Knots in Electromagnetism

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Abstract

Maxwell equations in vacuum allow for solutions with a non-trivial topology in the electric and magnetic field line configurations at any given moment in time. One example is a space filling congruence of electric and magnetic field lines forming circles lying on the surfaces of nested tori. In this example the electric, magnetic and Poynting vector fields are orthogonal everywhere. As time evolves the electric and magnetic fields expand and deform without changing the topology and energy, while the Poynting vector structure remains unchanged while propagating with the speed of light. The topology is characterized by the concept of helicity of the field configuration. Helicity is an important fundamental concept and for massless fields it is a conserved quantity under conformal transformations.

We will review several methods by which linked and knotted electromagnetic (spin-1) fields can be derived. A first method, introduced by A. Rañada, uses the formulation of the Maxwell equations in terms of differential forms combined with the Hopf map from the three-sphere S^3 to the two-sphere S^2 . A second method is based on spinor and twistor theory developed by R. Penrose in which elementary twistor functions correspond to the family of electromagnetic torus knots. A third method uses the Bateman construction of generating null solutions from complex Euler potentials. And a fourth method uses special conformal transformations, in particular conformal inversion, to generate new linked and knotted field configurations from existing ones. This fourth method

is often accompanied by shifting singularities in the field to complex space-time points.

Of course the various methods must be closely related to one another although they have been developed largely independently and they suggest different directions in which to expand the study of topologically non-trivial field configurations. It will be shown how the twistor formulation allows for a direct extension to massless fields of other spin values, such as spin-2 fields satisfying the linearized Einstein vacuum equation, and how the formulation by A. Rañada can be extended to fields for which the electric and magnetic fields are not orthogonal everywhere.

Underlying the various methods is the fact that electric and magnetic field lines can be described as the level curves of complex functions. Compactification of R^3 naturally leads to finite energy solutions because the fields at infinity in all directions should all converge towards zero. An intriguing question that is raised by the finite energy is whether there is a connection to the quantization of the classical electromagnetic field. We will review some issues related to this question.

Another interesting question is why the general formulation of topologically non-trivial solutions uses the electric and magnetic fields instead of the electromagnetic vector potentials. This leads to a discussion of the Clebsch representation of the electromagnetic field strength 2-form.

Finally, a topic of great interest is the possibility of experimentally generating and investigating linked and knotted field configurations. Since the non-trivial topological field solutions exploit the special conformal symmetry of the underlying vacuum wave-equations it will only be possible to approximate the solutions in an experiment, which necessarily introduces material objects that will break the special conformal symmetry. We will review the research on plasma configurations in which the magnetic field-line configuration approximates plasma torus knots leading to the prediction of topological solitons in plasma.

Keywords: Electromagnetic knots, topology, helicity, Hopf map, Clebsh representation, twistor, conformal symmetry, plasmas.

1. Introduction

Electric and magnetic fields were described by Faraday, Maxwell and others in the XIX century as vector fields that can be visualized using field lines or lines of force. A field line is a geometrical object, a curve in space, such that the
5 tangent line at each of its points is parallel to the vector field at that point. The lines of force, both electric and magnetic, were considered by Faraday to be real physical objects [1]. Faraday's original view was supported during a good part of the XIX century leading to attempts to explain the lines of force in terms of the streamlines and vorticity lines of the aether. It was in this context that in
10 1867 Kelvin proposed that atoms were knots or links of the vortex lines of the aether, a picture presented expressively in his paper called "On vortex atoms" [2]. This idea lost support in later years especially after the advances of the theory of relativity and quantum mechanics. In some sense string theory is a revival of the idea that elementary particles can be represented by linked and
15 knotted structures.

Topology have been, and still is, crucial in the development of the theory of electromagnetism. Topology concepts are already present even at very elementary stages. Gauss and Stokes' theorems are valid only if the surfaces are orientable. The property of orientability is a topological characteristic. In 1833
20 Gauss [3] considered two linked circuits and established the relation between the magnetic field induced by the currents along the circuits and a topological invariant known as the linking number. An extension of this concept led to the definition of magnetic helicity. Gauss had been fascinated by knots and in fact it was his student, Listing, who introduced the word "Topology" in the context
25 of the study of knots in 1847.

Topological arguments also played an essential role in Dirac's proposal of the monopole in 1931 [4], which implies a mechanism for the quantization of

the electric charge. Furthermore, since 1959, when Aharonov and Bohm [5] discovered the effect that bears their name, it is known that the description of certain electromagnetic phenomena does require topological considerations. Another interesting example of topology in physics is the Skyrme model [6]. Other developments from pure mathematics turned out to be related to Yang-Mills field theory, such as Witten's proposal of a topological quantum field theory [7].

In this review article we will be mainly dealing with developments in the application of topological concepts for the construction of linked and knotted electromagnetic (EM) field configurations. This development was largely initiated by two articles by A. F. Rañada on "Topological electromagnetism" in 1989 [8] and on "Knotted solutions of the Maxwell equations in vacuum" in 1990 [9] in which the Hopf map was used to formulate a linked EM field configuration. A precursor of this work was published by A. Trautman in 1977 in an article on "Solutions of the Maxwell and Yang-Mills equations associated with Hopf fibrings" [10]. As often happens in science, closely related earlier works written in a different context (here in the context of plasma physics [11]), or in a different mathematical formalism (here in the twistor formalism [12, 13, 14]) have been identified and will also be reviewed in this article.

Rañada considered solutions to the Maxwell equations in vacuum in which the magnetic and electric field lines are defined by the level curves of a pair of complex scalar fields $\phi(\mathbf{r}, t)$, $\theta(\mathbf{r}, t) : R^3 \times R \rightarrow C$:

$$\begin{aligned}\mathbf{B}(\mathbf{r}, t) &= \frac{\sqrt{a}}{2\pi i} \frac{\nabla\phi \times \nabla\bar{\phi}}{(1 + \bar{\phi}\phi)^2} = \frac{\sqrt{a}}{2\pi ic(1 + \bar{\theta}\theta)^2} \left(\frac{\partial\bar{\theta}}{\partial t} \nabla\theta - \frac{\partial\theta}{\partial t} \nabla\bar{\theta} \right), \\ \mathbf{E}(\mathbf{r}, t) &= \frac{\sqrt{ac}}{2\pi i} \frac{\nabla\bar{\theta} \times \nabla\theta}{(1 + \bar{\theta}\theta)^2} = \frac{\sqrt{a}}{2\pi i(1 + \bar{\phi}\phi)^2} \left(\frac{\partial\bar{\phi}}{\partial t} \nabla\phi - \frac{\partial\phi}{\partial t} \nabla\bar{\phi} \right),\end{aligned}\quad (1)$$

where $\bar{\phi}$ and $\bar{\theta}$ are the complex conjugates of ϕ and θ respectively, i is the imaginary unit, a is a constant introduced so that the magnetic and electric fields have correct dimensions, and c is the speed of light in vacuum.

Since compactified R^3 (R^3 combined with a single point representing infinity in all directions) is isomorphic to the three-sphere S^3 via stereographic projec-

tion and since the complex plane combined with a point at infinity is isomorphic to the two-sphere S^2 , there is a natural role for maps from S^3 to S^2 in describing the complex scalar fields $\phi(\mathbf{r}, t)$, $\theta(\mathbf{r}, t)$, provided two physical conditions are satisfied associated to the added point at infinity in R^3 and the added point at infinity in the complex plane. The first condition is that the scalar fields ϕ and θ in R^3 converge to a single value when approaching infinity in any direction. The natural choice is to take the value to be zero. This implies that such source-free fields have the desirable physical property of finite energy. The second condition is that field lines closest to the field line associated to the point at infinity in the complex plane must all become identical. This is possible if the field in R^3 has a central circle to which the nearby field lines degenerate down to. Hopf investigated maps from S^3 to S^2 in which closed loops are mapped to points. Different loops on S^3 can be linked to each other and knotted, leading after stereographic projection to linked and knotted fields in R^3 .

The Hopf-Rañada solution was obtained by the following complex scalar functions in which the Hopf map is implicit [15],

$$\begin{aligned}\phi &= \frac{(A X - T Z) + i(A Y + T(A - 1))}{(A Z + T X) + i(A(A - 1) - T Y)}, \\ \theta &= \frac{(A Y + T(A - 1)) + i(A Z + T X)}{(A X - T Z) + i(A(A - 1) - T Y)}.\end{aligned}\tag{2}$$

In these expressions, coordinates (X, Y, Z, T) are dimensionless and $A = (R^2 - T^2 + 1)/2$, with $R = X^2 + Y^2 + Z^2$. Per construction the EM field lines are linked loops filling the surfaces of nested tori. The electric field configuration is everywhere orthogonal to the magnetic field configuration. Two characteristic features, the energy distribution falling off to zero at infinity and the presence of a central circle, follow from the compactifications of R^3 and the complex plane as needed for the application of the Hopf map. Fig. 1 illustrates the field line configuration.

The Hopf-Rañada solution is a null field: the electric and magnetic fields are perpendicular and they have the same modulus in natural units. The topology is conserved during the time evolution, therefore the field lines remain linked

Figure 1: Structure of magnetic lines of the electromagnetic Hopfion. (a) Four magnetic lines at $T = 0$. Any pair of them is linked, and the linking number is always equal to 1. (b) The same situation occurs when $T = 0.2$.

and closed. The Gauss linking number for the fields coincides with the Hopf index, a number closely related to helicity which is an important concept in the characterization of fields with non-trivial topology [16].

85 The initial papers by Rañada and its subsequent refinements inspired the search for possible experimental routes for creating linked and knotted light beams, as well as the search for generalizations of the Hopf-Rañada field [17]. This search led to two new classes of torus knots. One class represents non-null fields in which helicity is exchanged between the electric and magnetic fields
90 [18]. The other class represents null-fields, characterized by the electric and magnetic fields being perpendicular and having the same modulus in natural units [19]. This class was obtained following the Bateman construction for generating EM null fields [20]. Furthermore, methods of deriving topologically non-trivial fields have been explored using conformal transformations of known
95 solutions combined with shifts of space-time parameters into the complex plane [21, 14].

Among the applications, the concept of linked and knotted magnetic field lines to magnetohydrodynamics has led to the suggestion that the ball lightning phenomenon could be explained in this way [22]. More recently self-organizing
100 plasma knots have been observed in numerical simulations, where the initial configuration contained helicity in the form of a few linked magnetic field rings [23]. It should be mentioned that the Hopf fibration and torus knots have also

generated a lot of activities in other research areas such as liquid crystals [24], Bose-Einstein condensates [25, 26] and hydrodynamics [27] but those topics are
105 outside the scope of this review.

The plan of this article is as follows. In Section 2, we review the concept of duality of the Maxwell theory in vacuum, which is important to understand the role played by the Bianchi identities in the derivation of the Hopf-Rañada field. In order to understand the relation between the equations that define an
110 electromagnetic knot and the field lines, we introduce the Clebsh representation of electromagnetic fields in vacuum. Then the helicity for the field lines is defined through the vector potentials. Helicity is a crucial concept for this review article and we have included Appendix A describing the relation between the Gauss integral, the linking number and the helicity. We show how the Euler
115 potentials provide a Clebsh representation of the fields and how they are defined in term of the scalar maps ϕ and θ . We present some examples, and show the local equivalence of the Rañada formulation and the solutions of the Maxwell equations in vacuum.

In Section 3 we discuss the general properties of the Rañada construction
120 when the scalar fields are restricted to applications from the three dimensional sphere S^3 to the two dimensional sphere S^2 . Topological properties of such maps are classified in homotopy classes using the Hopf index.

Section 4 gives a full account of the Hopf-Rañada field which was briefly presented in this introduction. We give the Euler potentials of the so-called
125 Hopfion, and study the properties of the magnetic surfaces which are the points where the Euler potentials take constant values. We will see the fibration of the Hopfion, that is the structure of all the magnetic and electric field lines at a given instant of time. We will see how the energy density spreads out and the motion of electric changes under its influence [28].

Section 5 is devoted to the non-null torus knotted class of electromagnetic
130 fields. They were found as an extension to the Hopf-Rañada solution [18], and in general the members of this class do not have electric and magnetic field perpendicular to each other (note as we will see that the Hopfion is a member

of this class and the Hopfion is a null field). Within this class, initially all the
135 magnetic lines and electric lines are linked torus knots. Torus knots are knotted
curves lying on the surface of a torus.

In Section 6 another class of fields that also has the torus topology for the
field lines is presented. However, this class is null, meaning that the electric and
magnetic fields are orthogonal. We present the Bateman construction, based on
140 complex Euler potentials in this section.

In Section 7 we provide a very different derivation of the Hopf-Rañada field
by applying special conformal transformations to very simple (but unphysical)
solutions, such as constant and orthogonal $\mathbf{E}$ and $\mathbf{B}$. The conformal trans-
formations form a 15 dimensional group which has the (6 dimensional) proper
145 Lorentz group and the (10 dimensional) Poincaré group as subgroups. The 5 spe-
cial conformal transformations are composed of a conformal inversion followed
by a (4-dimensional) translation followed by another conformal inversion. In
order to get a feeling for what special conformal transformations are let's look
at conformal inversion in R^3 . Conformal inversion for a vector in R^3 means
150 $\mathbf{x} \rightarrow \mathbf{x}/\mathbf{x} \cdot \mathbf{x}$. Let us take a plane in R^3 defined by $x = 1$ and a constant electric
field in this plane pointing in the z direction. After conformal inversion each
field line will turn into a circle that touches the origin and the collection of all
those circles will form a sphere that touches the origin and the point $(1, 0, 0)$.
Taking the same initial planes of constant $\mathbf{E}$ field but for different values of x
155 will produce a space filling congruence of 2 sets of nested spheres (for $x < 0$
and $x > 0$). All spheres touch in the origin at which the field strength becomes
infinite. If we had also started with an orthogonal constant $\mathbf{B}$ field in each
field (equal in magnitude in natural units to the $\mathbf{E}$ field and pointing in the
 y direction) we would get on each sphere a $\mathbf{B}$ field orthogonal to the $\mathbf{E}$ field
160 and equal in magnitude. If we consider $t = 0$ in Minkowski space we will have
the same situations as just described, up to a minus sign in the norm which
simply flips all field vectors. If we subsequently consider a time translation that
consists of a purely imaginary value, followed by another conformal inversion
one finds precisely the Hopf-Rañada electric field configuration [29, 21]. This

165 method of applying transformations that cannot be seen as just a change of
inertial reference frame combined with complex shifts is familiar to other disci-
plines. For example the Kerr solution can be obtained in a similar fashion from
the Schwarzschild solution and Gaussian beams can be generated from spherical
waves [30].

170 In Section 8 we review the twistor formulation of linked and knotted field
configurations. This method is part of the twistor theory developed by R.
Penrose and his colleagues over several decades. The applications of this theory
are wide and deep. The original goal of the theory is to unify general relativity
and quantum mechanics. Since light rays, following geodesics, define the space
175 time metric it would perhaps be advantageous to formulate the laws of physics
in a space, the twistor space, in which each point, a twistor, represents a light
ray (or a ray of another massless spin $n/2$ field). The norm of a twistor is its
helicity, which is invariant under the full conformal group. Solutions to the wave
equations can be derived from twistor functions of specific homogeneity followed
180 by the Penrose transform. In Appendix B we summarise the bare minimum of
the basics of twistor theory needed to understand the construction of the linked
and knotted massless field configurations. One of the strengths of the twistor
approach is that solutions to all massless wave equations are derived in the same
manner. Therefore we can directly identify linked and knotted solutions for the
185 spin 2 linearized Einstein equations in vacuum.

In Section 9 we make a connection between EM knots and plasma physics.
Since plasma is highly conductive, internal electric fields are generally small
while magnetic fields can be large and can strongly determine the dynamics
through the resulting Lorentz forces on moving charges. The equations of mag-
190 netohydrodynamics (MHD) describe the interplay between the magnetic forces
resulting from currents and pressure gradient forces. Kamchatnov showed how
a skeleton of a magnetic Hopfion (the magnetic part of the Hopf-Rañada field at
 $t = 0$) can result in a quasi-stable plasma configuration [11]. The configuration
is neither an equilibrium state nor a lowest energy ground state, however it is
195 still robust against decay because conservation of energy and angular momen-

tum combined with the topological invariance of magnetic helicity (in an ideal dissipationless plasma) prevents the configuration from expanding and radiating away its energy. This so-called topological soliton in plasma can be generalized to torus knot magnetic skeletons but only if the poloidal winding number of the magnetic field equals unity [31]. The toroidal winding number can be any integer. If the poloidal winding number is unequal to unity the configuration has no net angular momentum and this elevates one of the constraints on the dynamics allowing the solution to decay.

We end with a summary and concluding remarks in which we address the question of how to generate knotted EM fields in the laboratory.

2. Duality, Rañada construction, Clebsch representation and Euler potentials

In this section we present the original Rañada construction [8] of EM fields in vacuum whose magnetic and electric field lines coincide with level curves of a pair of complex scalar fields $\phi(\mathbf{r}, t)$, $\theta(\mathbf{r}, t)$. Rañada's attempt to understand the role played by topology in the classical electromagnetic theory [8], developed later in different references [9, 32, 33, 15, 34, 35, 17, 28, 36, 19, 18], led him to propose a topological model of electromagnetism. One of the main characteristics of his model is that it allows to obtain topological quantization rules [37, 38, 39, 40, 41].

The Rañada model makes use of the property of duality that the Maxwell equations in vacuum possesses and uses the language of tensors and differential forms in the Minkowski space-time. We first review briefly those concepts. Then we will present the construction and discuss some of the properties of this formulation in terms of Clebsch variables. We will see the connection between the vector potentials and the electric and magnetic helicities. This is important as the helicity concept is related to the topology of the fields. Electromagnetic fields in vacuum can be expressed in terms of Euler potentials which are useful to describe the topology of field lines. We will show that the fields defined in terms of the scalars introduced by Rañada are locally equivalent to the class of

225 electromagnetic fields in vacuum that satisfy the orthogonal condition $\mathbf{E} \cdot \mathbf{B} = 0$.

2.1. Maxwell theory in vacuum

Here we summarise some known facts of Maxwell theory of electromagnetism in vacuum making use of tensors and differential forms in a space-time with a Minkowski metric.

230 There is a duality between the magnetic and the electric aspects of electromagnetism in vacuum. Using the SI of units, Maxwell's equations in vacuum are written as

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \quad (3)$$

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = 0, \quad (4)$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic vector fields. The first two equations are identities when the electromagnetic potentials V , $\mathbf{A}$ are introduced. We will take Greek indices $\mu, \nu, \dots = 0, 1, 2, 3$, for four-dimensional quantities in Minkowski space-time, while Latin indices $i, j, \dots = 1, 2, 3$, will be three-dimensional space components. Unless otherwise stated, we will take the Minkowski metric as $\eta_{\mu\nu} = \eta^{\mu\nu} = \langle 1, -1, -1, -1 \rangle$, where the notation $\langle a, b, \dots \rangle$ means a diagonal matrix with elements $a, b, \dots$. We will follow Einstein summation notation, so repeated indices are implicitly summed over.

240 The four-vector electromagnetic potential has components

$$A^\mu = \left(\frac{V}{c}, \mathbf{A} \right), \quad (5)$$

where V and $\mathbf{A}$ are the usual scalar potential and vector potential, respectively, and the electromagnetic field tensor is defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (6)$$

We will use the notation $\partial_\mu f = \partial f / \partial x^\mu$, so that $\partial_0 f = (1/c) \partial f / \partial t$, $\partial_1 f = \partial f / \partial x$, etc. From the electromagnetic field tensor (6), the electric and magnetic field components are $\mathbf{E}_i = c F^{i0}$ and $\mathbf{B}_i = -\epsilon_{ijk} F^{jk} / 2$, being $\epsilon_{ijk} = \epsilon^{0ijk}$ and

$\epsilon^{\mu\nu\alpha\beta}$ the totally antisymmetric tensor with $\epsilon^{0123} = 1$. In three-dimensional quantities, equation (6) is written as the pair of equations

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}. \quad (7)$$

Using the language of differential forms (a good introduction to differential forms and their applications to electromagnetism can be found in [42]), we can write the four-vector potential as the 1-form $A = A_\mu dx^\mu$, and the electromagnetic field by the 2-form

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu, \quad (8)$$

where $\wedge$ denotes the exterior product between forms. The exterior derivative of the 1-form A satisfies $dA = F$. Now, given the property $d^2 = 0$ of the exterior derivative,

$$dF = d^2 A = 0, \quad (9)$$

or in tensor components,

$$\epsilon^{\mu\nu\alpha\beta} \partial_\nu F_{\alpha\beta} = 0. \quad (10)$$

In three-dimensional form, equation (10) is written as the pair of equations (3). Consequently, these equations are purely geometrical identities, without dynamical content. Equation (9) is known as a Bianchi identity (for its geometrical meaning see [43]).

The dynamics of the electromagnetic field in vacuum is given by the second pair of Maxwell equations (4), that can be written as the tensor equation

$$\partial_\mu F^{\mu\nu} = 0, \quad (11)$$

or in the language of differential forms as

$$d^*F = 0, \quad (12)$$

where $*F$ is the dual of the form (8),

$$*F = \frac{1}{4} \epsilon_{\mu\nu\alpha\beta} F^{\mu\nu} dx^\alpha \wedge dx^\beta. \quad (13)$$

We can write $*F$ as

$$*F = \frac{1}{2} *F_{\mu\nu} dx^\mu \wedge dx^\nu, \quad (14)$$

where the dual of the electromagnetic field $F_{\mu\nu}$ is defined as

$$*F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}. \quad (15)$$

245 The components of the tensor $*F_{\mu\nu}$ are given by $\mathbf{B}_i = *F^{i0}$, $\mathbf{E}_i = c\epsilon_{ijk} *F^{jk}/2$. Thus electromagnetism in vacuum is given by a closed two-form $F = dA$ in the Minkowski space-time, constrained by the dynamical equation $d*F = 0$.

Let us examine the meaning of the duality property. Maxwell equations in vacuum are invariant under the replacement of the pair of fields $(\mathbf{E}, c\mathbf{B})$ by the pair $(c\mathbf{B}, -\mathbf{E})$. The dynamical equation $d*F = 0$ implies that $*F$ is a closed two-form, so it is also exact in the Minkowski space-time. This means that we can write

$$*F = \frac{dC}{c}, \quad (16)$$

where C is a one-form and the factor $1/c$ appears because of dimensional requirements. The dynamical equation $d*F = 0$ becomes $d^2C = 0$, so it is again
250 a Bianchi identity. Note that the dual of the form $*F$ is $**F = -F$. Consequently, from the dual point of view, the dynamical equation $d*F = 0$ is a Bianchi identity, and the Bianchi identity $dF = 0$ is a dynamical equation [44].

As long as we are studying fields in vacuum we can define another four-potential

$$C^\mu = (cV', \mathbf{C}), \quad (17)$$

so that, in tensor components,

$$*F_{\mu\nu} = -\frac{1}{c} (\partial_\mu C_\nu - \partial_\nu C_\mu), \quad (18)$$

or, in three-dimensional components,

$$\mathbf{E} = \nabla \times \mathbf{C}, \quad \mathbf{B} = \nabla V' + \frac{1}{c^2} \frac{\partial \mathbf{C}}{\partial t}. \quad (19)$$

The identity $d*F = 0$, expressed in function of $\mathbf{E}$ and $\mathbf{B}$, is the pair of equations (4), and the dynamical equation $dF = 0$ is the pair of equations (3).

Exact solutions of Maxwell equations (3)-(4) in which magnetic and electric field lines coincide with level curves of a pair of complex scalar fields $\phi(\mathbf{r}, t)$, $\theta(\mathbf{r}, t) : R^3 \times R \rightarrow C$ can be written as functions of such scalars.

Let us discuss what can be deduced about the actual fields from the evolution of their magnetic and electric lines. We start by representing the magnetic lines by the equation $\phi(t, x, y, z) = \phi_0$ where ϕ is a complex function of space and time and ϕ_0 is a constant labelling each line. This means that the magnetic lines are the level curves of the complex scalar field $\phi(t, x, y, z)$. Since the magnetic field is tangent to the magnetic lines at each point, it has to satisfy

$$\mathbf{B}(\mathbf{r}, t) = g(\mathbf{r}, t) \nabla \phi \times \nabla \bar{\phi}. \quad (20)$$

The magnetic field has zero divergence. Applying that property to the expression (20), one gets

$$\nabla \cdot \mathbf{B} = \nabla g \cdot (\nabla \phi \times \nabla \bar{\phi}) = 0. \quad (21)$$

For this equation to be fulfilled, it is necessary that the function g depends on coordinates $(\mathbf{r}, t)$ only through ϕ and $\bar{\phi}$. Consequently, we get

$$\mathbf{B}(\mathbf{r}, t) = g(\phi, \bar{\phi}) \nabla \phi \times \nabla \bar{\phi}. \quad (22)$$

The magnetic field can also be written as $\mathbf{B}_i = -\epsilon_{ijk} F^{jk}/2$ if we take

$$F_{ij} = g(\phi, \bar{\phi}) (\partial_i \bar{\phi} \partial_j \phi - \partial_j \bar{\phi} \partial_i \phi). \quad (23)$$

Covariance implies that the electromagnetic tensor can be defined as

$$F_{\mu\nu} = g(\phi, \bar{\phi}) (\partial_\mu \bar{\phi} \partial_\nu \phi - \partial_\nu \bar{\phi} \partial_\mu \phi). \quad (24)$$

The electric field corresponding to (24) is

$$\mathbf{E}(\mathbf{r}, t) = g(\phi, \bar{\phi}) (\partial_0 \bar{\phi} \nabla \phi - \partial_0 \phi \nabla \bar{\phi}). \quad (25)$$

The electromagnetic field defined by the complex scalar field ϕ that gives the magnetic lines is then such that the electric and the magnetic fields are orthogonal.

If we consider electromagnetism in vacuum, there is a duality between the magnetic and electric parts of the electromagnetic field, so we can study the electric lines as level curves of another complex scalar field θ . The dual expressions to (20)-(25) are the following. The electric field is

$$\mathbf{E}(\mathbf{r}, t) = g'(\theta, \bar{\theta}) \nabla \bar{\theta} \times \nabla \theta, \quad (26)$$

and can be obtained from a dual electromagnetic tensor $*F_{\mu\nu}$ by making the identification $\mathbf{E}_i = c \epsilon_{ijk} *F^{jk}/2$, where

$$*F_{\mu\nu} = g'(\theta, \bar{\theta}) (\partial_\mu \bar{\theta} \partial_\nu \theta - \partial_\nu \bar{\theta} \partial_\mu \theta). \quad (27)$$

The magnetic field is then

$$\mathbf{B}(\mathbf{r}, t) = \frac{g'(\theta, \bar{\theta})}{c} (\partial_0 \bar{\theta} \nabla \theta - \partial_0 \theta \nabla \bar{\theta}). \quad (28)$$

To obtain a solution of Maxwell equations in vacuum, expressions (22) and (28) have to be equal for the magnetic field, and expressions (25) and (26) have to be equal for the electric field. This duality condition must be fulfilled.

265 A possible choice for the functions g and g' is [8]

$$\begin{aligned} g(\phi, \bar{\phi}) &= \frac{\sqrt{a}}{2\pi i(1 + \bar{\phi}\phi)^2} \\ g'(\theta, \bar{\theta}) &= \frac{\sqrt{ac}}{2\pi i(1 + \bar{\theta}\theta)^2}, \end{aligned} \quad (29)$$

where a and c are constants. The particular form of the dependence of g and g' with the complex fields ϕ and θ will become clear later when we consider the compactification of the physical space R^3 and the complex plane C . These compactifications will reveal some aspects of the rich topological structure hidden under the linear Maxwell equations. Before introducing topological tools, we must define these particular fields and study their properties as solutions of Maxwell equations in vacuum.

270 If $\phi(\mathbf{r}, t)$ and $\theta(\mathbf{r}, t)$ are two complex scalar fields (we will consider them dimensionless), a magnetic field $\mathbf{B}$ and an electric field $\mathbf{E}$ can be defined from

275 them as

$$\begin{aligned}\mathbf{B}(\mathbf{r}, t) &= \frac{\sqrt{a}}{2\pi i} \frac{\nabla\phi \times \nabla\bar{\phi}}{(1 + \bar{\phi}\phi)^2}, \\ \mathbf{E}(\mathbf{r}, t) &= \frac{\sqrt{a}c}{2\pi i} \frac{\nabla\bar{\theta} \times \nabla\theta}{(1 + \bar{\theta}\theta)^2},\end{aligned}\tag{30}$$

where $\bar{\phi}$ and $\bar{\theta}$ are the complex conjugates of ϕ and θ respectively, i is the imaginary unit, a is a constant introduced so that the magnetic and electric fields have correct dimensions, and c is the speed of light in vacuum. In the SI of units, a can be expressed as a pure number times the Planck constant $\hbar$ times the speed of light c times the vacuum permeability μ_0 . In natural units, 280 with $\hbar = c = \mu_0 = \varepsilon_0 = 1$, a is simply a real number.

