Note on predictions for $c\bar{s}$ quarkonia using a three-loop static potential

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We extend our treatment of the spectroscopy and decays of the charm-strange quarkonium system to include the effect of using the full three-loop QCD correction to the static short distance potential. As before, our potential model consists of the relativistic kinetic energy term, a scalar linear confining term including its relativistic corrections and the perturbative QCD spin-dependent terms. A set of unperturbed wave functions for the various states is obtained using a variational technique that is further constrained by requiring that the wave functions also satisfy the relativistic virial theorem. These are then used in a perturbative treatment of the potential to fit the mass spectrum of the $c\bar{s}$ system and calculate the radiative decay widths. Our results accurately describe the D_s spectrum and are compatible with the little data that is available for the radiative decays of the D_s states.

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I. INTRODUCTION

We revisit our earlier treatment [1] of the spectroscopy and radiative decays of the $c\bar{s}$ system by including the three-loop QCD short range static potential [2–4] in the unperturbed Hamiltonian. This is an extension to unequal mass quarkonia of the three-loop treatment of equal mass quarkonia in reference [5]. We use a variational approach [1, 5–7] to determine a set of unperturbed wave functions. This technique results in a set of linear equations for the coefficients of the trial wave functions. The associated energies are strong upper bounds on lowest $^{2S+1}L_J$ levels and accurate approximations of the excited state energies. These wave functions are used to calculate perturbative contributions to the spin-dependent energy levels of the $c\bar{s}$ spectrum. The resulting energies are then compared with a subset of experimental values of the $c\bar{s}$ spectrum using a χ^2 test that is further constrained by requiring that the relativistic version of the virial theorem [8, 9] is satisfied. Using minimization software [10, 11] to vary the parameters of the calculation, this procedure is repeated until a minimum in the χ^2 function is found.

Potential models have frequently been used to study the properties of the D_s system. These include a recent extensive treatment by Godfrey and Moats [12] as well as many others [13–18].

II. MODIFIED SEMIRELATIVISTIC MODEL

For the unperturbed Hamiltonian, we used a semirelativistic model of the form

$$H_0 = \sqrt{\vec{p}^2 + m_1^2} + \sqrt{\vec{p}^2 + m_2^2} + Ar + V(r), \quad (1)$$

where A is the linear coupling strength, and $V(r)$ is the complete three-loop QCD short range static potential. The expression for $V(r)$ can be found in reference [5]. The perturbative potential can be written as

$$H' = V_L + V_S. \quad (2)$$

V_L contains the v^2/c^2 corrections to the linear confining potential and is

$$V_L = -\frac{A}{4r} \left[\left(\frac{1}{m_1^2} + \frac{1}{m_2^2} \right) \vec{L} \cdot \vec{S} + \left(\frac{1}{m_1^2} - \frac{1}{m_2^2} \right) \vec{L} \cdot (\vec{S}_1 - \vec{S}_2) \right]. \quad (3)$$

V_S , which includes both the v^2/c^2 and one-loop corrections, has the form

$$V_S = V_{HF} + V_{LS} + V_T + V_{SI} + V_{MIX}. \quad (4)$$

These perturbative potentials are similar to those in [1], but the first order α_S terms have been transformed to be consistent with the $\overline{MS}$ renormalization scheme. This transformation between α_S and $\bar{\alpha}_S$ is given by

$$\alpha_S = \bar{\alpha}_S \left[1 + \frac{\bar{\alpha}_S}{4\pi} \left(\frac{49}{3} - \frac{10}{9}n_f \right) \right], \quad (5)$$

where n_f is the number of light quark flavors [19]. Hence, the terms of the short distance potential can be written as

