{d^2q}{(2\pi)^2} \frac{\vec{k}_\perp \times \vec{q}_\perp}{|\vec{k}_\perp \times \vec{q}_\perp|} \frac{(\vec{k}_\perp - \vec{q}_\perp) \cdot \vec{k}_\perp}{q_\perp^2 |\vec{k}_\perp - \vec{q}_\perp|^2} i[b_1(\vec{q}_\perp), b_2(\vec{k}_\perp - \vec{q}_\perp)] \end{aligned} \quad (73)$$

and

$$\begin{aligned}
& \mathfrak{B}_\perp^{(2)}(\vec{k}_\perp) + (\mathfrak{B}_\perp^{(2)}(-\vec{k}_\perp))^* = \\
& -\frac{2g}{k_\perp \pi} \int \frac{d^2 q}{(2\pi)^2} \frac{\vec{k}_\perp \times \vec{q}_\perp}{|\vec{k}_\perp \times \vec{q}_\perp|} \frac{1}{q_\perp^2 |\vec{k}_\perp - \vec{q}_\perp|^2} \\
& \left((k_\perp^2 + \vec{q}_\perp \cdot (\vec{q}_\perp - \vec{k}_\perp)) i [b_2(\vec{q}_\perp), b_2(\vec{k}_\perp - \vec{q}_\perp)] \right. \\
& \left. - \vec{q}_\perp \cdot (\vec{k}_\perp - \vec{q}_\perp) i [b_1(\vec{q}_\perp), b_1(\vec{k}_\perp - \vec{q}_\perp)] \right). \tag{74}
\end{aligned}$$

It is remarkable that the gauge field $\Lambda(\vec{k}_\perp)$ does not contribute to this expression. Both expressions in Eqs. (73) and (74) are non-local owing to the presence of the ratio

$$\frac{\vec{k}_\perp \times \vec{q}_\perp}{|\vec{k}_\perp \times \vec{q}_\perp|} = \text{sign} \left[\sin(\phi_{\angle(\vec{k}_\perp, \vec{q}_\perp)}) \right]. \tag{75}$$

Summing everything up, the odd contribution is given by the following

$$\begin{aligned}
& \frac{E_k}{2} \left(\frac{dN(\vec{k}_\perp)}{d^3 k} - \frac{dN(-\vec{k}_\perp)}{d^3 k} \right) = \\
& \Re \left(\frac{4ig}{\pi^2 k_\perp^2} b_2^*(\vec{k}_\perp) \int \frac{d^2 q}{(2\pi)^2} \frac{\vec{k}_\perp \times \vec{q}_\perp}{|\vec{k}_\perp \times \vec{q}_\perp|} \frac{1}{q_\perp^2 |\vec{k}_\perp - \vec{q}_\perp|^2} \right. \\
& \left((k_\perp^2 + \vec{q}_\perp \cdot (\vec{q}_\perp - \vec{k}_\perp)) i [b_2(\vec{q}_\perp), b_2(\vec{k}_\perp - \vec{q}_\perp)] \right. \\
& \left. - \vec{q}_\perp \cdot (\vec{k}_\perp - \vec{q}_\perp) i [b_1(\vec{q}_\perp), b_1(\vec{k}_\perp - \vec{q}_\perp)] \right) + \\
& \left. + \frac{8ig}{\pi^2 k_\perp^2} b_1^*(\vec{k}_\perp) \int \frac{d^2 q}{(2\pi)^2} \frac{\vec{k}_\perp \times \vec{q}_\perp}{|\vec{k}_\perp \times \vec{q}_\perp|} \frac{(\vec{k}_\perp - \vec{q}_\perp) \cdot \vec{k}_\perp}{q_\perp^2 |\vec{k}_\perp - \vec{q}_\perp|^2} i [b_1(\vec{q}_\perp), b_2(\vec{k}_\perp - \vec{q}_\perp)] \right) = \\
& = \Im \left\{ \frac{2g}{\pi^2 k_\perp^2} \int \frac{d^2 q}{(2\pi)^2} \frac{\vec{k}_\perp \times \vec{q}_\perp}{|\vec{k}_\perp \times \vec{q}_\perp|} \frac{1}{q_\perp^2 |\vec{k}_\perp - \vec{q}_\perp|^2} \right. \\
& \times f^{abc} \Omega_{ij}^a(\vec{q}_\perp) \Omega_{mn}^b(\vec{k}_\perp - \vec{q}_\perp) \Omega_{rp}^{c*}(\vec{k}_\perp) \times \\
& \left[\left(k_\perp^2 \epsilon^{ij} \epsilon^{mn} - \vec{q}_\perp \cdot (\vec{k}_\perp - \vec{q}_\perp) (\epsilon^{ij} \epsilon^{mn} + \delta^{ij} \delta^{mn}) \right) \epsilon^{rp} \right. \\
& \left. \left. + 2\vec{k}_\perp \cdot (\vec{k}_\perp - \vec{q}_\perp) \delta^{ij} \epsilon^{mn} \delta^{rp} \right] \right\}, \tag{76}
\end{aligned}$$