In order to obtain a solution of the Maxwell equations in vacuum from the fields given by Equations (30), they also have to satisfy

$$\begin{aligned}\mathbf{B}(\mathbf{r}, t) &= \frac{\sqrt{a}}{2\pi ic(1 + \bar{\theta}\theta)^2} \left(\frac{\partial\bar{\theta}}{\partial t} \nabla\theta - \frac{\partial\theta}{\partial t} \nabla\bar{\theta} \right), \\ \mathbf{E}(\mathbf{r}, t) &= \frac{\sqrt{a}}{2\pi i(1 + \bar{\phi}\phi)^2} \left(\frac{\partial\bar{\phi}}{\partial t} \nabla\phi - \frac{\partial\phi}{\partial t} \nabla\bar{\phi} \right).\end{aligned}\tag{31}$$

The magnetic and the electric fields resulting from (30) and (31) satisfy exactly 285 Maxwell equations in vacuum as it can be easily checked.

It is possible to write equations (30) and (31) in a covariant way. The electromagnetic tensor $F_{\mu\nu}$ satisfies

$$F_{\mu\nu} = \frac{\sqrt{a}}{2\pi i} \frac{\partial_\mu\bar{\phi}\partial_\nu\phi - \partial_\mu\phi\partial_\nu\bar{\phi}}{(1 + \bar{\phi}\phi)^2},\tag{32}$$

from which the usual relations $\mathbf{B}_i = -\epsilon_{ijk}F^{jk}/2$, $\mathbf{E}_i = cF^{i0}$ give the first equation in (30) and the second equation in (31). Another antisymmetric tensor $G_{\mu\nu}$ is defined so that

$$G_{\mu\nu} = \frac{\sqrt{a}}{2\pi i} \frac{\partial_\mu\theta\partial_\nu\bar{\theta} - \partial_\mu\bar{\theta}\partial_\nu\theta}{(1 + \bar{\theta}\theta)^2}.\tag{33}$$

It is necessary that $G_{\mu\nu}$ and $F_{\mu\nu}$ are dual so that they define an electromagnetic field in vacuum. The condition

$$G_{\mu\nu} = {}^*F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta},\tag{34}$$

is to be satisfied by the complex scalar fields ϕ and θ . Relations $\mathbf{B}_i = G^{i0}$, $\mathbf{E}_i = c \epsilon_{ijk} G^{jk}/2$ give the second equation in (30) and the first equation in (31). The duality condition can also be written as an equality of equations (30) and (31),

$$\begin{aligned} \frac{\nabla\phi \times \nabla\bar{\phi}}{(1 + \bar{\phi}\phi)^2} &= \frac{1}{c(1 + \bar{\theta}\theta)^2} \left(\frac{\partial\bar{\theta}}{\partial t} \nabla\theta - \frac{\partial\theta}{\partial t} \nabla\bar{\theta} \right), \\ \frac{\nabla\bar{\theta} \times \nabla\theta}{(1 + \bar{\theta}\theta)^2} &= \frac{1}{c(1 + \bar{\phi}\phi)^2} \left(\frac{\partial\bar{\phi}}{\partial t} \nabla\phi - \frac{\partial\phi}{\partial t} \nabla\bar{\phi} \right), \end{aligned} \quad (35)$$

290 that is a nonlinear pair of equations in the scalar fields.

As a consequence of their definition, those electromagnetic (EM) fields are exact solutions of Maxwell equations in vacuum and they satisfy $\mathbf{E} \cdot \mathbf{B} = 0$. The property $\mathbf{E} \cdot \mathbf{B} = 0$ is shared by null fields. The null fields are solutions of Maxwell equations in vacuum in which both the Lorentz invariant quantities $\mathbf{E} \cdot \mathbf{B}$ and $E^2 - c^2 B^2$ are zero for every time and every point. Since by construction the EM fields given by (30), (31) and (35) have a zero value for the first of these Lorentz invariants, one could ask if it is also true for the second invariant. The answer is negative in general. We will see some examples where the EM fields are also null electromagnetic fields, but there are other cases that satisfy $\mathbf{E} \cdot \mathbf{B} = 0$ but $E^2 - c^2 B^2 \neq 0$.

2.3. Clebsch representation

An immediate question that appears when equations (30)-(35) are examined is the need of introduce them. It is clear that these nonlinear equations are sometimes difficult to manage and it is not obvious that they present any advantage over standard Maxwell's equations in vacuum which are linear and can be treated by Fourier analysis. However giving an electromagnetic field through two complex scalars by equations (30)-(35) allows to study more directly the properties and evolution of electric and magnetic lines, something that is not so clear from the standard Maxwell theory [35]. There are applications in electromagnetism [45] where the field lines and their topology are central objects. To understand the relation between the study of field lines and the equations

that define Rañada's construction, we need to introduce canonical variables of an electromagnetic field in vacuum.

We begin by writing the electromagnetic 2-form F of an electromagnetic field in vacuum as

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{\mathbf{E}_i}{c} dx^0 \wedge dx^i - \varepsilon_{ijk} \mathbf{B}_i dx^j \wedge dx^k, \quad (36)$$

where $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$ are coordinates in the Minkowski space-time, and $\mathbf{E}_i$, $\mathbf{B}_i$ are the components of the electric and magnetic fields respectively. As long as we are considering fields in vacuum, we can also write the dual to the electromagnetic 2-form, that is given by

$$*F = \frac{1}{2} *F_{\mu\nu} dx^\mu \wedge dx^\nu = \mathbf{B}_i dx^0 \wedge dx^i + \varepsilon_{ijk} \frac{\mathbf{E}_i}{c} dx^j \wedge dx^k. \quad (37)$$

The representation given by equations (36)-(37) is not the simplest possible. Following [46], if ω is a 2-form in a 4-dimensional vector space, given in terms of a basis of 1-forms f^a ($a = 1, \dots, 4$) by

$$\omega = \frac{1}{2} \omega_{ab} f^a \wedge f^b, \quad (38)$$

then there is another basis of 1-forms g^a ($a = 1, \dots, 4$) such that ω can be written as

$$\omega = g^1 \wedge g^2 + g^3 \wedge g^4. \quad (39)$$

To see how this works for an electromagnetic field in vacuum, since $F_{\mu\nu}$ is an antisymmetric tensor we can take from (36),

$$F = \left(\frac{\mathbf{E}_1}{c} dx^0 + \mathbf{B}_3 dx^2 - \mathbf{B}_2 dx^3 \right) \wedge \left(dx^1 + \frac{\mathbf{E}_2}{\mathbf{E}_1} dx^2 + \frac{\mathbf{E}_3}{\mathbf{E}_1} dx^3 \right) - \left(\frac{\mathbf{E} \cdot \mathbf{B}}{\mathbf{E}_1} \right) dx^2 \wedge dx^3. \quad (40)$$

Now take the basis of 1-forms

$$\begin{aligned} e^0 &= \frac{\mathbf{E}_1}{c} dx^0 + \mathbf{B}_3 dx^2 - \mathbf{B}_2 dx^3, \\ e^1 &= dx^1 + \frac{\mathbf{E}_2}{\mathbf{E}_1} dx^2 + \frac{\mathbf{E}_3}{\mathbf{E}_1} dx^3, \\ e^2 &= - \left(\frac{\mathbf{E} \cdot \mathbf{B}}{\mathbf{E}_1} \right) dx^2, \\ e^3 &= dx^3. \end{aligned} \quad (41)$$

This basis can be defined except in the points of space-time in which $\mathbf{E}_1 = 0$ (in these points, we can define other basis similar to (41)). With this basis, F can be written as

$$F = e^0 \wedge e^1 + e^2 \wedge e^3. \quad (42)$$

Moreover, if the determinant of the matrix $F_{\mu\nu}$ is zero, then F is even simpler than (42), and there is a pair of 1-forms (f, g) such that

$$F = f \wedge g. \quad (43)$$

315 In this case, the 2-form F has only one term, and can be called decomposable. There are two Lorentz invariants of the electromagnetic field, given by

$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{\mathbf{E}^2 - c^2 \mathbf{B}^2}{2c^2}, \quad (44)$$

$$-\frac{1}{4} F_{\mu\nu} {}^*F^{\mu\nu} = \frac{\mathbf{E} \cdot \mathbf{B}}{c}. \quad (45)$$

The value of (45) determines the explicit way in which we can give the electromagnetic 2-form. If the value of (45) $\neq 0$, then the 2-form F is not decomposable and can be expressed as in (42). If the value is equal to 0, then $\mathbf{E} \cdot \mathbf{B} = 0$, so
320 that $e^2 = 0$ in (41). In this case, F is decomposable (the determinant of the matrix $F_{\mu\nu}$ is equal to zero if $\mathbf{E} \cdot \mathbf{B} = 0$) and can be written as in (43).

The local canonical representation of every electromagnetic field in vacuum can be mathematically described by means of a theorem due to Darboux [47]. Let ω be a k -form defined on a m -dimensional differentiable manifold M . The rank of ω is the minimum number r of linearly independent 1-forms such that F can be written as exterior products of them. Clearly $k \leq r \leq m$. Let us consider the case $k = 2$, so ω is an antisymmetric 2-form on a m -dimensional vector space M . The Darboux theorem establish that ω may have rank $r = m$ only if m is pair, so that $m = 2n$. If ω has rank $r = m = 2n$ then, in a convenient basis of the vector space M , we can write the matrix ω as

$$\omega = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \quad (46)$$

where I is the $n \times n$ unit matrix. Consequently, if ω is a symplectic 2-form (that is, it is closed so that $d\omega = 0$ and it has rank $r = 2n$) on a smooth manifold M , then for every point $x_0 \in M$ there is a local system of coordinates around x_0 in which the coefficients ω_{ij} are constant. This means that there is a local system of coordinates $(q^1, \dots, q^n, p^1, \dots, p^n)$ such that

$$\omega = dq^i \wedge dp^i. \quad (47)$$

The coordinates $(q^1, \dots, q^n, p^1, \dots, p^n)$ in (47) are called canonical coordinates or canonical variables of ω and they are not unique in general: there are canonical transformations of these coordinates that leave ω invariant.

For the electromagnetism in vacuum, the electromagnetic 2-form F has rank 4 in general, so that at least four 1-forms are needed to write it in the Minkowski space-time, as we have seen in equation (42). Moreover, F is a closed form due to the Bianchi identity $dF = 0$. In conclusion, the electromagnetic form F is symplectic. Then, due to the Darboux theorem it can be written locally as

$$F = dq^1 \wedge dp^1 + dq^2 \wedge dp^2. \quad (48)$$

The functions (q^1, q^2, p^1, p^2) are canonical coordinates of F and are defined except canonical transformations. When F is written as in (48), it is said that a Clebsch representation is being used [48]. In vacuum, the dual 2-form $*F$ is also symplectic, as can be seen by comparing equations (36) and (37), which means it can also be expressed in a Clebsch representation as

$$*F = dv^1 \wedge du^1 + dv^2 \wedge du^2. \quad (49)$$

The two sets of functions (q^1, q^2, p^1, p^2) and (v^1, v^2, u^1, u^2) are not independent, since they have to satisfy the duality condition

$$dv^1 \wedge du^1 + dv^2 \wedge du^2 = *(dq^1 \wedge dp^1 + dq^2 \wedge dp^2). \quad (50)$$

325 As we have seen above, if $\mathbf{E} \cdot \mathbf{B} = 0$, F is decomposable (and so is $*F$). In this case, the electromagnetic 2-forms F and $*F$ have rank 2. Thus a decomposable

electromagnetic field can be written locally in terms of canonical variables (q, p) and (v, u) as

$$\begin{aligned} F &= dq \wedge dp, \\ *F &= dv \wedge du, \end{aligned} \tag{51}$$

subjected to a duality condition $dv \wedge du = *(dq \wedge dp)$. Equations (51) also mean
 330 that the 2-form of a non-decomposable electromagnetic field in vacuum is locally the sum of two 2-forms of decomposable electromagnetic fields.

Suppose that a decomposable electromagnetic field is written globally as in equation (51). Then the same functions (q, p) and (v, u) can be used to write the electric and the magnetic fields in all the points of space-time,

$$\begin{aligned} \mathbf{B} &= \nabla p \times \nabla q = \left(\frac{\partial v}{\partial t} \nabla u - \frac{\partial u}{\partial t} \nabla v \right), \\ \mathbf{E} &= c \nabla v \times \nabla u = c \left(\frac{\partial q}{\partial t} \nabla p - \frac{\partial p}{\partial t} \nabla q \right). \end{aligned} \tag{52}$$

335 2.4. Vector potentials and helicities

It can be shown that, if a decomposable electromagnetic field can be written globally as $F = dq \wedge dp$ and the functions q and p are single-valued and well-defined in all the points of space, then the magnetic helicity has to be equal to zero [49, 34, 50]. The same happens with the electric helicity if the global
 340 relation $*F = dv \wedge du$ and the functions v and u are single-valued and well-defined in all the points of space.

Let us introduce the concept of helicity. This is an important quantity since the value of the helicity provides a measure of the topology associated with the field lines. The magnetic helicity is defined as

$$h_m = \frac{1}{2c\mu_0} \int_D \mathbf{A} \cdot \mathbf{B} d^3r, \tag{53}$$

where D is the spatial part of the 4-four space, which is a 3-dimensional manifold or volume. We will assume that its frontier ∂D is an orientable surface. We recall that $\mathbf{A}$ is defined such as $\mathbf{B} = \nabla \times \mathbf{A}$. The prefactor involving the speed
 345 of light c and the vacuum permeability μ_0 is introduced in order to have angular

momentum units for the helicity as done in particle physics. The definition of the magnetic helicity density $\mathbf{A} \cdot \mathbf{B}$ captures the idea of the rotation of the field $\mathbf{A}$ along itself. As the curl of a vector measures its rotation around a point, the integrand in (53) gives how much $\mathbf{A}$ rotates around itself times its own
350 modulus. The name of helicity was coined by Moffatt in the context of fluid dynamics [51]. The magnetic helicity provides a measure of the mean value of the linking number of the magnetic field lines (see Appendix A).

Since the electric field in vacuum is also a field with zero divergence, we can use the concept of duality to define the electric helicity of an electromagnetic field in vacuum as

$$h_e = \frac{\varepsilon_0}{2c} \int d^3r \mathbf{C} \cdot \mathbf{E} = \frac{1}{2c^3\mu_0} \int d^3r \mathbf{C} \cdot \mathbf{E}, \quad (54)$$

where $\mathbf{E} = \nabla \times \mathbf{C}$. The electric helicity in (54) has also dimensions of angular momentum and $\varepsilon_0 = 1/(c^2\mu_0)$ is the electric permittivity. By analogy with the
355 magnetic helicity, the electric helicity is also related to the linking number of the electric lines.

Let us consider the case of the magnetic helicity (the case of electric helicity is completely equivalent). If $F = dq \wedge dp$ globally, then the magnetic field can be written as $\mathbf{B} = \nabla p \times \nabla q$. If, moreover, the functions p and q are single-valued, a well-defined vector potential $\mathbf{A}$ could be taken so that

$$\mathbf{A} = \lambda p \nabla q - (1 - \lambda) q \nabla p, \quad (55)$$

where λ is a real number such that $0 \leq \lambda \leq 1$. Using (52) and (55), it is easy to see that $\mathbf{A} \cdot \mathbf{B} = 0$, so the magnetic helicity is zero.

Suppose now that any of the two vector fields ∇p or ∇q in (52), for example ∇q , diverges in certain set of points of the space R^3 , so the function q is not globally well-defined. Then, a well-defined vector potential can not be expressed as in (55). Instead, we can take

$$\mathbf{A} = p \nabla q + \nabla \chi, \quad (56)$$

where the term $\nabla \chi$ is added to correct the possible ill-definition of $p \nabla q$ (note that $\nabla \chi$ may be not globally well-defined either). The magnetic helicity is now

$$h_m = \frac{1}{2c\mu_0} \int d^3r \nabla\chi \cdot (\nabla p \times \nabla q). \quad (57)$$

If all the terms in equation (57) were well-defined in all the points of the space R^3 , then one could apply Stokes theorem to the integral and the magnetic helicity could be zero. Consequently, a necessary condition for the magnetic helicity to be non-zero in a decomposable field that can be written globally as in (52) is that one of the canonical variables (q, p) is not well-defined in all the points of R^3 . We will next see that one of the canonical variables associated to the Rañada construction is in an arctan function.

2.5. Euler potentials

The canonical representation of decomposable fields can be used to describe magnetic and electric lines. These lines are tangent to the magnetic and electric field, respectively, at any point in space. The magnetic line at time t_0 passing through a point $\mathbf{r}_0$ is the parametrized curve $\mathbf{r}(\tau)$ that satisfies the differential equation

$$\begin{aligned} \frac{d\mathbf{r}(\tau)}{d\tau} &= \mathbf{B}(\mathbf{r}(\tau), t_0), \\ \mathbf{r}(\tau)|_{\tau=0} &= \mathbf{r}_0. \end{aligned} \quad (58)$$

Here, τ is the natural parameter of the curve, related to the differential arc length of the curve $ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$ by the equation $d\tau = ds/B$. The magnetic lines may lie on surfaces or cover three-dimensional domains. The Euler potentials [52, 53] of the magnetic field are two real functions $\alpha_1(\mathbf{r}, t) \in R$ and $\alpha_2(\mathbf{r}, t) \in R$ such that

$$\mathbf{B} = \nabla\alpha_1 \times \nabla\alpha_2. \quad (59)$$

By comparing these equation with (52), we can see that the Euler potentials are canonical variables of the magnetic field and that they provide a Clebsch representation of this field. The points where the Euler potentials take a constant value determine a magnetic surface.

In vacuum, since the electric field also satisfies the condition $\nabla \cdot \mathbf{E} = 0$, we can in principle define also Euler potentials $\beta_1 \in R$ and $\beta_2 \in R$ through

$$\mathbf{E} = \nabla\beta_2 \times \nabla\beta_1, \quad (60)$$

so that the electric lines at time t_0 are the intersections of the electric surfaces $\beta_1(\mathbf{r}, t_0) = k'_1$, $\beta_2(\mathbf{r}, t_0) = k'_2$.

It can be seen that the complex scalar fields ϕ and θ (30) define Euler potentials (and canonical variables) of the magnetic and the electric field, respectively, and consequently give the magnetic and the electric lines at every instant of time [34]. Let us begin with the magnetic field equation in (30). Since ϕ is a complex field, we define its modulus $|\phi| = \sqrt{(\Re(\phi))^2 + (\Im(\phi))^2}$ and argument $\mathcal{A}_\phi = \arctan(\Im(\phi)/\Re(\phi))$, so that

$$\phi = |\phi| e^{i\mathcal{A}_\phi}. \quad (61)$$

Then, by (30),

$$\mathbf{B}(\mathbf{r}, t) = \frac{\sqrt{a}}{2\pi i} \frac{\nabla\phi \times \nabla\bar{\phi}}{(1 + \bar{\phi}\phi)^2} = \sqrt{a} \nabla \left(\frac{1}{1 + |\phi|^2} \right) \times \nabla \left(\frac{\mathcal{A}_\phi}{2\pi} \right). \quad (62)$$

This means that the real variables

$$\begin{aligned} \alpha_1(\mathbf{r}, t) &= \frac{1}{1 + |\phi|^2}, \\ \alpha_2(\mathbf{r}, t) &= \frac{\mathcal{A}_\phi}{2\pi}, \end{aligned} \quad (63)$$

are Euler potentials of the magnetic field. The constant $\sqrt{a}$ in (62) does not play any role in the definition of magnetic lines.

For the electric field, we define the modulus of the complex field θ as $|\theta| = \sqrt{(\Re(\theta))^2 + (\Im(\theta))^2}$ and its argument as $\mathcal{A}_\theta = \arctan(\Im(\theta)/\Re(\theta))$. Putting these into equation (30),

$$\mathbf{E}(\mathbf{r}, t) = \frac{\sqrt{ac}}{2\pi i} \frac{\nabla\bar{\theta} \times \nabla\theta}{(1 + \bar{\theta}\theta)^2} = -c\sqrt{a} \nabla \left(\frac{1}{1 + |\theta|^2} \right) \times \nabla \left(\frac{\mathcal{A}_\theta}{2\pi} \right). \quad (64)$$

The Euler potentials of this electric field are

$$\begin{aligned} \beta_1(\mathbf{r}, t) &= \frac{1}{1 + |\theta|^2}, \\ \beta_2(\mathbf{r}, t) &= \frac{\mathcal{A}_\theta}{2\pi}. \end{aligned} \quad (65)$$

The magnetic lines are then given by the equations $\alpha_1 = k_1$ and $\alpha_2 = k_2$. From (63), these two real equations can be written as the complex equation

$$\phi(\mathbf{r}, t) = \phi_0, \quad (66)$$

so that the complex scalar field ϕ gives all the magnetic lines as its level curves. The constant ϕ_0 may be chosen to take any value, in general, in the complex plane. Analogously, the equation

$$\theta(\mathbf{r}, t) = \theta_0, \quad (67)$$

gives all the electric lines when the constant θ_0 is set to take values in the complex plane. The complex scalar field θ is then the complex function whose level curves are the electric lines for every time.

385 Consequently, if an electromagnetic field can be written as functions of the two complex scalar fields ϕ and θ , it allows to study directly the magnetic and the electric lines for every time, and moreover define canonical variables and Euler potentials for the magnetic and the electric fields.

2.6. Topologically trivial examples

390 We will see now how some simple examples of electromagnetic fields can be written with the form of equations (30) and (31). It is easier to find the complex scalar fields ϕ and θ when we use the Euler potentials (63) and (65) to write the magnetic and electric fields as

$$\begin{aligned} \mathbf{B}(\mathbf{r}, t) &= \sqrt{a} \nabla \alpha_1 \times \nabla \alpha_2 = \frac{\sqrt{a}}{c} \left(\frac{\partial \beta_2}{\partial t} \nabla \beta_1 - \frac{\partial \beta_1}{\partial t} \nabla \beta_2 \right), \\ \mathbf{E}(\mathbf{r}, t) &= c \sqrt{a} \nabla \beta_2 \times \nabla \beta_1 = \sqrt{a} \left(\frac{\partial \alpha_2}{\partial t} \nabla \alpha_1 - \frac{\partial \alpha_1}{\partial t} \nabla \alpha_2 \right), \end{aligned} \quad (68)$$

where α_1 and α_2 are given by (63), and β_1 and β_2 are given by (65). From these Euler potentials, the scalar fields ϕ and θ are given by

$$\begin{aligned} \phi &= |\phi| e^{i\mathcal{A}_\phi} = \sqrt{\frac{1 - \alpha_1}{\alpha_1}} e^{2\pi i \alpha_2}, \\ \theta &= |\theta| e^{i\mathcal{A}_\theta} = \sqrt{\frac{1 - \beta_1}{\beta_1}} e^{2\pi i \beta_2}. \end{aligned} \quad (69)$$

The first example to consider will be the sinusoidal plane wave propagating along the x -axis. For a plane wave propagating along the x -axis, the electromagnetic field is given by the equations

$$\begin{aligned}\mathbf{E} &= cB_0 \sin k(x - ct) \mathbf{u}_y, \\ \mathbf{B} &= B_0 \sin k(x - ct) \mathbf{u}_z.\end{aligned}\tag{70}$$

From (68), the Euler potentials are [35]

$$\begin{aligned}\alpha_1 &= \frac{1}{2} [1 - \cos k(x - ct)], \\ \alpha_2 &= \frac{2B_0}{k\sqrt{a}} y, \\ \beta_1 &= \frac{1}{2} [1 - \cos k(x - ct)], \\ \beta_2 &= \frac{2B_0}{k\sqrt{a}} z.\end{aligned}\tag{71}$$

400 The magnetic surfaces at a given instant of time $t = t_0$ are given by $x = \text{constant}$ and $y = \text{constant}$, so that the magnetic lines are parallel to the z -axis. Each one of these magnetic lines is the intersection of a pair of magnetic surfaces. The electric surfaces at $t = t_0$ are given by $x = \text{constant}$ and $z = \text{constant}$, so the electric lines are parallel to the y -axis. Using equation (69), the complex scalar
405 fields are

$$\begin{aligned}\phi &= \sqrt{\frac{1 + \cos k(x - ct)}{1 - \cos k(x - ct)}} e^{i(4\pi B_0/k\sqrt{a})y}, \\ \theta &= \sqrt{\frac{1 + \cos k(x - ct)}{1 - \cos k(x - ct)}} e^{i(4\pi B_0/k\sqrt{a})z}.\end{aligned}\tag{72}$$

The second example to consider is the Coulomb electrostatic field produced by a point charge. The field produced by a point charge q at rest in the origin is given by

$$\begin{aligned}\mathbf{E} &= \frac{q}{4\pi\epsilon_0} \frac{x \mathbf{u}_x + y \mathbf{u}_y + z \mathbf{u}_z}{r^3}, \\ \mathbf{B} &= 0,\end{aligned}\tag{73}$$

where $r = \sqrt{x^2 + y^2 + z^2}$. Note that this Coulomb field is a solution of Maxwell
410 equations in vacuum in the space R^3 minus the point in which the charge q is,

which is diffeomorphic to $S^2 \times R$. As it was given in [35], the Euler potentials are

$$\begin{aligned}\alpha_1 &= \frac{r}{r+ct}, \\ \alpha_2 &= \frac{q}{4\pi\epsilon_0 c\sqrt{a}} \left(1 + \frac{ct}{r}\right) \ln\left(\frac{r}{r_0}\right), \\ \beta_1 &= \frac{r+z}{2r}, \\ \beta_2 &= \frac{q}{2\pi\epsilon_0 c\sqrt{a}} \arctan\left(\frac{y}{x}\right),\end{aligned}\tag{74}$$

where r_0 is a constant with length dimensions. The static Euler potentials β_1 and β_2 have been used in [38] to illustrate a topological mechanism of quantization of the electric charge. We will not discuss this mechanism but we can point out that β_2 is a phase function. To be well defined in all the space $R^3 - \{0\} = S^2 \times R$, the following condition can be imposed,

$$\frac{q}{\epsilon_0 c\sqrt{a}} = N,\tag{75}$$

where N is an integer number so that a quantized value of the electric charge at rest is obtained. As an example, when one takes $N = 1$ in the equation (75), the static complex scalar θ corresponding to a Coulomb field created by a point charge $q = \epsilon_0 c\sqrt{a}$ at rest in the origin is

$$\theta = \sqrt{\frac{r-z}{r+z}} \left(\frac{x}{\sqrt{x^2+y^2}} + i \frac{y}{\sqrt{x^2+y^2}} \right) = \frac{x+iy}{r+z}.\tag{76}$$

The electric surfaces of this example are given by the equations $x = k_1(r+z)$ and $y = k_2(r+z)$. The surfaces with $k_1 = k_2 = 1$ have been plotted in Fig. 2.

415 The intersection of both surfaces is the electric straight line $x = y = -2z$ and the z-axis.

2.7. Local equivalence of Rañada construction and decomposable solutions of Maxwell equations in vacuum

We have seen some examples in which the pair of complex scalar fields ϕ
420 and θ generate an electromagnetic field. We will see that any decomposable

Figure 2: Two views of two electric surfaces given by the static complex scalar field (76). These particular cases are given by the equations $x = r + z$ and $y = r + z$. They intersect along two straight lines, that are also the electric lines of the Coulomb field given by (73). The equation of these electric lines are $x = y = -2z$, and the $x = y = 0$, which is the z -axis.

electromagnetic field can be written locally in terms of this scalar pair, which has the consequence that the set of fields given by (30) and (31) is locally equivalent to the set of solutions of Maxwell equations in vacuum.

To understand the local equivalence, we follow the explanation in [35]. Let us consider the electromagnetic 2-form F^d of a decomposable solution of Maxwell equations in vacuum (the superindex d stands for decomposable in this notation, which implies that $\mathbf{E} \cdot \mathbf{B} = 0$). As we have seen above, this 2-form can be written locally in terms of canonical variables as in equation (51),

$$F^d = dq \wedge dp. \quad (77)$$

At the same time, from equation (68), the electromagnetic 2-form F^R of the EM field constructed from the complex scalar field ϕ can be written globally as

$$F^R = \sqrt{a} d\alpha_2 \wedge d\alpha_1, \quad (78)$$

where

$$\begin{aligned} \alpha_1 &= \frac{1}{1 + |\phi|^2}, \\ \alpha_2 &= \frac{\arctan(\Im(\phi)/\Re(\phi))}{2\pi}. \end{aligned} \quad (79)$$

425 The following property is satisfied by the 2-form F^d of a decomposable electromagnetic field in vacuum (the same could be proved for the dual 2-form $*F^d$).

Any decomposable electromagnetic field in vacuum, given by the 2-form F^d and regular in a bounded space-time domain D , coincides locally with a field given by (30) and (31) around any point $P \in D$ in the following sense: there is a field
430 given by (30) and (31) with two-form F^R , such that $F^d = F^R$ around P , except perhaps if P is in a zero measure set. This means that the difference between the set of decomposable solutions of Maxwell equations in vacuum and the set of fields (30) and (31) is not local but global.