$$V_{HF} = \frac{32\bar{\alpha}_S\vec{S}_1\cdot\vec{S}_2}{9m_1m_2} \left\{ \left[1 + \frac{\bar{\alpha}_S}{4\pi} \left(\frac{31}{3} - \frac{10}{9}n_f \right) \right] \delta(\vec{r}) - \frac{\bar{\alpha}_S}{24\pi^2} (33 - 2n_f) \nabla^2 \left[\frac{\ln \mu r + \gamma_E}{r} \right] - \frac{5\bar{\alpha}_S}{3\pi} \delta(\vec{r}) \right. \\ \left. - \frac{\bar{\alpha}_S}{8\pi} \left(8 \frac{m_1 - m_2}{m_1 + m_2} + \frac{m_1 + m_2}{m_1 - m_2} \right) \ln \left(\frac{m_2}{m_1} \right) \delta(\vec{r}) + \frac{21\bar{\alpha}_S}{16\pi^2} \nabla^2 \left[\frac{\ln(m_1m_2)^{1/2}r + \gamma_E}{r} \right] \right\} \quad (6a)$$

$$V_{LS} = \frac{\bar{\alpha}_S\vec{L}\cdot\vec{S}}{3m_1^2m_2^2r^3} \left\{ [(m_1 + m_2)^2 + 2m_1m_2] \left[1 + \frac{\bar{\alpha}_S}{4\pi} \left(\frac{31}{3} - \frac{10}{9}n_f \right) + \frac{\bar{\alpha}_S}{6\pi} (33 - 2n_f)(\ln(\mu r) + \gamma_E - 1) \right] \right. \\ \left. + \frac{\bar{\alpha}_S}{\pi} (m_1 + m_2)^2 \left[\frac{4}{3} - 3(\ln(m_1m_2)^{1/2}r + \gamma_E - 1) \right] - \frac{3\bar{\alpha}_S}{2\pi} (m_1^2 - m_2^2) \ln \left(\frac{m_2}{m_1} \right) \right\} \quad (6b)$$

$$V_T = \frac{4\bar{\alpha}_S(3\vec{S}_1\cdot\hat{r}\vec{S}_2\cdot\hat{r} - \vec{S}_1\cdot\vec{S}_2)}{3m_1m_2r^3} \left\{ 1 + \frac{\bar{\alpha}_S}{4\pi} \left(\frac{31}{3} - \frac{10}{9}n_f \right) + \frac{\bar{\alpha}_S}{6\pi} (33 - 2n_f) \left(\ln \mu r + \gamma_E - \frac{4}{3} \right) + \frac{17\bar{\alpha}_S}{6\pi} \right. \\ \left. - \frac{3\bar{\alpha}_S}{\pi} \left(\ln(m_1m_2)^{1/2}r + \gamma_E - \frac{4}{3} \right) \right\} \quad (6c)$$

$$V_{SI} = \frac{2\pi\bar{\alpha}_S}{3} \left(\frac{1}{m_1^2} + \frac{1}{m_2^2} \right) \left\{ \left[1 + \frac{\bar{\alpha}_S}{4\pi} \left(\frac{31}{3} - \frac{10}{9}n_f \right) \right] \delta(\vec{r}) - \frac{\bar{\alpha}_S}{24\pi^2} (33 - 2n_f) \nabla^2 \left[\frac{\ln \mu r + \gamma_E}{r} \right] \right. \\ \left. - \frac{\bar{\alpha}_S}{6\pi r^2} \left[\frac{9(m_1 + m_2)^2 - 8m_1m_2}{m_1m_2(m_1 + m_2)} \right] \right\} \quad (6d)$$

$$V_{MI} = -\frac{\bar{\alpha}_S\vec{L}\cdot(\vec{S}_1 - \vec{S}_2)}{3m_1^2m_2^2r^3} \left\{ (m_1^2 - m_2^2) \left[1 + \frac{\bar{\alpha}_S}{4\pi} \left(\frac{31}{3} - \frac{10}{9}n_f \right) + \frac{\bar{\alpha}_S}{6\pi} (33 - 2n_f)(\ln \mu r + \gamma_E - 1) \right] \right. \\ \left. + \frac{\bar{\alpha}_S}{\pi} (m_1^2 - m_2^2) \left[\frac{4}{3} - 3(\ln(m_1m_2)^{1/2}r + \gamma_E - 1) \right] - \frac{3\bar{\alpha}_S}{2\pi} (m_1 + m_2)^2 \ln \left(\frac{m_2}{m_1} \right) \right\}. \quad (6e)$$

Here, m_1 denotes the charm quark mass, m_2 denotes the strange quark mass, γ_E is the Euler constant and μ is the renormalization scale.

