SINGULAR PERTURBATION PROBLEM IN BOUNDARY/FRACTIONAL COMBUSTION

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ABSTRACT. Motivated by a nonlocal free boundary problem, we study uniform properties of solutions to a singular perturbation problem for a boundary-reaction-diffusion equation, where the reaction term is of combustion type. This boundary problem is related to the fractional Laplacian. After an optimal uniform Hölder regularity is shown, we pass to the limit to study the free boundary problem it leads to.

To our friend Juan-Luis Vázquez for his 70th birthday

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1. INTRODUCTION

In this paper we study nonnegative solutions for the semilinear boundary-reaction-diffusion problem:

$$(P_\varepsilon) \quad \begin{aligned} \mathcal{L}_s u_\varepsilon &:= \operatorname{div}(|x_n|^{1-2s} \nabla u_\varepsilon) = 0 \quad \text{in } B_1^+ = B_1 \cap \{x_n > 0\}, \\ - \lim_{x_n \rightarrow 0^+} x_n^{1-2s} \frac{\partial u_\varepsilon}{\partial x_n} &= -\beta_\varepsilon(u_\varepsilon) \quad \text{on } B_1' = B_1 \cap \{x_n = 0\}, \end{aligned}$$

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where B_1 is the unit ball in $\mathbb{R}^n$, $n \geq 2$, $s \in (0, 1)$, and ε is a small positive parameter. The nonlinear reaction term $\beta_\varepsilon(t)$ is of combustion type and is given by

$$(1.1) \quad \beta_\varepsilon(t) = \frac{1}{\varepsilon} \beta\left(\frac{t}{\varepsilon}\right), \quad t \in \mathbb{R},$$

with $\beta \in C_c^{0,1}(\mathbb{R})$ satisfying

$$(1.2) \quad \beta \geq 0, \quad \text{supp } \beta = [0, 1], \quad \text{and} \quad \int_0^1 \beta(t) dt = M.$$

Note that the solutions of (P_ε) are the critical points (including the local minimizers) of the energy functional

$$(1.3) \quad J_\varepsilon(u) = \int_{B_1^+} |\nabla u|^2 |x_n|^{1-2s} + \int_{B_1'} 2\mathcal{B}_\varepsilon(u)$$

among all functions in the weighed Sobolev space $W^{1,2}(B_1^+, |x_n|^{1-2s})$ with fixed trace on $(\partial B_1)^+ = \partial B_1 \cap \{x_n > 0\}$, where $\mathcal{B}_\varepsilon$ is the primitive of β_ε given by

$$\mathcal{B}_\varepsilon(t) = \int_0^t \beta_\varepsilon(s) ds.$$

Formally, as $\varepsilon \rightarrow 0+$, the functional J_ε converges to

$$J_0(u) = \int_{B_1^+} |\nabla u|^2 |x_n|^{1-2s} + \int_{B_1'} 2M \chi_{\{u>0\}},$$

which is the boundary (or thin) analogue of the Alt-Caffarelli [AC81] energy functional. The study of the minimizers of J_0 has been initiated in [CRS10] and by now there is a good understanding of the associated free boundary problem. Namely, it is known that the minimizers of J_0 solve (in the appropriate sense)

$$(P) \quad \begin{aligned} \text{div}(|x_n|^{1-2s} \nabla u) &= 0 \quad \text{in } B_1^+, \\ - \lim_{x_n \rightarrow 0+} x_n^{1-2s} \frac{\partial u}{\partial x_n} &= 0 \quad \text{on } \{u > 0\} \cap B_1', \\ \lim_{t \rightarrow 0+} \frac{u(x_0 + t\nu'_{x_0})}{t^s} &= \sqrt{\frac{2M}{c_0(s)}} \quad \text{for } x_0 \in \mathcal{F}_u, \end{aligned}$$

where

$$\mathcal{F}_u := \partial\{u(\cdot, 0) > 0\} \cap B_1'$$

is the *free boundary* in the problem, ν'_{x_0} is the in-plane, inner unit normal to $\{u(\cdot, 0) > 0\}$ and $c_0(s) > 0$ is a constant. The regularity properties of the free boundary $\mathcal{F}_u$ for the minimizers in the case $s = 1/2$ have been studied in the series of papers [DSR12, DSS12, DSS14], establishing the smoothness of flat free boundaries. For the general $s \in (0, 1)$, the $C^{1,\alpha}$ regularity of flat free boundaries has been established in [DSSS14].

One of our main objectives in this paper is to show that the solutions u_ε of the singular perturbation problem (P_ε) , also converge to a solution to the free boundary problem (P) , in a certain, weaker, sense. We show the uniform s -Hölder regularity of u_ε (Theorem 2.1), however, the passage to the limit u as $\varepsilon \rightarrow 0+$ is complicated by the fact that $\mathcal{B}_\varepsilon(u_\varepsilon)$ may not converge (in weakly- $*$ sense) to $M \chi_{\{u>0\}}$. Nevertheless, at free boundary points $x_0 \in \mathcal{F}_u$ with a measure-theoretical normal and a nondegeneracy condition on u , we can establish an asymptotic development of u , implying the free boundary condition in (P) (Theorem 4.1).

This kind of convergence results are very well known in combustion theory for the singular perturbation problems of the type

$$\Delta u_\varepsilon = \beta_\varepsilon(u_\varepsilon) \quad \text{in } B_1,$$

(even in time-dependent case) with β_ε as in (1.1), since the works of Zel'dovich and Frank-Kamenetskii [ZFK38]. Mathematically rigorous results, however, are much more recent. Here we cite some of the important ones for our paper: [BCN90, CV95, Vaz96, CLW97a, CLW97b, DPS03, Wei03, MT07, Tei08].

The singular-perturbation problem (P_ε) can be also viewed as the localized version of the global reaction-diffusion equation

$$(P'_\varepsilon) \quad \begin{aligned} (-\Delta_{x'})^s u_\varepsilon &= -\beta_\varepsilon(u_\varepsilon) \quad \text{in } \Omega \subset \mathbb{R}^{n-1} \\ u_\varepsilon &= g_\varepsilon \quad \text{on } \mathbb{R}^{n-1} \setminus \Omega \end{aligned}$$

for the fractional Laplacian $(-\Delta_{x'})^s$ in $x' = (x_1, \dots, x_{n-1})$ variables, where g_ε is a nonnegative function on $\mathbb{R}^{n-1} \setminus \Omega$ having the meaning of the boundary data. We recall that the fractional Laplacian is defined as the Fourier multiplier of symbol $|\xi'|^{2s}$ for $s \in (0, 1)$ (see [Lan72] for a treatment of these operators). Note that the solutions of (P'_ε) are the critical points of the energy functional

$$j(v) = c_{n,s} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{(u(x') - u(y'))^2}{|x' - y'|^{n-1+2s}} + \int_{\mathbb{R}^{n-1}} 2\mathcal{B}_\varepsilon(u),$$

among all functions such that $u = g_\varepsilon$ on $\mathbb{R}^{n-1} \setminus \Omega$. (Here $c_{n,s} > 0$ is a normalization constant.)

The connection between (P_ε) and (P'_ε) is then established through the so-called Caffarelli-Silvestre extension [CS07]: if for a given function u on $\mathbb{R}^{n-1}$ (with appropriate growth conditions at infinity) we consider the extension $\tilde{u}$ to $\mathbb{R}_+^n = \mathbb{R}^{n-1} \times (0, \infty)$ by solving the Dirichlet problem

$$\begin{aligned} \mathcal{L}_s \tilde{u} &= \operatorname{div}(x_n^{1-2s} \nabla \tilde{u}) = 0 \quad \text{in } \mathbb{R}_+^n \\ \tilde{u} &= u \quad \text{on } \mathbb{R}^{n-1} \times \{0\} \end{aligned}$$

then

$$-c_{n,s} \lim_{x_n \rightarrow 0+} x_n^{1-2s} \frac{\partial \tilde{u}}{\partial x_n} = (-\Delta_{x'})^s u \quad \text{on } \mathbb{R}^{n-1} \times \{0\}$$

for a positive constant $c_{n,s}$. Hence, if u_ε solves (P'_ε) , $x_0 \in \Omega$ and $R > 0$ are such that $B'_R(x_0) \subset \Omega$, then the extension of u_ε to $\mathbb{R}_+^n$ constructed as above will solve (P_ε) in $B_R^+(x_0)$. As a consequence, the singular perturbation problem (P'_ε) for the fractional Laplacian, becomes a boundary (or thin) singular perturbation problem (P_ε) for the operator $\mathcal{L}_s$ in one dimension higher.

Main results and the structure of the paper. In this paper, we will focus on the uniform estimate of the solutions to (P_ε) and the proof of the free boundary condition in (P) .

- In §2 we prove the uniform s -Hölder regularity for the solutions of (P_ε) , see Theorem 2.1. This allows to pass to the limit as $\varepsilon \rightarrow 0+$ and study the resulting solutions in the subsequent sections.

- In §3, we prove various results concerning the limits of u_ε , or, more precisely, the limits of the pairs $(u_\varepsilon, \mathcal{B}_\varepsilon(u_\varepsilon))$, which we denote (u, χ) . The results include the compactness lemma (Lemma 3.1), ensuring the convergence in the proper spaces and Weiss-type monotonicity formulas for u_ε and (u, χ) (Theorems 3.4 and 3.5).
- In §4 we prove that the free boundary condition in problem (P) is satisfied at free boundary points with measure-theoretical normal for $\{u(\cdot, 0) > 0\}$, under the additional nondegeneracy condition (Theorem 4.1). This is done by identifying the blowups with flat free boundaries (Proposition 4.3). We conclude the paper by proving two additional propositions related to the Weiss energy at nondegenerate points (Propositions 4.4 and 4.5).

Notations and preliminaries.

- We will use fairly standard notations in this paper.
 - $\mathbb{R}^n$ will stand for the n -dimensional Euclidean space;
 - For every $x \in \mathbb{R}^n$ we write $x = (x', x_n)$, where $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$. This identifies $\mathbb{R}^n$ with $\mathbb{R}^{n-1} \times \mathbb{R}$. We also don't distinguish between $(x', 0)$ and x' , thus identifying $\mathbb{R}^{n-1}$ with $\mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$.
 - $\mathbb{R}_\pm^n = \mathbb{R}^n \cap \{\pm x_n > 0\}$;
 - Balls and half-balls: $B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$, $B_r^\pm(x) = B_r(x) \cap \{\pm x_n > 0\}$,
 - 'Thin' balls: $B'_r(x) = B_r(x) \cap \{x_n = 0\}$.
 - Typically, we skip the center in the notation for balls if it is the origin. Thus, $B_1 = B_1(0)$, $B'_1 = B'_1(0)$, etc.
- For the functions β and β_ε , we make the following assumption throughout the paper. Besides (1.1)–(1.2), we fix a constant $A > 0$ such that

$$\max\{|\beta(s)|, |\beta'(s)|\} \leq A, \quad \text{for all } s \in \mathbb{R}.$$

We will also need to make a technical assumption that

$$\beta > 0 \quad \text{on } (0, 1).$$

- The functions $\mathcal{B}, \mathcal{B}_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$ are the primitives of β and β_ε given by

$$(1.4) \quad \mathcal{B}(t) = \int_0^t \beta(s) ds, \quad \mathcal{B}_\varepsilon(t) = \int_0^t \beta_\varepsilon(s) ds = \mathcal{B}(s/\varepsilon)$$

- *Even extension of u_ε and weak solutions of (P_ε) .* In what follows, we will be extending the functions u_ε in B_1^+ with even reflection to all of B_1 :

$$u_\varepsilon(x', -x_n) = u_\varepsilon(x', x_n), \quad \text{for } x \in B_1^+.$$

With such an extension in mind, u_ε is a *weak solution* of (P_ε) if for any test function $\varphi \in C_c^\infty(B_1)$

$$(1.5) \quad \int_{B_1} |x_n|^{1-2s} \nabla u_\varepsilon \cdot \nabla \varphi dx + \int_{B_1'} 2\beta_\varepsilon(u_\varepsilon) \varphi dx' = 0,$$

or, in other words,

$$\operatorname{div}(|x_n|^{1-2s} \nabla u_\varepsilon) = 2\beta_\varepsilon(u_\varepsilon) \chi_{\{x_n=0\}}$$

in the sense of distributions.