To show this property, take the following canonical transformation of the
435 variables (q, p) in (77),

$$\begin{aligned}\eta &= \frac{\pi}{\sqrt{a}}(p^2 + q^2), \\ \delta &= \frac{1}{2\pi} \arctan\left(\frac{q}{p}\right).\end{aligned}\tag{80}$$

Then, we can write

$$F^d = \sqrt{a} d\delta \wedge d\eta.\tag{81}$$

This means that η and δ give another choice of the canonical representation of the decomposable field. Since the 2-form F^R can be written as in (78), we can easily see that $F^d = F^R$ if there is a complex scalar field ϕ so that the following equations hold,

$$\begin{aligned}\delta = \alpha_2 &= \frac{\arctan(\Im(\phi)/\Re(\phi))}{2\pi}, \\ \eta = \alpha_1 &= \frac{1}{1 + |\phi|^2}.\end{aligned}\tag{82}$$

440 The first of these equations poses no problem because δ was defined as a phase function in (80). Let us concentrate in the second equation in (82).

It is easy to see that α_1 satisfies $0 \leq \alpha_1 \leq 1$. However, η is defined in (80) and satisfies $0 \leq \eta$. Consequently, if $\eta \leq 1$ for the decomposable field we are considering, the solution for α_1 will be trivial, and the decomposable field with
445 2-form F^d will be equal to F^R . The same happens if η is bounded, say if $\eta < A$, because we can then take as canonical variables the pair $\eta' = \eta/N$, $\delta' = N\delta$, N being an integer greater than A .

Let us consider the case in which η is not bounded in D . Let Σ be the set of points in which η diverges. Since the equation (81) is valid, Σ is a zero measure set. In general, $D - \Sigma$ consists of n connected open components D_j , so that there are n open subsets $D'_j \subset D_j$ in which η is bounded. In each one of them, we define canonical variables (η', δ') by the same method as before, with the consequence that $F^d = F^R$ in each D'_j . For the rest of the points in D , we notice that the volume of $D - \bigcup D'_j$ may be made as small as desired. This means that the decomposable field can be obtained by patching together all the fields given by F_j^R , each one defined in the corresponding D'_j , except for a set as small as required containing Σ . Thus every decomposable electromagnetic field in vacuum coincides locally with a field written as (30) and (31), except perhaps on a zero measure set.

3. Electromagnetic knots from maps $S^3 \rightarrow S^2$

Electromagnetic fields constructed from two complex scalar fields ϕ and θ through equations (30) and (31) allow to introduce topological properties for the field lines configuration. This is in fact what happens when the scalar fields $\phi(\mathbf{r}, t)$ and $\theta(\mathbf{r}, t)$ satisfy certain boundary conditions so that they can be interpreted as maps from the three-sphere S^3 to the two-sphere S^2 . We will review these conditions and the consequences of using maps $S^3 \rightarrow S^2$ to build electromagnetic knots. Those maps can be classified in homotopy classes characterised by a topological invariant called the Hopf index. The helicity of electromagnetic knots from $S^3 \rightarrow S^2$ maps are proportional to the Hopf index, so the electromagnetic helicity is topologically quantized and it can be identified as the classical limit of the difference between right- and left-handed photons.

In 1977 A. Trautman made use of the Hopf fibration to find solutions of the Maxwell and Yang-Mills equations [10]. Kamchatnov was able to find magnetic fields with topological properties from the Hopf map in the context of magnetohydrodynamics [11]. We will discuss Trautman solutions in this section, and Kamchatnov contribution in the section devoted to plasma knots.

3.1. Trautman solutions

One of the first applications to electromagnetism of maps from the three-sphere to the two-sphere is the paper by Trautman [10]. He showed that the static magnetic field corresponding to a magnetic monopole can be found using the natural connection associated to the principal bundle $S^3 \rightarrow S^2$. Furthermore, he proposed a generalization of the previous result to find topologically nontrivial solutions of the Maxwell and Yang-Mills equations.

Consider a magnetic monopole at rest in the Minkowski space-time. The magnetic field defined by the monopole is of Coulomb type and is not well defined at the point in which the monopole is placed as in the second example of Subsection 2.6. If this point, along with its worldline, is removed to get a space in which the magnetic field is finite, then the Minkowski space-time is diffeomorphic to $R^2 \times S^2$. Trautman then considers bundles over S^2 , and specially the Hopf fibration [54], that can be defined as follows. Take the sphere S^3 . The group $U(1)$ acts on this sphere in a way such that the fibers are circles. The quotient space by this action is the sphere S^2 . We will see the Hopf map in more detail in the following sections.

The Hopf bundle admits a connection in terms of Euler angles θ, ϕ . The curvature of this connection can be written as

$$F = \frac{1}{2} \sin \theta d\phi \wedge d\theta. \quad (83)$$

This 2-form is closed, so that $dF = 0$, and satisfies the Bianchi identity $d^*F = 0$, so it is a solution of the Maxwell equations in vacuum. If we take local coordinates $x = \sin \theta \cos \phi$, $y = \sin \theta \sin \phi$, $z = \cos \theta$, the electric field associated to the 2-form (83) is zero, and the magnetic field is of Coulomb type, thus it describes a magnetic monopole of strength $1/2$ (Trautman uses natural units in his work).

The nice relation between the Hopf fibration and the magnetic monopole found by Trautman [10] is generalized in the same work. He considers multi-dimensional spheres S^{2n+1} and groups of linear transformations that preserve

their definition in terms of Cartesian coordinates $u_1, \dots, u_{2n+2}$, i.e. the equation $u_1^2 + \dots + u_{2n+2}^2 = 1$. One example is the sphere S^5 and the group $U(1)$.
505 The quotient space by the action of this group is the 2-dimensional complex projective space CP_2 . The natural connection defines then, in local coordinates of CP_2 ,

$$\begin{aligned} z_1 &= e^{i\mu} \tan \theta \cos \phi, \\ z_1 &= e^{i\nu} \tan \theta \sin \phi, \end{aligned} \quad (84)$$

a closed 2-form F given by

$$F = \sin 2\theta d\theta \wedge (\cos^2 \phi d\mu + \sin^2 \phi d\nu) - \sin^2 \theta \sin 2\phi d\phi \wedge (d\mu - d\nu), \quad (85)$$

that satisfies the Bianchi identity and is at the same time self-adjoint, so $*F = F$. This solution of the Maxwell equations in vacuum is called electromagnetic
510 instanton.

3.2. Some properties of maps from $S^3 \rightarrow S^2$

Consider a complex field $\phi(\mathbf{r}, t)$. It is a map from the four-dimensional Minkowski space $R^3 \times R$, where R^3 is the physical space and R is the time axis, to the complex plane C . Suppose that we impose a restriction in the behaviour of the scalar ϕ at infinite distance from the origin of the physical space R^3 . Specifically we impose that $\phi = 0$ at infinity, independently of the direction that the infinity is approached. This means that

$$\lim_{|\mathbf{r}| \rightarrow \infty} \phi(\mathbf{r}, t) = 0. \quad (86)$$

Note that this condition is not fulfilled by the scalar fields (72) or (76). This is because the electromagnetic fields (70) or (73) have infinite electromagnetic energy. Physically, the condition (86) is taken so that the energy of the EM
515 fields (30) and (31) is finite. From a mathematical point of view, the condition (86) identifies all the points at the infinity as a unique point, which is called a compactification at infinity of the space R^3 by ϕ . The complex field ϕ with this condition can be seen as a map $S^3 \times R \rightarrow C$.

We will impose a second condition on ϕ which is related to the compactification of the complex plane C . We can compactify the complex plane at infinity imposing that the inverse images of $\phi = \infty$, which are points in R^3 , do not depend on the direction approaching to infinity in C . When this compactification is imposed, the scalar field ϕ take values in the sphere S^2 .

Consequently, the complex scalar field ϕ with the above restrictions can be seen as a map $S^3 \times R \rightarrow S^2$. Using stereographic projections we can achieve the compactification conditions.

- Compactifying R^3 to S^3 . The Cartesian coordinates (x, y, z) of a point $P \in R^3$ can be related to four real coordinates (u_1, u_2, u_3, u_4) which are Cartesian coordinates in S^3 if $u_1^2 + u_2^2 + u_3^2 + u_4^2 = 1$. The explicit expressions (projecting from the North pole of the sphere) are

$$u_1 = \frac{2x}{r^2 + 1}, \quad u_2 = \frac{2y}{r^2 + 1}, \quad u_3 = \frac{2z}{r^2 + 1}, \quad u_4 = \frac{r^2 - 1}{r^2 + 1}, \quad (87)$$

where $r^2 = x^2 + y^2 + z^2$.

- The complex plane C is compactified to S^2 . Any point in C can be given by a complex number ζ . This complex number is related to three real coordinates (n_1, n_2, n_3) which are Cartesian coordinates in S^2 if $n_1^2 + n_2^2 + n_3^2 = 1$. Again the projection from the North pole gives

$$n_1 = \frac{2\Re(\zeta)}{\bar{\zeta}\zeta + 1}, \quad n_2 = \frac{2\Im(\zeta)}{\bar{\zeta}\zeta + 1}, \quad n_3 = \frac{\bar{\zeta}\zeta - 1}{\bar{\zeta}\zeta + 1}. \quad (88)$$

3.3. The Hopf index and the homotopy classes

Consider a map $f : S^3 \rightarrow S^2$ written as a complex scalar field $f(\mathbf{r})$ using equations (87) and (88). According to the Hopf theorem [54, 55], if the map f is smooth, the inverse image of any two distinct points ζ_1 and ζ_2 of S^2 , $f^{-1}(\zeta_1)$ and $f^{-1}(\zeta_2)$, are two disjoint closed curves in S^3 (or $R^3 \cup \{\infty\}$). The linking number of the curves $f^{-1}(\zeta_1)$ and $f^{-1}(\zeta_2)$ does not depend on the particular pair of points, since by moving continuously from (ζ_1, ζ_2) to (ζ'_1, ζ'_2) the inverse images can neither untie nor tie any further to one another (since, for this to happen, they should have at a certain moment a common point with two

different images by f). This linking number is called the Hopf index of the map f ,

$$H(f) = L(f^{-1}(\zeta_1), f^{-1}(\zeta_2)), \forall \zeta_1 \neq \zeta_2 \in S^2. \quad (89)$$

If the map evolves continuously (in time or with any other continuous parameter), the Hopf index must keep a constant value. In other words, the linking number is a constant of the motion in any continuous evolution of the map f . Thus the set of maps $S^3 \rightarrow S^2$ (the set of complex scalar fields which are one-valued at infinity) can be classified in homotopy classes, each one labelled by an integer number $H(f)$, called the Hopf index, that is a topological invariant.

An equivalent definition of the Hopf index is the following. Let us take the inverse images of two distinct points ζ_1 and ζ_2 of S^2 . Since they are two disjoint closed curves, their linking number (the Hopf index) is equal to the number of times that one of them, say $f^{-1}(\zeta_2)$, cuts a surface S_1 bounded by the curve $f^{-1}(\zeta_1)$. This number is equal to the degree of the map f restricted to S_1 , because every point in S^2 has $H(f)$ inverse images in S_1 (see [56]). The degree of that map is equal to the integral over S_1 of the volume 2-form of S^2 . This allowed Whitehead [57] to write the Hopf index as an integral in S^3 , as we will see. The volume 2-form in S^2 can be written in terms of the Cartesian coordinates (n_1, n_2, n_3) with $n_1^2 + n_2^2 + n_3^2 = 1$ defined in (88) as

$$\sigma = \frac{1}{4\pi} \frac{dn_1 \wedge dn_2}{n_3}, \quad (90)$$

so that it is normalized to unity,

$$\int_{S^2} \sigma = 1. \quad (91)$$

The volume 2-form (90) can also be written in terms of the complex number ζ related to n_i through equations (88), as

$$\sigma = \frac{1}{2\pi i} \frac{d\zeta \wedge d\bar{\zeta}}{(1 + \bar{\zeta}\zeta)^2}. \quad (92)$$

Let us consider the pull-back by the map $f : S^3 \rightarrow S^2$, denoted by f^{pb} , that, in a natural way, maps forms in S^2 into forms in S^3 . Taking into account the

representation (92), the pull-back of the volume 2-form is simply

$$f^{pb}\sigma = \frac{1}{2\pi i} \frac{df \wedge d\bar{f}}{(1 + \bar{f}f)^2}. \quad (93)$$

The 2-form $f^{pb}\sigma$ is closed, since

$$d(f^{pb}\sigma) = f^{pb}(d\sigma) = 0, \quad (94)$$

where we have used that $d\sigma = 0$ because it is a volume form in S^2 . The cohomological properties [55] of S^3 imply then that $f^{pb}\sigma$ is also an exact form, what means that there is a 1-form g in S^3 such that

$$f^{pb}\sigma = dg. \quad (95)$$

Whitehead showed [57] that the Hopf index $H(f)$ can be written as the integral

$$H(f) = \int_{S^3} g \wedge f^{pb}\sigma. \quad (96)$$

When the stereographic projection (87) is used to characterize points in S^3 in terms of coordinates (x, y, z) of R^3 , the pull-back of the volume form σ can be written as

$$f^{pb}\sigma = \frac{1}{4\pi i} \frac{\partial_j f \partial_k \bar{f} - \partial_j \bar{f} \partial_k f}{(1 + \bar{f}f)^2} dx^j \wedge dx^k. \quad (97)$$

This can be expressed as

$$f^{pb}\sigma = \frac{1}{2} f_{jk} dx^j \wedge dx^k, \quad (98)$$

where the antisymmetric tensor f_{jk} is

$$f_{jk} = \frac{1}{2\pi i} \frac{\partial_j f \partial_k \bar{f} - \partial_j \bar{f} \partial_k f}{(1 + \bar{f}f)^2}. \quad (99)$$

Now one can simply define a vector field $\mathbf{b}$ in R^3 as $\mathbf{b}_i(\mathbf{r}) = \varepsilon_{ijk} f_{jk}/2$, so that

$$\mathbf{b}(\mathbf{r}) = \frac{1}{2\pi i} \frac{\nabla f \times \nabla \bar{f}}{(1 + \bar{f}f)^2}. \quad (100)$$

The fact that $f^{pb}\sigma$ is an exact form simply means that $\mathbf{b} = \nabla \times \mathbf{a}$, where $\mathbf{a}$ is a vector field in R^3 related to the 1-form g in (95) by

$$g = \mathbf{a}_i dx^i. \quad (101)$$

It is clear from (100) that the vector field $\mathbf{b}$ is tangent to the level curves of the map f . The field $\mathbf{b}$ is known as the Whitehead vector. The Hopf index (96) is given in terms of the Whitehead vector as an integral in R^3 ,

$$H(f) = \int d^3r \mathbf{a} \cdot \mathbf{b}, \quad (102)$$

535 We recognize the right-hand side of equations (100) and (102). Equation (100) is, except the constant $\sqrt{a}$, the definition (30) of the magnetic field $\mathbf{B}$ of an electromagnetic knot, and equation (102) is, except constants, its magnetic helicity. These similarities are the reason for the original Rañada's construction [8]. He was able to construct a way to find electromagnetic fields with nontrivial topological properties by choosing for the magnetic and electric fields of an electromagnetic knot equations similar to the Whitehead vector (100) of a map from S^3 to S^2 .
540

Let us summarise all that we have for EM fields constructed from maps $S^3 \rightarrow S^2$. We begin by considering two complex scalar fields $\phi(\mathbf{r}, t)$ and $\theta(\mathbf{r}, t)$.

545 These scalar fields satisfy the following conditions:

1. For each instant of time, the value of ϕ and θ at infinity is zero.
2. For each instant of time, the inverse image of the value $\phi = \infty$ does not depend on the particular direction to go to infinity in the complex plane. Same for the image inverse of the value $\theta = \infty$.

550 Then, both scalars can be interpreted, at each instant of time, as maps $S^3 \rightarrow S^2$. The equivalence between coordinates in C and coordinates in S^2 is given by the stereographic projection (88), and the equivalence between coordinates in R^3 and coordinates in S^3 is given by the stereographic projection (87).

If $\phi : S^3 \rightarrow S^2$ is a smooth function, the inverse image, at each instant of time, of every point $a \in C$ by ϕ is a closed curve $\phi^{-1}(a) \in R^3$. The linking number of two inverse images $\phi^{-1}(a)$ and $\phi^{-1}(b)$ is the same for all the pairs of points $a, b \in C$. This linking number is a homotopy invariant (and, in particular, a time invariant) called the Hopf index $H(\phi)$ of the map ϕ . Accordingly, if θ is a smooth function, the inverse images of every pair of points in the complex plane by θ are closed curves whose linking number is the Hopf index $H(\theta)$.
560

If we define the electric and magnetic field of an electromagnetic field as in equation (30), then all the magnetic (electric) lines are, at each instant of time, closed curves and they are the level curves of the complex field $\phi(\theta)$. The linking number of every pair of magnetic (electric) lines does not depend on the particular pair we choose, and it is given by the Hopf index $H(\phi)$ of the complex field ϕ , that is an integer number given by the Whitehead integral

$$H(\phi) = \frac{1}{a} \int d^3r \mathbf{A} \cdot \mathbf{B}, \quad (103)$$

where $\mathbf{B} = \nabla \times \mathbf{A}$. Similarly for the electric lines we have the Whitehead integral

$$H(\theta) = \frac{1}{ac^2} \int d^3r \mathbf{C} \cdot \mathbf{E}, \quad (104)$$

where $\mathbf{E} = \nabla \times \mathbf{C}$.

3.4. Topological quantization of the electromagnetic helicity

The magnetic helicity h_m and the electric helicity h_e of an EM field constructed through equations (30) and (31) from two complex scalar fields $\phi(\mathbf{r}, t)$ and $\theta(\mathbf{r}, t)$, which satisfy conditions to be considered as maps $S^3 \rightarrow S^2$ at each instant of time, are given by

$$\begin{aligned} h_m &= \frac{1}{2c\mu_0} \int d^3r \mathbf{A} \cdot \mathbf{B} = \frac{a}{2c\mu_0} H(\phi), \\ h_e &= \frac{1}{2c^3\mu_0} \int d^3r \mathbf{C} \cdot \mathbf{E} = \frac{a}{2c\mu_0} H(\theta), \end{aligned} \quad (105)$$

where (103) and (104) have been used. Equations (105) give also the meaning of the constant a . Since the Hopf index is dimensionless, the constant $a/(c\mu_0)$ is simply the unit of helicity for the EM field.

To investigate the conservation of magnetic and electric helicities we can take the time derivatives of the helicities. If it is imposed the condition that the fields have finite electromagnetic energy, we get

$$\begin{aligned} \frac{dh_m}{dt} &= -\frac{1}{c\mu_0} \int d^3r \mathbf{E} \cdot \mathbf{B}, \\ \frac{dh_e}{dt} &= \frac{1}{c\mu_0} \int d^3r \mathbf{E} \cdot \mathbf{B}. \end{aligned} \quad (106)$$

Since in the Rañada construction $\mathbf{E} \cdot \mathbf{B} = 0$ for every time, the magnetic and electric helicities will be constant.

575 The electromagnetic helicity h_{em} is defined only for electromagnetic fields in vacuum and it is the sum of the electric and magnetic ones. From expressions (106), we can see that $h_{em} = h_e + h_m$ is a constant independently of the value of $\mathbf{E} \cdot \mathbf{B}$.

Using a decomposition of electromagnetic fields in circularly polarised waves with components $a_R(\mathbf{k}), a_L(\mathbf{k})$, the electromagnetic helicity h_{em} can be written as [37]

$$h_{em} = h_m + h_e = \hbar \int d^3k (\bar{a}_R(\mathbf{k})a_R(\mathbf{k}) - \bar{a}_L(\mathbf{k})a_L(\mathbf{k})), \quad (107)$$

where $\hbar$ is the Planck constant. In Quantum Electrodynamics, the integral in
580 the right hand side of equation (107) is interpreted as the helicity operator, that subtracts the number of left-handed photons from the number of right-handed photons. From the usual expressions [58]

$$\begin{aligned} N_R &= \int d^3k \bar{a}_R(\mathbf{k})a_R(\mathbf{k}), \\ N_L &= \int d^3k \bar{a}_L(\mathbf{k})a_L(\mathbf{k}), \end{aligned} \quad (108)$$

we can write (107) as

$$h_{em} = \hbar (N_R - N_L). \quad (109)$$

Thus the electromagnetic helicity is the classical limit of the difference between the numbers of right-handed and left-handed photons [59, 60, 15]. Equation
585 (109) relates the wave and particle aspects of the electromagnetic helicity. The wave helicity is the sum of the electric and magnetic helicities and characterises the topology of the field lines as a function of the mean linking number of electric lines and magnetic lines. The particle aspect of the electromagnetic helicity is the classical limit of the difference between right- and left-handed
590 photons. An electromagnetic field with a trivial configuration of force lines (zero linking number), is such that the classical expression for the number of right-handed photons is equal to the classical expression for the number of left-handed photons. On the opposite, a non-zero linking number of field lines

indicates that the numbers of right-handed photons and left-handed photons
595 will be classically different.

The electromagnetic helicity for an electromagnetic knot constructed from maps $S^3 \rightarrow S^2$ is given by

$$h_{em} = h_m + h_e = \frac{a}{2c\mu_0} [H(\phi) + H(\theta)]. \quad (110)$$

This means that, for electromagnetic knots from maps $S^3 \rightarrow S^2$, we arrive at the relation

$$N_R - N_L = \frac{a}{2\hbar c\mu_0} [H(\phi) + H(\theta)]. \quad (111)$$

This equation has been called the topological quantization of the electromagnetic helicity [37] since it relates the helicity to a pair of integer numbers that are Hopf indices of maps $S^3 \rightarrow S^2$ used to build the magnetic and electric fields.

It is clear that not all the Rañada fields come from maps $S^3 \rightarrow S^2$. We
600 have previously seen examples in which ϕ and θ do not satisfy conditions to be considered as such kind of maps between spheres (see equations (72) and (76)). However, the Rañada fields that come from maps $S^3 \rightarrow S^2$ show very interesting topological properties that have a reflection in their physical behaviour. In particular, their (magnetic or electric) field lines form links and knots, and their
605 link and knot type appears to be the same for all the curves and for every time. This topological constraint can be used in interesting applications, as we will see later. And this is the main reason for the search of explicit examples of these fields.

4. The electromagnetic Hopfion

610 The first topologically nontrivial solution was based in the Hopf fibration and can be called the Hopf-Rañada electromagnetic knot [8, 9, 32, 33, 15, 35, 17, 28]. In a more general context, this solution is also known as the electromagnetic Hopfion. In 2009, Besieris and Shaarawi showed [61] that the Hopfion is equivalent to the Robinson congruence [62], used in gauge theories by Trautman [10]
615 and well studied by Bialynicki-Birula [63].

As mentioned previously, the term knot is a bit confusing for the Hopfion, since the field lines are isomorphic to linked circles, which can be considered rather trivial knots. However, those solutions are given by two complex scalar fields as defined by Rañada as we will see.

620 4.1. Scalar fields and explicit expression of the Hopf-Rañada knot

The explicit solution of the time-dependent scalar fields $\phi(\mathbf{r}, t)$, $\theta(\mathbf{r}, t)$ that satisfy equations (30) and (31) for the electromagnetic Hopfion was obtained in [15] and can be written as

$$\begin{aligned}\phi &= \frac{(AX - TZ) + i(AY + T(A - 1))}{(AZ + TX) + i(A(A - 1) - TY)}, \\ \theta &= \frac{(AY + T(A - 1)) + i(AZ + TX)}{(AX - TZ) + i(A(A - 1) - TY)}.\end{aligned}\quad (112)$$

In these expressions, coordinates (X, Y, Z, T) are dimensionless to be sure that the complex scalar fields ϕ and θ are also dimensionless. In this way, they can be used in applications covering different length scales. From the Minkowski space coordinates (x, y, z, t) , upper-case coordinates are defined as

$$X = \frac{x}{L_0}, Y = \frac{y}{L_0}, Z = \frac{z}{L_0}, T = \frac{ct}{L_0}, \quad (113)$$

where L_0 is a characteristic length size of the scalar fields. Equations (113) are supplemented with

$$R = \frac{r}{L_0} = \frac{\sqrt{x^2 + y^2 + z^2}}{L_0} = \sqrt{X^2 + Y^2 + Z^2}. \quad (114)$$

In the expressions (112), the quantity A (not to be confused with the modulus of a vector potential $\mathbf{A}$) is

$$A = \frac{R^2 - T^2 + 1}{2}. \quad (115)$$

The magnetic and electric fields corresponding to the scalar fields (112) are
625 computed using (30) and (31). The result is

$$\begin{aligned}\mathbf{B}(\mathbf{r}, t) &= \frac{\sqrt{a}}{\pi L_0^2} \frac{Q \mathbf{H}_1 + P \mathbf{H}_2}{(A^2 + T^2)^3}, \\ \mathbf{E}(\mathbf{r}, t) &= \frac{c\sqrt{a}}{\pi L_0^2} \frac{Q \mathbf{H}_2 - P \mathbf{H}_1}{(A^2 + T^2)^3}.\end{aligned}\quad (116)$$

where the quantities P, Q are defined by

$$P = T(T^2 - 3A^2), Q = A(A^2 - 3T^2), \quad (117)$$

and the vectors $\mathbf{H}_1$ and $\mathbf{H}_2$ are,

$$\begin{aligned} \mathbf{H}_1 &= (Y + T - XZ) \mathbf{u}_x - (X + (Y + T)Z) \mathbf{u}_y + \left(\frac{-1 - Z^2 + X^2 + (Y + T)^2}{2} \right) \mathbf{u}_z, \\ \mathbf{H}_2 &= \left(\frac{1 + X^2 - Z^2 - (Y + T)^2}{2} \right) \mathbf{u}_x + (-Z + X(Y + T)) \mathbf{u}_y + (Y + T + XZ) \mathbf{u}_z \end{aligned} \quad (118)$$

4.2. Euler potentials of the Hopfion

In order to study the magnetic and electric surfaces created by this electromagnetic field, and its magnetic and electric field lines, we use Euler potentials

as in the equations (68), so that

$$\begin{aligned} \mathbf{B}(\mathbf{r}, t) &= \sqrt{a} \nabla \alpha_1 \times \nabla \alpha_2, \\ \mathbf{E}(\mathbf{r}, t) &= c \sqrt{a} \nabla \beta_2 \times \nabla \beta_1. \end{aligned} \quad (119)$$

The Euler potentials for the electromagnetic knot (112) are

$$\begin{aligned} \alpha_1(\mathbf{r}, t) &= \frac{(AZ + TX)^2 + (A(A - 1) - TY)^2}{(A^2 + T^2)^2}, \\ \alpha_2(\mathbf{r}, t) &= \frac{1}{2\pi} \arctan \left(\frac{(A^2 - T^2)(YZ - (A - 1)X) + 2AT(XY + (A - 1)Z)}{(A^2 - T^2)(XZ + (A - 1)Y) + AT((A - 1)^2 + X^2 - Y^2 - Z^2)} \right), \\ \beta_1(\mathbf{r}, t) &= \frac{(AX - TZ)^2 + (A(A - 1) - TY)^2}{(A^2 + T^2)^2}, \\ \beta_2(\mathbf{r}, t) &= \frac{1}{2\pi} \arctan \left(\frac{(A^2 - T^2)(XZ - (A - 1)Y) + AT(X^2 + Y^2 - Z^2 - (A - 1)^2)}{(A^2 - T^2)(XY + (A - 1)Z) + 2AT((A - 1)X - YZ)} \right). \end{aligned} \quad (120)$$

The first two of these Euler potentials give the magnetic surfaces and the intersections of each pair of magnetic surfaces are the magnetic field lines. The last two Euler potentials give the electric surfaces and their intersections are the electric field lines. Using the alternative formalism of geometric algebra, Van Enk has shown this particular choice of the Euler potentials to be covariant [50].

Consider first the magnetic surfaces. In Fig. 3 we plot some magnetic surfaces given by $\alpha_1(\mathbf{r}, t) = k_1$ at instant of time $t = 0$, where $0 \leq k_1 \leq 1$. These surfaces satisfy the equation

$$(1 - k_1)R^4 - 2(1 + k_1)R^2 + 4Z^2 + (1 - k_1) = 0. \quad (121)$$

Figure 3: Structure of magnetic surfaces of the electromagnetic Hopfion at $T = 0$, given as solutions of the equation $\alpha_1(\mathbf{r}, t = 0) = k_1$, with $0 \leq k_1 \leq 1$. The surface $\alpha_1(\mathbf{r}, t = 0) = 0$ is singular, and it is the circle $X^2 + Y^2 = 1$. For $0 < k_1 < 1$ we have tori around the circle $x^2 + y^2 = 1$, each of them covering the previous ones. The points in these tori cover all the space R^3 . In the figure we have plotted the tori $k_1 = 0.001, 0.09, 0.5, 0.9995$. When k_1 approaches the value $k_1 = 1$ the radii of the torus approach infinity (see the surface $k_1 = 0.9995$), and we have another singular surface, that is the Z -axis.