where Ω is defined in Eq. (54).

7. Double inclusive gluon production in leading log

The double inclusive gluon production at the leading log approximation reads [24]

$$E_k E_q \frac{d \bar{N}}{d^3 k d^3 q} = \left\langle E_k \frac{dN}{d^3 k} E_q \frac{dN}{d^3 q} \right\rangle , \quad (77)$$

where the average is performed over the target and the projectile fields. In a Gaussian ensemble, the average removes all contributions odd in ρ_1 .

To simplify the notation we define

$$E_k \frac{dN}{d^3 k} = n^{(2)}(\vec{k}_\perp) + n^{(3)}(\vec{k}_\perp) + n^{(4)}(\vec{k}_\perp) + \dots , \quad (78)$$

where according to previously used definitions

$$n^{(2)}(\vec{k}_\perp) = \sum_{\gamma=\eta,\perp} |a_\gamma^{(1)}(\vec{k}_\perp)|^2 , \quad (79)$$

$$n^{(3)}(\vec{k}_\perp) = \sum_{\gamma=\eta,\perp} a_\gamma^{(1)}(\vec{k}_\perp) \left(a_\gamma^{(2)}(\vec{k}_\perp) \right)^* + \text{c.c.} , \quad (80)$$

$$n^{(4)}(\vec{k}_\perp) = \sum_{\gamma=\eta,\perp} \left[a_\gamma^{(1)}(\vec{k}_\perp) \left(a_\gamma^{(3)}(\vec{k}_\perp) \right)^* + a_\gamma^{(3)}(\vec{k}_\perp) \left(a_\gamma^{(1)}(\vec{k}_\perp) \right)^* + |a_\gamma^{(2)}(\vec{k}_\perp)|^2 \right] . \quad (81)$$

As we established earlier, to the leading order the cross-section is symmetric configuration-by-configuration

$$n^{(2)}(\vec{k}_\perp) = n^{(2)}(-\vec{k}_\perp). \quad (82)$$

In addition, the condition that

$$E_k E_q \frac{d \bar{N}}{d^3 k d^3 p}(\vec{k}_\perp, \vec{p}_\perp) = E_k E_q \frac{d \bar{N}}{d^3 k d^3 p}(-\vec{k}_\perp, -\vec{p}_\perp) \quad (83)$$

leads to

$$\left\langle \left(n_\gamma^{(4)}(\vec{k}_\perp) - n_\gamma^{(4)}(-\vec{k}_\perp) \right) n_\gamma^{(2)}(\vec{p}_\perp) \right\rangle = 0 . \quad (84)$$