Unless specified otherwise, by a solution of (P_ε) we will always understand a weak solution of (P_ε) .

We also note that for functions which are even symmetric in x_n , the energy functional (1.3) can be rewritten as

$$J_\varepsilon(v) = \frac{1}{2} \int_{B_1} |x_n|^{1-2s} |\nabla v|^2 + \int_{B'_1} 2\mathcal{B}_\varepsilon(v).$$

- *Rescalings.* Finally, throughout the paper we will make the extensive use of rescalings. For a given $x_0 \in B'_1$ and $\lambda > 0$ define

$$u_{\varepsilon,\lambda}(x) = u_{\varepsilon,\lambda}^{x_0}(x) := \frac{u(x_0 + \lambda x)}{\lambda^s}, \quad x \in B_{(1-|x_0|)/\lambda}.$$

A straightforward computation shows that $u_{\varepsilon,\lambda}$ satisfies

$$\operatorname{div}(|x_n|^{1-2s} \nabla u_{\varepsilon,\lambda}) = 2\beta_{\varepsilon/\lambda^s}(u_{\varepsilon,\lambda}) \mathcal{H}^{n-1} \llcorner \{x_n = 0\} \quad \text{in } B_{(1-|x_0|)/\lambda}.$$

2. UNIFORM $C^{0,s}$ REGULARITY

In this section we prove the following uniform Hölder regularity result for the solutions of (P_ε) .

Theorem 2.1 (Uniform s -Hölder estimate). *Let u_ε be a nonnegative solution of (P_ε) with $\|u_\varepsilon\|_{L^\infty(B_1)} \leq L$. Then $u_\varepsilon \in C^{0,s}(K)$ for any $K \Subset B_1$ with*

$$\|u_\varepsilon\|_{C^{0,s}(K)} \leq C(n, s, A, L, K)$$

uniformly for all $\varepsilon \in (0, 1)$.

Our proof follows the ideas from [DPS03] in the case of p -harmonic functions. One of the main steps is the following Harnack-type inequality.

Lemma 2.2 (Harnack-type inequality). *Let v be a locally bounded nonnegative weak solution of*

$$0 \leq \operatorname{div}(|x_n|^{1-2s} \nabla v) \leq A \chi_{\{0 < v < 1\}} \mathcal{H}^{n-1} \llcorner \{x_n = 0\} \quad \text{in } B_1$$

with $v(0) \leq 1$. Then there exists a constant $C = C(n, s, A)$ such that

$$\|v\|_{L^\infty(B_{1/4})} \leq C.$$

To prove this lemma, we will need the following interior Hölder estimate.

Lemma 2.3 (Interior s -Hölder estimate). *Let $|w| \leq M$ be a weak solution of*

$$|\operatorname{div}(|x_n|^{1-2s} \nabla w)| \leq \mu \mathcal{H}^{n-1} \llcorner \{x_n = 0\} \quad \text{in } B_1(x_0).$$

Here x_0 is not necessarily on $\{x_n = 0\}$.

Then $w \in C^{0,s}(B_{1/2}(x_0))$ with

$$\|w\|_{C^{0,s}(B_{1/2}(x_0))} \leq C(n, s, \mu, M).$$

Proof.

- (i) When $x_0 = 0$, or more generally, $(x_0)_n = 0$, we refer to Remark 5.2 and the proof of Theorem 5.1 in [ALP15].
- (ii) The case of general x_0 is obtained by considering the subcases
 - (a) $|(x_0)_n| > 3/4$, and
 - (b) $|(x_0)_n| \leq 3/4$.

In the subcase (a) the equation is uniformly elliptic in $B_{5/8}(x_0)$, and the estimate follows from standard interior estimates for uniformly elliptic equations. In the subcase (b), the s -Hölder continuity in $B_{1/2}(x_0) \cap \{|x_n| \leq 1/8\}$ is obtained from the case (i) above. The s -Hölder continuity in $B_{1/2}(x_0) \cap \{|x_n| > 1/8\}$ is obtained from the uniform ellipticity of the operator $\mathcal{L}_s$ in $B_1(x_0) \cap \{|x_n| > 1/8\}$. $\square$

Proof of Lemma 2.2. We start by an observation that the function v is continuous by Lemma 2.3. We will use this fact implicitly throughout the proof.

We argue by contradiction. Assuming that the conclusion of the lemma fails, there exists a sequence of nonnegative solutions v_k with

$$v_k(0) \leq 1 \quad \text{but} \quad \|v_k\|_{L^\infty(B_{1/4})} \geq k.$$

Let $\Omega_k := \{x \in B'_1 : v_k(x) \leq 1\}$, $O_k := \{x \in B_1 : \text{dist}(x, \Omega_k) \leq \frac{1}{3}(1 - |x|)\}$, and

$$m_k := \max_{O_k} (1 - |x|)v_k(x).$$

Observe that $B_{1/4} \subset O_k$. Then $m_k \geq \frac{3}{4} \sup_{B_{1/4}} v_k \geq \frac{3}{4}k$. Let $x_k \in O_k$ such that $(1 - |x_k|)v_k(x_k) = m_k$, then

$$(2.1) \quad v_k(x_k) \geq m_k \geq \frac{3}{4}k.$$

Consider the distance $\delta_k := \text{dist}(x_k, \Omega_k)$ and $y_k \in \Omega_k$ realize δ_k . By (2.1), $\delta_k > 0$. Using the fact that $\delta_k \leq \frac{1}{3}(1 - |x_k|)$ and the triangle inequality, we obtain that $B_{\delta_k/2}(y_k) \subset O_k$ and for any $z \in B_{\delta_k/2}(y_k)$,

$$(2.2) \quad v_k(z) \leq \frac{m_k}{1 - |z|} = \frac{1 - |x_k|}{1 - |z|} v_k(x_k) \leq 2v_k(x_k).$$

Since v_k satisfies the homogeneous equation $\text{div}(|x_n|^{1-2s}\nabla v_k) = 0$ in $B_{\delta_k}(x_k)$, by the Harnack inequality in [FKS82] there exists $c = c(n, s)$ such that

$$\inf_{B_{3\delta_k/4}(x_k)} v_k \geq c v_k(x_k).$$

In particular, since $\overline{B_{\delta_k/4}(y_k)} \cap \overline{B_{3\delta_k/4}(x_k)} \neq \emptyset$, then

$$(2.3) \quad \sup_{B_{\delta_k/4}(y_k)} v_k \geq c v_k(x_k).$$

Define

$$w_k(x) := \frac{v_k(y_k + \delta_k x)}{v_k(x_k)}, \quad x \in B_{1/2}.$$

From (2.2) and (2.3) we have

$$\sup_{B_{1/2}} w_k \leq 2 \quad \text{and} \quad \sup_{B_{1/4}} w_k \geq c.$$

Moreover, using (2.1) and recalling that $v_k(y_k) \leq 1$, w_k satisfies

$$0 \leq \text{div}(|x_n|^{1-2s}\nabla w_k) \leq \frac{4A\delta_k^{2-2s}}{3k} \mathcal{H}^{n-1}[\{x_n = 0\}] \quad \text{in } B_{1/2}$$

$$w_k \geq 0, \quad w_k(0) \leq \frac{4}{3k}.$$

Now, invoking Lemma 2.3, we obtain that w_k are uniformly s -Hölder continuous on compact subsets of $B_{1/2}$ and hence, over a subsequence, they will converge locally uniformly to a function w_0 which satisfies

$$\operatorname{div}(|x_n|^{1-2s}\nabla w_0) = 0 \quad \text{in } B_{1/2}, \quad \sup_{B_{1/4}} w_0 \geq c > 0, \quad w_0 \geq 0, \quad w_0(0) = 0$$

This is a contradiction to the strong maximum principle in [FKS82]. $\square$

Now we prove the uniform $C^{0,s}$ regularity of u_ε .

Proof of Theorem 2.1. Note that it will be sufficient to prove the uniform estimates for small $0 < \varepsilon < \varepsilon_0$, with universal ε_0 , as the estimate for $\varepsilon_0 < \varepsilon < 1$ will follow from Lemma 2.3. It will also be sufficient to give the proof for $K = B_{1/8}$. Throughout the proof, we let

$$\Omega_\varepsilon := \{x \in B'_1 : u_\varepsilon \leq \varepsilon\}.$$

Step 1. We will show that there exists a constant $C = C(n, s, A)$ such that

$$u_\varepsilon(x) \leq \varepsilon + C \operatorname{dist}(x, \Omega_\varepsilon)^s, \quad x \in B'_{1/4} \setminus \Omega_\varepsilon.$$

The proof is based on the construction of a proper lower barrier function. Given $x_0 \in B'_{1/4} \setminus \Omega_\varepsilon$, let

$$m_0 := u_\varepsilon(x_0) - \varepsilon, \quad \delta_0 := \operatorname{dist}(x_0, \Omega_\varepsilon).$$

We are going to show that

$$m_0 \leq C(n, s, A) \delta_0^s.$$

By the Harnack inequality (see [CS14]), there exists a constant $c_{n,s}$ such that

$$u_\varepsilon(x) - \varepsilon \geq c_{n,s} m_0, \quad \text{for any } x \in B_{\delta_0/2}(x_0).$$

Next, we construct an auxiliary function as follows: Let $A_{1/2,2} := B_2 \setminus \overline{B_{1/2}}$, $A'_{1,2} := B'_2 \setminus B'_1$, and $D := A_{1/2,2} \setminus A'_{1,2}$. Let $\varphi : D \rightarrow \mathbb{R}$ be the solution to the following Dirichlet problem

$$\begin{aligned} \operatorname{div}(|x_n|^{1-2s}\nabla \varphi) &= 0 \quad \text{in } D, \\ \varphi &= 1 \quad \text{on } \partial B_{1/2}, \\ \varphi &= 0 \quad \text{on } \partial B_2 \cup A'_{1,2}. \end{aligned}$$

By the boundary Hopf lemma and boundary growth estimate (see [CS14]) as well as the symmetry of φ , the function φ has the following asymptotics at $\bar{x} \in \partial B'_1$: there exists $c_0 = c_0(n) > 0$ such that

$$(2.4) \quad \lim_{t \rightarrow 0^+} \frac{\varphi(\bar{x} + t\nu'_{\bar{x}})}{t^s} = c_0.$$

Here $\nu'_{\bar{x}}$ is the in-plane outer unit normal of $A'_{1,2}$ at $\bar{x}$. Hence, we also have that

$$(2.5) \quad \frac{\varphi(\bar{x} + t\nu'_{\bar{x}})}{t^s} > c_0/2 \quad \text{for } 0 < t < t_0$$

for sufficiently small t_0 . Now let

$$\psi(x) := c_{n,s} m_0 \varphi\left(\frac{x - x_0}{\delta_0}\right),$$

and $D_{\delta_0, x_0} := \{x : \frac{x-x_0}{\delta_0} \in D\}$. Note that $D_{\delta_0, x_0} \subset B_1 \setminus (\Omega_\varepsilon \cap B'_1)$. Applying the comparison principle in D_{δ_0, x_0} we have

$$(2.6) \quad \psi(x) \leq u_\varepsilon(x) - \varepsilon, \quad x \in D_{\delta_0, x_0}.$$

Choose now $y_0 \in \partial\Omega_\varepsilon \cap \partial B_\delta(x_0)$ which realizes the distance δ_0 . By (2.6) and recalling the explicit expression of ψ , we have

$$(2.7) \quad \frac{c_{n,s} m_0 \varphi\left(\frac{y_0 - x_0}{\delta_0} + \frac{t}{\delta_0} \nu'_{y_0}\right)}{t^s} \leq \frac{u_\varepsilon(y_0 + t \nu'_{y_0}) - \varepsilon}{t^s}.$$