The magnetic surfaces for $k_1 = 0$ and $k_1 = 1$ are extreme cases, and they are singular. For $k_1 = 1$, we have the Z -axis, and for $k_1 = 0$ we have the curve $X^2 + Y^2 = 1$. Both are magnetic lines. For $0 < k_1 < 1$, the rest of magnetic
640 surfaces $\alpha_1(\mathbf{r}, t = 0) = k_1$ are nested tori between the singular extreme cases $k_1 = 0$ and $k_1 = 1$. This structure means that the magnetic lines of the magnetic field (116) at $t = 0$ lie on tori.

The magnetic surfaces $\alpha_2(\mathbf{r}, t) = k_2$ at $T = 0$ are given by

$$\frac{2YZ - X(R^2 - 1)}{2XZ + Y(R^2 - 1)} = \tan(2\pi k_2), \quad (122)$$

for different values of k_2 . The surfaces given by equation (122) are more complicated than the tori of equation (121). Some examples are plotted in Fig. 4.

645 The magnetic lines at $T = 0$ are the intersections of the surfaces $\alpha_1(\mathbf{r}, 0) = k_1$ with the surfaces $\alpha_2(\mathbf{r}, 0) = k_2$. Each one of these intersections is a closed curve in R^3 , and this curve is topologically equivalent to a circle (see left pannel in Fig. 6). The set of all these magnetic lines is known as the Hopf fibration [54].

Figure 4: The magnetic surface of the electromagnetic Hopfion at $T = 0$, given as solution of the equation $\alpha_2(\mathbf{r}, t = 0) = k_2$ with $k_2 = -1/2, 0, 1/2$.

The structure of the magnetic surfaces is topologically conserved as time evolves. In Fig. 5 we have plotted some magnetic surfaces $\alpha_1(\mathbf{r}, T = 0.4) = k_1$ as an example. The structure of the magnetic lines is also conserved as can be seen in the right pannel of Fig. 6.

At $t = 0$, the electric surfaces given by $\beta_1(\mathbf{r}, t = 0) = k'_1$, with $0 \leq k'_1 \leq 1$, satisfy

$$(1 - k'_1)R^4 - 2(1 + k'_1)R^2 + 4X^2 + (1 - k'_1) = 0. \quad (123)$$

Note that this equation is very similar to the equation (121) for the magnetic lines at $T = 0$, with the change $Z \rightarrow X$. This means that the electric surfaces $\beta_1(\mathbf{r}, t = 0) = k'_1$ can be given as a $\pi/2$ -rotation of the magnetic surfaces $\alpha_1(\mathbf{r}, t = 0) = k'_1$ around the Y -axis. Consequently, magnetic surfaces and electric surfaces are orthogonal at each point of R^3 . For $k'_1 = 1$, we have a singular electric surface that is the X -axis. For $k'_1 = 0$ we have another singular surface, that is the circle $Y^2 + Z^2 = 1$. For $0 < k'_1 < 1$, the rest of electric surfaces $\beta_1(\mathbf{r}, t = 0) = k'_1$ are nested tori. The overall structure is the same as in Fig. 3 with a $\pi/2$ rotation around the Y -axis. At $t = 0$, the electric surfaces $\beta_2(\mathbf{r}, t = 0) = k'_2$ are given by

$$\frac{2XZ - Y(R^2 - 1)}{2XY + Z(R^2 - 1)} = \tan(2\pi k'_2). \quad (124)$$

Figure 5: Structure of magnetic surfaces of the electromagnetic Hopfion at $T = 0.4$ as solutions of the equation $\alpha_1 = k_1$. The overall topological structure is the same as the one in Fig. 4 but the surfaces have been smoothly deformed. We plot the cases $k_1 = 0.001, 0.09, 0.5$.

Again, this equation is similar to (123) with the changes $X \rightarrow Y$, $Y \rightarrow Z$, $Z \rightarrow X$. This transformation is a positive permutation of coordinates (X, Y, Z) .
 655 An example of these electric surfaces is then the one given in Fig. 4 with a positive permutation of the R^3 -coordinates. The intersection of a pair of electric surfaces, one of them given by (123) and the other given by (124), is topologically equivalent to a circle, the set of all the electric lines is also a Hopf fibration, whose structure is conserved as time evolves, and the electric line at a given
 660 point is then orthogonal to the magnetic line at the same point.

However, note that not all the electric surfaces can be obtained from the magnetic surfaces by a positive permutation of coordinates (X, Y, Z) when $T \neq 0$. This only happens with $T = 0$. To verify it, one simply makes the permutation in α_1 and α_2 and compare the result with β_1 and β_2 in equation (120). Nev-
 665 ertheless at every time the magnetic and electric surfaces are similar and they are orthogonal to each other at every point of R^3 , so the magnetic lines of the electromagnetic Hopfion are always orthogonal to the electric lines.

Figure 6: Structure of magnetic lines of the electromagnetic Hopfion. The intersections of magnetic isosurfaces are magnetic lines. Left at $T = 0, \alpha_1 = 0.3, \alpha_2 = 0$. Right the same at $T = 0.2$.

4.3. The fibration of the Hopfion

The structure of all the magnetic lines at a given instant of time, and also the
670 structure of all the electric lines at a given instant of time, can be understood
by considering the scalar fields (112) as maps $S^3 \times R \rightarrow S^2$. To do that, we
write the scalar fields as

$$\begin{aligned}\phi &= \frac{u_1 + i u_2}{u_3 + i u_4}, \\ \theta &= \frac{u_2 + i u_3}{u_1 + i u_4},\end{aligned}\tag{125}$$

where the quantities

$$\begin{aligned}u_1(\mathbf{r}, t) &= \frac{AX - TZ}{(A^2 + T^2)}, \\ u_2(\mathbf{r}, t) &= \frac{AY + T(A - 1)}{(A^2 + T^2)}, \\ u_3(\mathbf{r}, t) &= \frac{AZ + TX}{(A^2 + T^2)}, \\ u_4(\mathbf{r}, t) &= \frac{A(A - 1) - TY}{(A^2 + T^2)},\end{aligned}\tag{126}$$

satisfy $u_1^2 + u_2^2 + u_3^2 + u_4^2 = 1$, and can be interpreted as time-dependent coordinates of the sphere S^3 , equation (126) being a time-dependent smooth deformation of the stereographic projection (87). In fact, the exact form of (87) can be obtained by taking $t = 0$ in (126). We can also write coordinates in S^2 from

the complex field ϕ by using the stereographic projection (88). We get

$$\phi = \frac{n_1 + i n_2}{1 - n_3}, \quad (127)$$

where $n_1^2 + n_2^2 + n_3^2 = 1$, so that (n_1, n_2, n_3) are coordinates of the sphere S^2 .

675 These coordinates are given by the inverse relation

$$\begin{aligned} n_1 &= \frac{\phi + \bar{\phi}}{\phi \bar{\phi} + 1}, \\ n_2 &= \frac{-i(\phi - \bar{\phi})}{\phi \bar{\phi} + 1}, \\ n_3 &= \frac{\phi \bar{\phi} - 1}{\phi \bar{\phi} + 1}. \end{aligned} \quad (128)$$

Using (125) and (126), the scalar field ϕ can be interpreted as a time-dependent map $S^3 \rightarrow S^2$ that can be expressed as

$$\begin{aligned} n_1 &= 2(u_1 u_3 + u_2 u_4) \\ &= \frac{2}{(A^2 + T^2)^2} [(A^2 - T^2)(XZ + Y(A - 1)) + AT(X^2 - Y^2 - Z^2 + (A - 1)^2)], \\ n_2 &= 2(u_2 u_3 - u_1 u_4) \\ &= \frac{2}{(A^2 + T^2)^2} [(A^2 - T^2)(YZ - X(A - 1)) + 2AT(XY + Z(A - 1))], \\ n_3 &= u_1^2 + u_2^2 - u_3^2 - u_4^2 \\ &= \frac{1}{(A^2 + T^2)^2} [(A^2 - T^2)(X^2 + Y^2 - Z^2 - (A - 1)^2) + 4AT(Y(A - 1) - XZ)]. \end{aligned} \quad (129)$$

This is the Hopf map. When $t = 0$ we get the more familiar expressions

$$\begin{aligned} n_1 &= \frac{4}{(R^2 + 1)^2} [2XZ + Y(R^2 - 1)], \\ n_2 &= \frac{4}{(R^2 + 1)^2} [2YZ - X(R^2 - 1)], \\ n_3 &= \frac{1}{(R^2 + 1)^2} [4X^2 + 4Y^2 - 4Z^2 - (R^2 - 1)^2]. \end{aligned} \quad (130)$$

The Hopf structure of the level curves of the scalar field ϕ determines the
680 properties of the magnetic lines of the field given(116). The magnetic lines are level curves of ϕ and all of them are closed curves topologically equivalent to circles. Moreover, any pair of these magnetic lines is a link, and the linking number of any pair of magnetic lines is equal to 1. In Fig. 1, we have plotted some magnetic lines at $T = 0$ and at $T = 0.2$.

The scalar field θ shows a similar behaviour than ϕ . It can be expressed as a time-dependent map $S^3 \rightarrow S^2$ where

$$\begin{aligned}
 n_1 &= 2(u_1 u_2 + u_3 u_4) \\
 &= \frac{2}{(A^2 + T^2)^2} [(A^2 - T^2)(XY + Z(A - 1)) + 2AT(X(A - 1) - YZ)], \\
 n_2 &= 2(u_1 u_3 - u_2 u_4) \\
 &= \frac{2}{(A^2 + T^2)^2} [(A^2 - T^2)(XZ - Y(A - 1)) + AT(X^2 + Y^2 - Z^2 - (A - 1)^2)], \\
 n_3 &= u_2^2 + u_3^2 - u_1^2 - u_4^2 \\
 &= \frac{1}{(A^2 + T^2)^2} [(A^2 - T^2)(Y^2 + Z^2 - X^2 - (A - 1)^2) + 4AT(Y(A - 1) + XZ)].
 \end{aligned} \tag{131}$$

Again this is the Hopf map. It can be obtained from the map ϕ (130) by the changes $u_1 \rightarrow u_2$, $u_2 \rightarrow u_3$, $u_3 \rightarrow u_1$. When $t = 0$, the map θ is given by

$$\begin{aligned}
 n_1 &= \frac{4}{(R^2 + 1)^2} [2XY + Z(R^2 - 1)], \\
 n_2 &= \frac{4}{(R^2 + 1)^2} [2XZ - Y(R^2 - 1)], \\
 n_3 &= \frac{1}{(R^2 + 1)^2} [4Y^2 + 4Z^2 - 4X^2 - (R^2 - 1)^2].
 \end{aligned} \tag{132}$$

By analogy with the magnetic case, the electric lines are level curves of θ and all of them are closed curves topologically equivalent to circles. Any pair of electric lines is a link, and the linking number is equal to 1.

The fact that the linking number of any pair of magnetic lines is equal to 1 and the linking number of any pair of electric lines is equal to 1 for the electromagnetic Hopfion has an easy explanation: both the magnetic lines and the electric lines are the level curves of Hopf maps, and the Hopf index of the Hopf map is equal to 1. Remember that the Hopf index of a map $S^3 \rightarrow S^2$ is the linking number of any pair of its level curves. Consequently, for the Hopfion we have that the Hopf indices of the maps ϕ and θ satisfy

$$H(\phi) = H(\theta) = 1. \tag{133}$$

According to (105), this means that the magnetic and electric helicities of the

electromagnetic Hopfion are given by

$$\begin{aligned} h_m &= \frac{1}{2c\mu_0} \int d^3r \mathbf{A} \cdot \mathbf{B} = \frac{a}{2c\mu_0} H(\phi) = \frac{a}{2c\mu_0}, \\ h_e &= \frac{1}{2c^3\mu_0} \int d^3r \mathbf{C} \cdot \mathbf{E} = \frac{a}{2c\mu_0} H(\theta) = \frac{a}{2c\mu_0}, \end{aligned} \quad (134)$$

and the electromagnetic helicity is

$$h_{em} = h_m + h_e = \frac{a}{2c\mu_0} [H(\phi) + H(\theta)] = \frac{a}{c\mu_0}, \quad (135)$$

so that the classical limit of the difference between right- and left-handed photons as discussed previously is

$$N_R - N_L = \frac{h_{em}}{\hbar} = \frac{a}{\hbar c\mu_0}. \quad (136)$$

4.4. Further properties of the Hopfion

695 Next we look for some properties of the electromagnetic Hopfion. First, it is easy to see that this solution verifies the null field conditions $\mathbf{E} \cdot \mathbf{B} = 0$ and $E^2 - c^2 B^2 = 0$. To show these null field conditions, we use equations (116). Then

$$\begin{aligned} \mathbf{E} \cdot \mathbf{B} &= \frac{ac}{\pi^2 L_0^4 (A^2 + T^2)^6} [(Q^2 - P^2) \mathbf{H}_1 \cdot \mathbf{H}_2 + PQ(H_2^2 - H_1^2)] = 0, \\ E^2 - c^2 B^2 &= \frac{ac^2}{\pi^2 L_0^4 (A^2 + T^2)^6} [(Q^2 - P^2)(H_2^2 - H_1^2) - 2PQ \mathbf{H}_1 \cdot \mathbf{H}_2] = 0, \end{aligned} \quad (137)$$

since, by equation (118), $H_2^2 - H_1^2 = 0$ and $\mathbf{H}_1 \cdot \mathbf{H}_2 = 0$.

700 The null field conditions satisfied by the electromagnetic Hopfion allows to compute easily the magnetic and electric helicities from the magnetic and electric fields and show that equations (134) hold. First we use equations (106) to show that, for the electromagnetic Hopfion,

$$\begin{aligned} \frac{dh_m}{dt} &= -\frac{1}{c\mu_0} \int d^3r \mathbf{E} \cdot \mathbf{B} = 0, \\ \frac{dh_e}{dt} &= \frac{1}{c\mu_0} \int d^3r \mathbf{E} \cdot \mathbf{B} = 0. \end{aligned} \quad (138)$$

This means that both the magnetic and the electric helicities are constant.

705 Then,

$$\begin{aligned} h_m &= h_m(t=0) = \frac{1}{2c\mu_0} \int d^3r \mathbf{A}(\mathbf{r}, t=0) \cdot \mathbf{B}(\mathbf{r}, t=0), \\ h_e &= h_e(t=0) = \frac{1}{2c^3\mu_0} \int d^3r \mathbf{C}(\mathbf{r}, t=0) \cdot \mathbf{E}(\mathbf{r}, t=0). \end{aligned} \quad (139)$$

The magnetic and electric fields at $t=0$ can be taken from (116). We get

$$\begin{aligned} \mathbf{B}(\mathbf{r}, t=0) &= \frac{8\sqrt{a}}{\pi L_0^2(1+R^2)^3} \left[(Y - XZ) \mathbf{u}_x - (X + YZ) \mathbf{u}_y + \left(\frac{-1 - Z^2 + X^2 + Y^2}{2} \right) \mathbf{u}_z \right], \\ \mathbf{E}(\mathbf{r}, t=0) &= \frac{8\sqrt{ac}}{\pi L_0^2(1+R^2)^3} \left[\left(\frac{1 + X^2 - Z^2 - Y^2}{2} \right) \mathbf{u}_x + (-Z + XY) \mathbf{u}_y + (Y + XZ) \mathbf{u}_z \right] \end{aligned} \quad (140)$$

From them, the vector potentials such that $\mathbf{B}(\mathbf{r}, t=0) = \nabla \times \mathbf{A}(\mathbf{r}, t=0)$ and

$\mathbf{E}(\mathbf{r}, t=0) = \nabla \times \mathbf{C}(\mathbf{r}, t=0)$ can be taken as [33]

$$\begin{aligned} \mathbf{A}(\mathbf{r}, t=0) &= \frac{2\sqrt{a}}{\pi L_0(1+R^2)^2} [Y \mathbf{u}_x - X \mathbf{u}_y - \mathbf{u}_z], \\ \mathbf{C}(\mathbf{r}, t=0) &= \frac{2\sqrt{ac}}{\pi L_0(1+R^2)^2} [\mathbf{u}_x - Z \mathbf{u}_y + Y \mathbf{u}_z]. \end{aligned} \quad (141)$$

Writing (140) and (141) into (139), we get

$$\begin{aligned} h_m &= \frac{1}{2c\mu_0} \int d^3r \mathbf{A} \cdot \mathbf{B} = \frac{a}{2c\mu_0}, \\ h_e &= \frac{1}{2c^3\mu_0} \int d^3r \mathbf{C} \cdot \mathbf{E} = \frac{a}{2c\mu_0}, \end{aligned} \quad (142)$$

710 for the electromagnetic Hopfion, in agreement with (134), that was obtained using only topological arguments.

The energy density of the electromagnetic Hopfion is given by

$$U(\mathbf{r}, t) = \frac{\varepsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} = \frac{a}{4\pi^2\mu_0 L_0^4} \frac{(1 + X^2 + (Y + T)^2 + Z^2)^2}{(A^2 + T^2)^3}. \quad (143)$$

The maximum of the energy density is located at $X = Z = 0$ during the time evolution [28], and U is symmetric in the coordinates X and Z . In Fig. 7, we show some level surfaces of the energy density U for different times. $T = 0, 1.5, 2.5$. The energy density levels represented are 0.5, 1 and 1.5 in $a/(4\pi^2\mu_0 L_0^4)$ units. It can be seen how the level surfaces spread as the energy density goes to zero. The total electromagnetic energy of the knot, that is

$$\mathcal{E} = \int U d^3r = \frac{2a}{\mu_0 L_0}, \quad (144)$$

Figure 7: (Colour online) Evolution of energy density levels of the electromagnetic Hopfion. From left to right $T = 0$, $T = 1.5$, $T = 2.5$. From outside to inside, the energy levels correspond to values 0.5, 1 and 1.5 (red, green, blue) of the energy density in $a/(4\pi^2\mu_0L_0^4)$ units.

remains constant. At $T = 0$, the energy density has spherical symmetry and its maximum is located at the origin. As time increases, the symmetry is broken and the maximum is numerically located at $X = Z = 0$ and Y close to T .

715 Approximately, the position of the maximum of the energy density can be found up to $T = 1$ at $X = Z = 0$, $Y = T(1 + 6T^2)/(2 + 6T^2)$.

Following [28], the energy density of the electromagnetic Hopfion extends to infinity, but we can define a mean quadratic radius of the energy distribution as

$$\langle r^2 \rangle = \frac{\int (\mathbf{r} - \mathbf{r}_{max})^2 U d^3r}{\int U d^3r}, \quad (145)$$

where $\mathbf{r}_{max}$ is the position of the maximum of the distribution. At $T = 0$, the distribution has spherical symmetry and the mean quadratic radius of the distribution is given by $\sqrt{\langle r^2 \rangle} = L_0$. The maximum has a value of the energy density equal to 4^3 in $a/(4\pi^2\mu_0L_0^4)$ units, and at a distance equal to the mean quadratic value, the energy density is 4 in $a/(4\pi^2\mu_0L_0^4)$ units. This means that, 720 initially, more than 70% of the energy is localized inside a sphere of radius L_0 centered at the origin. As time evolves the mean quadratic radius of the distribution spreads out (its value at $t = L_0/c$ or $T = 1$ is about $1.1 L_0$) and the 725 position of the center is at $(0, 7/8, 0)$. The distribution is not well characterized as a sphere when $T \neq 0$, taking an umbrella-like shape [17].

Figure 8: Evolution of the position of electrons under the influence of the electromagnetic Hopfion. From left to right, $T = 2$, $T = 4.5$. The energy level shown corresponds to value 0.1 of the density energy in $a/(\mu_0 L_0^4)$ units for the same times. The initial velocity of the electrons is 0.

The linear momentum of the Hopfion takes the value

$$\mathbf{p} = \int \varepsilon_0 \mathbf{E} \times \mathbf{B} d^3r = \frac{a}{c\mu_0 L_0} \mathbf{u}_y. \quad (146)$$

As can be seen, it has a single contribution along the y -axis. This explains why the maximum of the energy density moves along this axis. In Fig. 8 we have plot the motion of test charges under the Hopfion influence. For more details about the motion of the test charges we point the interested reader to [28].

5. Non-null torus knotted electromagnetic fields

A new range of solutions covering the topology of the whole torus knots set and having the Hopfion as a particular case, was found [18]. In those solutions initially all the magnetic lines and all the electric lines are linked torus knots. However, for some members of the class, the properties of the Rañada construction are modified so the solutions are non-null fields in general. Although we call them non-null fields, we want to point that in this class there are null fields, the Hopfion for instance, so the class is broader.

Torus knots are knotted curves lying on the surface of a torus. Any torus knot is defined by two coprime integer numbers (n, m) in such a way that the

Figure 9: Some (n, m) torus knots. These knotted curves are lying on the surface of a torus in the three-dimensional space R^3 . In the figures, all the torus knots are called nontrivial. If n or m were equal to 1, the knot is said to be trivial since its representation is a circle, that can be linked to other ones.

curve winds n times around a circle inside the torus and m times around a line through the hole in the torus. In Fig. 9 there are plotted some particular examples.

The initial topology of the field lines in the new solutions will determine the evolution of the field, and the value of quantities like the energy, the momentum and the magnetic and electric helicities.

5.1. Construction of the class

Let $\phi_0(\mathbf{r})$, $\theta_0(\mathbf{r})$, two complex scalar fields that depend on the position $\mathbf{r}$ in the physical space R^3 . We impose that, at some initial time $t = 0$, the level curves of the complex scalar fields (ϕ_0, θ_0) coincide with the magnetic and

electric lines, so the magnetic and electric fields at $t = 0$ will be

$$\mathbf{B}(\mathbf{r}, 0) = \frac{\sqrt{a}}{2\pi i} \frac{\nabla \phi_0 \times \nabla \bar{\phi}_0}{(1 + \bar{\phi}_0 \phi_0)^2}, \quad (147)$$

$$\mathbf{E}(\mathbf{r}, 0) = \frac{\sqrt{ac}}{2\pi i} \frac{\nabla \bar{\theta}_0 \times \nabla \theta_0}{(1 + \bar{\theta}_0 \theta_0)^2}. \quad (148)$$

The notation $\bar{\phi}$ means the complex conjugate of ϕ and i is the imaginary unit. The expressions (147)-(148) are such that they satisfy $\nabla \cdot \mathbf{B} = 0$ and $\nabla \cdot \mathbf{E} = 0$ for any scalar fields ϕ and θ , which is a necessary condition for $\mathbf{B}(\mathbf{r}, 0)$ and $\mathbf{E}(\mathbf{r}, 0)$ to be admissible initial values of an electromagnetic field in vacuum.

We take both scalar fields ϕ_0 and θ_0 as maps $S^3 \rightarrow S^2$ after identifying the physical space R^3 with S^3 as we did for the Hopfion, so by construction at $t = 0$, all the pairs of field lines of $\mathbf{B}(\mathbf{r}, 0)$ are linked, and the linking number is given by the Hopf index $H(\phi_0)$ (103) of the map ϕ_0 , being the same for all pairs of lines. Correspondingly, all pairs of lines of the field $\mathbf{E}(\mathbf{r}, 0)$ are linked, the linking number of all pairs of lines is the same and it is given by the Hopf index $H(\theta_0)$ (104) of the map θ_0 .

As for the Hopfion case, dimensionless coordinates (X, Y, Z, T) , related to the physical ones (x, y, z, t) in the SI of units by $(X, Y, Z, T) = (x, y, z, ct)/L_0$, being c the speed of light are introduced. We will also use the relation $r^2/L_0^2 = (x^2 + y^2 + z^2)/L_0^2 = X^2 + Y^2 + Z^2 = R^2$, where L_0 is a constant with dimensions of length that can be considered to be the characteristic size of the knot.

The complex scalar fields are chosen as,

$$\phi_0 = \frac{(X + iY)^{(n)}}{(Z + i(R^2 - 1)/2)^{(m)}}, \quad (149)$$

$$\theta_0 = \frac{(Y + iZ)^{(l)}}{(X + i(R^2 - 1)/2)^{(s)}}, \quad (150)$$

where n, m, l and s are positive integer numbers. These scalar fields are related to the Seifert fibrations [64]. The main difference is that, in equations (149) and (150), the notation $\eta^{(n)}$, η being a complex number, means to leave the modulus of η invariant while the phase of η is multiplied by n .

Level curves of the complex scalar fields given by expression (149) and (150) are (n, m) linked torus knots. If we choose any complex number, say $2 + i$, the

Figure 10: Time evolution of field lines for the $(4,3)$ torus knot. We plot two magnetic (up) and electric (down) lines for $T = 0$, $T = 0.015$ and $T = 0.02$ (left to right). Lines are obtained numerically by integrating the explicit time dependent expressions (151)-(152) of the magnetic and the electric field. At $T = 0$ all the field lines are linked $(4,3)$ torus knots and the linking number is $nm = 12$. When $T > 0$ we find $(4,3)$ linked torus knots.

775 equation $\phi = 2 + i$ gives a curve in the space R^3 , and this curve is a (n, m) torus knot. If we choose any other value, say $1 - i$, we get another (n, m) torus knot. The tangent vectors of such curves are parallel to $\nabla \Re(\phi) \times \nabla \Im(\phi)$. Moreover, both curves are linked and their linking number is precisely the Hopf index of ϕ , that is $H(\phi) = nm$ in this case. This occurs for any level curves of ϕ . Since
780 level curves of ϕ coincide, through equation (147), with magnetic lines at $t = 0$, we can say that any pair of magnetic lines at $t = 0$ is a linked pair of (n, m) torus knots, and that the linking number is nm . The same can be said about the complex scalar field θ given by equation (150) and the electric field at $t = 0$ given by equation (148), so that any pair of electric lines at $t = 0$ is a linked
785 pair of (l, s) torus knots and the linking number is $H(\theta) = ls$.

Now, proceeding like in the case of the Hopfion, it was found that the electric

and magnetic fields are given by [18]

$$\mathbf{B}(\mathbf{r}, t) = \frac{\sqrt{a}}{\pi L_0^2} \frac{Q \mathbf{H}_1 + P \mathbf{H}_2}{(A^2 + T^2)^3} \quad (151)$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{\sqrt{ac}}{\pi L_0^2} \frac{Q \mathbf{H}_4 - P \mathbf{H}_3}{(A^2 + T^2)^3} \quad (152)$$

where the quantities A , P , Q are defined by

$$A = \frac{R^2 - T^2 + 1}{2}, \quad (153)$$

$$P = T(T^2 - 3A^2), \quad (154)$$

$$Q = A(A^2 - 3T^2), \quad (155)$$

and the vectors $\mathbf{H}_1$, $\mathbf{H}_2$, $\mathbf{H}_3$ and $\mathbf{H}_4$ are

$$\begin{aligned} \mathbf{H}_1 &= (-nXZ + mY + sT) \mathbf{u}_x + (-nYZ - mX - lTZ) \mathbf{u}_y \\ &\quad + \left(n \frac{-1 - Z^2 + X^2 + Y^2 + T^2}{2} + lTY \right) \mathbf{u}_z, \\ \mathbf{H}_2 &= \left(s \frac{1 + X^2 - Y^2 - Z^2 - T^2}{2} - mTY \right) \mathbf{u}_x \\ &\quad + (sXY - lZ + mTX) \mathbf{u}_y + (sXZ + lY + nT) \mathbf{u}_z, \\ \mathbf{H}_3 &= (-mXZ + nY + lT) \mathbf{u}_x + (-mYZ - nX - sTZ) \mathbf{u}_y \\ &\quad + \left(m \frac{-1 - Z^2 + X^2 + Y^2 + T^2}{2} + sTY \right) \mathbf{u}_z, \\ \mathbf{H}_4 &= \left(l \frac{1 + X^2 - Y^2 - Z^2 - T^2}{2} - nTY \right) \mathbf{u}_x \\ &\quad + (lXY - sZ + nTX) \mathbf{u}_y + (lXZ + sY + mT) \mathbf{u}_z. \end{aligned} \quad (156)$$

790 Using the expressions (151) and (152), the field lines at any instant of
time can be numerically computed. We have plotted few lines for the case
(n, m, l, s) = (4, 3, 4, 3) in Fig. 10. At $T = 0$, all the lines are linked (4, 3) torus
knots and the linking number is $nm = 12$. These are the only kind of curves
that can be found since these lines are level curves of the scalar fields (149)-
795 (150). For $T > 0$ we find lines with the same topology. However, as T increases,
numerical evidence prevent us to rule out the existence of open field lines. A
similar behaviour was found also numerically for knot-like fields constructed
from Chandrasekhar-Kendall curl eigenstates [17].

