Boundedness of multi-parameter Fourier multiplier operators on Triebel-Lizorkin and Besov-Lipschitz spaces

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Abstract

The main purpose of this paper is three-fold. First, we prove that under the limited smoothness conditions, multi-parameter Fourier multiplier operators are bounded on multi-parameter Triebel-Lizorkin and Besov-Lipschitz spaces by the Littlewood-Paley decomposition and the strong maximal operator. Second, we offer a different and more direct method to deal with the boundedness instead of transforming Fourier multiplier operators into multi-parameter Calderón-Zygmund operators. Third, we also prove the boundedness of multi-parameter Fourier multiplier operators on weighted multi-parameter Triebel-Lizorkin and Besov-Lipschitz spaces when the Fourier multiplier is only assumed with limited smoothness.

1 Introduction and Statement of main Results

In the last century, many authors have worked on proving boundedness of the one-parameter Fourier multipliers. Mihlin [26] obtained the L^p -boundedness of the one-parameter Fourier multiplier operators with minimal smooth condition. Hörmander [21] reformulated and improved Mihlin's theorem using the Sobolev regularity of the multiplier. Peetre [30] considered the boundedness of the one-parameter Fourier multiplier operators on Triebel-Lizorkin spaces.

The multi-parameter singular integral and Hardy space theory play an important role in many aspects of harmonic analysis. Many authors have studied the pure

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product theory over the last decades. This theory includes the boundedness on multi-parameter L^p spaces ($1 < p < \infty$) and multi-parameter Hardy spaces H^p ($0 < p \leq 1$), etc. For more results related to multi-parameter theory, see [1, 2, 3, 4, 10, 11, 12, 13, 15, 16, 17, 18, 20, 22, 23, 24, 27, 28, 29, 31, 33, 34, 35, 36]. Recently, the authors of [9, 25] established the boundedness of singular integral operators on multi-parameter Triebel-Lizorkin and Besov-Lipschitz spaces. More recently, the atomic decomposition and dual spaces for multi-parameter Triebel-Lizorkin spaces associated with the composition of two singular operators studied by Phong and Stein [32] were given in [7, 6].

The main purpose of this paper is to establish the sufficient conditions for boundedness of multi-parameter Fourier multiplier operators on multi-parameter Triebel-Lizorkin and Besov-Lipschitz spaces (Theorem 1.8). Furthermore, we also consider the weighted cases (Theorem 1.10). For the sake of simplicity of presentations, we will restrict our consideration to the bi-parameter case of $\mathbb{R}^n \times \mathbb{R}^n$.

In this paper, we use the following notations.

We denote by $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ the n -dimension multi-index, by $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ the spaces of Schwartz functions on $\mathbb{R}^n \times \mathbb{R}^n$ and by $\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ the spaces of all tempered distributions on $\mathbb{R}^n \times \mathbb{R}^n$. The bi-parameter Fourier transform and the Fourier inverse transform of f are defined respectively by

$$\widehat{f}(\xi_1, \xi_2) = \int_{\mathbb{R}^{2n}} e^{-2\pi i(x_1 \cdot \xi_1 + x_2 \cdot \xi_2)} f(x_1, x_2) dx_1 dx_2$$

and

$$f^\vee(\xi_1, \xi_2) = \int_{\mathbb{R}^{2n}} e^{2\pi i(x_1 \cdot \xi_1 + x_2 \cdot \xi_2)} f(x_1, x_2) dx_1 dx_2$$

for $f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$.

Definition 1.1. A bi-parameter Fourier multiplier operator is defined as follows

$$T(f)(x_1, x_2) = \int_{\mathbb{R}^{2n}} m(\xi_1, \xi_2) e^{2\pi i(x_1 \cdot \xi_1 + x_2 \cdot \xi_2)} \widehat{f}(\xi_1, \xi_2) d\xi_1 d\xi_2,$$

where $m(\xi_1, \xi_2) \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$.