We now want to use the estimate in Lemma 2.2 to obtain the bound on m_0 . For that purpose, consider the following rescalings at y_0

$$u_{\varepsilon, \varepsilon^{1/s}}(x) = \frac{u_\varepsilon(y_0 + \varepsilon^{1/s} x)}{\varepsilon},$$

which satisfy

$$\operatorname{div}(|x_n|^{1-s} \nabla u_{\varepsilon, \varepsilon^{1/s}}) = 2\beta(u_{\varepsilon, \varepsilon^{1/s}}) \mathcal{H}^{n-1}[\{x_n = 0\}] \quad \text{in } B_{1/(2\varepsilon^{1/s})}.$$

Then, we can apply Lemma 2.2 to conclude that $|u_{\varepsilon, \varepsilon^{1/s}}| \leq C = C(n, s, A)$ in $B_{1/4}$. For the function u_ε this translates into having the bound

$$u_\varepsilon(y_0 + \varepsilon^{1/s} x) \leq C\varepsilon, \quad \text{for } |x| \leq 1/4.$$

In particular, this gives that

$$\frac{u_\varepsilon(y_0 + t \nu'_{y_0}) - \varepsilon}{t^s} \leq C, \quad \text{for } t = \varepsilon^{1/s}(\delta_0/4)$$

Hence, from (2.7), we obtain

$$\frac{c_{n,s} m_0 \varphi\left(\frac{y_0 - x_0}{\delta_0} + \tau \nu'_{y_0}\right)}{\tau^s} \leq C \delta_0^s, \quad \text{for } \tau = \varepsilon^{1/s}(1/4).$$

Then, using (2.5), we conclude that for small $0 < \varepsilon < \varepsilon_0$, necessarily

$$m_0 \leq C \delta_0^s$$

Step 2. We will show that for any $y_0 \in \Omega_\varepsilon \cap B'_{1/4}$

$$(2.8) \quad |u_\varepsilon(x) - u_\varepsilon(y_0)| \leq C|x - y_0|^s$$

for any $x \in B_1$, with a universal constant C .

- (i) Suppose first $|x - y_0| \leq (1/8)\varepsilon^{1/s}$. For this case, recall that for the rescaling $u_{\varepsilon, \varepsilon^{1/s}}$ at y_0 defined in *Step 1* above we have the estimate $\|u_{\varepsilon, \varepsilon^{1/s}}\|_{L^\infty(B_{1/4})} \leq C$. Then by Lemma 2.3 we also have the estimate $\|u_{\varepsilon, \varepsilon^{1/s}}\|_{C^{0,s}(B_{1/8})} \leq C$, which then implies that

$$|u_\varepsilon(x) - u_\varepsilon(y_0)| = \varepsilon |u_{\varepsilon, \varepsilon^{1/s}}((x - y_0)/\varepsilon^{1/s}) - u_{\varepsilon, \varepsilon^{1/s}}(0)| \leq C|x - y_0|^s$$

- (ii) Suppose now $x \in B'_1$ and $|x - y_0| \geq (1/8)\varepsilon^{1/s}$. Then from *Step 1* we have

$$|u_\varepsilon(x) - u_\varepsilon(y_0)| \leq 2\varepsilon + C|x - y_0|^s \leq C|x - y_0|^s.$$

Combining the estimates in (i)–(ii) above, we obtain that (2.8) holds for any $x \in B'_1$. It remains to establish (2.8) for $x \in B_1$ with $x_n \neq 0$. Note that it will be enough to show it for $x \in B_{1/2}^+$.

In order to do that, we first extend $u_\varepsilon(\cdot, 0)$ to all of $\mathbb{R}^{n-1}$ by putting it equal to zero outside B'_1 . Note that estimate (2.8) will continue to hold now for all $x \in \mathbb{R}^{n-1}$. Then, consider the convolution of the extended $u_\varepsilon(\cdot, 0)$ with the Poisson kernel

$$P_{x_n}(x') := C_{n,s} \frac{x_n^{2s}}{(|x'|^2 + x_n^2)^{\frac{n-1+2s}{2}}}$$

for the operator $\mathcal{L}_s$. We then have

$$\begin{aligned} & |(u_\varepsilon(\cdot, 0) * P_{x_n})(x') - u_\varepsilon(y_0)| \\ &= \left| \int_{\mathbb{R}^{n-1}} [u_\varepsilon(x' - z', 0) - u_\varepsilon(y'_0, 0)] P_{x_n}(z') dz' \right| \quad (\text{since } \int_{\mathbb{R}^{n-1}} P_{x_n} = 1, \forall x_n > 0) \\ &\leq C \int_{\mathbb{R}^{n-1}} |x' - y'_0 - z'|^s P_{x_n}(z') dz' \quad (\text{by (2.8)}) \\ &\leq C \int_{\mathbb{R}^{n-1}} (|x' - y'_0|^s + |z'|^s) P_{x_n}(z') dz' \quad (\text{triangle inequality}) \\ &\leq C (|x' - y'_0|^s + |x_n|^s) \leq C |x - y_0|^s, \end{aligned}$$

where in the second last inequality we have used $\int_{\mathbb{R}^{n-1}} |z'|^s P_{x_n}(z') dz' \leq C |x_n|^s$. Next, the difference

$$v(x) = u_\varepsilon(x) - (u_\varepsilon(\cdot, 0) * P_{x_n})(x')$$

satisfies

$$\operatorname{div}(|x_n|^{1-2s} \nabla v) = 0 \quad \text{in } B_1^+, \quad v = 0 \quad \text{on } B'_1.$$

By making the odd reflection in x_n , we can make v $\mathcal{L}_s$ -harmonic in B_1 . Hence, applying Lemma 2.3, we will have

$$\|v\|_{C^{0,s}(B_{1/2})} \leq C(n, s, L).$$

Combining the estimates above, we then conclude that (2.8) holds for all $x \in B_{1/2}$ and hence for all $x \in B_1$.

Step 3. In this step, we complete the proof that $u_\varepsilon \in C^{0,s}(B_{1/8})$ uniformly in ε . From *Step 2*, it is enough to show that for any $x_1, x_2 \in B_{1/8} \setminus \Omega_\varepsilon$,

$$|u_\varepsilon(x_1) - u_\varepsilon(x_2)| \leq C |x_1 - x_2|^s.$$

Let $d(x) := \operatorname{dist}(x, \Omega_\varepsilon)$. Then consider the following two subcases:

- (a) Suppose that $|x_1 - x_2| \leq \frac{1}{2} \max\{d(x_1), d(x_2)\}$. Without loss of generality, we assume that $d := d(x_1) \geq d(x_2)$. Let also $y_1 \in \Omega_\varepsilon$ be such that $|x_1 - y_1| = d$. Then, consider the rescaling of u_ε at y_1 by the factor of d

$$u_{\varepsilon,d}(x) := \frac{u_\varepsilon(y_1 + dx)}{d^s}.$$

From *Step 2*, $0 \leq u_{\varepsilon,d}(x) \leq C |x|^s$ for $x \in B_1(\xi)$, $\xi := (x_1 - y_1)/d$. Moreover, $u_{\varepsilon,d}$ satisfies the homogeneous equation $\operatorname{div}(|x_n|^{1-2s} \nabla u_{\varepsilon,d}) = 0$ in $B_1(\xi)$. By Lemma 2.3, $u_{\varepsilon,d} \in C^{0,s}(B_{3/4}(\xi))$. In particular, for $\eta := \frac{x_2 - y_1}{d} \in B_{3/4}(\xi)$ we have

$$|u_{\varepsilon,d}(\eta) - u_{\varepsilon,d}(\xi)| \leq C |\eta - \xi|^s.$$

Rescaling back to u_ε we obtain

$$|u_\varepsilon(x_1) - u_\varepsilon(x_2)| \leq C|x_1 - x_2|^s.$$

(b) Suppose now $|x_1 - x_2| > \frac{1}{2} \max\{d(x_1), d(x_2)\}$. In this case, by *Step 2*,

$$|u_\varepsilon(x_1) - u_\varepsilon(x_2)| \leq C(d(x_1)^s + d(x_2)^s) \leq C|x_1 - x_2|^s. \quad \square$$

We conclude this section with the following remark that $\{u_\varepsilon\}$ are uniformly bounded also in $W_{\text{loc}}^{1,2}(B_1, |x_n|^{1-2s})$.

Proposition 2.4 (Uniform $W^{1,2}$ bound). *Let u_ε be a nonnegative solution of (P_ε) with $\|u_\varepsilon\|_{L^\infty(B_1)} \leq L$. Then $u_\varepsilon \in W_{\text{loc}}^{1,2}(B_1, |x_n|^{1-2s})$ for any $K \Subset B_1$ with*

$$\|u_\varepsilon\|_{W^{1,2}(K, |x_n|^{1-2s})} \leq C(n, s, A, L, K)$$

uniformly for all $\varepsilon \in (0, 1)$.

Proof. Since u_ε is a nonnegative subsolution of $\mathcal{L}_s$, the proof follows from a standard energy inequality. $\square$

3. PASSAGE TO THE LIMIT AS $\varepsilon \rightarrow 0$

3.1. Compactness. We start the section with the following local compactness lemma. Recall that we always assume that the functions u_ε and u are evenly extended in x_n -variable.

Lemma 3.1 (Compactness and limit solutions). *Let u_ε be a nonnegative solution to (P_ε) . Then over a subsequence*

- (i) $\{u_\varepsilon\}$ converges uniformly on compact subsets of B_1 to a function $u \in C_{\text{loc}}^{0,s}(B_1)$.
- (ii) The limit function u in (i) solves $\text{div}(|x_n|^{1-2s}\nabla u) = 0$ in $\{u > 0\}$.
- (iii) $\beta_\varepsilon(u_\varepsilon) \xrightarrow{*} \mu$ in the space of measures $\mathcal{M}(B'_R)$ for any $0 < R < 1$.
- (iv) $|x_n|^{(1-2s)/2}\nabla u_\varepsilon \rightarrow |x_n|^{(1-2s)/2}\nabla u$ strongly in $L_{\text{loc}}^2(B_1)$.
- (v) $\mathcal{B}_\varepsilon(u_\varepsilon) \xrightarrow{*} \chi$ in $L^\infty(B'_1)$ for some $\chi \in L^\infty(B'_1)$, where $\mathcal{B}_\varepsilon$ are defined in (1.4).

We call the function u as above a limit solution of (P) , and the pair (u, χ) a limit solution pair.

Proof. (i) For any compact $K \Subset B_1$, we know by Theorem 2.1, u_ε are uniformly bounded in $C^{0,s}(K)$. By Ascoli-Arzelà's theorem up to a subsequence $\varepsilon_j \rightarrow 0$, we obtain a function $u \in C^{0,s}(K)$ such that $u_{\varepsilon_j} \rightarrow u$ in $C^{0,\alpha}(K)$ with $0 < \alpha < s$. Since $u_\varepsilon \geq 0$, it follows that $u \geq 0$.

(ii) From (i) we know that $\{u > 0\}$ is open. If $u(x_0) = c > 0$, from the uniform convergence of u_ε we obtain a small neighborhood U of x_0 such that $u_\varepsilon(x) \geq c/2 > \varepsilon$ in U for every $0 < \varepsilon \leq \varepsilon_0$ for some ε_0 small. Then u_ε solves $\text{div}(|x_n|^{1-2s}\nabla u_\varepsilon) = 0$ in U for any $0 < \varepsilon \leq \varepsilon_0$. The statement follows from the uniform convergence of u_ε to u .

(iii) Since u_ε are uniformly bounded in $W_{\text{loc}}^{1,2}(B_1, |x_n|^{1-2s})$, see Proposition 2.4, by plugging in a cut-off function into (1.5) it is straightforward to see that $\beta_\varepsilon(u_\varepsilon)$ are uniformly bounded in $L_{\text{loc}}^1(B'_1)$.