5.2. Properties of the class

800 We proceed to enumerate some properties of the class. The Lorentz invariants are

$$\begin{aligned}\mathbf{E} \cdot \mathbf{B} &= \frac{ac}{\pi^2 L_0^4} \frac{1}{(A^2 + T^2)^4} [(ms - nl)(A^2 + T^2) \\ &\quad + (ml - ns)(A^2 + T^2)(1 - A)Y \\ &\quad + 2(ls - mn)AT(T^2 - A^2)],\end{aligned}\tag{157}$$

$$\begin{aligned}E^2 - c^2 B^2 &= \frac{ac^2}{\pi^2 L_0^4} \frac{1}{(A^2 + T^2)^4} [(n^2 - m^2)(A^2 + T^2)(X^2 + Y^2) \\ &\quad + (s^2 - l^2)(A^2 + T^2)(Y^2 + Z^2) + 4(m^2 - s^2)A^2 T^2 \\ &\quad - (n^2 - l^2)(A^2 - T^2)^2].\end{aligned}\tag{158}$$

The particular choice of the set (n, m, l, s) determines the properties of the new torus knots class:

- If the positive integer numbers n, m, l, s are equal, i.e. satisfy the condition
805 $n = m = l = s$, then $\mathbf{E} \cdot \mathbf{B} = 0$ and $E^2 - c^2 B^2 = 0$. This is the case of the Hopfion.
- If the condition $n = m = l = s$ is not fulfilled, then $\mathbf{E} \cdot \mathbf{B} \neq 0$ and $E^2 - c^2 B^2 \neq 0$. We have a non-null electromagnetic field in vacuum. This field is non-null and non-decomposable.

810 With the help of equation (157), the time behaviour of the helicities of these electromagnetic fields can be exactly computed. The calculations yield

$$\begin{aligned}h_m(t) &= \frac{1}{2\mu_0} \int d^3r \mathbf{A} \cdot \mathbf{B} \\ &= \frac{a}{4\mu_0} \left[(mn + ls) + (mn - ls) \frac{1 - 6T^2 + T^4}{(1 + T^2)^4} \right],\end{aligned}\tag{159}$$

$$\begin{aligned}h_e(t) &= \frac{\varepsilon_0}{2} \int d^3r \mathbf{C} \cdot \mathbf{E} \\ &= \frac{a}{4\mu_0} \left[(mn + ls) - (mn - ls) \frac{1 - 6T^2 + T^4}{(1 + T^2)^4} \right],\end{aligned}\tag{160}$$

where we recall that $T = ct/L_0$, being t the time.

Figure 11: (Colour online) Time evolution of the magnetic h_m (blue solid line) and electric h_e (green dashed line) helicities in units of $a/(2\mu_0)$. Initially, for $t = 0$, $h_m = 1$ and $h_e = 0$. The sum of both is the electromagnetic helicity (red dash-dot line), that is equal to 1 in the same units. The interchange between magnetic and electric helicities is clear in the figure. Both helicities reach a common value for large values of time.

If, at $t = 0$, we have $h_m \neq h_e$, then both helicities are not constant but, for large time, both reach the same value. This property was first discovered and studied in [36], and called the exchange of helicities mechanism, as the total electromagnetic helicity

$$h_{em} = h_m(t) + h_e(t) = \frac{a}{2\mu_0} (nm + ls),$$

remains constant. An explicit example of the interchange of helicities is plotted in Fig. 11 in which $n = m = 1$ and $l = s = 0$. When t is large, both helicities coincide, i. e. $\lim_{t \rightarrow \infty} h_m(t) = \lim_{t \rightarrow \infty} h_e(t)$. This result suggests that there is an exchange of helicity between the electric and magnetic field until they reach a common value.

Another important property is that the electromagnetic energy content of the fields remains finite and is related to the square of the integer numbers

(m, n, s, l) by

$$\mathcal{E} = \int d^3r \left(\frac{\varepsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} \right) = \frac{a}{2\mu_0 L_0} (n^2 + m^2 + l^2 + s^2). \quad (161)$$

The linear momentum of these knotted solutions can also be obtained from the Poynting vector $\mathbf{E} \times \mathbf{B}/\mu_0$ and results

$$\mathbf{p} = \int \varepsilon_0 \mathbf{E} \times \mathbf{B} = \frac{a}{2c\mu_0 L_0} (ln + ms) \mathbf{u}_y. \quad (162)$$

In general, using vector identities it is easily proved that

$$\left| \int d^3r \varepsilon_0 \mathbf{E} \times \mathbf{B} \right| \leq \frac{1}{c} \int d^3r \left(\frac{\varepsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} \right), \quad (163)$$

The equality holds only when the field transport energy at the speed of light (see [65]), being the velocity at which the energy is transported defined as the time derivative of

$$\langle \mathbf{r} \rangle = \frac{\int \mathbf{r} U d^3r}{\int U d^3r}, \quad (164)$$

i.e. the equality for expression (163) is obtained when

$$v_u = \left| \frac{d \langle \mathbf{r} \rangle}{dt} \right| = c. \quad (165)$$

However, from expressions (161) and (162), we always have the strict inequality

$$p < \frac{\mathcal{E}}{c}, \quad (166)$$

being $p = |\mathbf{p}|$. Thus we can attribute a rest energy to our electromagnetic field given by the relation

$$\mathcal{E}^2 = p^2 c^2 + \mathcal{E}_0^2. \quad (167)$$

Taking into account results (161) and (162) for the energy and linear momentum of these knotted electromagnetic structures, $\mathcal{E}_0$ reads

$$\mathcal{E}_0 = \frac{a}{2\mu_0 L_0} \sqrt{(n^2 + m^2 + l^2 + s^2)^2 - (ln + ms)^2}. \quad (168)$$

This rest energy depends on the characteristic size of the knot L_0 , on the constant a included in the definition of this kind of knotted electromagnetic structures (147)-(148) and on the integer numbers (n, m, l, s) which determine the topological nature of the magnetic and electric torus knots.

Figure 12: Time evolution of the Poynting vector $\mathbf{E} \times \mathbf{B}/\mu_0$ for the electromagnetic field given by equations (151)-(152) in the case in which $n = 2, m = 3, s = 2, l = 3$. We plot the Poynting vectors at some points of space for $T = 0, T = 0.5, T = 1, T = 1.5$ and $T = 2.5$ (left to right, up to bottom).

A way to picture the evolution of these fields is to plot Poynting vectors at some points in space at different times. In Fig. 12, we see the behaviour of these electromagnetic fields for instants of time $T = 0, 0.5, 1, 1.5, 2.5$. In the figure we
825 have plotted the case $(n, m, s, l) = (2, 3, 2, 3)$ for the initial magnetic field lines, and $(l, s) = (2, 3)$ for the initial electric field lines. The magnetic and electric fields are given by equations (151) and (152) with $n = 3, m = 4, l = 2, s = 3$.

6. Null torus knotted electromagnetic fields

We have discussed how to build electromagnetic fields in vacuum covering
830 the whole torus set of knots in at least one instant of time. At that particular time they are globally described by two Rañada's scalar fields ϕ and θ . These solutions were non-null fields.

In this section we present a new class of solutions [19]. If the null condition is preserved during time evolution, the electric and magnetic field remain perpendicular which impose a strong geometric constrain on the topology of the field lines. By Robinson's theorem [62], there is a shear-free family of light rays, underlying each null electromagnetic field. The light rays are given by the Poynting vector, and the electromagnetic field evolution can be thought of as a fluid-like, being transported along the world lines of the light rays. The rays follow geodesics paths in free space. The angles between $\mathbf{E}$ and $\mathbf{B}$ are preserved along those paths as they remain perpendicular, so the fluid evolution is shear-free. The null condition imposes then a strong constrain on the configuration of the field lines, reducing the problem to find a triplet of mutually orthogonal fields that remain orthogonal under time evolution. It turns out that it is possible to build null solutions whose field lines also encode torus knots and links [19].

Bateman [20] developed a formalism for the construction of the null fields with the introduction of complex Euler potentials instead of real ones. Also one can use the spinor formalism to achieve the same results [12]. We will briefly review here the Bateman's construction, leaving the spinor formulation for a latter section.

6.1. Bateman's null fields

Bateman construction [20] is based on the introduction of two complex scalar fields α and β which are functions of space-time coordinates. In order that they generate electromagnetic fields, they have to fulfil the conditions

$$\begin{aligned}\partial_\mu \alpha \partial^\mu \alpha &= 0, \\ \partial_\mu \beta \partial^\mu \beta &= 0, \\ \partial_\mu \alpha \partial^\mu \beta &= 0.\end{aligned}\tag{169}$$

In the original work, Bateman introduced the complex vector [20] $\mathbf{M} = c\mathbf{B} \pm i\mathbf{E}$, so Maxwell's equation in vacuum (3) and (4) can be written as

$$\nabla \times \mathbf{M} = \mp \frac{i}{c} \frac{\partial \mathbf{M}}{\partial t}, \quad \nabla \cdot \mathbf{M} = 0.\tag{170}$$

Note that Silberstein [66] made use of the same vector without the $\pm$ sign, only choosing one sign and defining $c\mathbf{B} - i\mathbf{E}$ and its complex conjugate. Earlier, according to Weber [67], Riemann originally took $\mathbf{F} = \mathbf{E} + ic\mathbf{B}$. Some authors call this last one the Riemann-Silberstein vector $\mathbf{F}$ and we will follow this convention [68].

If we can find two functions (α, β) such that the conditions (169) are satisfied, we can write

$$\mathbf{M} = \nabla\alpha \times \nabla\beta. \quad (171)$$

as a solution to the equations (170). The scalar functions (α, β) play the same role as the Euler potentials but they are now complex functions. We can check that by construction we get null fields as $\mathbf{M}^2 = 0$.

6.2. Knotted fields based on Bateman's construction

We have already discussed the Hopf-Rañada solution or Hopfion and how it can be expressed in terms of Euler potentials. It was found that the Hopfion solution also can be written in terms of the two complex scalar fields (α, β) using Bateman's construction [61], as the Hopfion is a null electromagnetic field.

The Riemann-Silberstein complex vector for the Hopfion is easily calculated from (116) as

$$\mathbf{F}(\mathbf{r}, t) = \frac{\sqrt{ac}}{\pi L_0^2 (A + iT)^3} \begin{bmatrix} \frac{1+X^2-Z^2-(Y+T)^2}{2} + i(Y+T-XZ) \\ (X-iZ)(Y+T-i) \\ Y+T+XZ + i\left(\frac{-1+X^2-Z^2+(Y+T)^2}{2}\right) \end{bmatrix}. \quad (172)$$

We recall that A is given by (115). It can be easily checked that the Riemann vector can be written as

$$\mathbf{F}(\mathbf{r}, t) = \frac{c\sqrt{a}}{2\pi L_0^2} \nabla\alpha_h \times \nabla\beta_h \quad (173)$$

with the complex potentials

$$\alpha_h = -\frac{2Z + 2Xi}{Y + T - i}, \quad (174)$$

$$\beta_h = \left(\frac{Y + T - i}{X^2 + Y^2 + Z^2 - (T - i)^2} \right)^2. \quad (175)$$

870 Using Bateman's framework, and the fact that there is a natural connection
between knots and singular points of complex maps [69], a new class of knotted
solutions was found with the property that there is at least a singular line with
the topology of a torus knot and they are null fields [19]. In particular, the
intersection of the four dimensional sphere S^3 with the complex hypersurface
875 $z_1^n \pm z_2^m = 0$ (where $z_1, z_2 \in C$), gives a (n, m) torus knot when n and m are
coprime integers. This fact was used to construct knotted optical vortices in
light beams [70]. The trick is to take the following Bateman pair (α, β) based
on the stereographic projection of S^3 ,

$$\begin{aligned}\alpha &= \frac{X^2 + Y^2 + Z^2 - T^2 - 1 + 2Zi}{X^2 + Y^2 + Z^2 - (T - i)^2}, \\ \beta &= \frac{2X - 2Yi}{X^2 + Y^2 + Z^2 - (T - i)^2},\end{aligned}\quad (176)$$

so $|\alpha|^2 + |\beta|^2 = 1$, and $\alpha^n \pm \beta^m = 0$ gives torus knot lines if as said before n
and m are coprime. The potentials are taken as α^n and β^m , and consequently
the fields can be obtained using the Riemann vector as before

$$\mathbf{F} = \nabla \alpha^n \times \nabla \beta^m, \quad (177)$$

with (α, β) given by (176).

880 Specifically, the magnetic lines are organised around a set of core field lines
which are closed and form (n, m) torus knots or links when n and m are co-
primes, as plotted in Fig. 9. Again, as in the case of the class of non-null torus
fields, the Hopfion is contained in this class for the particular values $n = m = 1$.

The magnetic and electric helicities are equal for this new class. Normalised
by the energy, and in natural units, their values are [19]

$$\frac{h_e}{\mathcal{E}} = \frac{h_m}{\mathcal{E}} = \frac{1}{n + m}. \quad (178)$$

The linear momentum turns out to be in the z -direction, and normalized by the
energy reads

$$\frac{\mathbf{p}}{\mathcal{E}} = \left(0, 0, \frac{-n}{n + m}\right). \quad (179)$$

For more details of the properties of those solutions, we refer to [19].

885 7. Conformal transformations

There are several threads in the literature based on introducing complex space-time values in order to transform a familiar solution (e.g. of the Maxwell equations or Einstein equations) into a new solution. Two well know exam-
 890 z direction, from the field of a point charge at a location on the imaginary z axis, and the construction of the Kerr spacetime for a rotating black hole from the Schwarzschild spacetime solution via a shift of spacetime variables into the complex plane. This method is often combined with a change of the coordinate system, as is the case for the Kerr solution where cartesian coordinates are re-
 895 placed by oblate spherical coordinates. We will show that this technique can transform trivial solutions of the Maxwell equations, as simple as uniform and orthogonal $\mathbf{E}$ and $\mathbf{B}$ fields, into the topologically nontrivial solutions discussed in the previous sections.

The method of considering complex shifts dates back at least to the year
 900 1887 in which P. Appell wrote a remarkable letter to F. Klein. In order to familiarize ourselves with this concept we will briefly review this letter, as well as the constructions of Gaussian beams and the Kerr solution before addressing linked and knotted light field. Then we follow the construction given in [29] to obtain new fields from old ones using conformal inversion and complex shifts.
 905 This method is further generalized in [21].

7.1. Letter from Appell

In a letter from P. Appell to F. Klein [71] solutions to Laplace's equation $\Delta\psi = 0$ are considered. In 3 dimensions,

$$\psi(x, y, z) = \frac{A}{\sqrt{(x - \alpha)^2 + (y - \beta)^2 + (z - \sigma^2)}} \quad (180)$$

is a solution which is singular at (α, β, γ) for α, β, γ real. Appell considered solutions with complex values: $A = \lambda + i\mu$, $\alpha = a + ia'$, $\beta = b + ib'$, and $\gamma = c + ic'$, with $\lambda, \mu, a, a', b, b', c, c'$, real. A real field $\psi(x, y, z)$ can still

be obtained by simply adding the complex conjugate (which is also a solution).

This gives

$$\psi(x, y, z) = \frac{A}{\sqrt{S - iP}} + \frac{\bar{A}}{\sqrt{S + iP}}, \quad (181)$$

with

$$S = (x - a)^2 + (y - b)^2 + (z - c)^2 - a'^2 - b'^2 - c'^2,$$

and

$$P = 2a'(x - a) + 2b'(y - b) + 2c'(z - c).$$

$S = 0$ defines a sphere with radius $a'^2 + b'^2 + c'^2$ centered around (a, b, c) , and $P = 0$ is a plane through the center of the sphere $S = 0$. This means that $\psi(x, y, z)$ is singular on a circle (the intersection of $S = 0$ and $P = 0$). There is something very peculiar about this situation. The fact that we now deal with a singular circle implies that there is the possibility of following the field along a loop that is linked with the singular ring. Since P changes sign when crossing the plane $P = 0$ and S changes sign when crossing the surface of the sphere $S = 0$ the loop is topologically equivalent to a circle around the origin in the complex (S, iP) plane. Since the denominators of the two terms in (181) are complex square roots $\sqrt{z} = \sqrt{r}e^{i\theta/2}$ they pick up a minus sign by completing a loop ($\theta = 0$ to $\theta = 2\pi$) around the origin. This two-valuedness of $\psi(x, y, z)$ is of course the result of the Riemann surface of the complex square root. Still, by construction, we are considering only real valued solutions $\psi(x, y, x)$ to the Laplace equation.

7.2. Gaussian beams

In 1971 Deschamps showed that a Gaussian beam is equivalent in paraxial approximation to a spherical wave with a center at a complex location [30]. Consider the wave equation for

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \psi(x, y, z, t) - \nabla^2 \psi(x, y, z, t) = 0.$$

For a monochromatic solution with $\omega/c = k$ the equation for the spatial part of the wave function reduces to $(\nabla^2 + k^2)\psi(x, y, z) = 0$. The radial function

$$\psi(x, y, z) = \frac{e^{ikr}}{r}, \quad (182)$$

with r the distance of the point (x, y, z) from the origin, is a solution to the wave equation.

We now want to consider beam like solutions along the z -axis such that the amplitude variations in x, y directions are much larger than in the z direction. Therefore the second order derivative with respect to z can be neglected and the approximate paraxial wave equation becomes

$$\frac{1}{i} \frac{\partial}{\partial z} = k - \frac{1}{2k} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right). \quad (183)$$

Deschamps noticed that by taking the solution in (182) such that the origin is $(0, 0, ia)$ and assuming that $|x|$ and $|y|$ are much smaller than $|z - ia|$ the solution represent a Gaussian beam near the z -axis. This is seen by approximating r in the exponent as

$$r = \sqrt{(z - ia)^2 + x^2 + y^2} \approx z - ia + \frac{1}{2} \frac{x^2 + y^2}{z - ia},$$

and $r \approx z - ia$ in the denominator. The approximate solution then reads

$$\psi(x, y, z) = \frac{e^{ik(z-ia)}}{z - ia} \exp \left(\frac{ik}{2} \frac{x^2 + y^2}{z - ia} \right). \quad (184)$$

It is easily verified that (184) is indeed a Gaussian beam solution of the paraxial wave equation (183). Since lasers produce such beams to a good approximation this solution is highly relevant for practical applications.

7.3. Kerr solution

In Cartesian coordinates, the Schwarzschild geometry in Kerr-Schild form [72] is

$$g_{ab} = \eta_{ab} + \frac{2m}{r} l_a l_b, \quad (185)$$

where $\eta_{ab} = < -1, +1, +1, +1 >$ and $l_a = (1, x/r, y/r, z/r)$, with r the radial distance from the origin to the point (x, y, z) .

The Kerr solution for a rotating black hole is now obtained from the Schwarzschild solution by first considering (r, θ, ψ) to be oblate spheroidal coordinates, such that,

$$x + iy = (r + ia) \sin \theta e^{i\psi}, \quad z = r \cos \theta$$

and second, by replacing $2m/r$ by

$$m \left(\frac{1}{r + ia \cos \theta} + \frac{1}{r - ia \cos \theta} \right) = \frac{2mr}{r^2 + a^2 \cos^2 \theta}.$$

930 In the second step we recognize, similar to the Appell example above, a complex shift of the parameter r and adding the complex conjugate to obtain a real valued expression, and in the first step we see a coordinate transformation. This transformation is obviously not a Poincaré transformation and therefore does not simply represent a transformation of a solution to a different inertial
935 reference frame.

7.4. Conformal inversion

The full Maxwell's equations are invariant under Poincaré transformations $\mathcal{P}(1, 3)$. Poincaré transformations are part of the conformal group $\mathcal{C}(1, 3)$, under which the source free Maxwell equations are also invariant, as was discovered
940 by Bateman and Cunningham [73, 74]. In fact, requiring conformal symmetry $\mathcal{C}(1, 3)$ for a spin 1 vector field uniquely leads to source-free Maxwell equations [75].

When considering Poincaré transformations of electromagnetic fields, we do not generate new solutions of the Maxwell equations but rather describe the
945 same solution in a different inertial reference frame. However when one considers special conformal transformations of an electromagnetic field one does obtain genuine new solutions of the Maxwell equations.

Every element of the conformal group that does not belong to the Poincaré group (dilation and four special conformal transformations) can be composed
950 from Poincaré transformations and conformal inversions, which, like time reversal, parity and charge conjugation, form discrete symmetry operations.

A special conformal transformation $\mathcal{SCT}(b)$ is given by

$$x^\mu \rightarrow \tilde{x}^\mu = \frac{x^\mu + b^\mu (x_\nu x^\nu)}{1 + 2(b_\nu x^\nu) + (x_\nu x^\nu)(b_\nu b^\nu)}, \quad (186)$$

and can be built from a conformal inversion $\mathcal{CI}(a)$,

$$x^\mu \rightarrow \tilde{x}^\mu = a \frac{x^\mu}{x_\nu x^\nu}, \quad (187)$$

and a translation $\mathcal{T}(c)$,

$$x^\mu \rightarrow \tilde{x}^\mu = x^\mu + c^\mu, \quad (188)$$

as follows

$$\mathcal{SCT}(b) = \mathcal{CI}(1) \circ \mathcal{T}(b) \circ \mathcal{CI}(1).$$

Dilations $\mathcal{D}(a)$,

$$x^\mu \rightarrow ax^\mu, \quad (189)$$

can be obtained in the same way, but with different parameters,

$$\mathcal{D}(a) = \mathcal{CI}(a) \circ \mathcal{T}(0) \circ \mathcal{CI}(1) = \mathcal{CI}(a) \circ \mathcal{CI}(1).$$

Since any conformal transformation can be built from Poincaré transformations and conformal inversions, we are primarily interested in the role of conformal inversion in generating new solutions from existing solutions of the
955 Maxwell equations. Note that a conformal inversion (187) is singular for points on the light-cone. A general discussion of the conformal group should therefore include the compactification of Minkowski space [76]. However, for the following discussion it will suffice to consider the Minkowski space.

In order to arrive at new solutions from existing ones, we can proceed as follows [29]. Let us have two Minkowski spaces with coordinates $x^\mu \in M^4$ and $\tilde{x}^\mu \in \tilde{M}^4$, and a mapping between these manifolds given by a conformal inversion $\mathcal{CI}(1)$ (187),

$$f : x^\mu \rightarrow \tilde{x}^\mu(x) = \frac{x^\mu}{x \cdot x}.$$

Here we use the notation $x \cdot x$ for $x_\nu x^\nu$. For the inverse we get

$$f^{-1} : \tilde{x}^\mu \rightarrow x^\mu(\tilde{x}) = \frac{\tilde{x}^\mu}{(\tilde{x} \cdot \tilde{x})}.$$

We use f to pull back tensor fields to M^4 . In this way the metric tensor $\tilde{\eta}_{\mu\nu}$ is mapped to

$$g_{\mu\nu}(x) = (f^{pb}\tilde{\eta})_{\mu\nu}(x) = \frac{\partial \tilde{x}^\alpha(x)}{\partial x^\mu} \frac{\partial \tilde{x}^\beta(x)}{\partial x^\nu} \tilde{\eta}_{\alpha\beta} = \frac{1}{(x \cdot x)^2} \eta_{\mu\nu}$$

as $\tilde{\eta}_{\mu\nu} = \eta_{\mu\nu}$ and this shows that, by definition, f is a conformal map [46]. We will regard tensor fields in M^4 and therefore use $\eta_{\mu\nu}$ to lower or raise indices,

not $g_{\mu\nu}$. We proceed with pulling back a Faraday or electromagnetic field tensor satisfying source-free Maxwell equations ($\tilde{F}_{\mu\nu}$) from $\tilde{M}^4$,

$$F_{\mu\nu}(x) = (f^{pb}\tilde{F})_{\mu\nu}(x) = \frac{\partial\tilde{x}^\alpha(x)}{\partial x^\mu} \frac{\partial\tilde{x}^\beta(x)}{\partial x^\nu} \tilde{F}_{\alpha\beta}(\tilde{x}(x)), \quad (190)$$

and

$$F^{\mu\nu}(x) = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta} = \eta^{\mu\alpha}\eta^{\nu\beta} \frac{\partial\tilde{x}^\sigma}{\partial x^\alpha} \frac{\partial\tilde{x}^\kappa}{\partial x^\beta} \eta_{\sigma\lambda}\eta_{\kappa\rho} \tilde{F}^{\lambda\rho}(\tilde{x}(x)) = \left| \frac{\partial\tilde{x}}{\partial x} \right| \frac{\partial x^\mu}{\partial \tilde{x}^\lambda} \frac{\partial x^\nu}{\partial \tilde{x}^\rho} \tilde{F}^{\lambda\rho}, \quad (191)$$

in which we introduced the absolute value of the Jacobian in the last step, being $|\partial\tilde{x}/\partial x| = 1/(x \cdot x)^4$. These are tensor fields on M^4 . Since $\tilde{F}$ is a Faraday tensor on $\tilde{M}^4$, F is expected to be also a Faraday tensor on M^4 , satisfying the Bianchi identity (10). From (190),

$$\partial_\lambda F_{\mu\nu}(x) + \partial_\nu F_{\lambda\mu}(x) + \partial_\mu F_{\nu\lambda}(x) = 0,$$

follows by reshuffling of indices, systematic cancellation of terms and using the fact that $\tilde{F}$ is a Faraday tensor on $\tilde{M}^4$. Getting the dynamical equations (11) is more intricate, but again from (191),

$$\partial_\mu F^{\mu\nu}(x) = 0.$$

With this result we end up with a prescription for constructing new vacuum solutions to the Maxwell equations from known ones: in M^4 we can make an conformal inverse transform of coordinates. If F is a Faraday tensor then F' is also a Faraday tensor, in which

$$F'_{\mu\nu}(x) = (\partial_\mu f^\alpha)(\partial_\nu f^\beta)F_{\alpha\beta}(\tilde{x}(x)), \quad (192)$$

and

$$F'^{\mu\nu}(x) = \eta^{\mu\alpha}\eta^{\nu\beta}F'_{\alpha\beta}(x). \quad (193)$$

As pointerd out in Section 6 it can be advantageous to use the Riemann-Silberstein (RS) vector $\mathbf{F} = \mathbf{E} + i\mathbf{B}$. Instead of the Faraday tensor, $F_{\mu\nu}$, we should then use the self-dual or anti-self-dual form, $F_{\mu\nu}^{(sd)}$ or $F_{\mu\nu}^{(asd)}$, of it,

$$F_{\mu\nu}^{(asd)} = F_{\mu\nu} + i^*F_{\mu\nu}, \quad (194)$$

$$F_{\mu\nu}^{(sd)} = F_{\mu\nu} - i^*F_{\mu\nu}, \quad (195)$$

where ${}^*F_{\mu\nu}$ is defined in (15). We can get the complex conjugate of the RS vector $\bar{\mathbf{F}} = \mathbf{E} - i\mathbf{B}$ from the relation

$$F_{0i}^{(asd)} = \bar{\mathbf{F}}_i \quad (196)$$

From the different transformations of $F_{\mu\nu}$ (190) and $F^{\mu\nu}$ it is not immediately clear how to calculate $F^{(asd)'} from $F^{(asd)}$. But in the case of conformal inversions, we have some special properties like$

$$\frac{\partial \tilde{x}^\nu}{\partial x^\mu} = \left| \frac{\partial \tilde{x}}{\partial x} \right|^{\frac{1}{2}} \frac{\partial x^\nu}{\partial \tilde{x}^\mu} = \frac{\partial \tilde{x}_\mu}{\partial x_\nu},$$

so we get using (191),

$$F'^{\mu\nu}(x) = \left| \frac{\partial \tilde{x}}{\partial x} \right| \frac{\partial x^\mu}{\partial \tilde{x}^\lambda} \frac{\partial x^\nu}{\partial \tilde{x}^\rho} F^{\lambda\rho}(\tilde{x}(x)) = \frac{\partial \tilde{x}^\mu}{\partial x^\lambda} \frac{\partial \tilde{x}^\nu}{\partial x^\rho} F^{\lambda\rho}(\tilde{x}(x)),$$

and this leads using (190) to

$$F_{\mu\nu}^{(asd)'}(x) = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial \tilde{x}^\beta}{\partial x^\nu} F_{\alpha\beta}^{(sd)}(\tilde{x}(x)). \quad (197)$$

Thus the self-dual is transformed into an anti-self-dual form. Also

$$\begin{aligned} F'^{\rho\sigma}_{(asd)}(x) &= \eta^{\rho\mu} \eta^{\sigma\nu} F_{\mu\nu}^{(asd)'} = \frac{\partial \tilde{x}_\alpha}{\partial x_\rho} \frac{\partial \tilde{x}_\beta}{\partial x_\sigma} F^{\alpha\beta}_{(sd)} = \frac{\partial \tilde{x}^\rho}{\partial x^\alpha} \frac{\partial \tilde{x}^\sigma}{\partial x^\beta} F^{\alpha\beta}_{(sd)} \\ &= \left| \frac{\partial \tilde{x}}{\partial x} \right| \frac{\partial x^\rho}{\partial \tilde{x}^\alpha} \frac{\partial x^\sigma}{\partial \tilde{x}^\beta} F^{\alpha\beta}_{(sd)}(\tilde{x}(x)). \end{aligned}$$

From (197) and (196) the transformation of the RS vector follows [29],

$$\mathbf{F}'(x) = \frac{1}{(t^2 - r^2)^3} \left[-(t^2 - r^2) \bar{\mathbf{F}}(\tilde{x}(x)) + 2\mathbf{r} \times (\mathbf{r} \times \bar{\mathbf{F}}(\tilde{x}(x)) - 2it\mathbf{r} \times \bar{\mathbf{F}}(\tilde{x}(x))) \right]. \quad (198)$$

where we have used $x = (x^\mu) = (t, \mathbf{r})$. This expression can be used to obtain the transformation of the Poynting vector $\mathbf{P} = \frac{i}{2} \mathbf{F} \times \bar{\mathbf{F}}$ and the energy density $U = \frac{1}{2} \mathbf{F} \cdot \bar{\mathbf{F}}$.

