On the local boundedness of maximal H-monotone operators

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Abstract

In this paper we prove that maximal H-monotone operators $T : \mathbb{H}^n \rightrightarrows V_1$ whose domain is all the Heisenberg group $\mathbb{H}^n$ are locally bounded. This implies that they are upper semicontinuous. As a consequence, maximal H-monotonicity of an operator on $\mathbb{H}^n$ can be characterized by a suitable version of Minty's type theorem.

Keywords: Heisenberg group; H-monotonicity; maximal H-monotonicity; Minty theorem

MSC: 47H05; 49J53

1 Introduction

Maximal monotone maps in Euclidean spaces $\mathbb{R}^n$ and, more in general, in Hilbert spaces, play key roles in evolution equations and in other fields of functional analysis. The most notable example of a maximal monotone map in $\mathbb{R}^n$ is provided by the subdifferential map ∂f associated to a convex function $f : \mathbb{R}^n \rightarrow \mathbb{R}$.

The celebrated Minty theorem provides a characterization of maximal monotonicity (see [15]): given a monotone set-valued map $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, then T is maximal monotone if and only if $I + \lambda T$ is surjective onto $\mathbb{R}^n$, for every $\lambda > 0$; in this case, the resolvent map $(I + \lambda T)^{-1}$ is single-valued and 1-Lipschitz continuous on $\mathbb{R}^n$.

For operators defined on Carnot groups $\mathbf{G}$, a notion of H-monotonicity, and maximal H-monotonicity, has been introduced in [9]. This notion fits the monotonicity of maps in Euclidean spaces to the horizontal structure V_1 of $\mathbf{G}$. It arises naturally as the property fulfilled by the H-normal map $\partial_H f$ associated to an H-convex function $f : \mathbf{G} \rightarrow \mathbb{R}$.

In the classical case, well-known regularity properties enjoyed by maximal monotone maps $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ are upper semicontinuity and local boundedness in the interior of the domain of T ; in particular, the proof of the latter relies essentially on the fact that any given ball of $\mathbb{R}^n$ is contained in the convex hull of at most $n + 1$ points.

In this paper, we investigate maximal H-monotone operators $T : \mathbb{H}^n \rightrightarrows V_1$ defined on the Heisenberg group $\mathbb{H}^n$, where $V_1 \cong \mathbb{R}^{2n}$ denotes the first layer of the Lie algebra of $\mathbb{H}^n$. An important example of a such operator is the horizontal normal map $\partial_H f$ of an H-convex function $f : \mathbb{H}^n \rightarrow \mathbb{R}$. When dealing with these operators, one has to face a much more intricate situation, due to the lack of the Euclidean geometry of the underlying setting. More

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precisely, we say that $T : \mathbb{H}^n \rightrightarrows V_1$ is H-monotone if for every $\eta \in \mathbb{H}^n$, $\eta' \in H_\eta$, $v \in T_\lambda(\eta)$ and $v' \in T_\lambda(\eta')$ we have (see Definition 2.1)

$$\langle v - v', \xi_1(\eta) - \xi_1(\eta') \rangle \geq 0,$$

where H_η is the horizontal plane through η , ξ_1 is the canonical projection $\xi_1(x, y, t) = (x, y) \in V_1$, for every $(x, y, t) \in \mathbb{H}^n$, and $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on V_1 .

The restriction $\eta' \in H_\eta$ is an essential one, it implies that the notion of H-monotonicity provides information on the behaviour of the operator T at the point $\eta \in \mathbb{H}^n$ only along the horizontal directions through η . This restriction creates major difficulties in studying the properties of H-monotone maps. Despite the fact that several notions of convex hulls have been introduced in $\mathbb{H}^n$ ([8]), they seem not to be useful for our purpose.

The goal of this paper is to overcome the above indicated difficulties and study the local boundedness of maximal H-monotone maps. In Theorem 2.2 we show that, for an operator T with $\text{dom}(T) = \mathbb{H}^n$, upper semicontinuity is equivalent to local boundedness. Our main result is the following:

Theorem 1.1 *Let $T : \mathbb{H}^n \rightrightarrows V_1$ be a maximal H-monotone map, such that $\text{dom}(T) = \mathbb{H}^n$. Then T is locally bounded.*

This statement implies that T is upper semicontinuous. The proof of this theorem is considerably more involved when compared to the Euclidean framework. The statement recovers the same result as in the Euclidean case with considerably reduced assumptions as we can use information provided by the monotonicity only along horizontal directions. Our proofs require a deeper understanding of the horizontal geometry of $\mathbb{H}^n$; in particular, the non-integrability of the horizontal bundle, or the so-called *twirling effect* (see [5]) of horizontal planes, is used repeatedly in our considerations.

Theorem 1.1 sheds a new light on the regularity properties of a maximal H-monotone operator on $\mathbb{H}^n$ and leads to the proof that any maximal H-monotone operator on $\mathbb{H}^n$ can be characterized by a suitable version of Minty's type theorem, thereby improving a previous result by two of the authors [10].

Theorem 1.2 *Let $T : \mathbb{H}^n \rightrightarrows V_1$ be an H-monotone map with $\text{dom}(T) = \mathbb{H}$. Then the following two properties are equivalent:*

- i. T is maximal H-monotone;
- ii. for every fixed $\eta \in \mathbb{H}^n$ and $\lambda > 0$, the map $(\xi_1 + \lambda T)|_{H_\eta}$ is surjective onto V_1 .

As we will see in subsection 4.1, another application of our main theorem is the study of the regularity of the resolvent $(\xi_1 + \lambda T)^{-1} : V_1 \rightrightarrows \mathbb{H}^n$. Another forthcoming application (see [4]), following the line of investigation in [1], [2], will target the study of the Hausdorff dimension of singular sets $\Sigma^k(T) = \{\eta \in \mathbb{H}^n : \dim(T(\eta)) \geq k\}$, for H-monotone maps T and integers k .

2 Basic notions and preliminary results

2.1 The Heisenberg group $\mathbb{H}^n$.

The Heisenberg group $\mathbb{H}^n$ is the simplest Carnot group of step 2. In this section we will recall some of the necessary notation and background results used in the sequel. We will

focus only on those geometric aspects that are relevant to our paper. For a general overview of the subject we refer to [7].

The Lie algebra $\mathfrak{h}$ of $\mathbb{H}^n$ admits a stratification $\mathfrak{h} = V_1 \oplus V_2$ with $V_1 = \text{span}\{X_i, Y_i; 1 \leq i \leq n\}$ being the first layer of the so called horizontal vector fields, and $V_2 = \text{span}\{T\}$ being the second layer which is one-dimensional. We assume $[X_i, Y_i] = -4T$ and the remaining commutators of basis vectors vanish. The exponential map $\exp : \mathfrak{h} \rightarrow \mathbb{H}^n$ is defined in the usual way. By these commutator rules we obtain, using the Baker-Campbell-Hausdorff formula, that $\mathbb{H}^n$ can be identified with $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ endowed with the non-commutative group law given by

$$\eta \circ \eta' = (x, y, t) \circ (x', y', t') = (x + x', y + y', t + t' + 2(\langle x', y \rangle - \langle x, y' \rangle)),$$

where x, y, x' and y' are in $\mathbb{R}^n$, $t \in \mathbb{R}$, and for $z, z' \in \mathbb{R}^n$, we have $\langle z, z' \rangle = \sum_{j=1}^n z_j z'_j$ the inner product in $\mathbb{R}^n$. Let us denote by e the neutral element in $\mathbb{H}^n$. Transporting the basis vectors of V_1 from the origin to an arbitrary point of the group by a left-translation, we obtain a system of left-invariant vector fields written as first order differential operators as follows

$$\begin{aligned} X_j &= \partial_{x_j} + 2y_j \partial_t, & j &= 1, \dots, n, \\ Y_j &= \partial_{y_j} - 2x_j \partial_t, & j &= 1, \dots, n. \end{aligned} \tag{1}$$

Via the exponential map $\exp : \mathfrak{h} \rightarrow \mathbb{H}$ we identify the vector $\sum_{i=1}^n (\alpha_i X_i + \beta_i Y_i) + \gamma T$ in $\mathfrak{h}$ with the point $(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n, \gamma)$ in $\mathbb{H}^n$; the inverse $\xi : \mathbb{H}^n \rightarrow \mathfrak{h}$ of the exponential map has the unique decomposition $\xi = (\xi_1, \xi_2)$, with $\xi_i : \mathbb{H}^n \rightarrow V_i$. Since we identify V_1 with $\mathbb{R}^{2n}$ when needed, $\xi_1 : \mathbb{H}^n \rightarrow V_1 \cong \mathbb{R}^{2n}$ is given by $\xi_1(x, y, t) = (x, y)$.

Let $N(x, y, t) = ((\|x\|^2 + \|y\|^2) + t^2)^{\frac{1}{4}}$ be the gauge norm in $\mathbb{H}^n$. It is an interesting exercise (see [11]) to check that the expression

$$d_H(\eta, \eta') = N((\eta')^{-1} \circ \eta)$$

satisfies the triangle inequality defining a metric on $\mathbb{H}^n$: this metric is the so-called Korányi-Cygan metric which is by left-translation and dilation invariance bi-Lipschitz equivalent to the Carnot-Carathéodory metric. Here, the non-isotropic Heisenberg dilations $\delta_\lambda : \mathbb{H}^n \rightarrow \mathbb{H}^n$ for $\lambda > 0$ are defined by $\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t)$. The Korányi-Cygan ball of center $\eta_0 \in \mathbb{H}^n$ and radius $r > 0$ is given by $B_{\mathbb{H}^n}(\eta_0, r) = \{\eta \in \mathbb{H}^n : d_H(\eta_0, \eta) \leq r\}$.