(iv) For any $0 < R < 1$, plugging a test function $\varphi = u_\varepsilon \eta$ in (1.5), where $\eta \in C_c^\infty(\mathbb{R}^n)$, $\eta \geq 0$ and $\eta = 0$ outside B_R , we obtain that

$$(3.1) \quad \int_{B_R} |x_n|^{1-2s} |\nabla u_\varepsilon|^2 \eta + |x_n|^{1-2s} u_\varepsilon \nabla u_\varepsilon \cdot \nabla \eta = - \int_{B'_R} 2\beta_\varepsilon(u_\varepsilon) u_\varepsilon \eta$$

From (iii) and the fact that $\beta_\varepsilon(u_\varepsilon)$ is supported on the set $\{u_\varepsilon \leq \varepsilon\}$ we have

$$(3.2) \quad \text{RHS of (3.1)} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

From Proposition 2.4, we know that over a sequence $\varepsilon = \varepsilon_j \rightarrow 0$, $|x_n|^{(1-2s)/2} \nabla u_{\varepsilon_j}$ converges to $|x_n|^{(1-2s)/2} \nabla u$ weakly in $L^2(B_R)$. This together with (3.1), (3.2) and the uniform convergence in (i) gives us

$$(3.3) \quad \lim_{\varepsilon_j \rightarrow 0} \int_{B_R} |x_n|^{1-2s} |\nabla u_{\varepsilon_j}|^2 \eta = - \int_{B_R} |x_n|^{1-2s} u \nabla u \cdot \nabla \eta.$$

On the other hand, for every $\delta > 0$ consider the truncation $u^\delta = \max\{u - \delta, 0\}$. By (ii), u^δ solves the homogeneous equation in $\{u > \delta\}$. Taking the test function $\varphi = u^\delta \eta$ in (1.5) for $\varepsilon_j \in (0, \delta)$, where η is the same as above, and letting $\varepsilon_j \rightarrow 0$, we obtain

$$(3.4) \quad 0 = \int_{B_R} |x_n|^{1-2s} |\nabla u^\delta|^2 \eta + \int_{B_R} |x_n|^{1-2s} u^\delta \nabla u^\delta \cdot \nabla \eta$$

Letting $\delta \rightarrow 0+$ in (3.4), we obtain

$$(3.5) \quad \int_{B_R} |x_n|^{1-2s} |\nabla u|^2 \eta = - \int_{B_R} |x_n|^{1-2s} u \nabla u \cdot \nabla \eta.$$

Comparing (3.3) and (3.5), we conclude

$$(3.6) \quad \lim_{\varepsilon_j \rightarrow 0} \int_{B_R} |x_n|^{1-2s} |\nabla u_{\varepsilon_j}|^2 \eta = \int_{B_R} |x_n|^{1-2s} |\nabla u|^2 \eta.$$

This together with the weak L^2 convergence gives (iv).

- (v) Since $0 \leq \mathcal{B}_\varepsilon(u_\varepsilon) \leq M$, then there exists a subsequence $\mathcal{B}_{\varepsilon_j}(u_{\varepsilon_j})$ and χ such that

$$\mathcal{B}_{\varepsilon_j}(u_{\varepsilon_j}) \xrightarrow{*} \chi \quad \text{in } L^\infty(B'_1). \quad \square$$

Lemma 3.2. *Let χ be as in Lemma 3.1. Then*

$$\chi \in \{0, M\} \quad \text{for a.e. } x \in B'_1.$$

Proof. Given $0 < \delta \ll M/4$ and $K \Subset B'_1$, let $\tilde{\delta}_1$ and $\tilde{\delta}_2$ be the (unique) positive numbers satisfying

$$\int_0^{\tilde{\delta}_1} \beta(s) ds = \int_{1-\tilde{\delta}_2}^1 \beta(s) ds = \delta.$$

Then we have

$$(3.7) \quad \begin{aligned} |K \cap \{\delta < \mathcal{B}_{\varepsilon_j}(u_{\varepsilon_j}) < M - \delta\}| &= |K \cap \{\tilde{\delta}_1 < \frac{u_{\varepsilon_j}}{\varepsilon_j} < 1 - \tilde{\delta}_2\}| \\ &\leq |K \cap \{\beta_{\varepsilon_j}(u_{\varepsilon_j}) \geq \frac{1}{\varepsilon_j} \min_{[\tilde{\delta}_1, 1-\tilde{\delta}_2]} \beta\}| \\ &\leq \frac{\varepsilon_j}{\min_{[\tilde{\delta}_1, 1-\tilde{\delta}_2]} \beta} \int_K \beta_{\varepsilon_j}(u_{\varepsilon_j}) \rightarrow 0 \quad \text{as } \varepsilon_j \rightarrow 0, \end{aligned}$$

where we have used Lemma 3.1(iii) and the assumption that $\beta > 0$ in $(0, 1)$. Hence if we let $A_{\delta, K} := K \cap \{2\delta < \chi < M - 2\delta\}$, then

$$\begin{aligned} |A_{\delta, K}| &\leq |A_{\delta, K} \cap \{\mathcal{B}_{\varepsilon_j}(u_{\varepsilon_j}) \leq \delta \text{ or } \mathcal{B}_{\varepsilon_j}(u_{\varepsilon_j}) \geq M - \delta\}| \\ &\quad + |A_{\delta, K} \cap \{\delta < \mathcal{B}_{\varepsilon_j}(u_{\varepsilon_j}) < M - \delta\}| \\ &\leq |K \cap \{|\mathcal{B}_{\varepsilon_j}(u_{\varepsilon_j}) - \chi| \geq \delta\}| + |A_{\delta, K} \cap \{\delta < \mathcal{B}_{\varepsilon_j}(u_{\varepsilon_j}) < M - \delta\}|. \end{aligned}$$

By Lemma 3.1(v), $\mathcal{B}_{\varepsilon_j}(u_{\varepsilon_j}) \xrightarrow{*} \chi$ in $L^\infty(B'_1)$, and moreover $0 \leq \mathcal{B}_{\varepsilon_j} \leq M$ for all j , thus $\mathcal{B}_{\varepsilon_j}(u_{\varepsilon_j}) \rightarrow \chi$ in $L^1(B'_1)$. This implies that $|K \cap \{|\mathcal{B}_{\varepsilon_j}(u_{\varepsilon_j}) - \chi| \geq \delta\}| \rightarrow 0$ as $j \rightarrow \infty$. This combined with (3.7) yields that passing to the limit $j \rightarrow \infty$, $|A_{\delta, K}| = 0$. Because δ and K are arbitrary, we have $\chi \in \{0, M\}$ for a.e. $x \in B'_1$. $\square$

The following lemma will play a crucial role in the paper. The proof follows the lines of Lemma 3.2 in [CLW97b] and is therefore omitted.

Lemma 3.3 (Blowups at free boundary points). *Let $u_{\varepsilon_j} \rightarrow u$ uniformly on compact subsets of B_1 , and $\mathcal{B}_{\varepsilon_j} \xrightarrow{*} \chi$ in $L^\infty(B'_1)$, as in Lemma 3.1. For $x_0 \in \mathcal{F}_u = \partial\{u(\cdot, 0) > 0\} \cap B'_1$ and $\lambda > 0$, consider the following rescalings*

$$\begin{aligned} u_\lambda^{x_0}(x) &:= \frac{1}{\lambda^s} u(x_0 + \lambda x), \\ u_{\varepsilon, \lambda}^{x_0}(x) &:= \frac{1}{\lambda^s} u_\varepsilon(x_0 + \lambda x), \\ \chi_\lambda^{x_0}(x') &:= \chi(x_0 + \lambda x'). \end{aligned}$$

Assume that there exists $\lambda_k \rightarrow 0$ such that $u_{\lambda_k}^{x_0} \rightarrow U$ as $k \rightarrow \infty$ uniformly on compact subsets of $\mathbb{R}^n$ and $\chi_{\lambda_k}^{x_0} \xrightarrow{*} \chi_0$ in $L^\infty(\mathbb{R}^{n-1})$. Then there exists $j(k) \rightarrow \infty$ such that for every $j_k \geq j(k)$ we have that $(\varepsilon_{j_k}/\lambda_k^s) \rightarrow 0$ and

- (i) $u_{\varepsilon_{j_k}, \lambda_k}^{x_0} \rightarrow U$ uniformly on compact subsets of $\mathbb{R}^n$
- (ii) $|x_n|^{(1-2s)/2} \nabla u_{\varepsilon_{j_k}, \lambda_k}^{x_0} \rightarrow |x_n|^{(1-2s)/2} \nabla U$ in $L_{\text{loc}}^2(\mathbb{R}^n)$
- (iii) $\mathcal{B}_{\varepsilon_{j_k}/\lambda_k^s}(u_{\varepsilon_{j_k}, \lambda_k}^{x_0}) \xrightarrow{*} \chi_0$ in $L^\infty(\mathbb{R}^{n-1})$
- (iv) $|x_n|^{(1-2s)/2} \nabla u_{\lambda_k}^{x_0} \rightarrow |x_n|^{(1-2s)/2} \nabla U$ in $L_{\text{loc}}^2(\mathbb{R}^n)$.

We will call the function U (or the pair (U, χ_0)) a *blowup* of u (or the pair (u, χ)) at x_0 . Note that the above lemma says that (U, χ_0) is a limit solution pair on any ball B_R , $R > 0$.

3.2. Solutions in the sense of domain variation. We say that the function $u_\varepsilon \in W_{\text{loc}}^{1,2}(B_1, |x_n|^{1-2s})$ is a *domain-variation solution* of (P_ε) , if it satisfies

$$(3.8) \quad \int_{B_1} |x_n|^{1-2s} [-\nabla u_\varepsilon \otimes \nabla u_\varepsilon : \nabla \psi + \frac{1}{2} |\nabla u_\varepsilon|^2 \operatorname{div} \psi] + \int_{B'_1} 2\mathcal{B}_\varepsilon(u_\varepsilon) \operatorname{div} \psi = 0$$

for every smooth vector field $\psi \in C_c^\infty(B_1; \mathbb{R}^n)$ with $\psi(B'_1) \subset \mathbb{R}^{n-1}$. The name comes from the fact that the equation (3.8) is equivalent to the condition

$$\frac{d}{d\tau} J_\varepsilon(u(x + \tau \psi(x)))|_{\tau=0} = 0,$$

where

$$J_\varepsilon(v) = \frac{1}{2} \int_{B_1} |x_n|^{1-2s} |\nabla v|^2 + \int_{B'_1} 2\mathcal{B}_\varepsilon(v)$$

is the energy associated with (P_ε) . In particular, we see that the weak solutions of (1.5) are also domain-variation solutions.

Now, the advantage of the domain-variation solutions is as follows: if u_ε is a weak solution of (P_ε) , then by the compactness Lemma 3.1, the limit solution pair (u, χ) over any $\varepsilon = \varepsilon_j \rightarrow 0$ satisfies

$$(3.9) \quad \int_{B_1} |x_n|^{1-2s} [-\nabla u \otimes \nabla u : \nabla \psi + \frac{1}{2} |\nabla u|^2 \operatorname{div} \psi] + \int_{B'_1} 2\chi \operatorname{div} \psi = 0,$$

for every smooth vector field $\psi \in C_c^\infty(B_1; \mathbb{R}^n)$ with $\psi(B'_1) \subset \mathbb{R}^{n-1}$. While we could pass to the limit also in the weak formulation (1.5), the additional information on χ that we have from Lemma 3.2 will be important in the sequel.

3.3. Weiss-type monotonicity formula. In this section we prove monotonicity formulas for the solution u_ε of (P_ε) , and the limit solution pair (u, χ) for (P) . This kind of formula has been first used by Weiss [Wei03] in the “thick” counterpart of our problem, as well is the Alt-Caffarelli problem [Wei99].