This guarantees that the contribution to the odd asymmetry depends only on $n^{(3)}$ computed in the previous section. Indeed

$$\frac{E_k E_q}{2} \left(\frac{d\bar{N}}{d^3 k d^3 q}(\vec{k}_\perp, \vec{p}_\perp) - \frac{d\bar{N}}{d^3 k d^3 p}(-\vec{k}_\perp, \vec{p}_\perp) \right) = \frac{1}{2} \left\langle \left(n_\gamma^{(3)}(\vec{k}_\perp) - n_\gamma^{(3)}(-\vec{k}_\perp) \right) n_\gamma^{(3)}(\vec{p}_\perp) \right\rangle. \quad (85)$$

In this notation, the difference $\frac{1}{2} \left(n_\gamma^{(3)}(\vec{k}_\perp) - n_\gamma^{(3)}(-\vec{k}_\perp) \right)$ is given entirely by Eq. (76). This contribution is non-vanishing and gives rise to odd azimuthal anisotropy. It is obviously connected to the initial state distribution of the color charges, but has some non-local dependence on spatial points. Most importantly this contribution comes from the evolution of the field in the forward light cone and is not just defined by the initial conditions on the light-cone as at the leading order.

To proceed further we will consider an expression asymmetrized both with respect to $\vec{k}_\perp$ and $\vec{q}_\perp$:

$$\frac{1}{4} \left(n_\gamma^{(3)}(\vec{k}_\perp) - n_\gamma^{(3)}(-\vec{k}_\perp) \right) \left(n_\gamma^{(3)}(\vec{p}_\perp) - n_\gamma^{(3)}(-\vec{p}_\perp) \right). \quad (86)$$

As we established in the previous section, each difference is proportional to the imaginary part of some function f , i.e

$$\begin{aligned} \frac{1}{2} \left(n_\gamma^{(3)}(\vec{p}_\perp) - n_\gamma^{(3)}(-\vec{p}_\perp) \right) &= \Im f(\vec{p}_\perp) = \frac{1}{2i} (f(\vec{p}_\perp) - f^*(\vec{p}_\perp)) \\ &= \frac{1}{2i} (f(\vec{p}_\perp) - f(-\vec{p}_\perp)). \end{aligned}$$

Thus for our purpose, we can just equate $f(\vec{p}_\perp) = i n_\gamma^{(3)}(\vec{p}_\perp)$. This assumption is not true in general but is sufficient for the current calculations of the asymmetric part

$$\begin{aligned} \frac{1}{4} \left(n_\gamma^{(3)}(\vec{k}_\perp) - n_\gamma^{(3)}(-\vec{k}_\perp) \right) \left(n_\gamma^{(3)}(\vec{p}_\perp) - n_\gamma^{(3)}(-\vec{p}_\perp) \right) &= \\ -\frac{1}{4} (f(\vec{p}_\perp) - f(-\vec{p}_\perp)) \left(f(\vec{k}_\perp) - f(-\vec{k}_\perp) \right) &= \\ -\frac{1}{4} \left(\left[f(\vec{p}_\perp) f(\vec{k}_\perp) - (\vec{k}_\perp \rightarrow -\vec{k}_\perp) \right] - (\vec{p}_\perp \rightarrow -\vec{p}_\perp) \right). \end{aligned} \quad (87)$$

Therefore it is enough to consider only one term, e.g. $f(\vec{p}_\perp) f(\vec{k}_\perp)$; the rest of the terms can be obtained by changing the direction of momenta. In

Appendix C we derived the expression for $\langle f(\vec{p}_\perp) f(\vec{k}_\perp) \rangle_{\rho_1}$. It has fifteen different terms and must be further averaged with respect to the target field. This would generate over 125 terms only for $\langle f(\vec{p}_\perp) f(\vec{k}_\perp) \rangle_{\rho_1, \rho_2}$. At this point we see that the only reasonable resolution would be to perform numerical simulations where averages with respect to the projectile and target configurations are performed using Monte-Carlo technique. We postpone this for further publications.

8. Summary and Conclusions

Here we briefly summarize our results and provide some comments.