These terms satisfy the Gromes' constraints derived from infinitesimal Lorentz invariance [20]. In calculating the matrix elements in equations (6a) and (6d), the $\delta(\vec{r})$ terms are "softened" by using the quasistatic approximation from [14]. Specifically, the substitution

$$\delta(\vec{r}) \rightarrow \frac{\omega^2}{\pi r} e^{-2\omega r} \quad (7)$$

is made, where $\omega^2 = 2m_1^2m_2^2/(m_1^2 + m_2^2)$. This softening helps stabilize the eigenvalue calculation.

III. OVERVIEW OF CALCULATIONS

To obtain the $c\bar{s}$ mass spectrum, we first used a variational approach [5] to find a set of trial wave functions. These trial wave functions are given by

$$\psi_{j\ell s}^m(\vec{r}) = \sum_{k=0}^N C_k \left(\frac{r}{R} \right)^{k+\ell} e^{-r/R} \mathcal{Y}_{j\ell s}^m(\Omega), \quad (8)$$

where $\mathcal{Y}_{j\ell s}^m(\Omega)$ is the spin-angular wave function for total angular momentum j , orbital angular momentum ℓ , and total spin s . The C_k 's are found by approximating a solution to the Schrödinger equation, $H_0|\psi\rangle = E|\psi\rangle$. By inserting a complete set of eigenstates and multiplying from the left by $\langle\psi_k|$ we get the following

$$\sum_{k'=0}^{\infty} \langle\psi_k|H_0|\psi_{k'}\rangle \langle\psi_{k'}|\psi\rangle = E \sum_{k'=0}^{\infty} \langle\psi_k|\psi_{k'}\rangle \langle\psi_{k'}|\psi\rangle \quad (9)$$

This can be approximated by truncating the series to n terms, which gives

$$\sum_{k'=0}^N H_0(k, k') C_{k'} = E \sum_{k'=0}^N N(k, k') C_{k'}, \quad (10)$$

where the kinetic energy terms in $H_0(k, k')$ are calculated using the Fourier transform of Eq. (8) and $N(k, k')$ is the overlap integral of the radial wave functions. Our results were obtained using $N = 14$. Equation (10) can be solved as a generalized eigenvalue problem.

The normalized eigenvectors obtained from equation (10) represent the wave functions of the unperturbed Hamiltonian. These wave functions are, of course, sensitive to the value of R , which sets their length scale. To increase the reliability of the unperturbed wave functions, we minimize the wave function with respect to R , which is equivalent to requiring that the relativistic virial theorem [8, 9]

$$\langle p \frac{dH_0}{dp} \rangle - \langle r \frac{dH_0}{dr} \rangle = 0, \quad (11)$$

where H_0 is the unperturbed Hamiltonian given in equation (1), be satisfied [7]. In the present calculation, the virial theorem is evaluated for all wave functions after the χ^2 minimization. Its vanishing is satisfied to within 1%, which gives us confidence that our wave functions are fully optimized variational wave functions of the unperturbed Hamiltonian.

We then calculate the first order perturbations from equation (2) and add them to the unperturbed energies. We also take into account the mixing between the 1P_1 and 3P_1 as well as the 1D_2 and 3D_2 eigenstates of the $c\bar{s}$ Hamiltonian due to the $\vec{L} \cdot (\vec{S}_1 - \vec{S}_2)$ terms in Eqs. (3) and (6e) of the perturbative potential. This mixing yields the two $J = 1$ states D_{s1} and D'_{s1} for the p states and a similar pair for the d states. The details of this calculation can be found in reference [1].