965 7.5. The Hopf-Rañada field from a conformal transformation of a constant electric and magnetic field

Consider the following Riemann-Silberstein vector

$$\mathbf{F} = \mathbf{u}_x + i\mathbf{u}_y = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \quad (199)$$

representing a constant $\mathbf{E}$ field in the x direction and a constant $\mathbf{B}$ field in the y direction. Performing conformal inversion (198) to this simple $\mathbf{F}$ gives (taking $c = 1$ and (t, x, y, z) dimensionless variables)

$$\mathbf{F}' = \frac{1}{(t^2 - r^2)^3} \begin{pmatrix} (x - iy)^2 - (t - z)^2 \\ i(x - iy)^2 + i(t - z)^2 \\ -2(x - iy)(t - z) \end{pmatrix}, \quad (200)$$

or explicitly in $\mathbf{E}$ and $\mathbf{B}$ fields

$$\mathbf{E}' = \frac{1}{(t^2 - r^2)^3} \begin{pmatrix} x^2 - y^2 - (z - t)^2 \\ 2yx \\ -2x(t - z) \end{pmatrix}, \quad \mathbf{B}' = \frac{1}{(t^2 - r^2)^3} \begin{pmatrix} -2xy \\ x^2 - y^2 + (t - z)^2 \\ 2y(t - z) \end{pmatrix}. \quad (201)$$

These fields represent sets of field lines that at any time fill the surface of nested degenerate tori: degenerate in the sense that the central hole is reduced to a singular point where all the field lines touch. This singular point is the origin
970 at $t = 0$ and it then moves along the z axis with the speed of light. Fig. 13 illustrates one of the tori for $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{P}$, at $t = 0$ and at $t = 1$.

Now apply the imaginary time translation $t \rightarrow t - i$ (as in Appell's letter) to this Riemann-Silberstein vector. This will remove the singularity and make all the field lines become linked and revolve around a central circle as described by

$$\mathbf{F}'_{HR}(t, \mathbf{r}) = \frac{1}{((t - i)^2 - r^2)^3} \begin{pmatrix} (x - iy)^2 - (t - i - z)^2 \\ i(x - iy)^2 + i(t - i - z)^2 \\ -2(x - iy)(t - i - z) \end{pmatrix} \quad (202)$$

This gives precisely the electromagnetic Hopf-Rañada knot. Note that, up to a constant factor, the difference with (172) is the change of the $y \rightarrow z$ and the permutation of the vector components. Fig. 14 illustrates this solution (once
975 more) so that it can be easily compared with the solution before the complex time shift (Fig. 13). Several articles by I. Bialinicky-Birula and Z. Bialinicky-Birula are closely related to this topic and highly recommended [63, 77, 78].

Figure 13: Degenerate field configuration: time evolution of $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{P}$ field lines at $t = 0$ and at $t = 1$.

Figure 14: Hopf-Rañada field: time evolution of $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{P}$ field lines at $t = 0$ and at $t = 1$.

7.6. Generalization of complex shift method

In [21] the method of generating topologically non-trivial solutions of the source-free Maxwell equations from existing ones by applying conformal transformations with complex parameters is further investigated in terms of Bateman's variables. Bateman's construction presented in Section 6.1 for null electromagnetic fields, in terms of the Riemann-Silberstein vector $\mathbf{F} = \mathbf{E} + i\mathbf{B}$, reads

$$\mathbf{F} = \nabla\alpha \times \nabla\beta \quad (203)$$

and (170) is written now as $\nabla \cdot \mathbf{F} = 0$, and the dynamical part as $i\partial_t \mathbf{F} = \nabla \times \mathbf{F}$. Inserting in this equation the expression (203) yields

$$i\nabla \times (\partial_t \alpha \nabla \beta - \partial_t \beta \nabla \alpha) = \nabla \times (\nabla \alpha \times \nabla \beta), \quad (204)$$

which is satisfied if

$$i(\partial_t \alpha \nabla \beta - \partial_t \beta \nabla \alpha) = \nabla \alpha \times \nabla \beta, \quad (205)$$

which is equivalent to some of the conditions (169).

In the previous section it was shown that a constant $\mathbf{E}$ field and an orthogonal $\mathbf{B}$ field leads after conformal inversion and a complex time shifts to the Hopf-Rañada field. To reproduce this construction in terms of Bateman's variables we start with the representations for a constant electric and magnetic field in terms of the complex functions α and β .

$$\alpha = 2i(t + z) - 1, \quad \beta = 2(x - iy) \quad (206)$$

which give $\mathbf{E} = (-4, 0, 0)$, $\mathbf{B} = (0, 4, 0)$. Performing a special conformal transformation combined with a complex time shift,

$$x^\mu \rightarrow \frac{x^\mu - b^\mu x_\sigma x^\sigma}{1 - 2b_\sigma x^\sigma + b_\sigma b^\sigma x_\rho x^\rho}, \quad \text{with } b^\mu = (i, 0, 0, 0), \quad (207)$$

provides the Hopf-Rañada solution in Bateman's variables,

$$\alpha = \frac{a - 1 + iz}{a + it}, \quad \beta = \frac{(x - iy)}{a + it}, \quad \text{with } a = \frac{1}{2}(x^2 + y^2 + z^2 - t^2 + 1). \quad (208)$$

980 Reference [21] furthermore presents solutions to the source free Maxwell equations by considering complex shifts combined with conformal transformations performed on plane waves and Hopfions and characterizes the solutions by their conserved quantities.

8. The twistor formulation of linked and knotted null fields

985 An essential feature of the Hopfion is the fact that the Poynting vector is tangent to a Hopf fibration that propagates at the speed of light without deformation. This Poynting vector structure forms a so called Robinson congruence which plays a central role in twistor theory [62, 14].

Twistor theory was developed by Roger Penrose and his colleagues as an attempt to combine general relativity and quantum mechanics in a single formalism [13, 79, 80]. The underlying philosophy of the twistor formalism is that the structure of space time is determined by light rays, or, more general, by massless field configurations, and therefore it might be advantageous to reformulate the laws of physics in coordinates that are directly associated to massless fields. This leads to the concept of twistor space $\mathbb{T}$ in which a point, called a twistor Z^α , encodes both linear and angular momentum of a massless field. Z^α is an object with 4 complex variables ($\alpha = 0, 1, 2, 3$). We provide an introduction to twistor theory and twistor quantization in the Appendix B. There it is shown that the norm in twistor space, for which we also need dual twistors denoted by $\bar{Z}_\alpha$, is given by $\mathfrak{h} = \frac{1}{2} Z^\alpha \bar{Z}_\alpha$. This norm is preserved under conformal transformations (just like in Minkowski space the 4-interval is preserved under Lorentz transformations) and turns out to be the helicity of the massless field configuration associated to the twistor. Canonical quantization of twistor space can be carried out (in exact analogy with introducing non-commuting operators for position and momentum) leading to replacing Z^α and $\bar{Z}_\alpha$ by operators on twistor function: $(\hat{Z}^\alpha, \hat{\bar{Z}}_\beta) = (Z^\alpha, -\frac{\partial}{\partial \bar{Z}^\beta})$. After quantization the twistor norm becomes the helicity operator $\mathfrak{h} = -\frac{1}{2}(Z^\alpha \frac{\partial}{\partial \bar{Z}^\alpha} + 2)$ (see Appendix B). It is now a matter of finding the eigenvalues and eigenstates of the helicity operator.

This gives the so-called elementary states in twistor theory. After transforming
1010 those twistor functions into massless field solutions in Minkowski space, using
the Penrose transform, one obtains all the torus knots in electromagnetism as
well as a recipe for similar knots for fields with different spin values.

Next we show that the EM Hopf-Rañada knot is indeed equivalent to a
simple nontrivial solution to the spin-1 massless field equation in twistor space.
1015 Then we show that the entire family of torus knots is covered by elementary
twistor states. We use the power of the twistor formalism to derive knotted
field configurations for other massless fields, in particular for the spin 2 field
associated to the linearized Einstein equations in vacuum. We conclude by
analyzing the spin-2 solution within the framework of gravito-electromagnetism
1020 [81, 82]. By decomposing the spin-2 field into the spatial gravito-electric and
gravito-magnetic tensors we characterize their topological structure in terms of
the EM knots.

8.1. EM Hopf-Rañada knot from twistor theory

Classical electromagnetism [79] resides in the spin-1 sector of the source-free
spinor field equations (see (320) in Appendix B):

$$\begin{aligned}\nabla^{AA'}\varphi_{A'B'} &= 0, \\ F_{ab} &= F_{A'B'AB} = \varphi_{A'B'}\epsilon_{AB} + c.c.\end{aligned}\tag{209}$$

Indices a and b have values 1 to 4, and the spinor indices A and A' (and B
and B') have values 0, and 1. The standard electric and magnetic fields are
recovered by decomposing F_{ab} using a 4-velocity u^a so that

$$\begin{aligned}\mathcal{E}_b &= u^a F_{ab} \\ \mathcal{B}_b &= -u^a {}^*F_{ab}\end{aligned}\tag{210}$$

where $*$ denotes the Hodge dual. Taking $u^a = (1, 0, 0, 0)$ we have that

$$F_{ab} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}. \quad (211)$$

The corresponding twistor function with helicity value 1 leads after the Penrose transform (see (323) in Appendix B) to the solution,

$$\varphi_{A'_1 \dots A'_{2h}}(x) = \left(\frac{2}{\mu_B \sigma^B |x - y|^2} \right)^3 \mathcal{A}_{A'} \mathcal{A}_{B'} \quad (212)$$

where $\mathcal{A}_{A'}$ defines the doubly degenerate principle null direction of F_{ab} . Choosing

$$\begin{aligned} \bar{A}_a &= (-i\sqrt{2}, \sqrt{2}, -i, 1), \\ \bar{B}_\beta &= \frac{\pi^{1/3}}{2^{4/3}} (-\sqrt{2}, \sqrt{2}i, -1, i) \end{aligned} \quad (213)$$

reproduces precisely the Hopf-Rañada field. The expressions for the time dependent electromagnetic fields admit a simple Riemann-Silberstein vector,

$$\begin{aligned} \mathbf{F}_R &= \mathbf{E}_R + i\mathbf{B}_R \\ &= \frac{4}{\pi(-(t-i)^2 + r^2)^3} \begin{pmatrix} (x-iz)^2 - (t-i+y)^2 \\ 2(x-iz)(t-i+y) \\ i(x-iz)^2 + i(t-i+y)^2 \end{pmatrix}, \end{aligned} \quad (214)$$

where $r^2 = x^2 + y^2 + z^2$. The energy density and Poynting vector field are

$$U_R = \frac{16(1 + x^2 + (t+y)^2 + z^2)^2}{\pi^2(1 + 2(t^2 + r^2) + (t^2 - r^2)^2)^3}. \quad (215)$$

$$\mathbf{P}_R = \frac{U_R}{(1 + x^2 + (t+y)^2 + z^2)} \begin{pmatrix} 2(x(t+y) + z) \\ 1 + (t+y)^2 - x^2 - z^2 \\ 2(z(t+y) - x) \end{pmatrix}. \quad (216)$$

8.2. Null EM torus knots from twistor theory

1025 Now we show that the EM field configurations based on the null electromagnetic torus knots are contained within the elementary states of twistor theory.

Figure 15: (Colour online) Torus knots (green) wind (n_t, n_p) times around a torus (purple) in the toroidal and poloidal directions, respectively. Shown here are the cases of a) trefoil (2,3) knot, b) cinquefoil (2,5) knot, c) septafoil (2,7) knot, and d) nonafoil (2,9) knot.

The Hopf fibration appears as the degenerate case whereby the linked and knotted toroidal structure degenerates down to the linked hopfion configuration.

Torus knots are closed curves on the surface of a torus which wind an integer number of times about the toroidal direction n_t and poloidal direction n_p as in Fig. 15, where n_t and n_p are coprime and both greater than one. If n_t and n_p are not coprime, then there are $n_g = \gcd(n_t, n_p)$ linked curves, each corresponding to a $(n_t, n_p) \bmod n_g$ torus knot. If either n_t or n_p is equal to one, then the knot is trivial with n_g linked curves.

Following the twistor program, solutions to the massless spin- N equations in Minkowski space M are obtained via the Penrose transform of functions $f(Z)$ in twistor space [80]. Consider the twistor functions corresponding to the elementary states [13]

$$f(Z) = \frac{(\bar{C}_\gamma Z^\gamma)^c (\bar{D}_\delta Z^\delta)^d}{(\bar{A}_\alpha Z^\alpha)^a (\bar{B}_\beta Z^\beta)^b} \quad (217)$$

where $(\bar{A}_\alpha Z^\alpha)$ is the $SU(2, 2)$ twistor inner product. Choosing $a = 1$ yields null/Type N solutions and we must have $b = 2h + 1 + c + d$ to give the correct homogeneity $n = -2h - 2$ for a solution with helicity h . In this section it is shown that the twistor functions of the form

$$f(Z) = \frac{(\bar{C}_\gamma Z^\gamma)^{h(n_p-1)} (\bar{D}_\delta Z^\delta)^{h(n_t-1)}}{(\bar{A}_\alpha Z^\alpha) (\bar{B}_\beta Z^\beta)^{h(n_p+n_t)+1}}, \quad (218)$$

lead to field configurations with a torus knot topology where n_p and n_t correspond to the poloidal and toroidal winding numbers.

After applying the spin-1 Penrose transform to (218), the resulting spinor field is

$$\psi_{A'B'}(x) = \frac{(\mathcal{A}_{C'}\mathcal{C}^{C'})^{n_p-1}(\mathcal{A}_{D'}\mathcal{D}^{D'})^{n_t-1}}{(\mathcal{A}_{E'}\mathcal{B}^{E'})^{n_p+n_t+1}}\mathcal{A}_{A'}\mathcal{A}_{B'} \quad (219)$$

Note that the Latin script spinor variables are the spinors associated to the Latin twistor variable. Ergo, $\mathcal{A}_{A'}$ is defined implicitly by $\bar{A}_\alpha Z^\alpha = \mathcal{A}_{A'}\pi^{A'}$.

For the following choice of the dual twistors

$$\begin{aligned}\bar{A}_\alpha &= i(0, \sqrt{2}, 0, 1) \\ \bar{B}_\beta &= i(-\sqrt{2}, 0, -1, 0) \\ \bar{C}_\gamma &= (0, -\sqrt{2}, 0, 1) \\ \bar{D}_\delta &= i(-\sqrt{2}, 0, 1, 0),\end{aligned} \quad (220)$$

the spin-1 fields are null torus knots with a Poynting vector that is everywhere
1040 tangent to a Hopf fibration and propagates in the $\hat{z}$ -direction without deformation. The solutions have the same topology as the electromagnetic fields in [19]. The electric and magnetic vector fields each have the following topological structure as shown in Fig. 16. There are $2n_g$ core field lines, where $n_g = \gcd(n_t, n_p)$, which are linked (and knotted if $n_t, n_p > 1$). Each core
1045 line has the same configuration as the corresponding torus knot with (n_t, n_p) shown in Fig. 15. With the choice for C and D given in (220), the poloidal and toroidal winding numbers for the EM case are related to the exponents in (217) by $c = n_p - 1$ and $d = n_t - 1$. A single core field line is surrounded by nested, toroidal surfaces, each filled by one field line. A second core field line,
1050 also surrounded by nested surfaces, is linked with the first so that there are $2n_g$ sets of linked nested surfaces which fill all of space. A complete solution to the Maxwell equations consists of an electric and a magnetic field orthogonal to each other, both with this field line structure. The (1,1) case corresponds to the electromagnetic Hopfion.

1055 At $t = 0$ the electric and magnetic fields are tangent to orthogonal torus knots, as shown in Fig. 17 (first row). The fields will deform under time evolution, but the topology will be conserved since $\mathbf{E} \cdot \mathbf{B} = 0$.

Figure 16: (Colour online) The field line structure based on (2,3) trefoil knot. a) The core field line is a torus knot (green). b) Each field line except the core lies on the surface of a nested, deformed torus. c) One field line fills a complete surface (red). d) Another field line fills a second surface (blue) linked with the first. The two linked core field lines and the nested surfaces around them fill all of space.

8.3. Gravito-electromagnetic Hopf Knot

Linearized general relativity makes up the spin-2 sector of the spin- N equations [79]. The field equation and associated field strength spinor are given by

$$\begin{aligned}\nabla^{AA'}\varphi_{A'B'C'D'} &= 0, \\ C_{A'B'C'D'ABCD} &= \varphi_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} + c.c.\end{aligned}\tag{221}$$

In analogy with the spin-1 case, following [81] and [82], we decompose the Weyl tensor using a 4-velocity u^a into a tidal (gravito-electric) field and frame-drag (gravito-magnetic) field,

$$\begin{aligned}\mathcal{E}_{ac} &= u^b u^d C_{abcd} \\ \mathcal{B}_{ac} &= -u^b u^{d*} C_{abcd}.\end{aligned}\tag{222}$$

Taking $u^a = (1, 0, 0, 0)$ we define the spatial tensors E_{ij} and B_{ij}

$$\begin{aligned}E_{ij} &= C_{i0j0} \\ B_{ij} &= -{}^*C_{i0j0}.\end{aligned}\tag{223}$$

The symmetric traceless tensors E_{ij} and B_{ij} are called tidal and frame-drag fields since two orthogonal observers separated by a small spatial vector ξ will

Figure 17: (Colour online) A comparison of the spin-1 (EM) and spin-2 (gravity) trefoil knots at $t = 0$. The first row is the EM trefoil knot: **a** the electric field, **b** the magnetic field, and **c** the Poynting vector field. The second row is the gravito-electric trefoil knot: **d** the negative eigenvalue field $\vec{e}_-$, **e** the positive eigenvalue field $\vec{e}_+$, and **f** the zero eigenvalue field $\vec{e}_0$. The third row is the gravito-magnetic trefoil knot: **g** the negative eigenvalue field $\vec{b}_-$, **h** the positive eigenvalue field $\vec{b}_+$, and **i** the zero eigenvalue field $\vec{b}_0$. The color scale indicates magnitude of the eigenvalue, with lighter colors indicating a higher magnitude.

experience a relative tidal acceleration given by

$$\Delta a_i = -E_{ij}\xi^j \quad (224)$$

and a gyroscope at the tip of ξ will precess with angular velocity

$$\Delta\Omega_i = B_{ij}\xi^j \quad (225)$$

relative to inertial frames at the tail of ξ . Since the tidal and frame-drag
1060 fields are symmetric and traceless, each may be characterized entirely by its
eigenspaces. Thus, if $\mathbf{v}$ is an eigenvector of E or B then the integral curves of
 $\mathbf{v}$ are the gravitational analog of field lines. The tidal field stretches or com-
presses objects, and its associated field lines are referred to as tendex lines. The
frame-drag field rotates gyroscopes, and its associated field lines are referred
1065 to as vortex lines. The tidal field has an associated eigenvalue $E_{\mathbf{v}}$ which has a
physical interpretation given by the tidal acceleration equation (224). Thus, if
the tendex eigenvalue is negative (respectively, positive) then an object oriented
along the tendex line is stretched (compressed) along the tendex line. Similar
relations hold for the frame-drag field whose eigenvalues are interpreted using
1070 (225), where an object oriented along a vortex line observes counter-clockwise
(clockwise) precession of gyroscopes around the vortex line.

The gravito-electromagnetic knot (GEM knot) is constructed in the same
fashion as the electromagnetic knot since linearized gravitational fields are taken
to be spin-2 fields on M . Thus, taking $h = 2$ in (323), we have

$$\varphi_{A'_1 \dots A'_{2h}}(x) = \left(\frac{2}{\Omega|x-y|^2} \right)^5 \mathcal{A}_{A'} \mathcal{A}_{B'} \mathcal{A}_{C'} \mathcal{A}_{D'} \quad (226)$$

where $\mathcal{A}_{A'}$ defines the totally degenerate principal null directions of C_{abcd} mak-
ing it a Petrov type N linearized gravitational field. Taking the dual twistors $\bar{A}_\alpha$
and $\bar{B}_\beta$ to be the same as in the electromagnetic knot we construct the Weyl cur-
1075 vature from (221). Then we perform the gravito-electric and gravito-magnetic
decompositions as in (223).

The tidal and frame-drag fields are characterized by their spectral decom-
position which provides a three dimensional picture of the space-time via the

Figure 18: (Colour online) A comparison of the spin-1 (EM) and spin-2 (gravity) knots at $t = 0$. In the first row, we have the EM knot: **a** the electric field, **b** the magnetic field, and **c** the Poynting vector field. In the second row, we have the gravito-electric knot: **d** the negative eigenvalue field E_- , **e** the positive eigenvalue field E_+ , and **f** the zero eigenvalue field E_0 . In the third row, we have the gravito-magnetic knot: **g** the negative eigenvalue field B_- , **h** the positive eigenvalue field B_+ , and **i** the zero eigenvalue field B_0 . The color scale indicates magnitude of the eigenvalue, with lighter colors indicating a higher magnitude.

integral curves of their eigenvector fields and their various physical interpretations. Performing the spectral decomposition we find that both fields possess an eigenvalue structure $\{+\Lambda, -\Lambda, 0\}$ corresponding respectively to the eigenvectors $\{\mathbf{E}_+, \mathbf{E}_-, \mathbf{E}_0\}$ (and $\{\mathbf{B}_+, \mathbf{B}_-, \mathbf{B}_0\}$) where

$$\Lambda(x) = \frac{2^{8/3}(1+x^2+(y+t)^2+z^2)^2}{\pi^{5/3}(1+2(t^2+r^2)+(t^2-r^2)^2)^{5/2}} \quad (227)$$

is the positive eigenvalue which determines the strength of the interaction, for both the tidal and frame-drag fields.

Considering first the eigenvector fields which correspond to the zero eigenvalue, we find that

$$\begin{aligned} \mathbf{E}_0 = \mathbf{B}_0 &= \begin{pmatrix} 2(x(t+y)+z) \\ 1+(t+y)^2-x^2-z^2 \\ 2(z(t+y)-x) \end{pmatrix} \\ &= (1+x^2+(t+y)^2+z^2) \frac{\mathbf{P}_R}{|\mathbf{P}_R|} \end{aligned} \quad (228)$$

which is, up to an overall scaling function, the Poynting vector of the EM knot. Constructing the Riemann-Silberstein structures for the remaining fields $\{\mathbf{E}_-, \mathbf{E}_+\}$ and $\{\mathbf{B}_-, \mathbf{B}_+\}$

$$\mathbf{F}_{GE} = \mathbf{E}_- + i\mathbf{E}_+, \quad (229)$$

$$\mathbf{F}_{GB} = \mathbf{B}_- + i\mathbf{B}_+, \quad (230)$$

we find that

$$\mathbf{F}_{GE} = e^{i\pi/4} \mathbf{F}_{GB} \quad (231)$$

$$= e^{i\text{Arg}(\vartheta)} \mathbf{F}_R, \quad (232)$$

where

$$\vartheta = \sqrt{-(t-i)^2 + r^2}. \quad (233)$$

This defines the tidal and frame-drag fields in terms of the EM knot. Equation (231) shows that the frame-drag field is a rotation of the tidal field about the

Figure 19: (Colour online) A comparison of the spin-1 (EM) and spin-2 (gravity) knots at $t = 1$. In the first row, we have the EM knot: **a** the electric field, **b** the magnetic field, and **c** the Poynting vector field. In the second row, we have the gravito-electric knot: **d** the negative eigenvalue field E_- , **e** the positive eigenvalue field E_+ , and **f** the zero eigenvalue field E_0 . In the third row, we have the gravito-magnetic knot: **g** the negative eigenvalue field B_- , **h** the positive eigenvalue field B_+ , and **i** the zero eigenvalue field B_0 . The color scale indicates magnitude of the eigenvalue, with lighter colors indicating a higher magnitude.

Poynting vector of the EM knot. Thus in passing to spin-2 we obtain two independent gravito-knot structures: a gravito-electric knot and a gravito-magnetic knot which differ simply by a global rotation. Equation (232) shows that the tidal field is simply a local duality transformation of the EM knot. At $t = 0$,
1085 $Arg(\vartheta) = 0$ and thus provides no duality transformation. Hence the tendex structure of the tidal field is the same as the field line structure of the EM knot. Furthermore, the vortex lines are the same as the tendex lines but rotated by 45° in the xy -plane. Thus the $t = 0$ tendex and vortex configuration is given by the six Hopf fibrations of Fig. 18 (rows 2 and 3) so that for each fixed eigen-
1090 value the associated tendex (vortex) lines are closed and linked exactly once. For $t \neq 0$, ϑ is complex and hence the tendex lines differ from those of the EM knot by a local duality transformation as shown in Fig. 19 with the vortex structure different again by only a rotation of 45° .

For electromagnetic fields, there are two quantities that are invariant under a local duality rotation - the energy density and Poynting vector. For the EM knot, these are given in (215) and (216). In direct analogy with the local duality invariants of electromagnetism, a covariant super-energy density and super-Poynting vector that are invariant under a local duality rotation can be defined [81]. For the GEM knot, we find the super-energy density and super-Poynting vector are given by

$$U_G = \frac{1}{2}(E_{ab}E^{ab} + B_{ab}B^{ab}) \quad (234)$$

$$= \frac{64 \cdot 2^{1/3}(1 + x^2 + (t + y)^2 + z^2)^4}{\pi^{10/3}(1 + 2(t^2 + r^2) + (t^2 - r^2)^2)^5} \quad (235)$$

$$(P_G)_a = \epsilon_{abc}E_a^b B^{cd} \quad (236)$$

$$= \frac{U_G}{(1 + x^2 + (t + y)^2 + z^2)} \begin{pmatrix} 2(x(t + y) + z) \\ 1 + (t + y)^2 - x^2 - z^2 \\ 2(z(t + y) - x) \end{pmatrix}. \quad (237)$$

There is a striking similarity between the local duality invariants for the spin-1
1095 and spin-2 cases, which differ only by the power of the scalar factor which shows the energy falls off more rapidly for the higher spin fields.

8.4. Gravito-electromagnetic torus knots

Following [83] we applying the spin-2 Penrose transform to (218), the resulting spinor field is

$$\phi_{A'B'C'D'}(x) = \frac{(\mathcal{A}_{F'}\mathcal{C}^{F'})^{2(n_p-1)}(\mathcal{A}_{G'}\mathcal{D}^{G'})^{2(n_t-1)}}{(\mathcal{A}_{E'}\mathcal{B}^{E'})^{2(n_p+n_t)+1}}\mathcal{A}_{A'}\mathcal{A}_{B'}\mathcal{A}_{C'}\mathcal{A}_{D'}. \quad (238)$$

The source-free field equation and Weyl field strength spinor are

$$\begin{aligned} \nabla^{AA'}\varphi_{A'B'C'D'} &= 0, \\ C_{A'B'C'D'ABCD} &= \varphi_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} + c.c. \end{aligned}$$

The Weyl tensor can then be decomposed into the GEM components. For Type N, the eigenvalues for both the GE and GM tensors take the form $\{-\Lambda, 0, +\Lambda\}$, with $-\Lambda(x) \leq 0 \leq +\Lambda(x)$ for all points x in space-time. The magnitude of the eigenvalues are

$$\Lambda = \frac{2^{2n_p-3}(1+r^2+t^2-2tz)^2(r^2-z^2)^{n_p-1}(r^4-2r^2(1+t^2)+(1+t^2)^2+4z^2)^{n_t-1}}{(r^4-2r^2(-1+t^2)+(1+t^2)^2)^{\frac{5}{2}+n_t+n_p}} \quad (239)$$

We label the eigenvectors $\{\vec{e}_-, \vec{e}_0, \vec{e}_+\}$ and $\{\vec{b}_-, \vec{b}_0, \vec{b}_+\}$ corresponding to the eigenvalues for the tidal and frame-drag fields respectively. For the zero eigenvalue, the eigenvectors $\vec{e}_0$ and $\vec{b}_0$ are both aligned with the Poynting vector of the null EM torus knots. For the remaining eigenvectors, we can construct Riemann-Silberstein (RS) vectors $\vec{f}_e = \vec{e}_- + i\vec{e}_+$ and $\vec{f}_b = \vec{b}_- + i\vec{b}_+$ which are related to each other by

$$\vec{f}_e = e^{i\pi/4}\vec{f}_b. \quad (240)$$

At $t = 0$, the eigenvectors of the GE fields have precisely the same structure as the EM fields, and the GM eigenvector fields have the same structure but
1100 rotated by 45° . The surfaces of the $\vec{e}_-$ eigenvector, color-scaled according to the magnitude of the eigenvalue, for different values of (n_t, n_p) are shown in Fig. 20. The other GEM fields can be constructed by rotating $\vec{e}_-$ according to (240): $\vec{e}_+$ is found by rotating $\vec{e}_-$ by 90° about the Poynting vector. $\vec{b}_-$ and $\vec{b}_+$ are found by rotating $\vec{e}_-$ and $\vec{e}_+$ by 45° , respectively. The eigenvalues of the
1105 GEM fields for a given (n_t, n_p) have the same magnitude (color-scaling) given by $|\Lambda(x)|$ in (239).