We denote by $\psi^{(1)}(x)$ and $\psi^{(2)}(y)$ Schwartz functions whose Fourier transforms are supported in $\{\frac{1}{2} \leq |\xi| \leq 2\}$. Moreover, we require that they satisfy the vanishing moment condition

$$\int_{\mathbb{R}^n} x^\alpha \psi^{(1)}(x) dx = \int_{\mathbb{R}^n} y^\alpha \psi^{(2)}(y) dy = 0$$

for all multi-indices α and β . The test function defined on $\mathbb{R}^n \times \mathbb{R}^n$ can be given by

$$\psi(x, y) = \psi^{(1)}(x) \psi^{(2)}(y),$$

where $\psi^{(1)}(x)$ satisfies $\sum_{j \in \mathbb{Z}} |\widehat{\psi^{(1)}}(2^{-j}x)| = 1$ for all $x \in \{\mathbb{R}^n \setminus (0)\}$ and $\psi^{(2)}(y)$ satisfies $\sum_{k \in \mathbb{Z}} |\widehat{\psi^{(2)}}(2^{-k}y)| = 1$ for all $y \in \{\mathbb{R}^n \setminus (0)\}$.

Let $\mathcal{S}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ denote the space of Schwartz functions whose Fourier transform are supported away from origin and $(\mathcal{S}^\infty(\mathbb{R}^n \times \mathbb{R}^n))^*$ be its dual. For $f \in (\mathcal{S}^\infty(\mathbb{R}^n \times \mathbb{R}^n))^*$, the Littlewood-Paley-Stein square function of f is defined by

$$g(f)(x, y) = \left(\sum_{j,k} |\psi_{j,k} * f(x, y)|^2 \right)^{\frac{1}{2}},$$

where the function

$$\psi_{j,k}(x, y) = 2^{(j+k)n} \psi^{(1)}(2^j x) \psi^{(2)}(2^k y).$$

Definition 1.2. Let $0 < p, q < \infty$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$. The bi-parameter Triebel-Lizorkin space $\dot{F}_p^{\alpha,q}(\mathbb{R}^n \times \mathbb{R}^n)$ is defined by

$$\dot{F}_p^{\alpha,q}(\mathbb{R}^n \times \mathbb{R}^n) = \{f \in (\mathcal{S}^\infty(\mathbb{R}^n \times \mathbb{R}^n))^* : \|f\|_{\dot{F}_p^{\alpha,q}} < \infty\},$$

where

$$\|f\|_{\dot{F}_p^{\alpha,q}} = \left\| \left(\sum_{j,k} 2^{j\alpha_1 q + k\alpha_2 q} |\psi_{j,k} * f(\cdot)|^q \right)^{\frac{1}{q}} \right\|_{L^p}.$$

Definition 1.3. Let $0 < p, q < \infty$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$. The bi-parameter Besov-Lipschitz space $\dot{B}_p^{\alpha,q}(\mathbb{R}^n \times \mathbb{R}^n)$ is defined by

$$\dot{B}_p^{\alpha,q}(\mathbb{R}^n \times \mathbb{R}^n) = \{f \in (\mathcal{S}^\infty(\mathbb{R}^n \times \mathbb{R}^n))^* : \|f\|_{\dot{B}_p^{\alpha,q}} < \infty\},$$

where

$$\|f\|_{\dot{B}_p^{\alpha,q}} = \left(\sum_{j,k} 2^{j\alpha_1 q + k\alpha_2 q} \|\psi_{j,k} * f(\cdot)\|_{L^p}^q \right)^{\frac{1}{q}}.$$

Here we need to emphasize that the definitions of bi-parameter Besov-Lipschitz and Triebel-Lizorkin spaces are independent of $\psi^{(1)}$ and $\psi^{(2)}$. We refer the reader to [6] and [25] for the detailed proof. Therefore, we can choose $\psi^{(1)} = \psi^{(2)} = \psi$ in our proofs.

Definition 1.4. The strong maximal operator is defined as follows

$$M_s f(x, y) = \sup_{r_1, r_2 > 0} \frac{1}{(r_1)^n} \frac{1}{(r_2)^n} \int_R |f(u, v)| du dv,$$

where $R = \{(u, v) \in \mathbb{R}^n \times \mathbb{R}^n : |u - x| < r_1, |v - y| < r_2\}$, and f is a locally integrable function on $\mathbb{R}^n \times \mathbb{R}^n$.

Definition 1.5. A nonnegative locally integrable function w is said to be in the class $A_p(\mathbb{R}^n \times \mathbb{R}^n)$, $1 < p < \infty$ if there exists a constant $C > 0$ such that

$$\left(\frac{1}{|R|} \int_R w(x_1, x_2) dx_1 dx_2 \right) \left(\frac{1}{|R|} \int_R w(x_1, x_2)^{-\frac{1}{p-1}} dx_1 dx_2 \right)^{p-1} \leq C$$

for any rectangle $R = I \times J$.