These calculations resulted in a test spectrum, which is compared to the experimentally known spectrum [21] using a χ^2 function given by

$$\chi^2 = \sum_{i=1}^N \frac{(\mathcal{O}_{\text{exp } i} - \mathcal{O}_{\text{th}}(\alpha)_i)^2}{\sigma_i^2}, \quad (12)$$

where the $\mathcal{O}_i$ are experimental and theoretical values of some D_s observable with σ_i as the associated error, and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is the set of parameters in the Hamiltonian. An intrinsic theoretical error of 1.0 MeV was added in quadrature to the error terms in the χ^2 function to account for higher-order loop corrections. The test spectrum calculation was repeated as the parameters were varied by the search programs, MINUIT [10] and STEPIT [11], until an optimal set of parameters was found for a minimum χ^2 value. The final values of the parameters from these two searches are in good agreement.

IV. RESULTS

The procedure outlined above results in the set of parameters listed in Table I. The resulting spectrum using

A (GeV ²)	0.1118
$\bar{\alpha}_S$	0.3209
m_C (GeV)	1.855
m_S (GeV)	0.3378
μ (GeV)	0.8223
R (GeV ⁻¹)	0.5725

TABLE I. Fitted parameters for the $c\bar{s}$ system are shown. We also include an additive constant to fix the value of the ground state.

the fitted parameters in Table I is given in Table II, which includes those states of mass < 3.1 GeV. The $1s$ and $1p$ states are quite adequately described by the fit. We have taken into account the singlet-triplet mixing

resulting from Eqs. [3] and [6e] in the p and d states. The p state mixing angles are $\theta_1 = -60.8^\circ$, $\theta_2 = -57.9^\circ$, $\theta_3 = -57.2^\circ$ and the d state mixing angle is $\phi_1 = 40.2^\circ$. There are two other D_{sJ} states that have been observed but are not yet included in the Summary Tables of Ref.[21]. One is the $D_{sJ}(2860)$, which is known to have natural J^P and, if interpreted as a $c\bar{s}$ d -state [22], could be either a $1D_s(^3D_3)$ or a $1D_s(^3D_1)$ state. Recent LHCb results indicate that there are two overlapping states at this mass [23]. From Table II, both the $1D_s(^3D_1)$ and the $1D_s(^3D_3)$ masses are on the low side. The other state denoted as $D_{sJ}(3040)$ has no information on J^P . If J^P in this case is natural, this state could be identified with the $2D_s(^3P_2)$ in Table II. We have also examined the $^3S_1 - ^3D_1$ mixings from the tensor interaction Eq. [6c] and these turn out to be negligibly small.

$m_{c\bar{s}}$ (MeV)	Theory	[12]	Expt
$1D_s$	1968.4	1979	1968.30 ± 0.11
$1D_s^*$	2112.8	2129	2112.1 ± 0.4
$1D_{s0}(2317)$	2317.2	2484	2317.7 ± 0.6
$1D_{s1}(2460)$	2458.2	2549	2459.5 ± 0.6
$1D'_{s1}(2536)$	2534.1	2556	2535.10 ± 0.08
$1D_{s2}(2573)$	2575.8	2592	2571.9 ± 0.8
$1D_s(^3D_1)$	2830.2	2899	
$1D_s(^1-^3D_2)$	2751.6	2900	
$1D_s(^1-^3D'_2)$	2875.6	2926	
$1D_s(^3D_3)$	2776.1	2917	
$2D_s(^1S_0)$	2500.7	2673	
$2D_s(2700)$	2654.0	2732	$2709. \pm 4$
$2D_s(^3P_0)$	2634.4	3005	
$2D_s(^1-^3P_1)$	2799.1	3018	
$2D_s(^1-^3P'_1)$	2961.5	3038	
$2D_s(^3P_2)$	3009.3	3048	
$3D_s(^1S_0)$	2913.0	3154	
$3D_s(^3S_1)$	3051.7	3193	
$3D_s(^3P_0)$	2916.5	3412	
$3D_s(^1-^3P_1)$	3092.2	3416	

TABLE II. Our results for the D_s spectrum for $m_{c\bar{s}} < 3.1$ GeV are shown and compared with those of Ref.[12]. The fit uses the $n = 1$ s and p D_s states. Only one of the $n = 2$ states has been confirmed [21].