The horizontal structure relies on the notion of horizontal plane: given a point $\eta_0 \in \mathbb{H}^n$, the *horizontal plane* H_{η_0} associated to $\eta_0 = (x_0, y_0, t_0)$ is the plane in $\mathbb{H}^n$ defined by

$$H_{\eta_0} = \{\eta = (x, y, t) \in \mathbb{H}^n : t = t_0 + 2(\langle y_0, x \rangle - \langle x_0, y \rangle)\}.$$

This is the plane spanned by the horizontal vector fields $\{X_i, Y_i\}_i$ at the point η_0 . We note that $\eta' \in H_\eta$ if and only if $\eta \in H_{\eta'}$.

2.2 Multivalued maps on $\mathbb{H}^n$.

Let us consider a set-valued map $T : \mathbb{H}^n \rightrightarrows V_1$; we denote by $\text{dom}(T)$ the *effective domain* of T , i.e. the set $\{\eta \in \mathbb{H}^n : T(\eta) \neq \emptyset\}$, and by $\text{gr}(T)$ the *graph* of T , i.e. $\{(\eta, v) \in \mathbb{H}^n \times V_1 : \eta \in \text{dom}(T), v \in T(\eta)\}$.

Let $T : \mathbb{H}^n \rightrightarrows V_1$ be a set-valued map, with closed values, i.e. $T(\eta)$ is a closed set for every η . We recall (see [3] for this general setting) that T is *upper semicontinuous* (briefly *usc*) at $\eta \in \mathbb{H}^n$ if, for every positive ϵ , there exists $\delta > 0$ such that

$$T(\eta') \subseteq T(\eta) + B_{\mathbb{R}^{2n}}(0, \epsilon), \quad \forall \eta' \in \mathbb{H}^n, d_H(\eta', \eta) < \delta,$$

where $T(\eta) + B_{\mathbb{R}^{2n}}(0, \epsilon)$ denotes the Minkowski sum of the two sets in $\mathbb{R}^{2n}$. If the operator T is compact-valued, i.e. $T(\eta)$ is a compact for every η , then the usc of T can be equivalently given as follows: if $\eta_k \rightarrow \eta$, and $v_k \in T(\eta_k)$, then there exists a subsequence $\{v_{k_n}\}$ such that $v_{k_n} \rightarrow v \in T(\eta)$. We say that T is *closed* if $\text{gr}(T)$ is a closed subset of $\mathbb{H}^n \times V_1$.

Note that there is a gap between the dimension of the source and target spaces in this definition, unlike in the Euclidean case. Nevertheless, some basic properties follow in the same way as in the Euclidean setting. First, the properties of being upper semicontinuous, or closed, are related. Indeed,

Remark 2.1 (see [3], Th. 16.12) *Let $T : \mathbb{H}^n \rightrightarrows V_1$. Then the following statements hold:*

- i. if T is usc and closed-valued, then it is closed;*
- ii. if T is closed, and $\text{rge}(T)$ is compact, then T is upper semicontinuous.*

Single-valued continuous functions map compact sets to compact sets. This property is also true for upper semicontinuous compact-set valued maps:

Proposition 2.1 (see [3], Lemma 17.8) *Let $T : \mathbb{H}^n \rightrightarrows V_1$ be a compact-valued usc map. Then $T(K) \subset V_1$ is compact for every compact set $K \subset \mathbb{H}^n$.*

2.2.1 H-monotone and H-cyclical monotone maps.

We say that $A \subset \mathbb{H}^n \times V_1$ is *H-monotone* (see [10]) if

$$\langle \xi_1(\eta) - \xi_1(\eta'), v - v' \rangle \geq 0, \quad \forall (\eta, v) \in A, (\eta', v') \in A, \eta' \in H_\eta. \quad (2)$$

We stress that in the previous definition, for every point (ξ, v) in the set A , the H-monotonicity property gives us information about A *only* in the horizontal directions $\{X_i(\xi), Y_i(\xi)\}_i$ through ξ ; more precisely, (2) is equivalent to

$$\langle \xi_1(\eta) - \xi_1(\eta \circ \exp(tw)), v - v' \rangle \geq 0, \quad \forall (\eta, v) \in A, (\eta \circ \exp(tw), v') \in A, t \in \mathbb{R}, w \in V_1,$$

where, for every w fixed, $t \mapsto \eta \circ \exp(tw)$ is the so called horizontal segment. This restriction gives rise to the most difficulties of our study.

We say that A is *maximal H-monotone* if there are no H-monotone sets $B \subset \mathbb{H}^n \times V_1$ such that $A \subset B$ and there exists $(\eta, v) \in B$ such that $(\eta, v) \notin A$. As usual, such notions of monotonicity and maximality are inherited by the functions as follows:

Definition 2.1 *We say that a set-valued map $T : \mathbb{H}^n \rightrightarrows V_1$ is an H-monotone map if $\text{gr}(T)$ is an H-monotone set, i.e. for every $\eta \in \mathbb{H}^n$, $\eta' \in H_\eta$, $v \in T(\eta)$ and $v' \in T(\eta')$ we have*

$$\langle v - v', \xi_1(\eta) - \xi_1(\eta') \rangle \geq 0. \quad (3)$$

We say that T is strictly H-monotone, if for every $\eta \in \mathbb{H}^n$, $\eta' \in H_\eta$ with $\eta' \neq \eta$, $v \in T(\eta)$ and $v' \in T(\eta')$ in (3) we have a strict inequality. Moreover, we say that T is maximal H-monotone if the set $\text{gr}(T)$ is maximal H-monotone.

A stronger version of the concept of monotonicity is the notion of cyclical monotonicity: in our context we say that $A \subset \mathbb{H}^n \times V_1$ is an *H-cyclically monotone* set (see Definition 6.1

in [9]) if for every sequence $\{(\eta_i, v_i)\}_{i=0}^m \subset A$ such that $\{\eta_i\}_{i=0}^m$ is a closed H-sequence, i.e. $\eta_{i+1} \in H_{\eta_i}$ and $\eta_{m+1} = \eta_0$, we have that

$$\sum_{i=0}^m \langle \xi_1(\eta_{i+1}), v_i \rangle \leq \sum_{i=0}^m \langle \xi_1(\eta_i), v_i \rangle. \quad (4)$$

Moreover, we say that A is *maximal H-cyclically monotone* if there are no H-cyclically monotone sets $B \subset \mathbb{H}^n \times V_1$ such that $A \subset B$ and there exists $(\eta, v) \in B$ such that $(\eta, v) \notin A$. A set-valued map $T : \mathbb{H}^n \rightrightarrows V_1$ is a (maximal) H-cyclically monotone map if $\text{gr}(T)$ is a (maximal) H-cyclically monotone set.

Given a function $u : \mathbb{H}^n \rightarrow \mathbb{R}$ we define the horizontal normal map of u , $\partial_H u : \mathbb{H}^n \rightrightarrows V_1$, by

$$\partial_H u(\eta) = \{p \in V_1 : u(\eta') \geq u(\eta) + \langle p, \xi_1(\eta') - \xi_1(\eta) \rangle, \forall \eta' \in H_\eta\}.$$

It is well known that a function $u : \mathbb{H}^n \rightarrow \mathbb{R}$ is H-convex (see [12]) if and only if $\partial_H u(\eta)$ is non empty, for every η . Moreover, for an H-convex function u , we have that $\partial_H u$ is H-cyclically monotone.

A cyclically monotone map has a better regularity since essentially it coincides with the horizontal normal map of an H-convex function. More precisely, in [9] the authors proved that if $T : \mathbb{H}^n \rightrightarrows V_1$ is an H-cyclically monotone map with $\text{dom}(T) = \mathbb{H}^n$, then there exists an H-convex function $u : \mathbb{H} \rightarrow \mathbb{R}$ such that $\text{gr}(T) \subset \text{gr}(\partial_H u)$; if, in addition, T is maximal, then $\text{gr}(T) = \text{gr}(\partial_H u)$.

We have the following result (see [10]) of Minty type in the case $n = 1$:

Theorem 2.1 *Let $T : \mathbb{H} \rightrightarrows V_1$ be an H-monotone map with $\text{dom}(T) = \mathbb{H}$.*

- i. If T is maximal H-cyclically monotone, then the map $(\xi_1 + \lambda T)|_{H_\eta}$ is surjective onto V_1 for every $\eta \in \mathbb{H}$ and $\lambda > 0$.*
- ii. If the map $(\xi_1 + \lambda T)|_{H_\eta}$ is surjective onto V_1 for every $\eta \in \mathbb{H}$ for some $\lambda > 0$, then T is maximal H-monotone.*

Theorem 1.2 is a generalisation of Theorem 2.1, since we remove the H-cyclically monotone assumption in i., and show that the result holds in $\mathbb{H}^n$. We note here, that every H-cyclically monotone set/map is an H-monotone set/map: the following example will convince the reader that the contrary is false, i.e. there exist maps that satisfies the assumption in Theorem 1.2, but not the assumption i. in Theorem 2.1:

Example 2.1 *Let us consider $T : \mathbb{H}^1 \rightrightarrows V_1$ defined by*

$$T(x, y, t) = \tilde{T}(x, y) = (3x, -2x + 4y).$$

Then it follows (see Example 1 in [10] for the details) that T is maximal H-monotone, but not maximal H-cyclically monotone.