Figure 20: (Colour online) The eigenvector field $\vec{e}_-$ for the gravitational field based on the a) trefoil (2,3) knot, b) cinquefoil (2,5) knot, c) septafoil (2,7) knot, and d) nonafoil (2,9) knot. The color scaling is the same as in Fig. 17.

9. Plasma with a magnetic knot skeleton

The topologically non-trivial EM fields can be considered in the context of plasma physics where Maxwell's equations are combined with fluid dynamics, giving rise to magnetohydrodynamics (MHD). Kamchatnov [11] has explored the Hopf map in a similar way as in the case of electromagnetism but he restricted his attention to a static magnetic field configuration which forms the skeleton of a plasma configuration. The linked magnetic field structure defines a stationary plasma current and pressure configuration in the case of ideal (non-resistive) MHD. He showed that the conservation of energy, angular momentum and the topological quantity of magnetic helicity give stability to the configuration. Such a configuration is referred to as a topological soliton which is a meta stable state. It is not in an equilibrium, or lowest energy, state, but is shielded from decay by a conserved topological quantity. The energy and angular momentum are functions of scale factors, typically the size R of the soliton, and the magnetic field strength B_0 at the centre of the soliton. Combined with a third conserved quantity, here the magnetic helicity, the values of R and B_0 become uniquely defined so that the energy and angular momentum become pinned at a fixed value.

In this section the topological soliton of Kamchatnov is presented and generalized to a class of topological plasma configurations characterized by their

toroidal and poloidal winding numbers, n_t and n_p respectively [31]. It will be shown that for positive integers n_t and $n_p = 1$ these configurations represent stable, localized solutions to the MHD equations for an ideal incompressible fluid with infinite conductivity. The section concludes by considering plasma with finite conductivity and by estimating the soliton lifetime in such a medium as a function of the toroidal winding number.

9.1. Plasma knots

Because of the importance of topology in plasma dynamics, there has previously been interest in generalizing the Kamchatnov soliton [84]. The topology of field lines has been shown to be related to stability of flux tube configurations, with the helicity placing constraints on the relaxation of magnetic fields in plasma [85, 86]. Magnetic helicity gives a measure of the structure of a magnetic field, including properties such as twisting, kinking, knotting, and linking [87, 88]. Simulations have shown that magnetic flux tubes with linking possess a longer decay time than similar configurations with zero linking number [89, 90, 91]. Higher order topological invariants have also been shown to place additional constraints on the evolution of the system [85, 92, 93].

There are many applications where magnetic field topology has a significant effect on the stability and dynamics of plasma systems. For example, toroidal magnetic fields increase confinement in fusion reactors [94, 95], and solving for the behavior of some magnetic confinement systems is only tractable in a coordinate system based on a known parameterization of the nested magnetic surface topology [96, 97, 95]. In Astrophysics, the ratio of the toroidal and poloidal winding of the internal magnetic fields impacts many properties of stars, including the shape [98, 99] and momentum of inertia [100], as well as the gravity wave signatures [101] and disk accretion [102] of neutron stars. The novel class of quasi stable, analytic MHD solutions presented in this section may be of use in the study of fusion reactions, stellar magnetic fields, and plasma dynamics in general.

Within ideal MHD a plasma with a pressure field $p(\mathbf{x})$ and current $\mathbf{j}(\mathbf{x})$ in

Figure 21: One lobe of the field configuration for $n_p = 1$ and $n_t = 2$. (a) A single, closed core magnetic field line. (b) The core field line is surrounded by nested toroidal surfaces, shown in cross section. (c) A complete magnetic surface filled entirely by one field line.

static equilibrium can be described by

$$\begin{aligned}\nabla p &= \mathbf{j} \times \mathbf{B}, \\ \nabla \times \mathbf{B} &= \mathbf{j}, \\ \nabla \cdot \mathbf{B} &= 0.\end{aligned}$$

Each magnetic surface corresponds to a constant pressure surface, $\mathbf{B} \cdot \nabla p = 0$. The time independent magnetic field of the Kamchatnov MHD topological soliton is identical to the magnetic field of the Hopf-Rañada solution at $t = 0$.

$$\mathbf{B}_{soliton}(\mathbf{r}) = \mathbf{B}_{hopfion}(t = 0, \mathbf{r}). \quad (241)$$

The soliton field is then sourced by a stationary current

$$\mathbf{j}(\mathbf{x}) = \frac{1}{\mu_0} \nabla \times \mathbf{B}_{soliton}(\mathbf{r}). \quad (242)$$

Generalized topological soliton fields can be constructed using (241) and (242) applied to the null radiative torus knots. In other words the time-independent magnetic field of the soliton is identical to the magnetic field of the radiative torus knots at $t = 0$.

The torus knots are constructed from the Bateman potentials (176),

$$\alpha = \frac{(r^2 - t^2 - 1) + 2iz}{r^2 - (t - i)^2} \quad (243)$$

$$\beta = \frac{2(x - iy)}{r^2 - (t - i)^2}. \quad (244)$$

Figure 22: Topological solitons in MHD with $n_p = 1$ and (a) $n_t = 2$, (b) $n_t = 3$, and (c) $n_t = 4$. A single magnetic field line fills out each of the linked, toroidal surfaces.

The magnetic field of the torus knots is obtained from the expression

$$\mathbf{B} = \Im[\nabla\alpha^{n_t} \times \nabla\beta^{n_p}]|_{t=0}. \quad (245)$$

Each (n_t, n_p) with $n_t, n_p = 1, 2, 3, \dots$ represents a solution to the Maxwell equations. A single magnetic field line fills the entire surface of a torus. These tori
 1165 are nested and each degenerates down to a closed core field line that winds n_t times around the toroidal direction and n_p times around the poloidal direction, as illustrated in Fig. 21. A complete solution for a given (n_t, n_p) is composed of pairs of these nested surfaces that are linked and fill all of space as shown in Fig. 22. For $n_g = \gcd(n_t, n_p)$, the solution is a magnetic field with $2n_g$ linked core
 1170 field lines (knotted if $n_t > 1$ and $n_p > 1$). If $n_t = 1$ and $n_p = 1$, the solution is the Kamchatnov soliton.

9.2. Stability Analysis

Following [31] we will study the stability of those knotted structures. First the plasma is considered to be an ideal, perfectly conducting, incompressible
 1175 fluid, and subsequently the effect of finite conductivity will be considered.

Consider the case where the poloidal winding number $n_p = 1$ and the toroidal winding number n_t is any positive integer. To analyze the stability of the plasma configurations, following the stability analysis in [11], consider the two scaling parameters of the system - the length scale R which corresponds to the size of

the soliton and B_0 which is the magnetic field strength at the origin. The length scale R is also the radius of the sphere S^3 before stereographic projection. First change to coordinates given by:

$$\{x, y, z\} \rightarrow \left\{\frac{x}{R}, \frac{y}{R}, \frac{z}{R}\right\}, \quad \text{and} \quad |\mathbf{B}(0, 0, 0)| = B_0. \quad (246)$$

The stability depends on three quantities - energy, magnetic helicity, and angular momentum - which are functions of R and B_0 . For a perfectly conducting plasma, the magnetic helicity h_m is an integral of motion and is thus conserved. The magnetic helicity is also a topological invariant proportional to the linking
1180 number of the magnetic field lines. If the field can evolve into a lower energy state by a continuous deformation (therefore preserving the topological invariant) then it will be unstable. However, it will be shown that such a deformation does not exist because the angular momentum $\mathbf{L}$ is also conserved and serves to inhibit the spreading of the soliton.

Except for a constant factor, here the magnetic helicity is defined as (53),

$$h_m = \int \mathbf{A} \cdot \mathbf{B} d^3r, \quad (247)$$

where $\mathbf{A} = \Im[\alpha^{n_t} \nabla \beta^{n_p}]$ is the vector potential. From (243)-(245), it follows that

$$h_m = \frac{2n_t}{(n_t + 1)} \pi^2 B_0^2 R^4. \quad (248)$$

The MHD equations for stationary flow are satisfied for a fluid with velocity

$$\mathbf{v} = \pm \frac{\mathbf{B}}{(\mu_0 \rho)^{\frac{1}{2}}}. \quad (249)$$

The energy of the soliton is given by

$$\mathcal{E} = \int \left(\frac{\rho v^2}{2} + \frac{B^2}{2\mu_0} \right) d^3r = \int \frac{3B^2}{2\mu_0} d^3r, \quad (250)$$

so that

$$\mathcal{E} = \frac{3n_t \pi^2}{\mu_0} B_0^2 R^3 \propto \frac{h_m}{R}. \quad (251)$$

The angular momentum is

$$\mathbf{L} = \rho \int [\mathbf{r} \times \mathbf{v}] d^3r = \left(\frac{\rho}{\mu_0} \right)^{1/2} 4n_t \pi^2 B_0 R^4 h_m \mathbf{u}_y$$

Figure 23: (Colour online) The magnetic surfaces for $n_t = 1$ and (a) $n_p = 2$, (b) $n_p = 3$, and (c) $n_p = 4$. Solutions with $n_p \neq 1$ have zero angular momentum and are therefore not stable solitons. The magnetic field lines in each lobe wind in opposite directions (represented by the red and blue surfaces).

where the positive velocity solution was taken. The conserved quantities h_m and $\mathbf{L}$ fix the values of R and B_0 ,

$$R = \left[\frac{1}{8\pi^2 n_t (n_t + 1)} \left(\frac{\mu_0}{\rho} \right) \frac{|L|^2}{h_m} \right]^{\frac{1}{4}}, \quad (252)$$

$$B_0 = 2n_t (n_t + 1) \left(\frac{\rho}{\mu_0} \right)^{1/2} \frac{h_m}{|L|},$$

thus inhibiting energy dissipation. This shows that the solution given in (243)-
 (245) (and shown in Fig. 22) represents a class of topological solitons characterized by the positive integer n_t for $n_p = 1$.

For $n_p \neq 1$, the angular momentum for all n_t is zero. Some examples of fields with $n_t = 1$ and different n_p values are shown in Fig. 23. The field lines
 fill two sets of linked surfaces. For a given pair of linked surfaces, the field in each lobe wraps around the surface in opposite directions. In Fig. 23 the red and blue surfaces wind in opposite directions. This means that the contribution to the angular momentum of the two field lines cancels. In this case the length scale is not fixed by the conserved quantities. The energy can therefore decrease
 by increasing the radius and the fields are not solitons.

For realistic plasma finite conductivity has to be considered. The effect of

losses combined with plasma nonlinearities on the dynamics and plasma topology can be very complex because it will allow for magnetic field lines pinching off and reconnecting. Only a rough estimate of the soliton lifetime can be obtained by dividing the energy by $d\mathcal{E}/dt$, calculated before any energy dissipation [11]. Then a lower bound is obtained on the time it takes for the total energy to dissipate,

$$\frac{d\mathcal{E}}{dt} = \frac{1}{\sigma} \int j^2 d^3x = (3n_t + 7n_t^2 + 5n_t^3) \frac{\pi^2 B_0^2 R}{\mu_0^2 \sigma}. \quad (253)$$

The resulting lifetime is

$$t_{n_t} \geq \frac{3n_t}{3n_t + 7n_t^2 + 5n_t^3} \mu_0 \sigma R^2. \quad (254)$$

For higher n_t , the lifetime decreases although the helicity in (248) increases. This result is interesting as one would have expected from the results regarding flux tubes mentioned previously that the lifetime would increase with increasing helicity.

10. Conclusions and outlook

In this article we have reviewed the work on non-trivial topological solutions to Maxwell's equation in vacuum. An important example is the Hopf-Rañada field, which we have obtained using the Rañada formulation, the conformal transformation method, the Bateman construction and the twistor formalism. This solution is characterized by the property that all the electric and magnetic field lines are linked loops. The helicity of this field is related to the Gauss linking number which coincides with the Hopf index, which is a topological invariant. We have addressed two new classes of solutions having the topology of the torus knots. One is a non-null class, where the electric and magnetic fields are not orthogonal. This class present the phenomena of exchanging helicity between the magnetic and electric field. The other class is a null one. We have reviewed the constructions and properties of both classes.

Maxwell's equations are invariant under conformal transformations. The group of conformal transformations contains the Poincaré group plus dilatations

1215 and special conformal transformations. We showed how this group acting on
electromagnetic fields with trivial topology yield linked and knotted fields.

We provided a summary of twistor theory in the Appendix B. One strength of
the twistor formalism is that solutions to all massless wave equations are derived
in the same manner. We addressed in particular the spin 2 field associated to
1220 the linearized Einstein equations in vacuum. So we can directly identify linked
and knotted solutions for those equations.

We have also seen how topological non-trivial EM fields can be considered
in the context of plasma configurations in which the magnetic field-line configura-
tion approximates plasma torus knots leading to the prediction of solitons in
1225 plasma.

A question that naturally arises is: can we generate linked and knotted
forms of electromagnetic radiation in the laboratory? This question has been
addressed in [17], concluding that indeed it should be possible, but very chal-
lenging, to generate these fields to a good approximation. To see this first
consider the decomposition of the Hopf-Rañada field in vector spherical har-
monics (VSPHs) $\mathbf{A}_{lm}^{TE}(k, \mathbf{r})$, labelled by angular-momentum integers $l \geq 1$ and
 $-l \leq m \leq l$, wavevector k and polarization TE/TM (electric/magnetic field
transverse to the radial direction) [103]. A general free-space vector potential
can be expressed as:

$$\mathbf{A}(\mathbf{r}, t) = \int dk \sum_{l=1}^{\infty} \sum_{m=-l}^{m=l} [\alpha_{lm}^{TM}(k) \mathbf{A}_{lm}^{TM}(k, \mathbf{r}) + \alpha_{lm}^{TE}(k) \mathbf{A}_{lm}^{TE}(k, \mathbf{r})] e^{-i\omega t} + \text{c.c.}$$

The decomposition of the Hopf-Rañada field has the surprisingly simple form:

$$\mathbf{A}_{\text{hopfion}}(\mathbf{r}, t) = \sqrt{\frac{4}{3\pi}} \int dk k^3 e^{-k} [\mathbf{A}_{11}^{TE}(k, \mathbf{r}) - i\mathbf{A}_{11}^{TM}(k, \mathbf{r})] e^{-i\omega t} + \text{c.c.} \quad (255)$$

It only contains the $l = m = 1$ spherical harmonics and has an energy spec-
trum $S(\omega) \approx \omega e^{-\omega}$. Furthermore the superposition $\mathbf{A}^{TE} - i\mathbf{A}^{TM}$ is an eigen-
state of the curl operator with eigenvalue $+k$. Such eigenstates are known
as Chandrasekhar-Kendall states for constant k [104], and are examples of
1230 force-free fields that play an important role in plasma physics and fluid dy-

namics [98, 104]. The Hopf-Rañada field is therefore a pulsed version of the Chandrasekhar-Kendall states with a spectrum $S\omega e^{-\omega}$. The fact that $l = m$ is not too surprising since we are considering field configurations for which all the angular momentum is in the propagation direction. It has numerically been
1235 shown in [17] that a strongly focussed zeroth-order Gaussian beam with circular polarization converges towards a $l = m = 1$ multipole field as the focusing angle increases towards 90° . In order to also produce the appropriate spectrum ultra short pulses of order one wavelength are required. Attosecond lasers can in principle produce approximately the require spectrum for visible light. Tightly
1240 focussing light with a very broad bandwidth is very challenging and perhaps the best way to proceed experimentally is to investigate longer wavelength EM radiation. The above decomposition in vector spherical harmonics also directly leads to a generalization of the Hopf-Rañada field by taking $l = m > 1$. For the case $l = m = 2$ it has been numerically shown that there exists a pair of core
1245 field lines that form two intertwined trefoil knots [17].

The creation and detections of knotted electromagnetic fields present a challenge from an experimental point of view and we only can foresee the potential applications of those fields, for example transporting the angular momentum which is carried by those fields, or taking advantages of the quasiparticle be-
1250 haviour of such fields.

As a last remark we note that in the proof pages stage of this review three articles appeared online that generate more complex electromagnetic knots, such as cable knots [105, 106], and that describe collisions between knots [107].

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Figure 24: A magnetic tube forming a magnetic surface.

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Appendix A: The Gauss integral, the linking number and the helicity of flux tubes

The value of the magnetic helicity is related to the topological configuration of the field lines. The simplest way to see the connection of topology and helicity is through the concept of a magnetic tube. Consider a small closed circular curve C . We define its magnetic tube [109] to be the set of all magnetic lines that intersect C (Fig. 24). We will assume that that set of all the magnetic lines on C forms a magnetic surface. The consequence is that the internal flux of any magnetic tube does not depend on the section through we compute it. We define the magnetic tube strength to be the flux through any section, so since

Figure 25: A filamental tube C and two curves C' and C'' . The Gauss linking number of the curve C' with respect to the filamental tube C is $L(C, C') = 1$. For the curve C'' , the Gauss linking number is $L(C, C'') = 0$.

$\mathbf{B} = \nabla \times \mathbf{A}$ we have

$$\Phi_C = \int_{S_1} \mathbf{B} \cdot d\mathbf{S} = \oint_C \mathbf{A} \cdot d\mathbf{r}, \quad (256)$$

where S_1 is any surface such as $\partial S_1 = C$. A filamental tube is a magnetic tube with infinitesimal section, but its strength is non-vanishing.

Suppose that a simply-connected domain D , on which a magnetic field $\mathbf{B}$ is defined. The helicity is gauge invariant, so we can take the Coulomb gauge for the vector potential $\nabla \cdot \mathbf{A} = 0$, and using the Helmholtz theorem [110] we can write

$$\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi} \int_D \frac{\nabla' \times \mathbf{B}(\mathbf{r}')}{R} d^3 r', \quad (257)$$

being $\nabla' = (\partial/\partial x', \dots)$, $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ and $R = \|\mathbf{R}\|$. Taking $\mathbf{B} = 0$ in ∂D , it follows

$$\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi} \int_D \frac{\mathbf{B}(\mathbf{r}') \times \mathbf{R}}{R^3} d^3 r'. \quad (258)$$

Let us consider the particular case of a field created by a filamental tube C inside the region D and calculate the the circulation of the vector potential along a closed curve C' which does not have any common point with the tube, as we can see in Fig. 25,

$$\Gamma_{C'} = \oint_{C'} \mathbf{A} \cdot d\mathbf{r}.$$

As the field is created by the filamental tube, we can substitute $\mathbf{B}(\mathbf{r}) d\mathbf{r}^3 = \mathbf{B} \cdot d\mathbf{S} d\mathbf{r} = \delta\Phi d\mathbf{r}$ in the expression for the potential (258), so we get

$$\Gamma_{C'} = \delta\Phi \left(\frac{1}{4\pi} \oint_C \oint_{C'} (d\mathbf{r} \times d\mathbf{r}') \cdot \frac{\mathbf{r} - \mathbf{r}'}{\|\mathbf{r} - \mathbf{r}'\|^3} \right). \quad (259)$$

The integral between brackets in (259) is known as the Gauss integral. The Gauss integral is an integer number [111, 112] also called the linking number. To see that we write

$$\frac{1}{4\pi} \oint_C \oint_{C'} (d\mathbf{r} \times d\mathbf{r}') \cdot \frac{\mathbf{r} - \mathbf{r}'}{\|\mathbf{r} - \mathbf{r}'\|^3} = \frac{1}{4\pi} \oint_C \oint_{C'} \frac{\mathbf{r} - \mathbf{r}'}{\|\mathbf{r} - \mathbf{r}'\|^3} \times d\mathbf{r} \cdot d\mathbf{r}', \quad (260)$$

and using the Stokes theorem [112],

$$\oint_C \frac{\mathbf{r} - \mathbf{r}'}{\|\mathbf{r} - \mathbf{r}'\|^3} \times d\mathbf{r} = - \int_S d\mathbf{S} \cdot \nabla_{\mathbf{r}} \frac{\mathbf{r} - \mathbf{r}'}{\|\mathbf{r} - \mathbf{r}'\|^3} = \nabla_{\mathbf{r}'} \int_S d\mathbf{S} \cdot \frac{\mathbf{r} - \mathbf{r}'}{\|\mathbf{r} - \mathbf{r}'\|^3}. \quad (261)$$

The last integral is the solid angle of a surface whose boundary is the curve C as seen from any point $\mathbf{r}'$, so (260) reads

$$\frac{1}{4\pi} \oint_C \oint_{C'} (d\mathbf{r} \times d\mathbf{r}') \cdot \frac{\mathbf{r} - \mathbf{r}'}{\|\mathbf{r} - \mathbf{r}'\|^3} = \frac{1}{4\pi} \oint_{C'} \nabla_{\mathbf{r}'} \Omega = \frac{1}{4\pi} [\Omega]_{C'}. \quad (262)$$

The change in the solid angle subtended by the curve C when the curve C' is traveled all the way is an integer number times 4π , so

$$L(C, C') = \frac{1}{4\pi} \oint_C \oint_{C'} (d\mathbf{r} \times d\mathbf{r}') \cdot \frac{\mathbf{r} - \mathbf{r}'}{\|\mathbf{r} - \mathbf{r}'\|^3} \quad (263)$$

1275 defines the linking number. In the cases shown in Fig. 25, $L(C, C') = 1$ and $L(C, C'') = 0$.

We will make now the connection with the helicity. The magnetic helicity in the region D can be written as

$$h_m = \frac{1}{2c\mu_0} \frac{1}{4\pi} \int \int \mathbf{B}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}') \cdot \frac{\mathbf{R}}{R^3} d^3r' d^3r. \quad (264)$$

Consider now the special situation in which $\mathbf{B}$ is zero except in two filamental tubes whose axes are some oriented, closed and non self-knotted curves C y C' which can be linked (Fig. 26). The sections of the filamental tubes have zero limit, but as before they have non vanishing strengths $\delta\Phi$, $\delta\Phi'$, respectively. Moreover, we must suppose that, along each filamental tube, the magnetic lines run parallel to C and C' respectively. In this case, for each magnetic fields in (264) we can write $\mathbf{B}(\mathbf{r}) d\mathbf{r}^3 = \mathbf{B} \cdot d\mathbf{S} d\mathbf{r} = \delta\Phi d\mathbf{r}$ and we get

$$h_m = \frac{1}{2c\mu_0} \delta\Phi \delta\Phi' \left(\frac{2}{4\pi} \oint_C \oint_{C'} (d\mathbf{r} \times d\mathbf{r}') \cdot \frac{\mathbf{r} - \mathbf{r}'}{\|\mathbf{r} - \mathbf{r}'\|^3} \right). \quad (265)$$

Figure 26: Oriented, linked and non self-knotted filamental tubes, without internal contribution to the helicity.

Note that the factor 2 enters as when integrating we have the two possibilities $\mathbf{r} \in C, \mathbf{r}' \in C'$ or $\mathbf{r}' \in C, \mathbf{r} \in C'$. We can see that the helicity of the two filamental tubes is proportional to the linking number of the tubes,

$$h_m = \frac{1}{2c\mu_0} 2\delta\Phi\delta\Phi' L(C, C'). \quad (266)$$

This result was obtained by Moffatt [51]. In the computation, it is essential that each filamental tube has zero helicity by itself, and this condition is assured if the magnetic lines into each tube are not linked. In the case of having various closed filamental tubes, the total helicity will be

$$h_m = \frac{1}{2c\mu_0} \sum_i \sum_{j \neq i} L(C_i, C_j) \delta\Phi_i \delta\Phi_j, \quad (267)$$

$L(C_i, C_j)$ being the linking number of the tubes C_i, C_j .

The relationship between the linking number and the helicity of a self-knotted filamental tube T has been studied by Berger and Field [87], and Moffatt and Ricca [16]. In this case we are looking for an expression for the helicity in a filamental tube of strength $\delta\Phi$, built around a closed magnetic line C as before, but as the tube is self-knotted (see Fig. 27), then its axis C has a non-vanishing torsion, and the twisting of the magnetic lines inside the tube must be taken into account.

It can be shown that the magnetic helicity of a self-linked filamental tube reads

$$h_m = \frac{1}{2c\mu_0} (\delta\Phi)^2 [\mathcal{W}(C) + \mathcal{T}w(C)] = \frac{1}{2c\mu_0} (\delta\Phi)^2 [\mathcal{W}(C) + \mathcal{T}(C) + \mathcal{T}_0(C, C_m)]. \quad (268)$$

Figure 27: A filamental tube built around the trefoil knot.

The quantity $\mathcal{W}(C)$ is the spatial writhing number which is real number defined by the limiting form of the Gauss integral (265).

$$\mathcal{W}(C) = \frac{1}{4\pi} \oint_C \oint_C \frac{(d\mathbf{r} \times d\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')}{\|\mathbf{r} - \mathbf{r}'\|^3}. \quad (269)$$

1285 For the interpretation of $\mathcal{W}(C)$ see [16, 113].

The quantity $\mathcal{T}w$ is the total twist number of the ribbon with boundaries the axis curve C and any other field line on the filamental tube. It was introduced by Fuller [114]. The total twist number of a ribbon is defined as the number of times that the ribbon revolves around its own axis

$$\mathcal{T}w(C) = \frac{1}{2\pi} \oint_C \frac{d\hat{\mathbf{r}}}{ds} \times \hat{\mathbf{r}} \cdot \mathbf{t} ds = \frac{1}{2\pi} \oint_C \left(\tau + \frac{d\theta}{ds} \right) ds = \mathcal{T}(C) + \frac{1}{2\pi} [\theta]_C, \quad (270)$$

where $\mathbf{t}$ is the unitary tangent and s the arc parameter of the axial curve C . The second equality is obtained using Frenet's relations, τ is the torsion and

$$\mathcal{T}(C) = \frac{1}{2\pi} \oint_C \tau(s) ds \quad (271)$$

is the total torsion of the axial curve C normalized by $(2\pi)^{-1}$. Finally

$$\mathcal{T}_0(C, C_m) = \frac{1}{2\pi} [\theta]_C = N, \quad (272)$$

which is an integer number giving the number of times that the boundary C_m of the ribbon, given by a field line on the tube, rounds the other boundary C , which is in this case the axis of the tube.

None of the numbers $\mathcal{W}$, $\mathcal{T}$ and $\mathcal{T}_0$ is a topological invariant, and only the last one is an integer number. However Calugareanu proved that their sum

$$\mathcal{W}(C) + \mathcal{T}(C) + \mathcal{T}_0(C, C_m) = L(C, C_m). \quad (273)$$

is a topological invariant, equal to the Gauss linking number of the axis C and any magnetic line C_m that rounds the surface of the tube T . That is the Calugareanu's theorem [115]. Using this theorem, we conclude that the magnetic helicity of a filamental tube T whose axis is a closed, self-knotted curve C , is given by

$$h_m = \frac{1}{2c\mu_0} (\delta\Phi)^2 L(C, C_m), \quad (274)$$

where $L(C, C_m)$ is the linking number of the axis C and any magnetic line C_m that rounds the surface of the tube T . In conclusion, the helicity of a
1290 divergencelless vector field is a measure of the linking number of the field lines.

Appendix B: A brief survey of twistor theory

The starting point for the twistor formalism is the spinor formalism. Consider a 4-vector $x^a = (t, x, y, z)$ and represent it by a 2 by 2 complex matrix $x^{AA'}$, with $A = 0, 1$ and $A' = 0, 1$ in the following way,

$$x^{AA'} = \frac{1}{\sqrt{2}} \begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix}. \quad (275)$$

The advantage of this notation is that the determinant of $x^{AA'}$ is the 4-interval $t^2 - x^2 - y^2 - z^2$. Therefore a matrix transformation of $x^{AA'} \rightarrow Lx^{AA'}L^\dagger$, with L an element of $SL(2, C)$ (the group of complex 2x2 matrices with determinant 1), and $L^\dagger$ its transposed and complex conjugated matrix, is associated with a Lorentz transformation. Clearly $SL(2, C)$ is 6 (real) dimensional (4 complex numbers and two constraints $\det(L) = 1$) and it is 2 to 1 isomorphic to $SO(1, 3)$, the proper Lorentz group. It is instructive to check that for example

$$L = \begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{pmatrix} \quad (276)$$

corresponds to a rotation over 2ψ around the z -axis. Note that $\psi = \pi$ is minus the identity matrix and corresponds to a 2π rotation around the z -axis
1295 that leaves $x^{AA'}$ unchanged. Having made a connection between $SL(2, C)$ and Lorentz transformation of 2 by 2 matrices representing space time points, it

is natural to have the element of $SL(2, C)$ act on a two (complex) component vectors $\pi^A = (\pi^0, \pi^1)$. These are spinors and they do acquire a minus sign for elements of $SL(2, C)$ that represent a 2π rotation in space.