1. The surface contribution on light cone gives zero odd azimuthal anisotropy to all orders. It is T-even and can be written in a local form.
2. The odd harmonics originate from evolution in the forward light cone. They are non-local and not T-even. In single particle inclusive process they average out to zero for a Gaussian ensemble because they are proportional to ρ_1^3 . Essentially they are defined by odderon exchanges.
3. We were unable to establish the connection between our formulae and geometric anisotropy in the initial state ϵ_3 . From the equation it is obvious that the anisotropy is not defined by the global scales, but rather by the geometry on the scales of $1/Q_s$.
4. The argument presented in Ref. [25, 26] is valid only for the surface contribution in the dilute approximation. We showed that the bulk contribution for configuration-by-configuration single inclusive result is not symmetric under $\vec{k}_\perp \rightarrow -\vec{k}_\perp$.
5. Our results take into account the first saturation correction, which was also considered in Ref. [27].
6. We complement the numerical results of Refs. [16, 19] with an analytical proof beyond any doubts in numerics and uncertainty in the prescription of what is defined by a gluon at an intermediate state, τ , that CYM produces odd azimuthal anisotropy.
7. Our results can be potentially used to calculate v_3 without solving CYM numerically.

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10. Appendix A: Leading order results in coordinate space

Equation (53) can be rewritten in an alternative form. Lets consider the combination

$$\begin{aligned} & \frac{\delta_{ij}\delta_{lm} + \epsilon_{ij}\epsilon_{lm}}{k_{\perp}^2} \frac{p_{\perp,i}(k-p)_{\perp,j}}{p_{\perp}^2} \frac{q_{\perp,l}(k-q)_{\perp,m}}{q_{\perp}^2} \\ &= \frac{\vec{p}_{\perp} \cdot (\vec{k}_{\perp} - \vec{p}_{\perp}) \vec{q}_{\perp} \cdot (\vec{k}_{\perp} - \vec{q}_{\perp}) + (\vec{p}_{\perp} \times \vec{k}_{\perp}) \cdot (\vec{q}_{\perp} \times \vec{k}_{\perp})}{k_{\perp}^2 p_{\perp}^2 q_{\perp}^2}. \end{aligned} \quad (88)$$

The last expression can be further simplified using the identity

$$(\vec{p}_{\perp} \times \vec{k}_{\perp}) \cdot (\vec{q}_{\perp} \times \vec{k}_{\perp}) = (\vec{p}_{\perp} \cdot \vec{q}_{\perp}) k_{\perp}^2 - (\vec{p}_{\perp} \cdot \vec{k}_{\perp}) (\vec{q}_{\perp} \cdot \vec{k}_{\perp}) \quad (89)$$

which can be proven starting from the identity

$$(\vec{k}_{\perp} \times \vec{u}_{\perp})^2 = k_{\perp}^2 u_{\perp}^2 - (\vec{k}_{\perp} \cdot \vec{u}_{\perp})^2 \quad (90)$$

and proceeding by substituting $\vec{u}_{\perp} = \vec{p}_{\perp} + \vec{q}_{\perp}$. Thus

$$\begin{aligned} & \frac{\delta_{ij}\delta_{lm} + \epsilon_{ij}\epsilon_{lm}}{k_{\perp}^2} \frac{p_{\perp,i}(k-p)_{\perp,j}}{p_{\perp}^2} \frac{q_{\perp,l}(k-q)_{\perp,m}}{q_{\perp}^2} \\ &= \left(\frac{\vec{k}_{\perp}}{k_{\perp}^2} - \frac{\vec{p}_{\perp}}{p_{\perp}^2} \right) \cdot \left(\frac{\vec{k}_{\perp}}{k_{\perp}^2} - \frac{\vec{q}_{\perp}}{q_{\perp}^2} \right). \end{aligned} \quad (91)$$