2.2.2 USC and local boundedness for maximal H-monotone maps.

The purpose of this section is to establish the equivalence of usc and the local boundedness of maximal H-monotone maps. In the sequel, the next remark will be of some use:

Remark 2.2 Let $T : \mathbb{H} \rightrightarrows V_1$ be H -monotone. Then T is maximal H -monotone if and only if for all $(g, v) \notin \text{gph}(T)$, there exists $g' \in H_g$ and $v' \in T(g')$ such that

$$\langle v - v', \xi_1(g) - \xi_1(g') \rangle < 0.$$

Let us start with the following preliminary result.

Proposition 2.2 Let T be a maximal H -monotone operator; then

- i. $T(\eta)$ is closed and convex (possibly empty) for every $\eta \in \mathbb{H}^n$;
- ii. if, in addition, $\text{dom}(T) = \mathbb{H}^n$, then T is compact-valued.

Proof: Let $\{v_k\}_k \subset T(\eta)$, with $v_k \rightarrow v$; then

$$\langle \xi_1(\eta) - \xi_1(\eta'), v_k - v' \rangle \geq 0, \quad \forall \eta' \in H_\eta, v' \in T(\eta').$$

Taking the limit as $k \rightarrow \infty$, we obtain

$$\langle \xi_1(\eta) - \xi_1(\eta'), v - v' \rangle \geq 0, \quad \forall \eta' \in H_\eta, v' \in T(\eta'),$$

and the maximality implies that $v \in T(\eta)$. This proves the closedness of $T(\eta)$. To show the convexity, consider v_1 and v_2 in $T(\eta)$ and $\lambda \in (0, 1)$; clearly

$$\langle \xi_1(\eta) - \xi_1(\eta'), \lambda v_1 + (1 - \lambda)v_2 - v' \rangle = \lambda \langle \xi_1(\eta) - \xi_1(\eta'), v_1 - v' \rangle + (1 - \lambda) \langle \xi_1(\eta) - \xi_1(\eta'), v_2 - v' \rangle \geq 0,$$

for every $\eta' \in H_\eta$, $v' \in T(\eta')$. Again the maximality of T implies that $\lambda v_1 + (1 - \lambda)v_2 \in T(\eta)$. Hence the proof of i. is finished.

Let us prove ii. Fix $\eta \in \mathbb{H}^n$. We know that $T(\eta)$ is closed: we have to show that $T(\eta)$ is bounded. Assuming the contrary, let us suppose that there exists $\{v_k\} \subset T(\eta)$, such that $\|v_k\| \rightarrow +\infty$. Since $\{v_k\} \subset V_1$, there exists $w \in V_1$ and a subsequence $\{v_{k_m}\}$ such that $\langle w, v_{k_m} \rangle \rightarrow +\infty$. Considering the point $\eta \circ \exp w \in H_\eta$, and any $v \in T(\eta \circ \exp w)$ we obtain that $\langle w, v - v_{k_m} \rangle \rightarrow -\infty$, contradicting the H -monotonicity of T . $\square$

In particular, from the previous proposition, and from Proposition 2.1, we immediately get that

Corollary 2.1 Let $T : \mathbb{H}^n \rightrightarrows V_1$ be a usc maximal H -monotone operator with $\text{dom}(T) = \mathbb{H}^n$. Then T is closed and maps compact sets into compact sets. In particular, it is locally bounded.

As a converse to the above Corollary, we will show that, under suitable assumptions, local boundedness implies upper semicontinuity. Let us first state the following technical lemma:

Lemma 2.1 Let us consider $\eta, \eta' \in \mathbb{H}^n$ with $\eta \neq \eta'$ and $\eta' \in H_\eta$, and a sequence $\{\eta_k\}_k \subset \mathbb{H}^n$ with $\eta_k \rightarrow \eta$ and $\eta' \notin H_{\eta_k}$. Then there exists a sequence $\{\eta'_k\}_k \subset \mathbb{H}^n$ with the following properties:

- a. $\eta'_k \in H_{\eta'} \cap H_{\eta_k}$;
- b. $\eta'_k \rightarrow \eta'$;

$$c. \frac{\xi_1(\eta'_k) - \xi_1(\eta')}{\|\xi_1(\eta'_k) - \xi_1(\eta')\|} \rightarrow \frac{(\xi_1(\eta) - \xi_1(\eta'))}{\|\xi_1(\eta) - \xi_1(\eta')\|}.$$

Proof: Let us suppose, without loss of generality, that

$$\eta = e := (0, 0, 0), \quad \eta' = (x', y', 0) \neq \eta;$$

moreover, $\eta_k = (x_k, y_k, t_k)$. Since $\eta' \in H_e$ and $\eta_k \rightarrow e$, we have that $\xi_1(\eta') \neq (0, 0)$; we will suppose that $x' \neq 0$. Moreover, $\xi_1(\eta_k) \neq \xi_1(\eta')$, for large k ; hence $H_{\eta'} \cap H_{\eta_k} \neq \emptyset$. In addition, $\eta_k \notin H_{\eta'}$, therefore,

$$t_k + 2(\langle x', y_k \rangle - \langle y', x_k \rangle) \neq 0.$$

Our aim is to construct a sequence η'_k satisfying conditions *a.*, *b.* and *c.* Set $\eta'_k = (x'_k, y'_k, t'_k)$, where

$$x'_k = (1 + \epsilon_k)x', \quad y'_k = (1 + \epsilon_k)y' + A_k\epsilon_k^2x', \quad A_k = -\text{sgn}(t_k + 2(\langle x', y_k \rangle - \langle y', x_k \rangle)). \quad (5)$$

We will show that there exists a sequence $\{\epsilon_k\}_k$, with $\epsilon_k > 0$ and $\epsilon_k \rightarrow 0$, such that *a* – *c.* hold. Indeed, for such sequence $\{\epsilon_k\}_k$, condition *c.* is satisfied; indeed,

$$\frac{(\epsilon_k x', \epsilon_k y' + A_k \epsilon_k^2 x')}{\|(\epsilon_k x', \epsilon_k y' + A_k \epsilon_k^2 x')\|} = \frac{(x', y' + A_k \epsilon_k x')}{\|(x', y' + A_k \epsilon_k x')\|} \rightarrow \frac{(x', y')}{\|(x', y')\|}.$$

Let us show that such a sequence does exist. The condition $\eta'_k \in H_{\eta'} \cap H_{\eta_k}$ is equivalent to the following:

$$t'_k = 2(\langle y', x'_k \rangle - \langle x', y'_k \rangle) = t_k + 2(\langle y_k, x'_k \rangle - \langle x_k, y'_k \rangle). \quad (6)$$

Taking into account (5), the second equality in (6) becomes

$$a_k A_k \epsilon_k^2 + b_k \epsilon_k + c_k = 0,$$

where

$$a_k = (\|x'\|^2 - \langle x', x_k \rangle), \quad b_k = (\langle x', y_k \rangle - \langle y', x_k \rangle), \quad c_k = (t_k/2 + \langle x', y_k \rangle - \langle y', x_k \rangle).$$

For every k , sufficiently large, $a_k > 0$; moreover $c_k \neq 0$ since $\eta_k \notin H_{\eta'}$. Hence we have two solutions

$$\epsilon_{k,\pm} = \frac{-b_k \pm \sqrt{b_k^2 + 4a_k|c_k|}}{2A_k a_k}.$$

Since $c_k \rightarrow 0$, we have $\epsilon_{k,\pm} \rightarrow 0$. For every k , we define

$$\epsilon_k = \begin{cases} \epsilon_{k,+} & \text{if } A_k > 0 \\ \epsilon_{k,-} & \text{if } A_k < 0 \end{cases}$$

The sequence $\{\epsilon_k\}$ satisfies the condition $\epsilon_k > 0$ and $\epsilon_k \rightarrow 0$, therefore the sequence $\{\eta'_k\}$ defined in (5) proves the assertion. $\square$

Theorem 2.2 *Let $T : \mathbb{H}^n \rightrightarrows V_1$ be maximal H -monotone, with $\text{dom}(T) = \mathbb{H}^n$. Then T is locally bounded if and only if T is usc.*

Proof: By Corollary 2.1 we need to prove only the “if ” part. We argue by contradiction. Suppose that T is not usc. Then there exists $\{(\eta_k, v_k)\}_k \subset \mathbb{H}^n \times V_1$ with $(\eta_k, v_k) \rightarrow (\eta, v)$ with $v_k \in T(\eta_k)$, but with $v \notin T(\eta)$. Since T is maximal, there exists a point $\eta' \in H_\eta$ and exists $v' \in T(\eta')$ such that

$$\langle \xi_1(\eta) - \xi_1(\eta'), v - v' \rangle < 0. \quad (7)$$

Suppose now that there is a subsequence $\{\eta_{k_j}\}_j$ of $\{\eta_k\}_k$ such that $\eta_{k_j} \in H_{\eta'}$: then

$$\langle \xi_1(\eta_{k_j}) - \xi_1(\eta'), v_{k_j} - v' \rangle \geq 0, \quad \forall j;$$

taking the limit, we obtain $\langle \xi_1(\eta) - \xi_1(\eta'), v - v' \rangle \geq 0$ which contradicts (7). Hence, for large k_0 , $\eta_k \notin H_{\eta'}$ for $k \geq k_0$. In particular

$$\eta' \notin H_{\eta_k}, \quad \forall k \geq k_0.$$

Now we define the sequence $\{\eta'_k\}_k \subset \mathbb{H}^n$ as in Lemma 2.1. By the local boundedness of T , up to considering a subsequence, there exists $\{v'_k\}_k$, with $v'_k \in T(\eta'_k)$ and $v'_k \rightarrow v'' \in V_1$. Since $\eta'_k \in H_{\eta_k}$, by the H-monotonicity of T we have

$$\langle \xi_1(\eta_k) - \xi_1(\eta'_k), v_k - v'_k \rangle \geq 0, \quad \forall k \geq k_0;$$

passing to the limit we obtain

$$\langle \xi_1(\eta) - \xi_1(\eta'), v - v'' \rangle \geq 0. \quad (8)$$

This last inequality and (7) imply that $v'' \neq v'$.