Theorem 3.4 (Monotonicity formula for (P_ε)). *Let $x_0 \in B'_1$ and u_ε be a solution to (P_ε) with $\|u_\varepsilon\|_{L^\infty(B_2)} \leq L$. For $0 < r < 1 - |x_0|$, let*

$$\begin{aligned} \Psi_\varepsilon^{x_0}(u_\varepsilon, r) &= \frac{1}{r^{n-1}} \int_{B_r(x_0)} |x_n|^{1-2s} |\nabla u_\varepsilon|^2 - \frac{s}{r^n} \int_{\partial B_r(x_0)} |x_n|^{1-2s} u_\varepsilon^2 \\ &\quad + \frac{1}{r^{n-1}} \int_{B'_r(x_0)} 4\mathcal{B}_\varepsilon(u_\varepsilon). \end{aligned}$$

Then $r \mapsto \Psi_\varepsilon^{x_0}(u_\varepsilon, r)$ is a nondecreasing function of r .

Proof. For $0 < r < 1 - |x_0|$ consider the rescalings

$$u_{\varepsilon,r}(x) := \frac{u_\varepsilon(x_0 + rx)}{r^s}.$$

Then

$$\Psi_\varepsilon^{x_0}(u_\varepsilon, r) = \Psi_\varepsilon^0(u_{\varepsilon,r}, 1) = \int_{B_1} |x_n|^{1-2s} |\nabla u_{\varepsilon,r}|^2 - s \int_{\partial B_1} |x_n|^{1-2s} u_{\varepsilon,r}^2 + \int_{B'_1} 4\mathcal{B}_\varepsilon(r^s u_{\varepsilon,r}).$$

Thus

$$\begin{aligned} \frac{d}{dr} \Psi_\varepsilon^{x_0}(u_\varepsilon, r) &= \int_{B_1} 2|x_n|^{1-2s} \nabla u_{\varepsilon,r} \cdot \nabla \frac{d}{dr} u_{\varepsilon,r} - 2s \int_{\partial B_1} |x_n|^{1-2s} u_{\varepsilon,r} \frac{d}{dr} u_{\varepsilon,r} \\ &\quad + \int_{B'_1} 4\beta_\varepsilon(r^s u_{\varepsilon,r}) \left(sr^{s-1} u_{\varepsilon,r} + r^s \frac{d}{dr} u_{\varepsilon,r} \right). \end{aligned}$$

Now, noting that $u_{\varepsilon,r}$ solves

$$\operatorname{div}(|x_n|^{1-2s} \nabla u_{\varepsilon,r}) = 2r^s \beta_\varepsilon(r^s u_{\varepsilon,r}) \mathcal{H}^{n-1}[\{x_n = 0\}] \quad \text{in } B_{1/r}$$

and integrating by parts, using also the nonnegativity of $\beta_\varepsilon(u_\varepsilon)u_\varepsilon$, we have

$$\begin{aligned} \frac{d}{dr} \Psi_\varepsilon^{x_0} &= 2 \int_{\partial B_1} |x_n|^{1-2s} (\partial_\nu u_{\varepsilon,r} - s u_{\varepsilon,r}) \frac{d}{dr} u_{\varepsilon,r} + \int_{B'_1} 4\beta_\varepsilon(r^s u_{\varepsilon,r}) sr^{s-1} u_{\varepsilon,r} \\ &\geq 2 \int_{\partial B_1} |x_n|^{1-2s} (\partial_\nu u_{\varepsilon,r} - s u_{\varepsilon,r}) \frac{d}{dr} u_{\varepsilon,r}. \end{aligned}$$

Observing that for $x \in \partial B_1$,

$$\begin{aligned} \frac{d}{dr} u_{\varepsilon,r}(x) &= r^{-(1+s)} ((rx) \cdot \nabla u_{\varepsilon}(x_0 + rx) - s u_{\varepsilon}(x_0 + rx)), \\ \partial_{\nu} u_{\varepsilon,r}(x) &= x \cdot \nabla u_{\varepsilon,r}(x) = r^{-s} (rx) \cdot \nabla u_{\varepsilon}(x_0 + rx), \end{aligned}$$

we then obtain

$$\frac{d}{dr} \Psi_{\varepsilon}^{x_0}(u_{\varepsilon}, r) \geq \frac{2}{r^{n+1}} \int_{\partial B_r(x_0)} |x_n|^{1-2s} ((x - x_0) \cdot \nabla u_{\varepsilon} - s u_{\varepsilon})^2 \geq 0.$$

This implies that $r \mapsto \Psi_{\varepsilon}^{x_0}(u_{\varepsilon}, r)$ is monotonically nondecreasing. $\square$

By the compactness Lemma 3.1, passing to the limit in a subsequence ε_j we get the following monotonicity formula for the limit pair (u, χ) . Similar monotonicity formula was used in the thin and fractional Alt-Caffarelli problems in [AP12] and [All12], respectively.

Theorem 3.5 (Monotonicity formula for (P)). *Let (u, χ) be a limit solution pair and $x_0 \in B'_1$. For $0 < r < 1 - |x_0|$, let*

$$\begin{aligned} \Psi^{x_0}(u, r) &= \frac{1}{r^{n-1}} \int_{B_r(x_0)} |x_n|^{1-2s} |\nabla u|^2 - \frac{s}{r^n} \int_{\partial B_r(x_0)} |x_n|^{1-2s} u^2 \\ &\quad + \frac{1}{r^{n-1}} \int_{B'_r(x_0)} 4\chi. \end{aligned}$$

Then $r \mapsto \Psi^{x_0}(u, r)$ is monotonically nondecreasing. More precisely, for $0 < \rho < \sigma < 1 - |x_0|$,

$$\Psi^{x_0}(u, \sigma) - \Psi^{x_0}(u, \rho) \geq \int_{\rho}^{\sigma} \frac{2}{r^{n+1}} \int_{\partial B_r(x_0)} |x_n|^{1-2s} ((x - x_0) \cdot \nabla u - s u)^2 d\sigma_r dr \geq 0.$$

In particular, the limit $\Psi^{x_0}(u, 0+) = \lim_{r \rightarrow 0+} \Psi^{x_0}(u, r)$ exists.

We will call the quantity $\Psi^{x_0}(u, 0+)$ the *Weiss energy of u at x_0* .

Next we prove a corollary of the monotonicity formula above. For notation convenience, sometimes we write the dependence of Ψ on (u, χ) explicitly, i.e. we write $\Psi(u, \chi, r)$ instead of $\Psi(u, r)$.

Corollary 3.6. *Let (u, χ) be a limit solution pair and $x_0 \in \mathcal{F}_u = \partial\{u(\cdot, 0) > 0\} \cap B'_1$. Then*

- (i) $\Psi^{x_0}(u, r) \geq -C$ for any $r > 0$.
- (ii) Suppose for $\lambda_j \rightarrow 0$ a blowup sequence $(u_{\lambda_j}, \chi_{\lambda_j})$ (as in Lemma 3.3) satisfies

$$\begin{aligned} u_{\lambda_j} &\rightarrow u_0 \quad \text{uniformly on compact subsets of } \mathbb{R}^n, \\ \chi_{\lambda_j} &\xrightarrow{*} \chi_0 \quad \text{in } L^{\infty}(\mathbb{R}^{n-1}). \end{aligned}$$

Then $\Psi^0(u_0, \chi_0, r)$ is constant in r . Moreover,

$$\Psi^0(u_0, \chi_0, r) = \Psi^{x_0}(u, \chi, 0+) = \int_{B'_1} 4\chi_0.$$

In particular, $\Psi^{x_0}(u, 0+) \geq 0$.

- (iii) u_0 is a homogeneous function of degree s , i.e.

$$u_0(\lambda x) = \lambda^s u_0(x), \quad \text{for a.e. } x \in \mathbb{R}^n, \lambda \geq 0.$$

Proof. (i) Since $u \in C_{\text{loc}}^{0,s}(\mathbb{R}^n)$ and $u(x_0) = 0$, then

$$\frac{s}{r^n} \int_{\partial B_r(x_0)} |x_n|^{1-2s} u^2 \leq C,$$

uniformly in r . Thus $\Psi^{x_0}(u, r) \geq -C$ for any $r > 0$.

(ii) We use the following scaling property of Ψ :

$$(3.10) \quad \Psi^0(u_{\lambda_j}, \chi_{\lambda_j}, r) = \Psi^{x_0}(u, \chi, \lambda_j r).$$

From Theorem 3.5, the limit $\Psi^{x_0}(u, \chi, \lambda_j r)$ as $j \rightarrow \infty$ exists, and is equal to $\Psi^{x_0}(u, \chi, 0+)$. Thus, passing to $j \rightarrow \infty$ in (3.10) we have

$$\Psi^0(u_0, \chi_0, r) = \Psi^{x_0}(u, \chi, 0+), \quad \text{for all } 0 < r < 1.$$

(iii) By (ii) and Theorem 3.5, for all $0 < s_1 < s_2 < 1$,

$$\begin{aligned} \Psi^{x_0}(u, \chi, \lambda_j s_1) - \Psi^{x_0}(u, \chi, \lambda_j s_2) &= \Psi^0(u_{\lambda_j}, \chi_{\lambda_j}, s_1) - \Psi^0(u_{\lambda_j}, \chi_{\lambda_j}, s_2) \\ &\geq \int_{s_1}^{s_2} \frac{2}{r^{n+1}} \int_{\partial B_r} |x_n|^{1-2s} (x \cdot \nabla u_{\lambda_j} - s u_{\lambda_j})^2 d\sigma_r dr \end{aligned}$$

By Theorem 3.5, the left hand side goes to zero as $j \rightarrow \infty$. Thus passing to the limit we have

$$\begin{aligned} \lim_{j \rightarrow \infty} \int_{s_1}^{s_2} \frac{2}{r^{n+1}} \int_{\partial B_r} |x_n|^{1-2s} (x \cdot \nabla u_{\lambda_j} - s u_{\lambda_j})^2 d\sigma_r dr \\ = \int_{s_1}^{s_2} \frac{2}{r^{n+1}} \int_{\partial B_r} |x_n|^{1-2s} (x \cdot \nabla u_0 - s u_0)^2 d\sigma_r dr = 0, \quad \text{for all } 0 < s_1 < s_2 < 1. \end{aligned}$$

This yields the desired homogeneity of u_0 .

Finally, we show that $\Psi^0(u_0, \chi_0, r) = \int_{B'_1} 4\chi_0$. In fact,

$$\Psi^0(u_0, \chi_0, r) = \int_{B_1} |x_n|^{1-2s} |\nabla u_0|^2 - s \int_{\partial B_1} u_0^2 + \int_{B'_1} 4\chi_0$$

By (ii), u_0 is homogeneous of degree s and satisfies $u_0 \operatorname{div}(|x_n|^{1-2s} \nabla u_0) = 0$, since $\beta_\varepsilon(u_\varepsilon) \in L_{\text{loc}}^1(\mathbb{R}^{n-1} \times \{0\})$ and $u_\varepsilon \beta_\varepsilon(u_\varepsilon) \rightarrow 0$ in $L_{\text{loc}}^1(\mathbb{R}^{n-1} \times \{0\})$ as $\varepsilon \rightarrow 0$. Then

$$\int_{B_1} |x_n|^{1-2s} |\nabla u_0|^2 - s \int_{\partial B_1} u_0^2 = 0$$

and consequently

$$\Psi^0(u_0, \chi_0, r) = \int_{B'_1} 4\chi_0 \geq 0. \quad \square$$

4. ASYMPTOTIC BEHAVIOR OF LIMIT SOLUTIONS

Assume (u, χ) is a limit solution pair in the sense of Lemma 3.1. In this section we will show the asymptotic behavior of u around the regular free boundary points of $\mathcal{F}_u = \partial\{u(\cdot, 0) > 0\} \cap B'_1$. By *regular free boundary point* we mean a point $x_0 \in \mathcal{F}_u$, where $\mathcal{F}_u$ has an *inward unit normal* ν in the *measure-theoretic sense*. More precisely, by this we understand $\nu \in \mathbb{R}^{n-1}$, $|\nu| = 1$, such that

$$(4.1) \quad \lim_{r \rightarrow 0} \frac{1}{r^{n-1}} \int_{B'_r(x_0)} \left| \chi_{\{u>0\}} - \chi_{\{(x', 0): \langle x' - x'_0, \nu \rangle > 0\}} \right| dx' = 0.$$

The main result of this section is as follows.