The predicted radiative decays of the D_s system can be found in Table III where they are compared with our earlier results [1] and those of reference [12]. They were calculated in the same manner as those in reference [1]. In particular, the mixed 1P_1 and 3P_1 states that define the $D_{s1}(2460)$ and $D'_{s1}(2536)$ states are

$$|D_{s1}(2460)\rangle = \sin(\theta)|^3P_1\rangle + \cos(\theta)|^1P_1\rangle \quad (13a)$$

$$|D'_{s1}(2536)\rangle = \cos(\theta)|^3P_1\rangle - \sin(\theta)|^1P_1\rangle. \quad (13b)$$

The corresponding electric dipole decay rates will contain an additional $\cos^2(\theta)$ or $\sin^2(\theta)$ factor depending on the final state.

V. CONCLUSION

We have shown that a potential model consisting of the relativistic kinetic energy, a linear long-range confining potential together with its scalar exchange v^2/c^2 relativistic corrections, and the full v^2/c^2 plus three-loop QCD corrected short distance potential is capable of providing extremely good fits to the spectra of the D_s states by treating them as states of the $c\bar{s}$ system. Our approach is based on determining the parameters of the potential using the well established $n = 1$ s and p states by minimizing χ^2 and constraining the radius R using the virial theorem. Over all, the $n = 1$ d states are predicted to be less massive than those of Ref. [12]. Our $n = 2$ and

Γ_γ (keV)	[1]	3loop	[12]	Expt
$D_s^* \rightarrow D_s \gamma$	1.12	2.39	1.03	$< 1.9 \times 10^3$
$D_{s0}(2317) \rightarrow D_s^* \gamma$	3.37	5.46	9.01	
$D_{s1}(2460) \rightarrow D_s \gamma$	8.30	13.2	15.2	BR = 0.18 ± 0.04
$D_{s1}(2460) \rightarrow D_s^* \gamma$	11.0	17.4	5.40	BR < 0.08
$D_{s1}(2460) \rightarrow D_{s0}(2317) \gamma$	4.70	5.63		
$D'_{s1}(2536) \rightarrow D_s \gamma$	37.7	61.2	9.23	
$D'_{s1}(2536) \rightarrow D_s^* \gamma$	5.74	9.21	9.61	possibly seen
$D'_{s1}(2536) \rightarrow D_{s0}(2317) \gamma$	5.62	7.75		
$D_{s2}(2575) \rightarrow D_s^* \gamma$	30.5	49.6	18.9	
$D_s^*(2658) \rightarrow D_{s0}(2317) \gamma$	5.64	8.77	0.91	
$D_s^*(2658) \rightarrow D_{s1}(2460) \gamma$	2.66	4.25	0.56	
$D_s^*(2658) \rightarrow D'_{s1}(2536) \gamma$	0.26	0.41	0.85	
$D_s^*(2658) \rightarrow D_{s2}(2573) \gamma$	0.48	0.71	1.55	
$D_s(2503) \rightarrow D_{s1}(2460) \gamma$	0.04	0.05	1.18	

TABLE III. The the radiative decays of the D_s mesons are shown. These widths are computed using the mass values obtained directly from our calculation. This includes the $n = 2$ pseudoscalar and vector states, the latter of which has recently been observed with a higher mass [21]. The experimental widths are from [21].

$n = 3$ hyperfine splittings are about the same as the ground state hyperfine splitting and the $n = 3$ levels have several states with masses below 3.1 GeV, unlike Ref. [12]. The mixing angles differ probably because we include the v^2/c^2 spin-orbit corrections from the scalar confining potential.

The single photon widths can be obtained from the variational wave functions, but, apart from some branching ratio measurements, there are relatively little data available. A comparison to our previous one-loop results [1] shows that the energy-level calculations and the electric dipole (E_1) matrix elements are not greatly affected, though there are minor differences. However, the calculated radiative decay widths show a systematic increase, in some cases dramatic. The E_1 decay rates are particularly sensitive to the m_1/m_2 ratio due to the coupling factor $(q_1 m_1 - q_2 m_2)^2 / (m_1 + m_2)^2$, where $q_1 = 2/3$ and $q_2 = 1/3$. The fit using the three loop potential results in a larger ratio that accounts for the differences in the E_1 decay rates.

In every case, efforts to model these states will be greatly improved by the availability of additional data.

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