A nonnegative locally integrable function w is said to be in $A_1(\mathbb{R}^n \times \mathbb{R}^n)$ if there exists a constant $C > 0$ such that

$$M_s w(x_1, x_2) \leq C w(x_1, x_2)$$

for almost every $(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n$, where M_s denotes the strong maximal operator.

The class $A_\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is defined by

$$A_\infty(\mathbb{R}^n \times \mathbb{R}^n) = \bigcup_{1 \leq p < \infty} A_p(\mathbb{R}^n \times \mathbb{R}^n).$$

Definition 1.6. Let $0 < p, q < \infty$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$. The weighted bi-parameter Triebel-Lizorkin space $\dot{F}_p^{\alpha, q}(w, \mathbb{R}^n \times \mathbb{R}^n)$ is defined by

$$\dot{F}_p^{\alpha, q}(w, \mathbb{R}^n \times \mathbb{R}^n) = \{f \in (\mathcal{S}^\infty(\mathbb{R}^n \times \mathbb{R}^n))^* : \|f\|_{\dot{F}_p^{\alpha, q}(w)} < \infty\},$$

where

$$\|f\|_{\dot{F}_p^{\alpha, q}(w)} = \left\| \left(\sum_{j, k} 2^{j\alpha_1 q + k\alpha_2 q} |\psi_{j, k} * f(\cdot)|^q \right)^{\frac{1}{q}} \right\|_{L^p(w)}.$$

Definition 1.7. Let $0 < p, q < \infty$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$. The weighted bi-parameter Besov-Lipschitz space $\dot{B}_p^{\alpha, q}(w, \mathbb{R}^n \times \mathbb{R}^n)$ is defined by

$$\dot{B}_p^{\alpha, q}(w, \mathbb{R}^n \times \mathbb{R}^n) = \{f \in (\mathcal{S}^\infty(\mathbb{R}^n \times \mathbb{R}^n))^* : \|f\|_{\dot{B}_p^{\alpha, q}(w)} < \infty\},$$

where

$$\|f\|_{\dot{B}_p^{\alpha, q}(w)} = \left(\sum_{j, k} 2^{j\alpha_1 q + k\alpha_2 q} \|\psi_{j, k} * f(\cdot)\|_{L^p(w)}^q \right)^{\frac{1}{q}}.$$

We are now ready to state our first main theorem of this paper.

Theorem 1.8. *Suppose that T is a Fourier multiplier operator defined on $\mathbb{R}^n \times \mathbb{R}^n$. Let $\rho = (\rho_1, \rho_2) \in \mathbb{R}^2$, $0 < p, q < \infty$, $N = \left\lceil \frac{n}{2} + \frac{n}{\min(p,q)} \right\rceil + 1$. Assume that $m(\xi, \eta)$ is a $L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ function that satisfies*

$$|\partial_\xi^\alpha \partial_\eta^\beta m(\xi, \eta)| \leq A \frac{1}{|\xi|^{|\alpha|} |\eta|^{|\beta|}}$$

for all $|\alpha| \leq N$, $|\beta| \leq N$ and $(\xi, \eta) \in (\mathbb{R}^n \times \mathbb{R}^n)$ with $|\xi||\eta| \neq 0$. Then there exists a constant C such that

$$\|T(f)\|_{\dot{B}_p^{\rho,q}} \leq C \|f\|_{\dot{B}_p^{\rho,q}}$$

and

$$\|T(f)\|_{\dot{F}_p^{\rho,q}} \leq C \|f\|_{\dot{F}_p^{\rho,q}},$$

where the constant C is independent of f .

If $\rho = (0, 0)$ and $q = 2$, then $\dot{F}_p^{0,2}(\mathbb{R}^n \times \mathbb{R}^n) = L^p(\mathbb{R}^n \times \mathbb{R}^n)$ when $1 < p < \infty$ and $\dot{F}_p^{0,2}(\mathbb{R}^n \times \mathbb{R}^n) = H^p(\mathbb{R}^n \times \mathbb{R}^n)$ when $0 < p < 1$. Then we can easily obtain the boundedness of Fourier multiplier operators on $L^p(\mathbb{R}^n \times \mathbb{R}^n)$ and $H^p(\mathbb{R}^n \times \mathbb{R}^n)$ from Theorem 1.8.