Theorem 4.1 (Limit solutions at regular points). *Let (u, χ) be a limit solution pair in the sense of Lemma 3.1. Let $x_0 \in \mathcal{F}_u$ be such that*

- (i) $\mathcal{F}_u$ has at x_0 an inward unit normal ν in the measure theoretic sense,
- (ii) u is nondegenerate at x_0 in the sense that there exist $c, r_0 > 0$ such that

$$(4.2) \quad \frac{1}{r^{n-1}} \int_{B'_r(x_0)} u dx' \geq c r^s, \quad \text{for any } 0 < r < r_0.$$

Then we have the following asymptotic development

$$u(x) = \sqrt{\frac{2M}{c_0(s)}} 2^{-s} \left(\sqrt{\langle x' - x'_0, \nu \rangle^2 + x_n^2} + \langle x' - x'_0, \nu \rangle \right)^s + o(|x - x_0|^s),$$

with

$$c_0(s) = s^2 2^{-1-2s} \frac{\sqrt{\pi}(7 + 4s(s-2))\Gamma(1-s)}{\Gamma(\frac{7}{2}-s)}.$$

Moreover, we have $\Psi^{x_0}(u, \chi, 0+) = 2M|B'_1|$.

Remark 4.2. In the case $s = \frac{1}{2}$, $c_0(\frac{1}{2}) = \frac{\pi}{8}$.

We start by identifying the limit solutions u of the form

$$u(x) = \frac{\alpha}{2^s} \left(\sqrt{x_1^2 + x_n^2} + x_1 \right)^s$$

for some $\alpha > 0$. The free boundary in this case is $\mathcal{F}_u = \{x_1 = 0, x_n = 0\}$. This is the first step of understanding the asymptotic development of limit solutions around the ‘regular’ free boundary points.

Proposition 4.3. *Let (u, χ) be a limit solution pair as in Lemma 3.1 such that*

$$u = \frac{\alpha}{2^s} \left(\sqrt{x_1^2 + x_n^2} + x_1 \right)^s \quad \text{for some } \alpha > 0.$$

Then

$$\chi = M \chi_{\{x_1 > 0\}} \quad \mathcal{H}^{n-1}\text{-a.e. in } \mathbb{R}^{n-1}$$

and the constant α is given by

$$\alpha = \sqrt{\frac{2M}{c_0(s)}}, \quad \text{with } c_0(s) \text{ as in Theorem 4.1.}$$

Proof. In the proof below, we identify $\mathbb{R}^{n-2}$ with $\{0\} \times \mathbb{R}^{n-2}$, and denote $x'' = (x_2, \dots, x_{n-1})$, $B''_r = B'_r \cap \{x_1 = 0\}$.

Step 1. We will show that for any $\varphi \in C_c^\infty(B'_1; \mathbb{R}^{n-1})$, $\varphi = (\varphi_1, \dots, \varphi_{n-1})$,

$$(4.3) \quad c_0(s) \alpha^2 \int_{B''_1} \varphi_1(0, x'') dx'' + \int_{B'_1} 2\chi \operatorname{div} \varphi = 0.$$

If we take $\psi \in C_c^\infty(B_1; \mathbb{R}^n)$, with $\psi = (\varphi, 0)$ on B'_1 as a test function in (3.9), then

$$\int_{B_1} \left[\frac{1}{2} |\nabla u|^2 \operatorname{div}(|x_n|^{1-2s} \psi) - (\nabla u)^t D\psi \nabla u |x_n|^{1-2s} \right] + \int_{B'_1} 2\chi \operatorname{div} \varphi = 0.$$

Since $|x_n|^{\frac{1-2s}{2}}|\nabla u| \in L^2_{\text{loc}}(\mathbb{R}^n)$ by Lemma 3.1 (iv) and $|\psi_n(x)| \leq C|x_n|$, then for any small $\delta > 0$,

$$(4.4) \quad \int_{\sqrt{x_1^2+x_n^2} \geq \delta} \left[\frac{1}{2} |\nabla u|^2 \operatorname{div}(|x_n|^{1-2s}\psi) - (\nabla u)^t D\psi \nabla u |x_n|^{1-2s} \right] + \int_{B'_1} 2\chi \operatorname{div} \varphi = O(\delta).$$

We next rewrite the first integrand in the LHS of (4.4) as follows.

$$(4.5) \quad \begin{aligned} & \frac{1}{2} |\nabla u|^2 \operatorname{div}(|x_n|^{1-2s}\psi) - (\nabla u)^t D\psi \nabla u |x_n|^{1-2s} \\ &= \frac{1}{2} \operatorname{div}(|\nabla u|^2 \psi |x_n|^{1-2s}) - \frac{1}{2} \nabla(|\nabla u|^2) \cdot \psi |x_n|^{1-2s} \\ & \quad - (\nabla u)^t D\psi \nabla u |x_n|^{1-2s}. \end{aligned}$$

To estimate the last two terms above, we observe that

$$(\nabla u \cdot \psi) \operatorname{div}(|x_n|^{1-2s} \nabla u) = 0 \quad \text{in } \mathbb{R}^n \setminus \left\{ \sqrt{x_1^2 + x_n^2} < \delta \right\},$$

since $\operatorname{div}(|x_n|^{1-2s} \nabla u) = 0$ in $\mathbb{R}^n \setminus \{x_1 \leq 0, x_n = 0\}$ and $\psi \cdot e_n = 0$, $\nabla' u = 0$ on $\{x_1 < 0, x_n = 0\}$. Thus, in $\{\sqrt{x_1^2 + x_n^2} \geq \delta\}$,

$$(4.6) \quad \begin{aligned} & \frac{1}{2} \nabla(|\nabla u|^2) \cdot \psi |x_n|^{1-2s} + (\nabla u)^t D\psi \nabla u |x_n|^{1-2s} \\ &= \operatorname{div}((\nabla u \cdot \psi) \nabla u |x_n|^{1-2s}). \end{aligned}$$

Combining (4.5) and (4.6), we rewrite (4.4) as

$$\begin{aligned} & \int_{\sqrt{x_1^2+x_n^2} \geq \delta} \frac{1}{2} \operatorname{div}(|\nabla u|^2 \psi |x_n|^{1-2s}) - \operatorname{div}((\nabla u \cdot \psi) \nabla u |x_n|^{1-2s}) \\ & \quad + \int_{B'_1} 2\chi \operatorname{div} \varphi = O(\delta). \end{aligned}$$

An integration by parts gives

$$(4.7) \quad \begin{aligned} & \int_{\sqrt{x_1^2+x_n^2} = \delta} \frac{1}{2} |\nabla u|^2 (\psi \cdot \nu) |x_n|^{1-2s} - \int_{\sqrt{x_1^2+x_n^2} = \delta} (\nabla u \cdot \psi) (\nabla u \cdot \nu) |x_n|^{1-2s} \\ & \quad + \int_{B'_1} 2\chi \operatorname{div} \varphi = O(\delta), \end{aligned}$$

where ν is the unit outer normal to $\{\sqrt{x_1^2 + x_n^2} = \delta\}$. Next we estimate the first two integrals in the LHS of the above equality. Since u is homogeneous of degree s (and depends only on x_1 and x_n), $\nabla u \cdot \nu = su$. Thus,

$$\int_{\sqrt{x_1^2+x_n^2} = \delta} (\nabla u \cdot \psi) (\nabla u \cdot \nu) |x_n|^{1-2s} = \int_{\sqrt{x_1^2+x_n^2} = \delta} (\nabla u \cdot \psi) (su) |x_n|^{1-2s} = O(\delta),$$

where we have used the growth estimate of u around the zero set as well as $|\nabla u||x_n|^{\frac{1-2s}{2}} \in L^2_{\text{loc}}(\mathbb{R}^n)$. To estimate the first integral we use the polar coordinates in (x_1, x_n) -plane: $x_1 = r \cos(\theta)$, $x_n = r \sin(\theta)$. Then on $\{\sqrt{x_1^2 + x_n^2} = \delta\}$,

$$\begin{aligned} & \frac{1}{2} |\nabla u|^2 (\psi \cdot \nu) |x_n|^{1-2s} \\ &= \frac{s^2 \alpha^2}{2\delta} (\cos(\theta/2))^{-1+2s} (|\sin(\theta)|)^{1-2s} [\cos(\theta)(\psi \cdot e_1) + \sin(\theta)(\psi \cdot e_n)] \end{aligned}$$

Since $\psi \cdot e_n = 0$ on B'_1 ,

$$\begin{aligned} \lim_{\delta \rightarrow 0} \int_{\sqrt{x_1^2 + x_n^2} = \delta} \frac{1}{2} |\nabla u|^2 (\psi \cdot \nu) |x_n|^{1-2s} dx &= \lim_{\delta \rightarrow 0} \int_{B''_1} \frac{s^2 \alpha^2}{2\delta} \cdot 2\delta \times \\ &\times \int_0^\pi (\cos(\theta/2))^{-1+2s} (\sin(\theta))^{1-2s} (\cos(\theta))^2 \psi_1(\delta \cos(\theta), x'', \delta \sin(\theta)) d\theta dx'' \\ &= c_0(s) \alpha^2 \int_{B''_1} \psi_1(0, x'', 0) dx'', \end{aligned}$$

where

$$\begin{aligned} c_0(s) &= s^2 \int_0^\pi (\cos(\theta/2))^{-1+2s} (\sin(\theta))^{1-2s} (\cos(\theta))^2 d\theta \\ &= s^2 2^{-1-2s} \frac{\sqrt{\pi}(7 + 4s(s-2))\Gamma(1-s)}{\Gamma(\frac{7}{2}-s)}. \end{aligned}$$

Combining the above estimates and letting $\delta \rightarrow 0$, we have for all $\varphi \in C_c^\infty(B'_1; \mathbb{R}^{n-1})$

$$c_0(s) \alpha^2 \int_{B''_1} \psi_1(0, x'', 0) dx'' + \int_{B'_1} 2\chi \operatorname{div} \varphi = 0$$

Recalling $\psi_1 = \varphi_1$ on B'_1 , we have proved (4.3).

Step 2. We now show that $\alpha = \sqrt{\frac{2M}{c_0(s)}}$ and $\chi = M\chi_{\{x_1 > 0\}}$ $\mathcal{H}^{n-1}$ -a.e. in $\mathbb{R}^{n-1}$.

(a) $\chi \equiv M$ in $\{x_1 > 0\}$.

In fact, if $y \in \{x_1 > 0\}$, then $u(y', 0) = \alpha(y_1)^s > 0$. Hence by the uniform Hölder convergence of u_{ε_j} to u , we have $u_{\varepsilon_j} \geq \varepsilon_j$ in a neighborhood of $(y', 0)$ for any $j \geq j_0$ for some $j_0 = j_0(\alpha, y_1)$ large enough. Thus for $j > j_0$ and $(x', 0)$ in the neighborhood of $(y', 0)$ we have

$$\mathcal{B}_{\varepsilon_j}(u_{\varepsilon_j})(x') = \int_0^{u_{\varepsilon_j}(x')/\varepsilon_j} \beta(s) ds = M.$$

Letting $j \rightarrow \infty$ and since y' is arbitrary, we get $\chi \equiv M$ in $\{x_1 > 0\}$.

(b) $\chi \equiv 0$ in $\{x_1 < 0\}$.

In fact, we can take any φ in (4.3) such that $\operatorname{supp} \varphi_1 \subset \mathbb{R}^{n-1} \cap \{x_1 < 0\}$. Then the LHS of (4.3) will vanish. This implies that $\chi = \text{const}$ in $\{x_1 < 0\}$. By Lemma 3.2, $\chi \equiv M$ or $\chi \equiv 0$ in $\{x_1 < 0\}$. If $\chi \equiv M$ in $\{x_1 < 0\}$, then $\chi \equiv M$ in $\mathbb{R}^{n-1}$ by (a). Thus from (4.3) $\int_{\mathbb{R}^{n-2}} \varphi_1(0, x'') dx'' = 0$ for any compactly supported vector field φ , which is a contradiction.