Substituting this into Eq. (53) we get

$$\begin{aligned} E_k \frac{dN}{d^3k} &= \frac{g^2}{(2\pi)^3} \int \frac{d^2 p_{\perp}}{(2\pi)^2} \frac{d^2 q_{\perp}}{(2\pi)^2} \left(\frac{\vec{k}_{\perp}}{k_{\perp}^2} - \frac{\vec{p}_{\perp}}{p_{\perp}^2} \right) \cdot \left(\frac{\vec{k}_{\perp}}{k_{\perp}^2} - \frac{\vec{q}_{\perp}}{q_{\perp}^2} \right) \\ &\quad \times \rho_a^*(\vec{p}_{\perp}) \left[W^\dagger(\vec{k}_{\perp} - \vec{p}_{\perp}) W(\vec{k}_{\perp} - \vec{q}_{\perp}) \right]^{ab} \rho_b(\vec{q}_{\perp}) \end{aligned} \quad (92)$$

or in coordinate space

$$E_k \frac{dN}{d^3k} = \frac{\alpha_s}{\pi} \int_u \int_v \int_x \int_y e^{i\vec{k}_{\perp}(\vec{u}_{\perp} - \vec{v}_{\perp})} \frac{\vec{v} - \vec{y}}{|\vec{v} - \vec{y}|^2} \frac{\vec{x} - \vec{u}}{|\vec{v} - \vec{x}|^2} \times \quad (93)$$

$$\rho^a(\vec{x}_{\perp}) \left([W^\dagger(\vec{x}_{\perp}) - W^\dagger(\vec{u}_{\perp})] [W(\vec{y}_{\perp}) - W(\vec{v}_{\perp})] \right)^{ab} \rho^b(\vec{y}_{\perp}). \quad (94)$$

11. Appendix B: List of useful integrals and relations

Here we collect the list of useful integrals and relations. The some integrals are adopted from more general ones of Ref. [28]

$$\int d\tau \tau J_\nu(p_\perp \tau) J_\nu(k_\perp \tau) = \frac{\delta(p_\perp - k_\perp)}{k_\perp}, \quad (95)$$

$$\int d\tau \tau J_0(p_\perp \tau) Y_0(k_\perp \tau) = \frac{2}{\pi} \frac{1}{k_\perp^2 - p_\perp^2}, \quad (96)$$

$$\int d\tau \tau J_1(p_\perp \tau) Y_1(k_\perp \tau) = \frac{2}{\pi} \frac{p_\perp}{k_\perp} \frac{1}{k_\perp^2 - p_\perp^2}, \quad (97)$$

$$\int d\tau \tau H_0^{(1)}(k_\perp \tau) J_0(|\vec{q}_\perp - \vec{k}_\perp| \tau) J_0(q_\perp \tau) = \frac{1}{\pi} \frac{1}{|\vec{k}_\perp \times \vec{q}_\perp|}, \quad (98)$$

$$\int d\tau \tau J_1(q_\perp \tau) J_0(|\vec{q}_\perp - \vec{k}_\perp| \tau) Y_1(k_\perp \tau) = -\frac{1}{\pi} \frac{1}{q_\perp k_\perp}, \quad (99)$$

$$\int d\tau \tau J_1(q_\perp \tau) J_1(|\vec{q}_\perp - \vec{k}_\perp| \tau) Y_0(k_\perp \tau) = \frac{1}{\pi} \frac{1}{q_\perp |\vec{k}_\perp - \vec{q}_\perp|}, \quad (100)$$

$$\int d\tau \tau J_1(q_\perp \tau) J_1(k_\perp \tau) J_0(|\vec{k}_\perp - \vec{q}_\perp| \tau) = \frac{1}{\pi q k} \frac{\vec{q}_\perp \cdot \vec{k}_\perp}{|\vec{q}_\perp \times \vec{k}_\perp|}, \quad (101)$$

$$\begin{aligned} & \int d\tau \tau J_1(q_\perp \tau) J_1(|\vec{k}_\perp - \vec{q}_\perp| \tau) J_0(k_\perp \tau) \\ &= \frac{1}{\pi q |\vec{k}_\perp - \vec{q}_\perp|} \frac{\vec{q}_\perp \cdot (\vec{q}_\perp - \vec{k}_\perp)}{|\vec{q}_\perp \times \vec{k}_\perp|}. \end{aligned} \quad (102)$$