Since $\eta'_k \in H_{\eta'}$, the monotonicity of T again gives

$$\langle \xi_1(\eta'_k) - \xi_1(\eta'), v'_k - v' \rangle \geq 0, \quad \forall k \geq k_0;$$

dividing by $\|\xi_1(\eta'_k) - \xi_1(\eta')\|$ and passing to the limit, condition *c.* in Lemma 2.1 guarantees

$$\langle \xi_1(\eta) - \xi_1(\eta'), v'' - v' \rangle \geq 0. \quad (9)$$

Now summing the inequalities in (8) and in (9), we obtain an inequality in contradiction with (7). This concludes the proof. $\square$

3 Local boundedness of maximal H-monotone operators.

It is well known that a maximal monotone operator $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is locally bounded. The proof relies essentially on the fact that, given any ball, there exist $n+1$ points whose convex hull contains the ball. In the case of operators $T : \mathbb{H}^n \rightrightarrows V_1 \sim \mathbb{R}^{2n}$ the situation is much more involved. This section is essentially devoted to the proof Theorem 1.1. We first show that a maximal H-monotone operator defined on all $\mathbb{H}^n$ is locally bounded on every vertical segment (see Proposition 3.1). We consider that this step is really the bulk of the paper. Secondly, we show that T inherits the local boundedness on every horizontal segment from the local boundedness of the vertical ones following an idea from [6].

Proposition 3.1 *Let $T : \mathbb{H}^n \rightrightarrows V_1$ be a maximal H-monotone map such that $\text{dom}(T) = \mathbb{H}^n$. Then the restriction of T to any vertical line is locally bounded, i.e. for every set of the type $L := \{\eta = (x, y, t) \in \mathbb{H}^n : t \in I\}$, with x and y fixed and $I \subseteq \mathbb{R}$, I compact interval, there exists $K = K(I)$ such that $\text{diam}_{\mathbb{R}^{2n}}\{T(L)\} \leq K$.*

Proof: The proof is by contradiction. Assume that there exists one of these vertical segments on which T is not bounded. Without loss of generality we can assume that the segment is contained in the t axis. Moreover, we can assume that there exists a sequence of points on the t axis of the form $\eta_k = (0, 0, h_k)$ such that $h_k \rightarrow 0$ and

$$\lim_{k \rightarrow \infty} \text{diam}_{\mathbb{R}^{2n}} \{T(\eta_k)\} = \infty. \quad (10)$$

To obtain a contradiction we use a measure-theoretical argument as follows. Consider the sets:

$$A_k = \{\eta \in B_{\mathbb{H}^n}(e, 5) : \exists u \in T(\eta) \text{ such that } \|u\| > \frac{k}{2^n}\}.$$

We will construct measurable subsets $S_k \subset A_k$ with the property that there exists a constant $c > 0$ such that for any k we have

$$\mathcal{L}^{2n+1}(S_k) > c, \quad (11)$$

where $\mathcal{L}^{2n+1}$ is the Lebesgue measure in $\mathbb{H}^n$.

Assuming the existence of S_k let us show how to get the desired contradiction. We consider the sets

$$U_k = \bigcup_{m \geq k} S_m.$$

Then U_k is measurable and it is decreasing: $U_{k+1} \subset U_k$ and $\mathcal{L}^{2n+1}(U_k) > c$. Let $S := \bigcap_k U_k$, then S is measurable and $\mathcal{L}^{2n+1}(S) \geq c$. Let $\eta \in S$, then η lies in infinitely many sets S_k . In particular there exists a sequence $h(k)$ of indexes $h(k) \rightarrow \infty$ such that $\eta \in S_{h(k)}$ for each $h(k)$. This implies that there exists $u_{h(k)} \in T(\eta)$ such that $\|u_{h(k)}\| \geq h(k)$. On the other hand $T(\eta)$ is compact by Proposition 2.2, which is a contradiction.

In the following we will construct the sets $S_k \subset A_k$ (first step) and will show the existence of a constant $c > 0$ (independent on k) for which (11) holds for any k (second step).

First step. The construction of S_k uses the measurable selection theorem (see e.g. [16]). Let us observe first that by (10) it follows that $A_k \neq \emptyset$ for any k . Moreover, for $k \geq 1$ there exists $h(k) \in \mathbb{N}$ such that

$$\text{diam}_{\mathbb{R}^{2n}} \{T(\eta_{h(k)})\} \geq 10k.$$

To ease the notation we can assume that $h(k) = k$. We obtain a sequence $\{u_k\}_k$ with $u_k \in T(\eta_k)$ such that

$$\|u_k\| \geq 10k. \quad (12)$$

Let us consider the unit vector in V_1

$$\omega_k = \frac{u_k}{\|u_k\|}, \quad (13)$$

and the horizontal segment

$$L_k = \{\nu_k(t) := \eta_k \circ \exp(t\omega_k) \in \mathbb{H}^n : t \in [1, 2]\}.$$

We claim that $L_k \subset A_{10k}$. Indeed, let $\nu_k(t) \in L_k$ and $v_k \in T(\nu_k(t))$. By the H -monotonicity of T we have

$$\langle \xi_1(\nu_k(t)) - \xi_1(\eta_k), v_k - u_k \rangle \geq 0$$

and hence, by (12) and (13)

$$\langle v_k, t\omega_k \rangle \geq \langle u_k, t\omega_k \rangle \geq 10k.$$

Since ω_k is a unit vector by the Cauchy-Schwarz inequality we obtain

$$\|v_k\| \geq \langle v_k, \omega_k \rangle \geq 10k. \quad (14)$$

The idea of the proof is to enlarge the segment L_k by glueing $2n$ -dimensional sectors in the horizontal plane of each of its points. We will prove that by this construction we obtain an enlarged $(2n+1)$ -dimensional set which is still a subset of A_k and whose Lebesgue measure is bounded below by a uniform constant.

Let us consider $I \subset \mathbb{R}^{2n-1}$ given by

$$I = [0, \pi] \times [0, \pi] \times \cdots \times [0, \pi] \times [0, 2\pi]$$

and the spherical coordinates $\omega : I \rightarrow S^{2n-1}$ given by $\omega(\Phi) = (\omega^1(\Phi), \dots, \omega^{2n}(\Phi))$, for $\Phi = (\phi^1, \dots, \phi^{2n-1})$,

$$\begin{cases} \omega^1(\Phi) = \cos \phi^1 \\ \omega^2(\Phi) = \sin \phi^1 \cos \phi^2 \\ \dots\dots\dots \\ \omega^{2n-1}(\Phi) = \sin \phi^1 \sin \phi^2 \dots \sin \phi^{2n-2} \cos \phi^{2n-1} \\ \omega^{2n}(\Phi) = \sin \phi^1 \sin \phi^2 \dots \sin \phi^{2n-2} \sin \phi^{2n-1} \end{cases} \quad (15)$$

To carry out the proposed construction let us select for each $t \in [1, 2]$ a vector $\tilde{v}_k(t) \in T(\nu_k(t))$. Here we apply the measurable selection theorem (see [16]) to obtain, for every k , a measurable map $t \rightarrow \tilde{v}_k(t)$. In the following consideration we will fix the index k . However we note that, by (14),

$$\|\tilde{v}_k(t)\| \geq 10k. \quad (16)$$

For each $t \in [1, 2]$ let us write

$$\tilde{v}_k(t) = \|\tilde{v}_k(t)\| \omega(\tilde{\Phi}_k(t)), \quad (17)$$

for a suitable $\tilde{\Phi}_k(t) = (\tilde{\phi}_k^1(t), \tilde{\phi}_k^2(t), \dots, \tilde{\phi}_k^{2n-1}(t)) \in I$. Since the mapping $t \rightarrow \tilde{v}_k(t)$ is measurable we obtain that the function $t \rightarrow (\tilde{\Phi}_k(t))$ is measurable as well. Set $\underline{i} = (i_1, i_2, \dots, i_{2n-1})$, where $i_j \in \{1, 2, 3, 4\}$ if $j \in \{1, 2, \dots, 2n-2\}$, and $i_{2n-1} \in \{1, 2, \dots, 8\}$, and denote by $I^{\underline{i}}$ the set

$$I^{\underline{i}} = \left[(i_1 - 1) \frac{\pi}{4}, i_1 \frac{\pi}{4} \right) \times \left[(i_2 - 1) \frac{\pi}{4}, i_2 \frac{\pi}{4} \right) \times \cdots \times \left[(i_{2n-1} - 1) \frac{\pi}{4}, i_{2n-1} \frac{\pi}{4} \right).$$