This then implies the claim that $\chi = M\chi_{\{x_1 > 0\}}$. Next, applying an integration by parts to the RHS of (4.3) we have

$$(4.8) \quad c_0(s) \alpha^2 \int_{B''_1} \varphi_1(0, x'') dx'' = 2M \int_{B'_1} \varphi_1(0, x'') dx''.$$

This implies

$$\alpha = \sqrt{\frac{2M}{c_0(s)}}. \quad \square$$

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Without loss of generality we assume $x_0 = 0$ and $\nu = e_1$. We also extend u by the even reflection with respect to x_n .

Consider the rescalings u_λ and χ_λ at the origin. By Theorem 2.1 given $\rho > 0$, u_λ is uniformly bounded in $C^{0,s}(\overline{B_{\rho/\lambda}})$. Therefore, there exists a sequence $\lambda_j \rightarrow 0$, $u_0 \in C^{0,s}(\mathbb{R}^n)$ and $\chi_0 \in L^\infty(\mathbb{R}^n)$ such that $u_{\lambda_j} \rightarrow u_0$ uniformly on compact subsets of $\mathbb{R}^n$ and $\chi_{\lambda_j} \xrightarrow{*} \chi_0$ in $L^\infty(\mathbb{R}^{n-1})$.

Now, rescaling (4.1), we see that for every $R > 0$,

$$|\{u_\lambda(\cdot, 0) > 0\} \cap \{x_1 < 0\} \cap B'_R| \rightarrow 0 \quad \text{as } \lambda \rightarrow 0,$$

and we deduce that $u_0 = 0$ $\mathcal{H}^{n-1}$ -a.e. in $\{x_1 < 0, x_n = 0\}$. By continuity, u_0 vanishes on all of $\{x_1 \leq 0, x_n = 0\}$. Besides, we readily have that u_0 satisfies $\operatorname{div}(|x_n|^{1-2s} \nabla u_0) = 0$ in $\{u_0 > 0\}$. Thus, we can apply Corollary A.2 in Appendix to obtain an asymptotic development

$$(4.9) \quad u_0(x) = \alpha 2^{-s} \left((x_1^2 + x_n^2)^{1/2} + x_1 \right)^s + o(|x|^s)$$

with $\alpha \geq 0$. Next, note that by rescaling the nondegeneracy condition (4.2) and passing to the limit, we have

$$\frac{1}{r^{n-1}} \int_{B'_r} u_0 dx' \geq c r^s, \quad \text{for any } r > 0,$$

which implies that $\alpha > 0$. On the other hand, by Corollary 3.6(iii), u_0 is homogeneous of degree s and hence

$$u_0(x) = \alpha 2^{-s} \left((x_1^2 + x_n^2)^{1/2} + x_1 \right)^s.$$

Now by Lemma 3.3, (u_0, χ_0) is a limit solution pair and we can apply Proposition 4.3 to conclude that $\alpha = \sqrt{\frac{2M}{c_0(s)}}$, which is a constant independent of the sequence λ_j . This proves the asymptotic development for u .

Finally, by Proposition 4.3, we also have that $\chi_0 = M \chi_{\{x_1 > 0\}}$ and hence by Corollary 3.6(ii),

$$\Psi^0(u, \chi, 0+) = \int_{B'_1} 4\chi_0 = 2M|B'_1|. \quad \square$$

We conclude the paper with two propositions regarding some additional properties of limit solutions and the value of $\Psi^{x_0}(u, \chi, 0+)$, which may be useful in the further treatment of the free boundary. At the end we also give an alternative proof of Theorem 4.1, relying on these results, rather than on asymptotic developments in Appendix.

The first proposition says that the nondegeneracy condition (4.2) at $x_0 \in \mathcal{F}_u$ implies also a nondegeneracy for $\{u(\cdot, 0) = 0\}$. The proof uses the dimension reduction argument (see e.g. [Wei03]).

Proposition 4.4. *Let (u, χ) be a limit solution pair, and $x_0 \in \mathcal{F}_u$ such that u is nondegenerate at x_0 in the sense of (4.2). Then $0 \leq \Psi^{x_0}(u, \chi, 0+) < 4M|B'_1|$. In particular, $|B'_r(x_0) \cap \{u(\cdot, 0) = 0\}| > 0$ for any $r > 0$.*

Proof. Without loss of generality we assume that $x_0 = 0$. By Corollary 3.6(ii) and the inequality $0 \leq \chi_0 \leq M$, to prove the first part of the proposition, we essentially have to exclude the possibility that $\Psi^0(u, \chi, 0+) = 4M|B'_1|$, which is equivalent to having $\chi_0 = M$ a.e. on $\mathbb{R}^{n-1}$ (recall that (u_0, χ_0) is a blow-up limit at 0 along a subsequence $(u_{\lambda_j}, \chi_{\lambda_j})$). We want to show that this implies that u_0 depends only on one variable x_n .

To this end, take $\hat{x}_0 \in \mathcal{F}_{u_0} \subset \mathbb{R}^{n-1}$ such that $\hat{x}_0 \neq 0$. Note that such point exists, otherwise $u_0 > 0$ in $\mathbb{R}^n \setminus \{0\}$ ($u_0 \equiv 0$ on $\mathbb{R}^{n-1}$ is excluded by the nondegeneracy assumption), which would imply that $\operatorname{div}(|x_n|^{1-2s}\nabla u_0) = 0$ in $\mathbb{R}^n \setminus \{0\}$. Since u_0 is locally bounded, by the removability of point singularities¹ we would have that $\operatorname{div}(|x_n|^{1-2s}\nabla u_0) = 0$ in all of $\mathbb{R}^n$. Then, by the Harnack inequality [FKS82] (which implies Liouville theorem) we would have that $u_0 \equiv 0$, which is a contradiction. Since u_0 is homogeneous, then $\lambda\hat{x}_0 \in \mathcal{F}_{u_0}$ for each $\lambda > 0$. Moreover, due to the equality $\chi_0 = M$ a.e. in $\mathbb{R}^{n-1}$ we will have $\Psi^{\lambda\hat{x}_0}(u_0, \chi_0, 0+) = 4M|B'_1|$ for each $\lambda > 0$. Applying the monotonicity formula (Theorem 3.5) to u_0 at $\hat{x}_0$ we get

$$\begin{aligned}
 (4.10) \quad & \int_0^R \frac{2}{r^{n+1}} \int_{\partial B_r(\hat{x}_0)} |x_n|^{1-2s} ((x - \hat{x}_0) \cdot \nabla u_0 - su_0)^2 d\sigma_r dr \\
 & \leq \Psi^{\hat{x}_0}(u_0, \chi_0, R) - \Psi^{\hat{x}_0}(u_0, \chi_0, 0+) \\
 & = \Psi^{\hat{x}_0}(u_0, \chi_0, R) - \Psi^0(u_0, \chi_0, 0+) \\
 & = \Psi^{\hat{x}_0}(u_0, \chi_0, R) - \Psi^0(u_0, \chi_0, R) \\
 & \rightarrow 0 \quad \text{as } R \rightarrow \infty,
 \end{aligned}$$

where the last line is due to the fact that u_0 is homogeneous, more precisely, since u_0 is homogeneous of degree s , then one has

$$(u_0)_{\hat{x}_0}^{\hat{x}_0}(x) = u_0(x + (\hat{x}_0/R)) \rightarrow u_0(x) \quad \text{as } R \rightarrow \infty.$$

Therefore, by (4.10) and $x \cdot \nabla u_0 = su_0$ we have $\nabla u_0 \cdot \hat{x}_0 = 0$ in $\mathbb{R}^n$. This implies that u_0 is constant in $\hat{x}_0/|\hat{x}_0|$ direction. An induction argument gives that $u_0(x) \equiv u_0(x_n)$. Then from $u_0(0) = 0$ we obtain that $u_0 = 0$ on all of $\mathbb{R}^{n-1}$, which contradicts to the nondegeneracy assumption (4.2).

The last statement in the proposition follows immediately from the observation that $\chi = M$ in $\{u(\cdot, 0) > 0\}$. $\square$

Our last proposition says that among all nonzero blowups at free boundary points, the one with a flat free boundary as in Proposition 4.3 has the smallest Weiss energy.

Proposition 4.5. *Let (u, χ) be a limit solution pair. Let $x_0 \in \mathcal{F}_u$ and assume that u is nondegenerate at x_0 in the sense of (4.2). Then $\Psi^{x_0}(u, \chi, 0+) \geq 2M|B'_1|$. Moreover, $\Psi^{x_0}(u, \chi, 0+) = 2M|B'_1|$ iff any blow-up limit u_0 at x_0 is $u_0(x) = \alpha 2^{-s}(\sqrt{x_1^2 + x_n^2} + x_1)^s$ up to a rotation, with $\alpha = \sqrt{2M/c_0(s)}$.*

¹*Proof.* Suppose $|u_0| \leq L$ and let w be such that $\mathcal{L}_s w = 0$ in B_1 and $w = u_0$ on ∂B_1 . By comparing the difference $w - u_0$ with $\frac{2L}{\Phi_s(\delta)} \Phi_s(x)$ in $B_1 \setminus B_\delta$, where $\Phi_s(x) = C_s|x|^{-(n-1-2s)}$ is the fundamental solution of $\mathcal{L}_s$, and letting $\delta \rightarrow 0+$, we conclude that $w = u_0$. $\square$

This result is the analogue of Theorem 9.4 in [AP12], with a similar proof.

Proof. Suppose there is $x_0 \in \mathcal{F}_u$ such that $\Psi^{x_0}(u, \chi, x_0) < 2M|B'_1|$. Let (u_0, χ_0) be a blow-up limit at x_0 along a sequence $(u_{\varepsilon_j, \lambda_j}^{x_0}, \chi_{\varepsilon_j, \lambda_j}^{x_0})$ with $\varepsilon_j/\lambda_j^s \rightarrow 0$ (by Lemma 3.3). Then by Corollary 3.6(ii)(iii), u_0 is homogeneous of degree s , and

$$(4.11) \quad \Psi^0(u_0, \chi_0, r) \equiv \int_{B'_1} 4\chi_0 = \Psi^{x_0}(u, \chi, 0+) < 2M|B'_1|.$$

Since u satisfies the nondegeneracy assumption at x_0 , then arguing as in Theorem 4.1 we have that u_0 is nontrivial.

Let $\Lambda := \{u_0(\cdot, 0) = 0\}$. Arguing as in *Step 2(a)* of Proposition 4.3 we have $\chi_0(x) = M$ in $\mathbb{R}^{n-1} \setminus \Lambda$. Thus (4.11) implies that

$$(4.12) \quad |\Lambda \cap B'_1| > |B'_1|/2.$$

We write the homogeneous blow-up limit as $u_0(x) = r^s f(\omega)$, with $r = |x|$ and $\omega = x/|x| \in \partial B_1$. Since u_0 solves $\operatorname{div}(|x_n|^{1-2s} \nabla u_0) = 0$ in $\mathbb{R}^n \setminus \Lambda$, $u_0 > 0$ in $\mathbb{R}^n \setminus \Lambda$ and $u_0 = 0$ on Λ , the function $f(\omega)$ satisfies

$$\begin{aligned} \omega_n^{2s-1} \nabla_\omega \cdot (\omega_n^{1-2s} \nabla_\omega) f &= s(s-n+1)f \quad \text{in } \partial B_1 \setminus \Lambda, \\ f &= 0 \quad \text{on } \partial B_1 \cap \Lambda, \\ f &> 0 \quad \text{on } \partial B_1 \setminus \Lambda. \end{aligned}$$

Thus f is the principal Dirichlet eigenfunction for the weighted spherical Laplacian on $\partial B_1 \setminus \Lambda_\omega$ with $\lambda_0 = s(s-n+1)$, where $\Lambda_\omega := \Lambda \cap \partial B_1 \subset \partial B_1 \cap \{\omega_n = 0\}$. From the variational formulation of the principal eigenvalue and using the symmetrization we have, among all Λ_ω with $|\Lambda_\omega|$ constant, λ takes the minimum iff Λ_ω is a spherical cap (here and later by spherical cap we mean the ‘thin’ spherical cap lying on $\{\omega_n = 0\}$, i.e. the classical spherical cap with center on $\partial B_1 \cap \{\omega_n = 0\}$ intersected with $\{\omega_n = 0\}$). This follows from the fact that the Steiner symmetrization in any spherical variable on ∂B_1 , orthogonal to x_n will decrease the principal eigenvalue, see Lemma 9.5 in [AP12], by adding the weight of $\omega_n^{1-2s} = (\cos \theta_{n-1})^{1-2s}$ (independent of $\theta_1, \dots, \theta_{n-2}$) in the energy functional in the proof.