For completeness we also list the limits used in the main text

$$\lim_{\tau \rightarrow 0} \frac{J_1(k_\perp \tau)}{k_\perp \tau} = \frac{1}{2}, \quad (103)$$

$$\lim_{\tau \rightarrow 0} \tau \partial_\tau H_0^{(1,2)}(k_\perp \tau) = \pm i \frac{2}{\pi}, \quad (104)$$

$$\lim_{\tau \rightarrow 0} \tau^3 \partial_\tau \tau^{-1} H_1^{(1,2)}(k_\perp \tau) = \pm i \frac{4}{\pi k_\perp} \quad (105)$$

and the identity connecting the transverse projector and antisymmetric symbols

$$a_i b_j \mathbf{t}_{ij}(\vec{k}_\perp) = \frac{\epsilon_{ij} k_j a_i}{k_\perp} \frac{\epsilon_{nm} k_n b_m}{k_\perp}. \quad (106)$$

12. Appendix C: Average with respect to projectile configurations in MV model

Lets consider the following combination averaged with respect to the projectile field in the MV model

$$\begin{aligned}
\Omega_{ij,lm}^{a,b}(\vec{p}_\perp, \vec{q}_\perp) &\equiv \langle \Omega_{ij}^a(\vec{p}_\perp) \Omega_{lm}^b(\vec{q}_\perp) \rangle_{\rho_1} = \\
&= g^2 \int \frac{d^2 u}{(2\pi)^2} \int \frac{d^2 v}{(2\pi)^2} \frac{u_i(p-u)_j v_l(q-v)_m}{u_\perp^2 v_\perp^2} \langle \rho_1^\alpha(\vec{u}_\perp) \rho_1^\beta(\vec{v}_\perp) \rangle_{\rho_1} \\
&\quad \times W_{a\alpha}(\vec{p}_\perp - \vec{u}_\perp) W_{b\beta}(\vec{q}_\perp - \vec{v}_\perp) = \\
&= g^2 \int \frac{d^2 u}{(2\pi)^2} \mu^2(\vec{u}_\perp) \frac{u_i(u+p)_j u_l(u-q)_m}{u_\perp^4} \\
&\quad \times [W(\vec{u}_\perp + \vec{p}_\perp) W^\dagger(\vec{u}_\perp - \vec{q}_\perp)]_{ab}, \tag{107}
\end{aligned}$$

where we used the MV correlator

$$\langle \rho_1^\alpha(\vec{u}_\perp) \rho_1^\beta(\vec{v}_\perp) \rangle_{\rho_1} = (2\pi)^2 \mu^2(\vec{u}_\perp) \delta(\vec{u}_\perp + \vec{v}_\perp). \tag{108}$$

We also use the notation from Sec. 7

$$\begin{aligned}
f(\vec{k}_\perp) &= \frac{2g}{\pi^2 k_\perp^2} \int \frac{d^2 q}{(2\pi)^2} \frac{\vec{k}_\perp \times \vec{q}_\perp}{|\vec{k}_\perp \times \vec{q}_\perp|} \frac{1}{q_\perp^2 |\vec{k}_\perp - \vec{q}_\perp|^2} \\
&\quad \times f^{abc} \Omega_{ij}^a(\vec{q}_\perp) \Omega_{mn}^b(\vec{k}_\perp - \vec{q}_\perp) \Omega_{rp}^{c*}(\vec{k}_\perp) \times \\
&\quad \left[\left(k_\perp^2 \epsilon^{ij} \epsilon^{mn} - \vec{q}_\perp \cdot (\vec{k}_\perp - \vec{q}_\perp) (\epsilon^{ij} \epsilon^{mn} + \delta^{ij} \delta^{mn}) \right) \epsilon^{rp} \right. \\
&\quad \left. + 2 \vec{k}_\perp \cdot (\vec{k}_\perp - \vec{q}_\perp) \delta^{ij} \epsilon^{mn} \delta^{rp} \right]. \tag{109}
\end{aligned}$$