Fix any $\underline{i}$ as above, and consider the set

$$T_k^{\underline{i}} = \left\{ t \in [1, 2] : \tilde{\Phi}_k(t) \in I^{\underline{i}} \right\}.$$

Then $\{T_k^{\underline{i}}\}$, for k fixed, are disjoint measurable sets with the property that $\bigcup_{\underline{i}} T_k^{\underline{i}} = [1, 2]$ up to a set of measure 0. This implies that there exists $\underline{i}_0(k)$ such that

$$\mathcal{L}^1 \left(T_k^{\underline{i}_0(k)} \right) \geq \frac{1}{2 \cdot 4^{2n-1}}.$$

Let us consider the subset of L_k defined by

$$L_k^{i_0(k)} = \{\nu_k(t) := \eta_k \circ \exp(t\omega_k) \in \mathbb{H}^n : t \in T_k^{i_0(k)}\}, \quad (18)$$

and to each $\underline{i}$ we associate the sector

$$S^{\underline{i}} = \{\rho\omega(\Phi) : \rho \in [1, 2], \Phi = (\phi^1, \phi^2, \dots, \phi^{2n-1}) \in I^{\underline{i}}\}.$$

These sectors are $2n$ -dimensional and disjoint. We define the desired set S_k by

$$S_k = \{\nu := \nu_k(t) \circ \exp(\rho\omega(\Phi)) \in \mathbb{H}^n : t \in T_k^{i_0(k)}, \rho\omega(\Phi) \in S^{i_0(k)}\}. \quad (19)$$

It is clear that for k sufficiently large, by the construction, we have $S_k \subset B_{\mathbb{H}^n}(e, 5)$. We claim first that $S_k \subseteq A_k$. To see this let $\nu = \nu_k(t) \circ \exp(\rho\omega(\Phi))$ be an arbitrary point in S_k , and let $v \in T(\nu)$.

We intend to prove that $\|v\| \geq \frac{k}{2^n}$.

This will be done, using the fact that $\nu \in H_{\nu_k(t)}$ and the monotonicity of T by comparing (ν, v) to the point $(\nu_k(t), \tilde{v}_k(t))$, i.e.

$$\langle \xi_1(\nu) - \xi_1(\nu_k(t)), v - \tilde{v}_k(t) \rangle \geq 0$$

which implies

$$\rho \langle \omega(\Phi), v - \tilde{v}_k(t) \rangle \geq 0 \quad (20)$$

Let us note first that, if $\Phi = (\phi^1, \phi^2, \dots, \phi^{2n-1})$ and $\Psi = (\psi^1, \psi^2, \dots, \psi^{2n-1})$ belong to the same $(2n-1)$ -cube $I^{\underline{i}}$, then

$$\langle \omega(\Phi), \omega(\Psi) \rangle \geq 2^{-(2n-1)/2}. \quad (21)$$

Indeed, from the expression of the left hand side of the previous inequality and taking into account the equation for the spherical coordinates, we have

$$\begin{aligned} \sum_{i=1}^{2n} \omega^i(\Phi) \omega^i(\Psi) &= \cos \phi^1 \cos \psi^1 + \\ &+ \sin \phi^1 \cos \phi^2 \sin \psi^1 \cos \psi^2 + \\ &+ \dots + \\ &+ \sin \phi^1 \sin \phi^2 \dots \sin \phi^{2n-2} \cos \phi^{2n-1} \sin \psi^1 \sin \psi^2 \dots \sin \psi^{2n-2} \cos \psi^{2n-1} + \\ &+ \sin \phi^1 \sin \phi^2 \dots \sin \phi^{2n-2} \sin \phi^{2n-1} \sin \psi^1 \sin \psi^2 \dots \sin \psi^{2n-2} \sin \psi^{2n-1}; \end{aligned}$$

if we take the last two lines of the sum above we have

$$\begin{aligned} \omega^{2n-1}(\Phi) \omega^{2n-1}(\Psi) + \omega^{2n}(\Phi) \omega^{2n}(\Psi) &= \\ &= \sin \phi^1 \sin \phi^2 \dots \sin \phi^{2n-2} \sin \phi^{2n-1} \sin \psi^1 \sin \psi^2 \dots \sin \psi^{2n-2} \cos(\phi^{2n-1} - \psi^{2n-1}) \\ &\geq \frac{1}{\sqrt{2}} \sin \phi^1 \sin \phi^2 \dots \sin \phi^{2n-2} \sin \phi^{2n-1} \sin \psi^1 \sin \psi^2 \dots \sin \psi^{2n-2}, \end{aligned}$$

noticing that to obtain the previous inequality we use the fact that $\sin \psi^i$ and $\sin \phi^i$ are nonnegative. Iterating this argument, we finally get (21).

Hence, by (16), (17) and (20), and recalling that by definition of the set S_k in (19) we have that $\tilde{\Phi}_k(t)$ and Φ lie in the same $(2n-1)$ -cube $I_{\mathbf{z}_0}^{(k)}$,

$$\begin{aligned}\|v\| &\geq \langle v, \omega(\Phi) \rangle \\ &\geq \langle \tilde{v}_k(t), \omega(\Phi) \rangle \\ &= \|\tilde{v}_k(t)\| \langle \omega(\tilde{\Phi}_k(t)), \omega(\Phi) \rangle \\ &\geq 10k2^{-(2n-1)/2} > \frac{k}{2^n}.\end{aligned}$$

Second step. Our second claim is, that there exists a constant $c > 0$ with the property that

$$\mathcal{L}^{2n+1}(S_k) \geq c.$$

To prove this fact let us consider, for every k , the mapping

$$F_k = (F_k^1, \dots, F_k^{2n+1}) : [1, 2] \times [1, 2] \times I_{\mathbf{z}_0}^{(k)} \rightarrow \mathbb{H}^n,$$

given by

$$F_k(t, \rho, \Theta) = \nu_k(t) \circ \exp(\rho\omega(\Theta)),$$

where $\nu_k(t)$ is as in (18) and $\Theta = (\theta^1, \dots, \theta^{2n-1})$. Let $\Phi_k \in I$ be such that $\omega_k = \omega(\Phi_k)$.

Our aim is to show that if Θ is suitably chosen with respect to Φ_k , then $|\det(JF_k(t, \rho, \Theta))|$ is bounded from below by a positive constant, where JF is the Jacobian of the function F_k . Since ω_k is fixed, we can assume, without loss of generality and to simplify the computations, that $\omega_k = (1, 0, \dots, 0)$, i.e. $\Phi_k = (0, \dots, 0)$.

Recalling that $\eta_k = (0, 0, h_k)$, we obtain the formula

$$F_k(t, \rho, \Theta) = (F_k^1, \dots, F_k^{2n+1})(t, \rho, \Theta) = \begin{pmatrix} t + \rho \cos \theta^1 \\ \rho \sin \theta^1 \cos \theta^2 \\ \rho \sin \theta^1 \sin \theta^2 \sin \theta^3 \cos \theta^4 \\ \dots \\ \rho \sin \theta^1 \sin \theta^2 \sin \theta^3 \dots \sin \theta^{2n-1} \\ \rho \sin \theta^1 \sin \theta^2 \sin \theta^3 \dots \cos \theta^{2n-1} \\ h_k - 2t\rho \sin \theta^1 \sin \theta^2 \dots \sin \theta^n \cos \theta^{n+1} \end{pmatrix}$$

Let us consider the Jacobian JF_k of the function F_k . If $n = 1$, trivial computations show that $|\det(JF_k(t, \rho, \theta))| = 2\rho^2 |\sin \theta|$. In the general case, we note that the first three columns of JF_k are

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ \dots \\ 0 \\ 0 \\ -2\rho \prod_{i=1}^n \sin \theta^i \cos \theta^{n+1} \end{pmatrix}, \begin{pmatrix} \cos \theta^1 \\ \sin \theta^1 \cos \theta^2 \\ \sin \theta^1 \sin \theta^2 \cos \theta^3 \\ \dots \\ \prod_{i=1}^{2n-2} \sin \theta^i \cos \theta^{2n-1} \\ \prod_{i=1}^{2n-2} \sin \theta^i \sin \theta^{2n-1} \\ -2t \prod_{i=1}^n \sin \theta^i \cos \theta^{n+1} \end{pmatrix}, \begin{pmatrix} -\rho \sin \theta^1 \\ \rho \cos \theta^1 \cos \theta^2 \\ \rho \cos \theta^1 \sin \theta^2 \cos \theta^3 \\ \dots \\ \rho \cos \theta^1 \prod_{i=2}^{2n-2} \sin \theta^i \cos \theta^{2n-1} \\ \rho \cos \theta^1 \prod_{i=2}^{2n-2} \sin \theta^i \sin \theta^{2n-1} \\ -2t\rho \cos \theta^1 \prod_{i=2}^n \sin \theta^i \cos \theta^{n+1} \end{pmatrix}$$