Corollary 1.9. *Let $0 < p < \infty$, $N = \left\lceil \frac{n}{2} + \frac{n}{\min(p,2)} \right\rceil + 1$. Assume that $m(\xi, \eta)$ is a $L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ function that satisfies*

$$|\partial_\xi^\alpha \partial_\eta^\beta m(\xi, \eta)| \leq A \frac{1}{|\xi|^{|\alpha|} |\eta|^{|\beta|}}$$

for all $|\alpha| \leq N$, $|\beta| \leq N$ and $(\xi, \eta) \in (\mathbb{R}^n \times \mathbb{R}^n)$ with $|\xi||\eta| \neq 0$. Then T is bounded on L^p for $1 < p < \infty$ and on H^p for $0 < p \leq 1$.

We need to point out that we offer a different way to deal with the boundedness of Fourier multiplier operators on bi-parameter Triebel-Lizorkin and Besov-Lipschitz spaces (mainly with index $0 < p \leq 1$) instead of transforming Fourier multiplier operators into bi-parameter Calderón-Zygmund operators. This way also allows us to avoid using atomic decomposition and Journé's covering lemma. Moreover, we can obtain the boundedness of the multi-parameter Fourier multiplier operators with limited smoothness assumption on the multiplier $m(\xi, \eta)$. We also remark here that using the multi-parameter Littlewood-Paley theory to establish the L^p ($p > 1$) boundedness of the multi-parameter and multilinear Fourier multipliers with limited smoothness has been done in [5].

Next, we will establish the boundedness of Fourier multiplier operators on weighted bi-parameter Triebel-Lizorkin and Besov-Lipschitz spaces when the multiplier has only *limited smoothness*. We only require the bi-parameter weight w to be in

$A_\infty(\mathbb{R}^n \times \mathbb{R}^n)$ (see also [8] for weighted boundedness for Calderón-Zygmund operators and [19] for Journé type operators and [25] on weighted Triebel-Lizorkin and Besov spaces in product spaces).

Theorem 1.10. *Suppose that T is a bi-parameter Fourier multiplier operator defined on $\mathbb{R}^n \times \mathbb{R}^n$. Let $\rho = (\rho_1, \rho_2) \in \mathbb{R}^2$, $0 < p, q < \infty$, $N = \left\lfloor \frac{n}{2} + \frac{n}{\min(p,q)} \right\rfloor + 1$. Assume that $m(\xi, \eta)$ is a $L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ function that satisfies*

$$|\partial_\xi^\alpha \partial_\eta^\beta m(\xi, \eta)| \leq A \frac{1}{|\xi|^{|\alpha|} |\eta|^{|\beta|}}$$

for all $|\alpha| \leq N$, $|\beta| \leq N$ and $(\xi, \eta) \in (\mathbb{R}^n \times \mathbb{R}^n)$ with $|\xi||\eta| \neq 0$. If $w \in A_\infty(\mathbb{R}^n \times \mathbb{R}^n)$, then the following inequalities are valid with the constant C independent of f :

- (1) $\|T(f)\|_{\dot{B}_p^{\rho,q}(w)} \leq C \|f\|_{\dot{B}_p^{\rho,q}(w)},$
- (2) $\|T(f)\|_{\dot{F}_p^{\rho,q}(w)} \leq C \|f\|_{\dot{F}_p^{\rho,q}(w)}.$

Finally, we extend the above theorems to the cases where the multipliers $m(\xi, \eta)$ have weaker decay condition and worse singularity at the origin.

Theorem 1.11. *Let $\rho = (\rho_1, \rho_2) \in \mathbb{R}^2$, $m = (m_1, m_2) \in \mathbb{R}^+ \times \mathbb{R}^+$, $0 < p, q < \infty$ and $N = \left\lfloor \frac{n}{2} + \frac{n}{\min(p,q)} \right\rfloor + 1$. Assume that $m(\xi, \eta)$ satisfies*

$$|\partial_\xi^\alpha \partial_\eta^\beta m(\xi, \eta)| \leq A \frac{1}{|\xi|^{|\alpha|+m_1} |\eta|^{|\beta|+m_2}}$$

for all $|\alpha| \leq N$, $|\beta| \leq N$ and $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$ with $|\xi||\eta| \neq 0$. Then there exists a constant C such that

$$\|T(f)\|_{\dot{B}_p^{\rho+m,q}} \leq C \|f\|_{\dot{B}_p^{\rho,q}}$$

and

$$\|T(f)\|_{\dot{F}_p^{\rho+m,q}} \leq C \|f\|_{\dot{F}_p^{\rho,q}},$$

where the constant C is independent of f .