Using Eq. (107) we can rewrite the projectile averaged combination $\langle f(\vec{p}_\perp) f(\vec{k}_\perp) \rangle_{\rho_1}$ as follows

$$\begin{aligned}
\langle f(\vec{p}_\perp) f(\vec{k}_\perp) \rangle_{\rho_1} &= \left(\frac{2g}{\pi^2} \right)^2 \frac{1}{p_\perp^2 k_\perp^2} \int \frac{d^2 q}{(2\pi)^2} \int \frac{d^2 q'}{(2\pi)^2} \frac{\vec{p}_\perp \times \vec{q}_\perp}{|\vec{p}_\perp \times \vec{q}_\perp|} \frac{\vec{k}_\perp \times \vec{q}'_\perp}{|\vec{k}_\perp \times \vec{q}'_\perp|} \times \\
&\quad \frac{f^{abc}}{q_\perp^2 |\vec{p}_\perp - \vec{q}_\perp|^2} \frac{f^{a'b'c'}}{q_\perp'^2 |\vec{k}_\perp - \vec{q}'_\perp|^2} \times \\
&\quad \left[(p_\perp^2 \epsilon^{ij} \epsilon^{mn} - \vec{q}_\perp \cdot (\vec{p}_\perp - \vec{q}_\perp) (\epsilon^{ij} \epsilon^{mn} + \delta^{ij} \delta^{mn})) \epsilon^{rp} \right. \\
&\quad \left. + 2\vec{p}_\perp \cdot (\vec{p}_\perp - \vec{q}_\perp) \delta^{ij} \epsilon^{mn} \delta^{rp} \right] \times \\
&\quad \left[\left(k_\perp^2 \epsilon^{i'j'} \epsilon^{m'n'} - \vec{q}'_\perp \cdot (\vec{k}_\perp - \vec{q}'_\perp) (\epsilon^{i'j'} \epsilon^{m'n'} + \delta^{i'j'} \delta^{m'n'}) \right) \epsilon^{r'p'} \right. \\
&\quad \left. + 2\vec{k}_\perp \cdot (\vec{k}_\perp - \vec{q}'_\perp) \delta^{i'j'} \epsilon^{m'n'} \delta^{r'p'} \right] \times \\
&\quad \left[\Omega_{ij,mn}^{a,b}(\vec{q}_\perp, \vec{k}_\perp - \vec{q}_\perp) \left(\Omega_{rp,i'j'}^{c,a'}(-\vec{p}_\perp, \vec{q}'_\perp) \Omega_{m'n',r'p'}^{b',c'}(\vec{k}_\perp - \vec{q}'_\perp, -\vec{k}_\perp) + \right. \right. \\
&\quad \Omega_{rp,m'n'}^{c,b'}(-\vec{p}_\perp, \vec{k}_\perp - \vec{q}'_\perp) \Omega_{i'j',r'p'}^{a',c'}(\vec{q}'_\perp, -\vec{k}_\perp) + \\
&\quad \left. \Omega_{rp,r'p'}^{c,c'}(-\vec{p}_\perp, -\vec{k}_\perp) \Omega_{i'j',m'n'}^{a',b'}(\vec{q}'_\perp, \vec{k}_\perp - \vec{q}'_\perp) \right) \\
&\quad \Omega_{ij,rp}^{a,c}(\vec{q}_\perp, -\vec{p}_\perp) \left(\Omega_{mn,i'j'}^{b,a'}(\vec{p}_\perp - \vec{p}_\perp, \vec{q}'_\perp) \Omega_{m'n',r'p'}^{b',c'}(\vec{k}_\perp - \vec{q}'_\perp, -\vec{k}_\perp) + \right. \\