In particular, the second and the third ones can be written as

$$\sin \theta^1 \begin{pmatrix} \cos \theta^1 / \sin \theta^1 \\ \cos \theta^2 \\ \sin \theta^2 \cos \theta^3 \\ \vdots \\ \prod_{i=2}^{2n-2} \sin \theta^i \cos \theta^{2n-1} \\ \prod_{i=2}^{2n-2} \sin \theta^i \sin \theta^{2n-1} \\ -2t \prod_{i=2}^n \sin \theta^i \cos \theta^{n+1} \end{pmatrix}, \quad \frac{\rho}{\cos \theta^1} \begin{pmatrix} -\sin \theta^1 / \cos \theta^1 \\ \cos \theta^2 \\ \sin \theta^2 \cos \theta^3 \\ \vdots \\ \prod_{i=2}^{2n-2} \sin \theta^i \cos \theta^{2n-1} \\ \prod_{i=2}^{2n-2} \sin \theta^i \sin \theta^{2n-1} \\ -2t \prod_{i=2}^n \sin \theta^i \cos \theta^{n+1} \end{pmatrix},$$

therefore, if we remove the first entry, we get two dependent columns. This means that, when computing the determinant of JF_k starting from the first column, we have actually only one term to consider, namely

$$\det(JF_k(t, \rho, \Theta)) = -2\rho \prod_{i=1}^n \sin \theta^i \cos \theta^{n+1} \cdot \det(Jw^\rho(\rho, \Theta)),$$

where $w^\rho(\rho, \Theta) = \rho w(\Theta)$ denotes the $2n$ -dimensional spherical coordinates (see (15)). By known computations,

$$\det(Jw^\rho(\rho, \Theta)) = \rho^{2n-1} \sin^{2n-2} \theta^1 \sin^{2n-3} \theta^2 \dots \sin \theta^{2n-2};$$

thus

$$|\det(JF_k(t, \rho, \Theta))| = 2\rho^{2n} \left| \prod_{i=1}^n \sin \theta^i \cos \theta^{n+1} \sin^{2n-2} \theta^1 \sin^{2n-3} \theta^2 \dots \sin \theta^{2n-2} \right|.$$

We note that $\det(JF_k(t, \rho, \Theta)) \neq 0$ for a.e. $\Theta \in I_0^{(k)}$. Let us consider the set

$$C_k = T_k^{I_0^{(k)}} \times [1, 2] \times I_0^{(k)}.$$

Since $S_k = F([1, 2] \times [1, 2] \times I_0^{(k)})$, we have that $F(C_k) \subseteq S_k$. By the change of variable formula we have that

$$\mathcal{L}^{2n+1}(S_k) \geq \int_{C_k} |\det(JF_k(t, \rho, \Theta))| dt d\rho d\theta^1 d\theta^2 \dots d\theta^{2n-1} \geq c_0 \mathcal{L}^{2n+1}(C_k) := c.$$

It is an exercise to show that c is a uniform constant which does not depend on k , finishing the proof. $\square$

We are now able to prove Theorem 1.1:

Proof of Theorem 1.1: We show that T is bounded in a suitable neighbourhood of the origin. Let us consider the $4n$ segments in $\mathbb{H}^n$:

$$\begin{aligned} I_j^+ &:= \{(e_j, 0, s) \in \mathbb{H}^n : -1 \leq s \leq 1\}, & I_j^- &:= \{(-e_j, 0, s) \in \mathbb{H}^n : -1 \leq s \leq 1\}, \\ I_{j+n}^+ &:= \{(0, e_j, s) \in \mathbb{H}^n : -1 \leq s \leq 1\}, & I_{j+n}^- &:= \{(0, -e_j, s) \in \mathbb{H}^n : -1 \leq s \leq 1\}, \end{aligned}$$

where $j = 1, 2, \dots, n$. Here e_j denotes the n -tuple with 1 in the j position, and 0 otherwise.

From Proposition 3.1, there is $K > 0$ such that $T(I_j^+), T(I_j^-) \subseteq B_{\mathbb{R}^{2n}}(0, K)$, for every $j = 1, \dots, 2n$. Let $r \in (0, 1)$ small enough such that, for every $\xi \in B_{\mathbb{H}^n}(0, r)$ and for every $j = 1, \dots, 2n$, we have $H_\xi \cap I_j^+, H_\xi \cap I_j^- \neq \emptyset$: we note that by the continuity

of the map $\xi \mapsto H_\xi$ such $r > 0$ exists since the claim holds for $\xi = 0$. Now, for any $\xi = (x, y, t) \in B_{\mathbb{H}^n}(0, r)$ we define ξ^+, ξ^-, v_j^+ and v_j^- by

$$\xi_j^+ = \xi \circ \exp(v_j^+(\xi)) = H_\xi \cap I_j^+, \quad \xi_j^- = \xi \circ \exp(v_j^-(\xi)) = H_\xi \cap I_j^-, \quad j = 1, 2, \dots, 2n.$$

Straightforward computations show that $v_j^\pm$ coincide with one of the vectors from the following list

$$(e_j - x, -y), \quad (-e_j - x, -y), \quad (-x, e_j - y), \quad (-x, -e_j - y),$$

thus $\|v_j^\pm\| \leq 2$, for every j , and for every $\xi \in B_{\mathbb{H}^n}(0, r)$.

From the H-monotonicity of T , we have that

$$\langle u, v_j^\pm(\xi) \rangle \leq \langle u_j, v_j^\pm(\xi) \rangle \leq K \|v_j^\pm(\xi)\| \leq 2K, \quad (22)$$

for every $u \in T(\xi)$, $u_j \in T(\xi_j)$, and for every $j = 1, \dots, 2n$. The inequalities (22) imply that $T(\xi)$ is contained in the polyhedron $P(\xi)$ defined by:

$$P(\xi) := \{u \in V_1 : \langle u, v_j^+(\xi) \rangle \leq 2K, \langle u, v_j^-(\xi) \rangle \leq 2K \quad j = 1, \dots, 2n\}.$$

Note that there is no $v \in \mathbb{R}^{2n} \setminus \{0\}$ such that the half-space $\{u \in \mathbb{R}^{2n} : \langle v, u \rangle \leq 0\}$ contains all the vectors $\{v_j^\pm\}_{j=1, \dots, 2n}$; as a consequence, the set $P(\xi)$ turns out to be a polytope, i.e. it is bounded. Indeed, on the contrary, if $v \in \mathbb{R}^{2n} \setminus \{0\}$ is such that $tv \in P(\xi)$, for every $t \geq 0$, then $\langle v, v_j^\pm(\xi) \rangle \leq 0$, i.e. the set $\{v_j^\pm(\xi)\}_j$ belongs to the half-space $\{u : \langle v, u \rangle \leq 0\}$, a contradiction. The continuity of the maps $\xi \mapsto v_j^\pm(\xi)$, for every j , entails, in particular, that the set-valued map $\xi \mapsto P(\xi)$ is upper semicontinuous; thus, if r is small enough, there exists $K' \geq 2K$ such that

$$P(\xi) \subseteq B_{\mathbb{R}^{2n}}(0, K'), \quad \forall \xi \in B_{\mathbb{H}^n}(0, r).$$

This implies that $T(\xi) \subseteq B_{\mathbb{R}^{2n}}(0, K')$, for all $\xi \in B_{\mathbb{H}^n}(0, r)$, therefore T is locally bounded at the origin. $\square$

Clearly, Theorem 2.2 and Theorem 1.1 give

Corollary 3.1 *Let $T : \mathbb{H}^n \rightrightarrows V_1$ be a maximal H-monotone map, such that $\text{dom}(T) = \mathbb{H}^n$. Then T is locally bounded and upper semicontinuous.*

4 On Minty's theorem.

This section we apply our main result in Theorem 1.1 in order to prove a horizontal version of Minty's theorem. In the following, for a given operator $T : \mathbb{H}^n \rightrightarrows V_1$ and $\lambda > 0$, we denote by $T_\lambda : \mathbb{H}^n \rightrightarrows V_1$ the operator

$$T_\lambda = \xi_1 + \lambda T.$$

It is clear that if T is H-monotone, then T_λ is strictly H-monotone. We recall that in [10] the authors prove Theorem 2.1, a result of Minty type in the case $n = 1$. Now, our aim is to prove Theorem 1.2. In comparison to Theorem 2.1, we will remove the H-cyclically monotone assumption in i., and we also show that the result holds for $\mathbb{H}^n$. We note that in

the Example 2.1 we have a map that satisfies the assumption in Theorem 1.2, but not the assumption i. in Theorem 2.1.

In order to prove the following result, we will follow the idea in [5] by using degree-theoretical arguments for set valued maps [14]; the results needed in the proof are collected in the Appendix of [5].

Proof of Theorem 1.2:

Let us first prove that *i.* implies *ii.*, which is the more difficult part. Let T be maximal H -monotone with $\text{dom}(T) = \mathbb{H}^n$. Let us fix $\eta \in \mathbb{H}^n$ and $\lambda > 0$. We consider the linear projection map $\pi : H_\eta \rightarrow V_1 = \mathbb{R}^{2n}$ defined by $\pi(x, y, t) = (x, y)$. Note that since we restricted the projection to a hyperplane we have that π is bijective and we denote by $\pi^{-1} : \mathbb{R}^{2n} \rightarrow H_\eta$ its inverse. We introduce the following notations: $\widetilde{T}_\lambda$ is the operator $\widetilde{T}_\lambda = T_\lambda \circ \pi^{-1} : \mathbb{R}^{2n} \rightrightarrows V_1$ and $\pi(\zeta) = \tilde{\zeta}$, $\forall \zeta \in H_\eta$. We have to prove that $\widetilde{T}_\lambda$ is surjective.