Let λ^* denote the minimum eigenvalue associated with the spherical cap Λ_ω^* with $|\Lambda_\omega^*| = |\Lambda_\omega|$. We immediately have $\lambda^* \leq \lambda_0$. On the other hand, however, by (4.12) we have $\lambda^* > \lambda_0$. This is due to the fact that $\lambda_0 = s(s-n+1)$ is the principal eigenvalue associated with the half sphere, which by (4.12) is contained in some Λ_ω^* after a rotation. Hence we arrive at a contradiction.

Finally, note that the eigenspace associated with half thin-sphere, which without of generality we assume to be $\partial B_1 \cap \{\omega_1 \leq 0, \omega_n = 0\}$, is generated by $u(x) = (\sqrt{x_1^2 + x_n^2} + x_1)^s$. By the above argument, if $\Psi^{x_0}(u, \chi, 0+) = 2M|B'_1|$, then any blow-up limit u_0 is of the form $u_0 = c(\sqrt{x_1^2 + x_n^2} + x_1)^s$, $c > 0$ after a rotation. By Proposition 4.3, $c = 2^{-s} \sqrt{2M/c_0(s)}$. $\square$

At the end of the paper we would like to give an alternative proof of Theorem 4.1, without relying on Corollary A.2 in Appendix, but rather using Proposition 4.4 and some ideas from Proposition 4.5.

Alternative Proof of Theorem 4.1. We start again by rescaling (4.1), to obtain that for every $R > 0$,

$$\{u_\lambda(\cdot, 0) > 0\} \cap \{x_1 < 0\} \cap B'_R \rightarrow 0 \quad \text{as } \lambda \rightarrow 0,$$

which implies that $u_0 = 0$ $\mathcal{H}^{n-1}$ -a.e. in $\{x_1 < 0, x_n = 0\}$ and hence $u_0 = 0$ everywhere on $\{x_1 \leq 0, x_n = 0\}$, by continuity. Next, using that $\chi = M$ when $u(\cdot, 0) > 0$, we also have

$$|\{\chi_\lambda < M\} \cap \{x_1 > 0\} \cap B'_R| \rightarrow 0 \quad \text{as } \lambda \rightarrow 0,$$

implying that $\chi_0 = M$ $\mathcal{H}^{n-1}$ -a.e. in $\{x_1 > 0, x_n = 0\}$. Hence, by Proposition 4.4, necessarily $u_0 > 0$ or $u_0 \equiv 0$ in all of $\{x_1 > 0, x_n = 0\}$. Indeed, if $\mathcal{F}_{u_0} \cap \{x_1 > 0, x_n = 0\} \neq \emptyset$, then there exists $\hat{x}_0 \in \mathcal{F}_{u_0} \cap \{x_1 > 0, x_n = 0\}$ such that $\Psi^{\hat{x}_0}(u_0, \chi_0, 0+) = 4M|B'_1|$. Since u_0 is homogeneous by Corollary 3.6(iii), then $\lambda\hat{x}_0 \in \mathcal{F}_{u_0}$ for each $\lambda > 0$ and $\Psi^{\lambda\hat{x}_0}(u_0, \chi_0, 0+) = 4M|B'_1|$. By the upper semicontinuity of the map $x \mapsto \Psi^x(u_0, \chi_0, 0+)$ we necessarily have $\Psi^0(u_0, \chi_0, 0+) = 4M|B'_1|$. However, this contradicts Proposition 4.4. Thus, $u_0 > 0$ in $\{x_1 > 0, x_n = 0\}$ and hence, u_0 satisfies $\operatorname{div}(|x_n|^{1-2s}\nabla u_0) = 0$ in $\mathbb{R}^n \setminus \{x_1 \leq 0, x_n = 0\}$. Since u_0 is also homogeneous of degree s , writing it as $u_0(x) = r^s f_0(\omega)$, with $r = |x|$ and $\omega = x/|x|$, we see that f_0 is a nonnegative eigenfunction of the weighted spherical Laplacian as in Proposition 4.5 in $\partial B_1 \setminus \{\omega_1 \leq 0, \omega_n = 0\}$. Hence, f_0 is a positive multiple of the explicitly given eigenfunction $2^{-s}((\omega_1^2 + \omega_n^2)^{1/2} + \omega_1)^s$ and hence

$$u_0(x) = \alpha 2^{-s} \left((x_1^2 + x_n^2)^{1/2} + x_1 \right)^s,$$

for some $\alpha > 0$. Then Theorem 4.1 follows directly by applying Proposition 4.3, as in the first proof of the theorem. $\square$

APPENDIX A

In this appendix we prove the asymptotic development for nonnegative solutions of $\operatorname{div}(|x_n|^{1-2s}\nabla u) = 0$ near the ‘flat’ boundary points. The proof uses ideas similar to those in Lemma A.1 and Corollary A.1 in [CLW97b].

Below, we will denote,

$$\begin{aligned} \Lambda &:= \{x \in \mathbb{R}^n : x_n = 0, x_1 \leq 0\}. \\ P(x) &:= \frac{1}{2^s} \left(\sqrt{x_1^2 + x_n^2} + x_1 \right)^s. \end{aligned}$$

Lemma A.1. *Let $u \in C^{0,s}(B_1)$ be nonnegative, $u = 0$ on $\Lambda \cap B_1$ and satisfy $\operatorname{div}(|x_n|^{1-2s}\nabla u) \leq 0$ in $B_1 \setminus \Lambda$. Then u has an asymptotic development at the origin*

$$u(x) = \alpha P(x) + o(|x|^s)$$

with a constant $\alpha \geq 0$.

Proof. Let

$$\varepsilon(r) := \sup\{\varepsilon : u(x) \geq \varepsilon P(x) \text{ in } B_r\}, \quad 0 < r < 1.$$

Then $\varepsilon(r)$ is a nonincreasing function of r , and moreover it is bounded above by the $C^{0,s}$ norm of u . Let $\alpha := \lim_{r \rightarrow 0+} \varepsilon(r)$. From this definition of α , we immediately have

$$(A.1) \quad u(x) \geq \alpha P(x) + o(|x|^s) \quad \text{in } B_1.$$

Claim. We have $u(x) = \alpha P(x) + o(|x|^s)$.

We argue by contradiction. Assume that there are some $\delta_0 > 0$ and a sequence $x_k \in B_1$ with $r_k := |x_k| \rightarrow 0$ such that

$$(A.2) \quad u(x_k) - \alpha P(x_k) \geq \delta_0 r_k^s.$$

Consider the rescalings

$$u_k(x) := \frac{u(r_k x)}{r_k^s} \quad \text{for } x \in B_{1/r_k}, \quad \bar{x}_k := \frac{1}{r_k} x_k \in \partial B_1.$$

Since $u \in C^{0,s}(B_1)$, then there exists a subsequence which we still denote by u_k and $v \in C_{\text{loc}}^{0,s}(\mathbb{R}^n)$ such that $u_k \rightarrow v$ uniformly on compact subsets in $\mathbb{R}^n$. We can also assume $\bar{x}_k \rightarrow \bar{x} \in \partial B_1$. From (A.1) and (A.2) we have

$$v - \alpha P \geq 0 \text{ in } \overline{B_1}, \quad v(\bar{x}) - \alpha P(\bar{x}) \geq \delta_0.$$

By the uniform convergence of u_k to v and the Hölder regularity of v , there exists $\eta > 0$ such that $B_\eta(\bar{x}) \cap \Lambda = \emptyset$ and

$$v - \alpha P \geq \frac{\delta_0}{2}, \quad u_k - \alpha P \geq \frac{\delta_0}{2} \quad \text{for } k > k_0 \text{ large enough, on } B_\eta(\bar{x}).$$

Now let w be a solution to $\operatorname{div}(|x_n|^{1-2s}\nabla w) = 0$ in $B_1 \setminus \Lambda$ with smooth boundary data, such that

$$\begin{aligned} w &= 0 && \text{on } \partial(B_1 \setminus \Lambda) \setminus B_{\eta/2}(\bar{x}), \\ w &= \frac{\delta_0}{4} && \text{on } \partial(B_1 \setminus \Lambda) \cap B_{\eta/4}(\bar{x}), \\ 0 &\leq w \leq \frac{\delta_0}{4} && \text{on } \partial(B_1 \setminus \Lambda) \cap B_{\eta/2}(\bar{x}). \end{aligned}$$

By the maximum principle, w is nonnegative in $B_1 \setminus \Lambda$. By the boundary Harnack principle (see [CSS08]), there exist small $\mu, \gamma > 0$ which depend on δ_0 and ε such that

$$w(x) \geq \mu P(x) \quad \text{on } B_\gamma.$$

Now, each u_k satisfies $\operatorname{div}(|x_n|^{1-2s}\nabla u_k) \leq 0$ in $B_1 \setminus \Lambda$. Let w_k be the solution to $\operatorname{div}(|x_n|^{1-2s}\nabla w_k) = 0$ in $B_1 \setminus \Lambda$ and $w_k = \min(0, u_k - \alpha P) + w$ on $\partial(B_1 \setminus \Lambda)$. By the comparison principle, $u_k - \alpha P \geq w_k$ in $B_1 \setminus \Lambda$. Moreover, by (A.1), we see $w_k \rightarrow w$ uniformly on $\partial(B_1 \setminus \Lambda)$ and hence, by the maximum principle, also on B_1 . By the boundary Harnack principle, we can assume therefore that

$$w(x) - w_k(x) \leq \frac{\mu}{2} P(x), \quad \text{in } B_\gamma,$$

for large k , which then gives

$$u_k(x) - \alpha P(x) \geq w_k(x) + \mu P(x) - w(x) \geq \frac{\mu}{2} P(x) \quad \text{in } B_\gamma.$$

Scaling back, we therefore have

$$u(x) \geq \left(\alpha + \frac{\mu}{2}\right) P(x) \quad \text{in } B_{\gamma r_k},$$

implying that $\varepsilon(\gamma r_k) \geq \alpha + \frac{\mu}{2}$, which in turn leads to the absurd $\alpha \geq \alpha + \frac{\mu}{2}$. This proves the lemma. $\square$

Corollary A.2. *Let $u \in C^{0,s}(B_1)$ be nonnegative, $u = 0$ on $\Lambda \cap B_1$ and satisfy $\operatorname{div}(|x_n|^{1-2s}\nabla u) = 0$ in $\{u > 0\}$. Then u has an asymptotic development at the origin*

$$u(x) = \alpha P(x) + o(|x|^s)$$

with a constant $\alpha \geq 0$.

Proof. Note that from the conditions above $\operatorname{div}(|x_n|^{1-s}\nabla u) \geq 0$ in $B_1 \setminus \Lambda$. Then, we claim that there exists $C \geq 0$ such that

$$u \leq CP(x) \quad \text{in } B_{1/2}.$$

Indeed, if w is a solution of the Dirichlet problem $\operatorname{div}(|x_n|^{1-2s}\nabla w) = 0$ in $B_{3/4} \setminus \Lambda$, $w = u$ on $\partial(B_{3/4} \setminus \Lambda)$, then by the boundary Harnack principle

$$w \leq CP(x) \quad \text{in } B_{1/2}$$

and our claim follows from the comparison $u \leq w$ in $B_{3/4}$. Now,

$$U(x) = CP(x) - u(x)$$

will satisfy the conditions of Lemma A.1 (in $B_{1/2}$ instead B_1) and the asymptotic development of u will follow from that of U . $\square$

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