&\quad \Omega_{mn,m'n'}^{b,b'}(\vec{p}_\perp - \vec{q}_\perp, \vec{k}_\perp - \vec{q}'_\perp) \Omega_{i'j',r'p'}^{a',c'}(\vec{q}'_\perp, -\vec{k}_\perp) + \\
&\quad \left. \Omega_{mn,r'p'}^{b,c'}(\vec{p}_\perp - \vec{q}_\perp, -\vec{k}_\perp) \Omega_{i'j',m'n'}^{a',b'}(\vec{q}'_\perp, \vec{k}_\perp - \vec{q}'_\perp) \right) + \\
&\quad \Omega_{ij,i'j'}^{a,a'}(\vec{q}_\perp, \vec{q}'_\perp) \left(\Omega_{mn,rp}^{b,c}(\vec{p}_\perp - \vec{p}_\perp, -\vec{p}_\perp) \Omega_{m'n',r'p'}^{b',c'}(\vec{k}_\perp - \vec{q}'_\perp, -\vec{k}_\perp) + \right. \\
&\quad \Omega_{mn,m'n'}^{b,b'}(\vec{p}_\perp - \vec{q}_\perp, \vec{k}_\perp - \vec{q}'_\perp) \Omega_{rp,r'p'}^{c,c'}(-\vec{p}_\perp, -\vec{k}_\perp) + \\
&\quad \left. \Omega_{mn,r'p'}^{b,c'}(\vec{p}_\perp - \vec{q}_\perp, -\vec{k}_\perp) \Omega_{rp,m'n'}^{c,b'}(-\vec{p}_\perp, \vec{k}_\perp - \vec{q}'_\perp) \right) + \\
&\quad \Omega_{ij,m'n'}^{a,b'}(\vec{q}_\perp, \vec{k}_\perp - \vec{q}'_\perp) \left(\Omega_{mn,rp}^{b,c}(\vec{p}_\perp - \vec{p}_\perp, -\vec{p}_\perp) \Omega_{i'j',r'p'}^{a',c'}(\vec{q}'_\perp, -\vec{k}_\perp) + \right. \\
&\quad \Omega_{mn,i'j'}^{b,a'}(\vec{p}_\perp - \vec{q}_\perp, \vec{q}'_\perp) \Omega_{rp,r'p'}^{c,c'}(-\vec{p}_\perp, -\vec{k}_\perp) + \\
&\quad \left. \Omega_{mn,r'p'}^{b,c'}(\vec{p}_\perp - \vec{q}_\perp, -\vec{k}_\perp) \Omega_{rp,i'j'}^{c,a'}(-\vec{p}'_\perp, \vec{q}'_\perp) \right) + \\
&\quad \Omega_{ij,r'p'}^{a,c'}(\vec{q}_\perp, -\vec{k}_\perp) \left(\Omega_{mn,rp}^{b,c}(\vec{p}_\perp - \vec{p}_\perp, -\vec{p}_\perp) \Omega_{i'j',m'n'}^{a',b'}(\vec{q}'_\perp, \vec{k}_\perp - \vec{q}'_\perp) + \right. \\
&\quad \Omega_{mn,i'j'}^{b,a'}(\vec{p}_\perp - \vec{q}_\perp, \vec{q}'_\perp) \Omega_{rp,m'n'}^{c,b'}(-\vec{p}_\perp, \vec{k}_\perp - \vec{q}'_\perp) + \\
&\quad \left. \Omega_{mn,m'n'}^{b,b'}(\vec{p}_\perp - \vec{q}_\perp, \vec{k}_\perp - \vec{q}'_\perp) \Omega_{rp,i'j'}^{c,a'}(-\vec{p}'_\perp, \vec{q}'_\perp) \right) \Big]. \tag{110}
\end{aligned}$$

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