Let us fix $p_0 \in V_1 \cong \mathbb{R}^{2n}$: we show that it is possible to find $R_0 > 0$ large enough such that

$$p_0 \in \widetilde{T}_\lambda(B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0)) : \quad (23)$$

in particular, we show this for

$$R_0 > \|p_0\| + \|\xi_1(\eta)\| + \lambda \sup\{\|v_\eta\| : v_\eta \in T(\eta)\}. \quad (24)$$

Note, that the fact that the expression on the right in the above inequality is finite follows from local boundedness of T .

Step 1. In order to prove (23), we show first that

$$\deg_{SV}(\widetilde{T}_\lambda - p_0, B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0), 0) = 1, \quad (25)$$

where $\deg_{SV}$ denotes the degree function for set-valued maps. We consider the parametric set-valued map $\mathcal{F} : [0, 1] \times \overline{B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0)} \rightrightarrows V_1$ defined by

$$\mathcal{F}(\alpha, \tilde{\zeta}) = \tilde{\zeta} - p_0 + \lambda \alpha T(\pi^{-1}(\tilde{\zeta})),$$

for all $\alpha \in [0, 1]$, $\tilde{\zeta} \in \overline{B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0)}$.

First we note that, by Proposition 2.2, the map $\mathcal{F}$ is convex-valued and compact-valued, i.e. for every fixed $(\alpha, \tilde{\zeta}) \in [0, 1] \times \overline{B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0)}$, the set $\mathcal{F}(\alpha, \tilde{\zeta})$ is compact and convex in $\mathbb{R}^{2n}$. Moreover, Corollary 2.1 and Corollary 3.1 imply that

$$\overline{\{\cup \mathcal{F}(\alpha, \tilde{\zeta}) : (\alpha, \tilde{\zeta}) \in [0, 1] \times \overline{B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0)}\}}$$

is compact in $\mathbb{R}^{2n}$. Finally, Corollary 2.1 implies that the map $(\alpha, \tilde{\zeta}) \mapsto \mathcal{F}(\alpha, \tilde{\zeta})$ is usc from $[0, 1] \times \overline{B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0)}$ into $2^{\mathbb{R}^{2n}} \setminus \{\emptyset\}$.

Now we are in the position to apply the mentioned degree-theoretical arguments for set valued maps. According to the above discussion, it follows that our map $\mathcal{F}(\alpha, \cdot)$ is a homotopy of class (P) (see [14] and also Appendix in [5]). The argument is based on the application of Theorem 6.2 in [5]. In order to apply this statement we need to show that the constant curve $\gamma : [0, 1] \rightarrow \mathbb{R}^{2n}$, defined by $\gamma(\alpha) = 0$, is such that

$$\gamma(\alpha) \notin \mathcal{F}(\alpha, \partial B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0)), \quad \forall \alpha \in [0, 1]. \quad (26)$$

We show (26) through arguing by contradiction: suppose that for some α there exists $\tilde{\zeta} \in \partial B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0)$ such that

$$0 \in \mathcal{F}(\alpha, \tilde{\zeta}) = \tilde{\zeta} - p_0 + \lambda \alpha T(\pi^{-1}(\tilde{\zeta})),$$

i.e. $p_0 = \xi_1(\zeta) + \lambda\alpha w_\zeta$ for some $w_\zeta \in T(\zeta)$, $\zeta \in H_\eta$ and $\zeta \in \partial B_{\mathbb{H}^n}(\eta, R_0)$. This implies that, for every $v_\eta \in T(\eta)$, we have

$$p_0 - \xi_1(\eta) - \lambda\alpha v_\eta = \xi_1(\zeta) - \xi_1(\eta) + \lambda\alpha(w_\zeta - v_\eta).$$

Multiplying the previous vector equality by $(\xi_1(\zeta) - \xi_1(\eta))$ we obtain

$$\langle \xi_1(\zeta) - \xi_1(\eta), p_0 - \xi_1(\eta) - \lambda\alpha v_\eta \rangle = \|\xi_1(\zeta) - \xi_1(\eta)\|^2 + \lambda\alpha \langle \xi_1(\zeta) - \xi_1(\eta), w_\zeta - v_\eta \rangle.$$

The H-monotonicity of T implies

$$\|\xi_1(\zeta) - \xi_1(\eta)\|^2 \leq \|\langle \xi_1(\zeta) - \xi_1(\eta), p_0 - \xi_1(\eta) - \lambda\alpha v_\eta \rangle\| \leq \|\xi_1(\zeta) - \xi_1(\eta)\| \cdot \|p_0 - \xi_1(\eta) - \lambda\alpha v_\eta\|;$$

hence

$$R_0 \leq \|p_0 - \xi_1(\eta) - \lambda\alpha v_\eta\|$$

This contradicts (24) and hence (26) holds. The homotopy invariance property for $\mathcal{F}$ (see Theorem 6.2 in [5]) gives that

$$\deg_{SV}(\mathcal{F}(\alpha, \cdot), B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0), \gamma(\alpha))$$

does not depend on α : hence,

$$\begin{aligned} \deg_{SV}(\widetilde{T}_\lambda - p_0, B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0), 0) &= \deg_{SV}(\mathcal{F}(1, \cdot), B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0), 0) \\ &= \deg_{SV}(\mathcal{F}(0, \cdot), B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0), 0) \\ &= \deg_{SV}(I_{\mathbb{R}^{2n}} - p_0, B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0), 0) \\ &= \deg_B(I_{\mathbb{R}^{2n}} - p_0, B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0), 0) \\ &= \deg_B(I_{\mathbb{R}^{2n}}, B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0), p_0), \end{aligned} \quad (27)$$

where $\deg_B$ denotes the degree function for single-valued maps. Note that (24) implies that $p_0 \in B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0)$: hence $\deg_B(I_{\mathbb{R}^{2n}}, B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0), p_0) = 1$ (see Theorem 6.1 in [5]) and hence, by (27), we have that (25) is true.

Step 2. By Step 1 and the definition of $\deg_{SV}$, for small $\varepsilon > 0$, one has that

$$\deg_B(f_\varepsilon - p_0, B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0), 0) = 1, \quad (28)$$

where $f_\varepsilon : \overline{B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0)} \rightarrow \mathbb{R}^{2n}$ is a continuous approximate selector of the upper semicontinuous set-valued map $\widetilde{T}_\lambda$ such that

$$f_\varepsilon(\tilde{\zeta}) \in \widetilde{T}_\lambda \left(B_{\mathbb{R}^{2n}}(\tilde{\zeta}, \varepsilon) \cap \overline{B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0)} \right) + B_{\mathbb{R}^{2n}}(0, \varepsilon), \quad \forall \tilde{\zeta} \in \overline{B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0)}, \quad (29)$$

see Proposition 6.1 in [5]. Let $\varepsilon = \frac{1}{k}$ and let $\phi_k := f_{1/k}$, $k \in \mathbb{N}$. First of all, from (28) and the properties of the Brouwer degree function $\deg_B$ (see Theorem 6.1 in [5]), we have that for every $k \in \mathbb{N}$ there exists $\tilde{\zeta}_k \in B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0)$ such that $p_0 = \phi_k(\tilde{\zeta}_k)$. Up to a subsequence, we may assume that $\tilde{\zeta}_k \rightarrow \tilde{\nu} \in \overline{B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0)}$. On the other hand, by relation (29), we have that

$$p_0 = \phi_k(\tilde{\zeta}_k) \in \widetilde{T}_\lambda \left(B_{\mathbb{R}^{2n}}(\tilde{\zeta}_k, 1/k) \cap \overline{B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0)} \right) + B_{\mathbb{R}^{2n}}(0, 1/k),$$

i.e., there exists $\tilde{\nu}_k \in B_{\mathbb{R}^{2n}}(\tilde{\zeta}_k, \frac{1}{k}) \cap \overline{B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0)}$ and $p_k \in B_{\mathbb{R}^{2n}}(0, \frac{1}{k})$ such that $p_0 \in \widetilde{T}_\lambda(\tilde{\nu}_k) + p_k$. Clearly, $\tilde{\nu}_k \rightarrow \tilde{\nu}$ and $p_k \rightarrow 0$ as $k \rightarrow \infty$. Let us consider the sequence $\{(p_0 - p_k, \tilde{\nu}_k)\} \in \text{graph}(\widetilde{T}_\lambda)$; by the usc of $\widetilde{T}_\lambda$ we get that $p_0 \in \widetilde{T}_\lambda(\tilde{\nu})$.

Finally, we claim that $\tilde{v} \in B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0)$. To see this, let us assume, by contradiction, that $\tilde{v} \in \partial B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0)$. Then, $p_0 \in \widetilde{T_\lambda}(\tilde{v})$ is equivalent to $0 \in \widetilde{T_\lambda}(\tilde{v}) - p_0 = \mathcal{F}(1, \tilde{v})$, which contradicts relation (26). Consequently, $\tilde{v} \in B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0)$; therefore we obtain (23), which concludes the proof of the first implication $i. \Rightarrow ii$ of Theorem 1.2.