If we restrict our attention to points on the light cone ($t^2 - x^2 - y^2 - z^2 = 0$ or equivalently $\det(x^{AA'}) = 0$) we see that the components of a single spinor $\pi^A = (\pi^0, \pi^1)$ and their complex conjugate values suffice to defines such points

$$x^{AA'} = \begin{pmatrix} \bar{\pi}^0 \pi^0 & \bar{\pi}^0 \pi^1 \\ \bar{\pi}^1 \pi^0 & \bar{\pi}^1 \pi^1 \end{pmatrix}. \quad (277)$$

1300 In fact multiplying the spinor by a phase factor corresponds to the same points on the light cone. This point is therefore called the flag pole associated to a spinor (the flag itself is the phase that can “freely rotate around the pole”). All possible flagpoles at a given time t represent a 2-sphere S^2 with radius t .

We can raise and lower indices on spin spaces using

$$\epsilon_{AB} = \epsilon_{A'B'} = \epsilon^{AB} = \epsilon^{A'B'} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (278)$$

Dual spin spaces and hence raising and lowering of indices are defined by

$$\xi^A = \epsilon^{AB} \xi_B, \text{ and } \xi_B = \xi^A \epsilon_{AB}, \quad (279)$$

with analogue expressions for the primed spin space. ϵ_{AB} is preserved under $SL(2, C)$ transformations and therefore plays a similar role as the metric η_{ab} which is preserved under Lorentz transformations. In fact $\epsilon_{AB} \epsilon_{A'B'}$ corresponds to η_{ab} . Note that ϵ_{AB} is the only, up to a complex multiplication factor, antisymmetric bispinor (rank two spinor), and therefore the anti-symmetric component of any bispinor can be expressed by ϵ_{AB} . We will use the following notation for symmetrization and anti-symmetrization of indices 1310 $T_{...(AB)...} = \frac{1}{2}(T_{...AB...} + T_{...BA...})$, and $T_{...[AB]...} = \frac{1}{2}(T_{...AB...} - T_{...BA...})$. Since spin space is two dimensional any anti-symmetrization over more than 2 indices has to be zero.

To get familiar with the spinor notation consider the spinor equivalent of a general valence 2 tensor $T_{ab} \rightarrow T_{AA'BB'} = T_{AB A'B'}$ (the order of primed and

unprimed indices is irrelevant). Using symmetrization and anti-symmetrization on the unprimed pair we obtain

$$T_{ABA'B'} = T_{(AB)A'B'} + T_{[AB]A'B'} = T_{(AB)A'B'} + \frac{1}{2}\epsilon_{AB}T_C{}^C{}_{A'B'}. \quad (280)$$

In the last step we use the fact that an antisymmetric pair of indices can be removed as an ϵ_{AB} with a contraction. In this sense, only symmetric spinors count and any spinor can be written in terms of symmetric spinors and the special spinors ϵ_{AB} , $\epsilon_{A'B'}$ and their inverses. Using also symmetrization and anti-symmetrization on the primed pair we obtain

$$\begin{aligned} T_{ABA'B'} = & T_{(AB)(A'B')} + \frac{1}{2}\epsilon_{A'B'}T_{(AB)C'}{}^{C'} + \frac{1}{2}\epsilon_{AB}T_C{}^C{}_{(A'B')} \\ & + \frac{1}{4}\epsilon_{AB}\epsilon_{A'B'}T_{CC'}{}^{CC'} \end{aligned}$$

The first term is the trace free symmetric part of T_{ab} and the last term is the symmetric term $\frac{1}{4}\eta_{ab}T_c{}^c$. The middle two terms are anti-symmetric in a and b . If we now restrict our attention to the important case of a real bivector (antisymmetric in a and b) which we call F_{ab} (for example the Faraday tensor in electromagnetism or the relativistic total angular momentum tensor) one concludes that

$$F_{ab} \rightarrow F_{AA'BB'} = \Phi_{AB}\epsilon_{A'B'} + \bar{\Phi}_{A'B'}\epsilon_{AB}, \quad (281)$$

where Φ_{AB} is symmetric. If we now consider the dual of the bivector F_{ab} we have

$${}^*F_{ab} = \frac{1}{2}\epsilon_{ab}{}^{cd}F_{cd} = -i\Phi_{AB}\epsilon_{A'B'} + i\bar{\Phi}_{A'B'}\epsilon_{AB}. \quad (282)$$

The self-dual part of F_{ab} (${}^*W_{ab} = iW_{ab}$) is now given by $\bar{\Phi}_{A'B'}\epsilon_{AB}$, and the anti-self-dual part of F_{ab} (${}^*W_{ab} = -iW_{ab}$) is $\Phi_{AB}\epsilon_{A'B'}$.

Since Φ_{AB} in (281) is symmetric it can be written as $\alpha_{(A}\beta_{B)}$. In other words Φ_{AB} can be factorized into two spinors (defining two principal null directions via the corresponding flag poles). Combining the above spinor manipulations we now have arrived at the following spinor notation for a real bivector

$$F_{ab} \rightarrow F_{AA'BB'} = i\omega_{(A}\bar{\pi}_{B)}\epsilon_{A'B'} + c.c., \quad (283)$$

where the first term is the self-dual part of F and the second term the anti-self-dual part.

The source-free Maxwell equations in terms of the Faraday tensor (a real bivector) F_{ab} are

$$\nabla_{[a} F_{bc]} = 0 ; \quad \nabla^a F_{ab} = 0. \quad (284)$$

In terms of the anti-self-dual part of F_{ab} , $W_{ab} = F_{ab} + i^*F_{ab}$, the Maxwell equations are compactly written as

$$\nabla^a W_{ab} = 0. \quad (285)$$

In spinor terms this becomes

$$\nabla^{AA'} \psi_{AB} \epsilon_{A'B'} = \nabla_{A'}^A \psi_{AB} = 0, \quad (286)$$

with ψ_{AB} a symmetric spinor, and

$$\nabla_{A'}^A = \frac{1}{\sqrt{2}} \begin{pmatrix} -\partial_x - i\partial_y & -\partial_t + \partial_z \\ \partial_t + \partial_z & \partial_x - i\partial_y \end{pmatrix}. \quad (287)$$

In a similar way the zero rest mass free field equations of arbitrary half integer spin can be formulated in terms of symmetric spinor fields. In particular

1325 the equations

$$\nabla_{A'}^A \psi_A = 0, \quad (288)$$

$$\nabla_{A'}^A \psi_{AB} = 0, \quad (289)$$

$$\nabla_{A'}^A \psi_{ABCD} = 0, \quad (290)$$

are the spin $\frac{1}{2}$ massless neutrino equation, the spin 1 Maxwell vacuum equations, and the spin 2 linearized Einstein vacuum equations, respectively.

In the following we review how twistor theory allows for solving these equations such that linked and knotted field solutions emerge as the most natural

1330 solutions.

One can think of twistor space $\mathbb{T}$ as the total momentum space for massless particles [12, 79] in which the linear and angular momenta are combined

into a single object called a twistor Z^α , an object with 4 complex variables ($\alpha = 0, 1, 2, 3$). This is accomplished by realizing that the combined linear and angular momentum possess two spinorial degrees of freedom denoted by ω^A and $\pi_{A'}$. Since the linear 4-momentum of a massless field p^a is light-like it can be decomposed as the flagpole of a first spinor $\pi_{A'}$,

$$p^{AA'} = \bar{\pi}^A \pi^{A'}. \quad (291)$$

In general the center of mass Lorentz boosts and the orbital angular momentum of a rotating object are combined into a four dimensional bivector $M = X \wedge P$ which in components reads $M^{\alpha\beta} = X^\alpha P^\beta - X^\beta P^\alpha$, with X^α the 4-position and P^α the 4-momentum. The second spinor ω^A is related to the relativistic angular momentum tensor M^{ab} in that $i\omega^{(A}\bar{\pi}^{B)}\epsilon^{A'B'}$ and its conjugate represent the self-dual (SD) and anti-self-dual (ASD) parts (see (283)) of M^{ab} respectively,

$$M^{ab} = M^{AA'BB'} = i\omega^{(A}\bar{\pi}^{B)}\epsilon^{A'B'} + c.c. \quad (292)$$

The pair $(\omega^A, \pi_{A'})$ is a twistor Z^α , with components

$$(Z^0, Z^1) \equiv (\omega^0, \omega^1) \quad (293)$$

$$(Z^2, Z^3) \equiv (\pi_{0'}, \pi_{1'}). \quad (294)$$

The dependence of the momentum structure (M^{ab}, p^c) on the origin induces a position dependence in $(\omega^A, \pi_{A'})$, making them spinor fields on space-time defined by

$$\begin{aligned} \pi_{A'}(x) &= \pi_{A'}, \\ \omega^A(x) &= \omega^A - ix^{AA'}\pi_{A'}. \end{aligned} \quad (295)$$

In order to define a norm in twistor space we have to introduce dual twistors given by $\bar{Z}_\alpha = (\bar{\pi}_A, \bar{\omega}^{A'})$. With this definition it follows that the norm in twistor space is twice the helicity $\mathfrak{h}$ associate to the massless field configuration represented by a given twistor Z^α ,

$$\mathfrak{h} = \frac{1}{2} Z^\alpha \bar{Z}_\alpha. \quad (296)$$

Figure 28: (Colour online) The flagpole relation is the central relation in the spin geometry of M . Here we see that to every 2-spinor $\pi^{A'}$ there corresponds a light-like 4-vector y^a called its flagpole. **a** When the lightcone of the origin (violet) is intersected with the space-like hyperplane $t = \tau$ (dark grey), we obtain the Minkowski space representation of an expanding sphere of light emitted at the origin, called the celestial sphere (blue). Any light-like 4-vector y^a (white) may be represented by the unique point where it meets this sphere (yellow). **b** Analogous to the pure state representation of the Bloch sphere, the celestial sphere may be put in 1-1 correspondence with the set of 2×2 Hermitian matrices with trace equal to $\sqrt{2}\tau$. The 2-spinor $\pi^{A'}$ is taken as a homogenous coordinate on the complex plane (light grey) of the sphere $\zeta = \pi^{1'}/\pi^{0'}$ (green). The flagpole of $\pi^{A'}$ is then identified with the 4-vector $y^{AA'}$ written in flagpole form as a 2×2 Hermitian matrix.

Helicity, the twistor norm, is invariant under conformal transformations and therefore it plays a similar role as the 4-interval in Minkowski space-time.

Helicity provides for a natural geometric partition of twistor space $\mathbb{T}$ into three parts, $\mathbb{T}^+$, $\mathbb{T}^-$, and $\mathbb{N}$, according to whether the helicity is positive, negative, or zero. Twistors from $\mathbb{N}$ are said to be null, otherwise they are non-null.

Twistor geometry in M can be defined by the vanishing of $\omega^A(x)$ in (295), yielding the incidence relation

$$\omega^A = ix^{AA'}\pi_{A'}. \quad (297)$$

This relation can be solved for Hermitian $x^{AA'}$, and thus points in M , only

Figure 29: (Colour online) A null twistor Z^α corresponds to a light-like world-line in M . A non-null twistor A^α possesses only an indirect correspondence in M constructed via the direct correspondence for null twistors. **a** If $Z^\alpha = (\omega^A, \pi_{A'})$ is a null twistor, then it corresponds to a light-like world-line (null geodesic) in M , parallel to the flagpole of $\pi_{A'}$, which meets the lightcone of the origin along the light-like ray parallel to the flagpole of ω^A . Here $\Omega = x^{AA'}(0)$. **b** The dual of a non-null twistor $\bar{A}_\alpha$ is a plane in $\mathbb{T}$ which is uniquely defined by its intersection with $\mathbb{N}$ (red). **c** The Minkowski space representation of the non-null twistor $\bar{A}_\alpha$ wherein each of the null twistors in the intersection with $\mathbb{N}$ appear as the null geodesics in M which comprise the Robinson congruence. **d** Viewed on a hyperplane of constant time, the Robinson congruence defines a vector field whose integral curves are the fibers of a Hopf map projected down stereographically onto the hyperplane.

when Z^α is null. In that case the solution is

$$x^{AA'}(r) = \frac{\omega^A \bar{\omega}^{A'}}{i \bar{\omega}^{B'} \pi_{B'}} + r \bar{\pi}^A \pi^{A'}, \quad (298)$$

for arbitrary real r . The geometry of equation (298) is shown in Fig. 29 and represents a light-like worldline in M parallel to the flagpole of $\pi_{A'}$, which meets the lightcone of the origin along the flagpole of ω^A at the point $x^{AA'}(0)$.

For non-null twistors, (297) possesses no Hermitian solutions, and thus its direct geometric interpretation is in terms of complexified Minkowski space. However, as shown in Fig. 29, we can obtain an indirect geometric correspondence by exploiting the more direct relation for null twistors. If A^α is non-null then its dual $\bar{A}_\alpha$ is a plane in $\mathbb{T}$. This plane is uniquely defined by its intersection with $\mathbb{N}$. By (298), each point in this intersection represents a null geodesic. The set of all null twistors in the intersection corresponds to a space-filling set of null geodesics in M called the Robinson congruence. If we project the tangent vector field of one of these congruences onto a hyperplane of constant t then we find that it forms the tangent vector field of a Hopf fibration. The integral curves of

Figure 30: (Colour online) A summary of the geometric correspondences in terms of $\mathbb{PT}$ and M . A point in either $\mathbb{PT}^+$ (red) or $\mathbb{PT}^-$ (green) corresponds to a Robinson congruence in M . An entire $\mathbb{CP}^1$ (celestial sphere) in $\mathbb{PN}$ (blue) corresponds to a point in M . Any two points which lie on a $\mathbb{CP}^1$ in $\mathbb{PN}$ correspond to light-like world-lines which lie on the lightcone of the point corresponding to that $\mathbb{CP}^1$.

the vector field are a special family of circles which lie on a set of space-filling nested tori. Each circle, called a Villarceau circle, is linked with every other one exactly once, and the entire structure propagates with the speed of light along its axis of symmetry without deformation. In fact, it was the Hopf structure of the Robinson congruence which inspired Penrose with the name twistor.

The momentum correspondences (291) and (292) clearly possess $U(1)$ invariance. Yet the geometric correspondence defined by the incidence relation (297) is projective. That is, a twistor Z^α multiplied by any non-zero complex number corresponds to the same object in M as Z^α itself. This leads us to consider as most fundamental the projective twistor space $\mathbb{PT}$ defined as $\mathbb{T}$ under the equivalence relation $Z^\alpha \sim \lambda Z^\alpha$ for $\lambda \neq 0$. The partitions of $\mathbb{T}$ naturally extend to partitions of $\mathbb{PT}$, denoted by $\mathbb{PT}^+$, $\mathbb{PT}^-$, and $\mathbb{PN}$. The geometric correspondences are most succinctly characterized in terms of the partitions of $\mathbb{PT}$, as shown in Fig. 30.

Twistor Quantization and Helicity Eigenfunctions

Now we will convert helicity, as given in (296), into an operator, via the standard quantization procedure, and show that this operator and its eigenfunctions and eigenvalues are related to the massless spin- N field equations and their solutions.

The canonical commutation relations

$$[x^a, x^b] = 0, \quad [p^a, p^b] = 0 \quad [p_b, x^a] = i\hbar\delta_b^a \quad (299)$$

lead to a choice of representation: either the position representation

$$(\hat{x}^a, \hat{p}_b) = (x^a, -i\hbar\frac{\partial}{\partial x^b}), \quad (300)$$

or the momentum representation

$$(\hat{x}^a, \hat{p}_b) = (i\hbar\frac{\partial}{\partial p_a}, p_b). \quad (301)$$

The operators act on wave functions $\Psi(x)$, or $\Psi(p)$ respectively. Since a twistor is directly related to position and momentum the above commutation relations translate into the twistor commutation relations

$$[Z^\alpha, Z^\beta] = 0, \quad [\bar{Z}_\alpha, \bar{Z}_\beta] = 0, \quad [Z^\alpha, \bar{Z}_\beta] = \delta_\beta^\alpha, \quad (302)$$

leading to a choice of representation

$$(\hat{Z}^\alpha, \hat{\bar{Z}}_\beta) = (Z^\alpha, -\frac{\partial}{\partial \bar{Z}^\beta}), \quad (303)$$

$$\text{or } (\hat{Z}^\alpha, \hat{\bar{Z}}_\beta) = (\frac{\partial}{\partial \bar{Z}_\alpha}, \bar{Z}_\beta). \quad (304)$$

The operators act on functions $f(Z^\alpha)$, and $f(\bar{Z}^\alpha)$ respectively. Taking for example the first choice (the Z^α representation) implies that $\frac{\partial}{\partial \bar{Z}} f(Z) = 0$ what in turn implies that f has to be a holomorphic (complex analytic) function in Z^α (4 complex variables) [13].

For massless free fields, helicity is the only quantum number and so the dynamics is entirely determined by the eigenfunctions of the helicity operator.

Polarizing to avoid factor ordering issues and converting to our chosen representation, we have that

$$\mathfrak{h} = \frac{1}{4}(Z^\alpha \bar{Z}_\alpha + \bar{Z}_\alpha Z^\alpha) = -\frac{1}{2}(Z^\alpha \frac{\partial}{\partial Z^\alpha} + 2). \quad (305)$$

The operator $z \frac{\partial}{\partial z}$ is known as the Euler homogeneity operator and its eigenfunctions satisfy

$$\begin{aligned} z \frac{\partial}{\partial z} f(z) &= n f(z), \\ f(tz) &= t^n f(z), \end{aligned}$$

with n an integer. We recognize (305) as representing a shifted Euler homogeneity operator on twistor space, and thus its eigenfunctions are homogeneous twistor functions where the relation between the homogeneity n and the eigenvalue h is given by

$$n = -2h - 2. \quad (306)$$

If we want to consider say massless fields with helicity $h = 1$ we have to use twistor functions of homogeneity $n = -4$. All we then have to do is to “translate” this twistor function into a field in Minkowski space. This is accomplished by the Penrose transform.

The Penrose Transform

The Penrose transform is an integral transform, dependent on helicity, which maps the holomorphic eigenfunctions of the helicity operator onto self dual (SD) or and anti self dual (ASD) solutions of the spin- N field equations on M [13]. Since we are interested only in classical fields, and thus fields whose SD and ASD components are conjugate, we require only the Penrose transform for positive helicity.

Let’s introduce the Penrose transform for a simple case namely a $h = 0$ field for which the twistor function must have homogeneity $n = -2$. The simplest of such functions would be

$$f(Z^\alpha) = \frac{1}{(\bar{A}_\alpha Z^\alpha)^2}, \quad (307)$$

where $\bar{A}_\alpha = (\mu_A, \lambda^{A'})$ is a constant dual twistor. As will become clear below, this function will result in a trivial solution (field is zero) because it has only a single (doubly degenerate) pole. Next consider

$$f(Z^\alpha) = \frac{1}{(\bar{A}_\alpha Z^\alpha)(\bar{B}_\beta Z^\beta)}, \quad (308)$$

where both $\bar{A}_\alpha$ and $\bar{B}_\beta$ are constant dual twistors. We aim to find an expressions for the corresponding zero-rest-mass field at a given point $x^{AA'}$ in (complexified) Minkowski space. Through the incidence relation (297) we need to restrict $Z^\alpha = (\omega^A, \pi_{A'})$ to L_x what implies $\omega^A = ix^{AA'}\pi_{A'}$. Let $\bar{A}_\alpha = (\mu_A, \lambda^{A'})$, $\bar{B}_\alpha = (\sigma_A, \psi^{A'})$ be the dual twistors then,

$$\begin{aligned} \bar{A}_\alpha Z^\alpha &= i\mu_A x^{AA'} \pi_{A'} + \lambda^{A'} \pi_{A'} \\ &\equiv \mathcal{A}^{A'} \pi_{A'}, \end{aligned} \quad (309)$$

$$\begin{aligned} \bar{B}_\beta Z^\beta &= i\sigma_B x^{BB'} \pi_{B'} + \psi^{B'} \pi_{B'} \\ &\equiv \mathcal{B}^{A'} \pi_{B'}. \end{aligned} \quad (310)$$

When the point $x^{AA'}$ is in real Minkowski space (which we will be interested in here) Z^α is restricted to all twistors that correspond to null geodesics going through $x^{AA'}$. Each such twistor corresponds to a point on the celestial sphere of $x^{AA'}$.

The spin 0 field at x is now given by the contour integral on this celestial sphere ($\mathbb{CP}^1$):

$$\varphi(x) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{1}{(\mathcal{A}^{A'} \pi_{A'})(\mathcal{B}^{A'} \pi_{A'})} \pi_{C'} d\pi^{C'} \quad (311)$$

Note that

$$\begin{aligned} \pi_{C'} d\pi^{C'} &= \pi_{C'} d\pi_{D'} \epsilon^{D'C'} \\ &= \pi_{0'} d\pi_{1'} - \pi_{1'} d\pi_{0'} \\ &= (\pi_{0'})^2 d\left(\frac{\pi_{1'}}{\pi_{0'}}\right), \end{aligned} \quad (312)$$

and introducing the canonical spin bases $\{o_{A'}, \iota_{A'}\}$ into the primed spin space

S' one obtains

$$\begin{aligned}\pi_{A'} &= \pi_{0'} o_{A'} + \pi_{1'} \iota_{A'} \\ &= \pi_{0'} (o_{A'} + (\frac{\pi_{1'}}{\pi_{0'}}) \iota_{A'}).\end{aligned}\quad (313)$$

Together with the relations

$$\frac{1}{\pi_{0'}} \mathcal{A}^{A'} \pi_{A'} = \mathcal{A}^{0'} + \mathcal{A}^{1'} (\frac{\pi_{1'}}{\pi_{0'}}), \quad (314)$$

$$\frac{1}{\pi_{0'}} \mathcal{B}^{A'} \pi_{A'} = \mathcal{B}^{0'} + \mathcal{B}^{1'} (\frac{\pi_{1'}}{\pi_{0'}}), \quad (315)$$

this indeed implies that (308) becomes an integral over the celestial sphere $\mathbb{CP}^1$.

Thus,

$$\begin{aligned}\varphi(x) &= \frac{1}{2\pi i} \oint_{\Gamma} \frac{1}{(\mathcal{A}^{0'} + \mathcal{A}^{1'} (\frac{\pi_{1'}}{\pi_{0'}})) (\mathcal{B}^{0'} + \mathcal{B}^{1'} (\frac{\pi_{1'}}{\pi_{0'}}))} d(\frac{\pi_{1'}}{\pi_{0'}}) \\ &= \frac{1}{2\pi i \mathcal{A}^{1'} \mathcal{B}^{1'}} \oint_{\Gamma} \frac{1}{(\mu + \zeta)(\nu + \zeta)} d\zeta.\end{aligned}\quad (316)$$

where $\zeta = \pi_{1'}/\pi_{0'}$, $\mu = \mathcal{A}^{0'}/\mathcal{A}^{1'}$, and $\nu = \mathcal{B}^{0'}/\mathcal{B}^{1'}$ represent the projective coordinate and poles respectively. Since the contour integral is over a sphere the contour can always be contracted to a point if there is only a single (possibly degenerate) pole as is the case of (307). Only if there are two (or more) distinct poles, that is $\mu \neq \nu$ for the case discussed here, the integral will be nonzero.

After the variable substitutions, the integral is straightforward. Taking Γ to enclose $-\mu$ we have

$$\begin{aligned}\varphi(x) &= \frac{1}{\mathcal{A}^{1'} \mathcal{B}^{1'}} \text{Res}_{\zeta=-\mu} \frac{1}{(\mu + \zeta)(\nu + \zeta)} \\ &= \frac{1}{\mathcal{A}^{1'} \mathcal{B}^{1'} (\nu - \mu)} \\ &= \frac{2}{(\mu_B \sigma^B (x^a - y^a)(x_a - y_a))},\end{aligned}\quad (317)$$

where the point y is given by

$$y^{AA'} = i \frac{\sigma^A \lambda^{A'} - \mu^A \psi^{A'}}{\mu_B \sigma^B}. \quad (318)$$

We now turn to the general case which includes the spin 1 linked and knotted solutions and the related spin 2 solutions.

The general Penrose transform is given by

$$\varphi_{A'_1 \dots A'_{2h}}(x) = \frac{1}{2\pi i} \oint_{\Gamma} \pi_{A'_1} \dots \pi_{A'_{2h}} f(Z) \pi_{B'} d\pi^{B'} \quad (319)$$

where Γ is a contour on the Celestial sphere of x (Fig. 31) which separates the poles of $f(Z)$. The result is a spinor field which satisfies the spin- N massless field equation [79]

$$\nabla^{AA'} \varphi_{A'_1 \dots A'_{2h}}(x) = 0. \quad (320)$$

From (306) we see that the positive helicity eigenfunctions are twistor functions of negative homogeneity. The simplest twistor function which is homogenous of degree $-2h - 2$ and has two poles is given by

$$f(Z) = (\bar{A}_\alpha Z^\alpha)^p (\bar{B}_\beta Z^\beta)^q,$$

1400 where $p, q < 0$ and $p + q = -2h - 2$.

Consider the Penrose transform for $f(Z)$ given by

$$f(Z) = \frac{1}{(\bar{A}_\alpha Z^\alpha)(\bar{B}_\beta Z^\beta)^{2h+1}} \quad (321)$$

which is homogeneous of degree $-2h - 2$,

$$\begin{aligned} \varphi_{A'_1 \dots A'_{2h}}(x) &= \frac{1}{2\pi i} \oint_{\Gamma} \frac{(o_{A_1} + (\frac{\pi_{1'}}{\pi_{0'}})\iota_{A'_1}) \dots (o_{A_{2h}} + (\frac{\pi_{1'}}{\pi_{0'}})\iota_{A'_{2h}})}{(\mathcal{A}^{0'} + \mathcal{A}^{1'}(\frac{\pi_{1'}}{\pi_{0'}}))(\mathcal{B}^{0'} + \mathcal{B}^{1'}(\frac{\pi_{1'}}{\pi_{0'}}))^{2h+1}} d(\frac{\pi_{1'}}{\pi_{0'}}) \\ &= \frac{1}{2\pi i \mathcal{A}^{1'}(\mathcal{B}^{1'})^{2h+1}} \oint_{\Gamma} \frac{(o_{A_1} + \zeta \iota_{A'_1}) \dots (o_{A_{2h}} + \zeta \iota_{A'_{2h}})}{(\mu + \zeta)(\nu + \zeta)^{2h+1}} d\zeta, \end{aligned} \quad (322)$$

where $\zeta = \pi_{1'}/\pi_{0'}$, $\mu = \mathcal{A}^{0'}/\mathcal{A}^{1'}$, and $\nu = \mathcal{B}^{0'}/\mathcal{B}^{1'}$ represent again the projective coordinate and poles respectively.

Taking Γ to enclose $-\mu$ we have

$$\begin{aligned} \varphi_{A'_1 \dots A'_{2h}}(x) &= \frac{1}{\mathcal{A}^{1'}(\mathcal{B}^{1'})^{2h+1}} \text{Res}_{\zeta=-\mu} \frac{(o_{A_1} + \zeta \iota_{A'_1}) \dots (o_{A_{2h}} + \zeta \iota_{A'_{2h}})}{(\mu + \zeta)(\nu + \zeta)^{2h+1}} \\ &= \frac{1}{\mathcal{A}^{1'}(\mathcal{B}^{1'})^{2h+1}} \frac{(o_{A'_1} - \mu \iota_{A'_1}) \dots (o_{A'_{2h}} - \mu \iota_{A'_{2h}})}{(\nu - \mu)^{2h+1}} \\ &= \frac{(\mathcal{A}^{1'} o_{A'_1} - \mathcal{A}^{0'} \iota_{A'_1}) \dots (\mathcal{A}^{1'} o_{A'_{2h}} - \mathcal{A}^{0'} \iota_{A'_{2h}})}{(\mathcal{A}^{1'} \mathcal{B}^{0'} - \mathcal{A}^{0'} \mathcal{B}^{1'})^{2h+1}} \\ &= \frac{\mathcal{A}_{A'_1} \dots \mathcal{A}_{A'_{2h}}}{(\epsilon_{A'B'} \mathcal{A}^{A'} \mathcal{B}^{B'})^{2h+1}} \\ &= 2^{2h+1} \frac{(i\mu_A x_{A'_1}^A + \lambda_{A'_1}) \dots (i\mu_A x_{A'_{2h}}^A + \lambda_{A'_{2h}})}{(\mu_B \sigma^B(x^a - y^a)(x_a - y_a))^{2h+1}}, \end{aligned} \quad (323)$$

Figure 31: Geometry of the integral in the Penrose transform. **a** In $\mathbb{PT}$, A and B are given as planes on which $f(Z)$ is singular, and they intersect at a $\mathbb{CP}^1$ describing a point $y \in \mathbb{CM}$. The point x at which we are computing φ is represented as a $\mathbb{CP}^1$ which meets A and B at the poles $-\mu$ and $-\nu$ of $f(Z)$. **b** The celestial sphere of x , where the contour Γ separates the poles.

where the point y is given by (318). The contour integral, whose geometry is shown in Fig. 31, thus turns the pole structure of $f(Z)$ at the point x into the specific spin- n massless field configuration in (323).

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