Let us prove that $ii.$ implies $i.$ The proof is essentially in [10], where the case $n = 1$ is considered; however, for the sake of completeness, we include it. Let $T : \mathbb{H}^n \rightrightarrows V_1$ be a set-valued H-monotone map, with domain $\mathbb{H}^n$, such that, for every $\eta_0 \in \mathbb{H}^n$,

$$\text{rge}(T_\lambda)|_{H_{\eta_0}} = V_1.$$

We argue by contradiction and suppose that T is not maximal H-monotone. Then there exist $\eta_0 \in \mathbb{H}^n$, and $w \notin T(\eta_0)$ such that, for every $\eta \in H_{\eta_0}$, and $v \in T(\eta)$,

$$\langle w - v, \xi_1(\eta_0) - \xi_1(\eta) \rangle \geq 0. \quad (30)$$

Without loss of generality, we assume that $\eta_0 = e$: in fact, via a left translation, the map $\eta \mapsto T(\eta_0 \circ \eta)$ has the same properties of T . From the assumptions, for $\lambda = 1$ we have $\text{rge}(T + \xi_1)|_{H_e} = V_1$; therefore

$$w = \tilde{v} + \xi_1(\tilde{\eta}), \quad (31)$$

for some $\tilde{\eta} \in H_e$ and $\tilde{v} \in T(\tilde{\eta})$. From (31), choosing $\eta = \tilde{\eta}$ in (30), we obtain

$$-\langle \xi_1(\tilde{\eta}), \xi_1(\tilde{\eta}) \rangle \geq 0,$$

i.e., $\xi_1(\tilde{\eta}) = 0$. Since $\tilde{\eta} \in H_e$, we deduce that $\tilde{\eta} = 0$, and $w = \tilde{v} \in T(e)$, contradicting our assumption on w . This concludes the proof of Theorem 1.2. $\square$

4.1 Lipschitz continuity of the resolvent operator in the Hausdorff metric.

In this subsection we are interested in studying the regularity of the resolvent Q_λ of a maximal H-monotone operator T defined by

$$Q_\lambda = (\xi_1 + \lambda T)^{-1} : V_1 \rightrightarrows \mathbb{H}^n.$$

First, we have to recall that if T is maximal H-monotone and $\eta \in \mathbb{H}^n$, then the map $T_\lambda|_{H_\eta}$ is not injective, in general, and hence $(T_\lambda|_{H_\eta})^{-1} : V_1 \rightrightarrows H_\eta$ is not single-valued (see Example 4.1 below). Using the strictly H-monotonicity of the operator T_λ , the only information we have is that, for every $\eta' \in H_\eta$,

$$T_\lambda(\eta) \cap T_\lambda(\eta') = \emptyset.$$

We note that, for every fixed v , $Q_\lambda(v)$ is a closed subset of $\mathbb{H}^n$, since it is the inverse image via the usc map T_λ of a point. Moreover, Theorem 1.2 implies that for every fixed $v \in V_1$ and $\eta \in \mathbb{H}^n$, there exists at least one point $\eta' \in H_\eta$ such that $v \in T_\lambda(\eta')$, i.e. $\eta' \in Q_\lambda(v)$. Therefore $H_{(0,0,h)} \cap Q_\lambda(v) \neq \emptyset$, for every $h \in \mathbb{R}$. Hence $Q_\lambda(v)$ is unbounded for every fixed $v \in V_1$. We summarize this discussion in the following:

Remark 4.1 *Let $T : \mathbb{H}^n \rightrightarrows V_1$ be a maximal H-monotone map with $\text{dom}(T) = \mathbb{H}^n$. Then, for every $\lambda > 0$, the resolvent $Q_\lambda : V_1 \rightrightarrows \mathbb{H}^n$ is closed-valued, and $Q_\lambda(v)$ is unbounded for every $v \in V_1$.*

As we mentioned in the introduction, if we consider the resolvent in our context, we are very far from the Euclidean situation where the resolvent map $(I + \lambda T)^{-1}$ of a maximal monotone set-valued map $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is single-valued on $\mathbb{R}^n$ and 1-Lipschitz continuous. However, in this line of investigation, it is useful to think about the notion of multivalued Lipschitz map.

Let $Q : V_1 \rightrightarrows \mathbb{H}^n$ be a closed-valued multivalued map. We recall (see Definition 9.26 in [15]) that Q is *Lipschitz continuous in the Hausdorff metric*, if $\text{dom}(Q) = V_1$ and there exists a positive k such that

$$Q(v') \subseteq Q(v) + B_{\mathbb{H}^n}(0, k\|v' - v\|), \quad \forall v, v' \in V_1.$$

We have the following regularity result for our resolvent:

Proposition 4.1 *Let $T : \mathbb{H}^n \rightrightarrows V_1$ be a maximal H -monotone map with $\text{dom}(T) = \mathbb{H}^n$. Then, for every $\lambda > 0$, the resolvent Q_λ is 1-Lipschitz continuous in the Hausdorff metric.*

Proof: Let us consider v and v' in V_1 , with $v \neq v'$. For every $\eta \in Q_\lambda(v)$, i.e.

$$v \in \xi_1(\eta) + \lambda T(\eta), \tag{32}$$

Theorem 1.2 guarantees that there exists $\eta' \in H_\eta$ such that $\eta' \in Q_\lambda(v')$, i.e.

$$v' \in \xi_1(\eta') + \lambda T(\eta'). \tag{33}$$

Relations (32) and (33) give that $v - \xi_1(\eta) \in \lambda T(\eta)$ and $v' - \xi_1(\eta') \in \lambda T(\eta')$. Since $\lambda T : \mathbb{H} \rightrightarrows V_1$ is H -monotone, we have

$$\langle v - \xi_1(\eta) - v' + \xi_1(\eta'), \xi_1(\eta) - \xi_1(\eta') \rangle \geq 0$$

and hence

$$\begin{aligned} \|v - v'\| &\geq \langle v - v', \frac{\xi_1(\eta) - \xi_1(\eta')}{\|\xi_1(\eta) - \xi_1(\eta')\|} \rangle \\ &\geq \|\xi_1(\eta) - \xi_1(\eta')\|. \end{aligned}$$

The previous inequality implies that for every $\eta \in Q_\lambda(v)$ there exists $\eta' \in Q_\lambda(v')$ such that $d_H(\eta, \eta') \leq \|v - v'\|$. This implies 1-Lipschitz continuity of Q_λ in the Hausdorff metric. The claim is proved. $\square$

Let us conclude with the following example presented in [10]:

Example 4.1 *Let us consider the gauge function $N : \mathbb{H} \rightarrow \mathbb{R}$ defined as*

$$N(x, y, t) := ((x^2 + y^2)^2 + t^2)^{1/4}.$$

It is known that this function is H -convex (see [12]). The associated horizontal subgradient map $\partial_H N$ is given by

$$\partial_H N(x, y, t) = \begin{cases} \overline{B_{\mathbb{R}^2}(0, 1)} & (x, y, t) = (0, 0, 0) \\ \frac{1}{N^3(x, y, t)} (x(x^2 + y^2) + yt, y(x^2 + y^2) - xt) & (x, y, t) \neq (0, 0, 0). \end{cases}$$

For every fixed $\lambda > 0$, let the map $T_\lambda := \xi_1 + \lambda \partial_H N : \mathbb{H} \rightrightarrows V_1$ that is maximal strictly H -monotone. First, it is possible to prove that there exist $\eta'' \in \mathbb{H}$, and $\eta, \eta' \in H_{\eta''}$, $\eta \neq \eta'$, such that

$$T_\lambda(\eta) \cap T_\lambda(\eta') \neq \emptyset.$$

Secondly, it is clear that T_λ is not a Lipschitz continuous map, i.e. it does not exist a positive k such that

$$T_\lambda(\eta') \subseteq T_\lambda(\eta) + B_{\mathbb{R}^2}(0, k d_H(\eta, \eta')) , \quad \forall \eta, \eta' \in \mathbb{H}.$$

In fact, for $\eta' = (0, 0, 0)$ and $\eta = (x, y, 0)$ the previous inclusion is false.

If we are interested in $Q_\lambda = (\xi_1 + \lambda \partial_H N)^{-1} : V_1 \rightrightarrows \mathbb{H}$, an easy calculation gives

$$Q_\lambda(0, 0) = \{(0, 0, t) \in \mathbb{H}; t \in \mathbb{R}\}.$$

Now, let us consider $(x, y, t) \neq (0, 0, 0)$ and $v \neq (0, 0)$ with $(x, y, t) \in Q_\lambda(v)$, i.e. $v = T_\lambda(x, y, t)$; straightforward computations lead to the following

$$\epsilon^2 \geq \|v\|^2 = \|T_\lambda(x, y, t)\|^2 = (x^2 + y^2) \left(1 + \frac{\lambda^2}{N^2(x, y, t)} + \frac{2\lambda}{N^3(x, y, t)}(x^2 + y^2) \right);$$

hence $(x, y, t) \in Q_\lambda(v)$ implies $\|(x, y)\| \leq \epsilon$. Moreover, since T_λ is surjective on every horizontal plane H_η and in particular on $H_{(0,0,t)}$, we obtain that for every $t \neq 0$ there exists (x, y) such that $v \in T_\lambda(x, y, t)$. These prove that $0 \neq \|v\| \leq \epsilon$ gives

$$Q_\lambda(v) \text{ is unbounded,} \quad Q_\lambda \subset \{(x, y, t) \in \mathbb{H} : \|(x, y)\| \leq \epsilon\}.$$

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