Theorem 1.12. *Let $\rho = (\rho_1, \rho_2) \in \mathbb{R}^2$, $m = (m_1, m_2) \in \mathbb{R}^+ \times \mathbb{R}^+$, $0 < p, q < \infty$ and $N = \left\lfloor \frac{n}{2} + \frac{n}{\min(p,q)} \right\rfloor + 1$. Assume that $m(\xi, \eta)$ satisfies*

$$|\partial_\xi^\alpha \partial_\eta^\beta m(\xi, \eta)| \leq A \frac{1}{|\xi|^{|\alpha|+m_1} |\eta|^{|\beta|+m_2}}$$

for all $|\alpha| \leq N$, $|\beta| \leq N$ and $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$ with $|\xi||\eta| \neq 0$. If $w \in A_\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Then there exists a constant C such that

$$\|T(f)\|_{\dot{B}_p^{\rho+m,q}(w)} \leq C \|f\|_{\dot{B}_p^{\rho,q}(w)}$$

and

$$\|T(f)\|_{\dot{F}_p^{\rho+m,q}(w)} \leq C \|f\|_{\dot{F}_p^{\rho,q}(w)},$$

where the constant C is independent of f .

This paper is organized as follows. In Section 2, we prove that the bi-parameter Fourier multiplier operators are bounded on bi-parameter Besov-Lipschitz and Triebel-Lizorkin spaces (Theorem 1.8). In Section 3, we are concerned with the weighted cases (Theorem 1.10). In Section 4, we extend the results of Section 2 and Section 3 to more general cases where the multipliers have the worse singularity at the origin (see Theorems 1.11 and 1.12).

2 Proof of Theorem 1.8.

In this section, we need the following lemmas to prove Theorem 1.8.

Lemma 2.1. ([21]) *Let $1 < p, q < \infty$. Then there exists a constant $C > 0$ such that*

$$\left\| \left\{ \sum_{k \in \mathbb{Z}} (M_s(f_k))^q \right\}^{\frac{1}{q}} \right\|_{L^p} \leq C \left\| \left\{ \sum_{k \in \mathbb{Z}} |f_k|^q \right\}^{\frac{1}{q}} \right\|_{L^p}$$

for all sequences $\{f_k\}$ of locally integrable functions on $\mathbb{R}^n \times \mathbb{R}^n$.

The following lemma is an extension of one-parameter version due to Peetre [30] (see also [14]).

Lemma 2.2. *Let $0 < c < \infty$ and $0 < r < \infty$. For any $C^1(\mathbb{R}^n \times \mathbb{R}^n)$ function u whose Fourier transform is supported in $\{|\xi| \leq ct_1, |\eta| \leq ct_2\}$, assume that $|u(x, y)| \leq B(1 + |x|)^{\frac{n}{r}}(1 + |y|)^{\frac{n}{r}}$ for some $B > 0$. Then there exist two constants C_1 and C_2 such that the following inequality is valid.*

$$\begin{aligned} & \sup_{z_1 \in \mathbb{R}^n, z_2 \in \mathbb{R}^n} \left\{ \frac{1}{t_1} \frac{|\nabla_x u(x - z_1, y - z_2)|}{(1 + t_1|z_1|)^{\frac{n}{r}}(1 + t_2|z_2|)^{\frac{n}{r}}} + \frac{1}{t_2} \frac{|\nabla_y u(x - z_1, y - z_2)|}{(1 + t_1|z_1|)^{\frac{n}{r}}(1 + t_2|z_2|)^{\frac{n}{r}}} \right\} \\ & \leq C_1 \sup_{z_1 \in \mathbb{R}^n, z_2 \in \mathbb{R}^n} \frac{|u(x - z_1, y - z_2)|}{(1 + t_1|z_1|)^{\frac{n}{r}}(1 + t_2|z_2|)^{\frac{n}{r}}} \\ & \leq C_2 M_s(|u|^r)(x, y)^{\frac{1}{r}}, \end{aligned}$$

where M_s denotes the strong maximal operator and the constants C_1 and C_2 depend only on n, c and r .

Proof of Lemma 2.2. This proof is divided into two parts.

Part 1. We first show that

$$\begin{aligned} (2.1) \quad & \sup_{z_1 \in \mathbb{R}^n, z_2 \in \mathbb{R}^n} \left\{ \frac{1}{t_1} \frac{|\nabla_x u(x - z_1, y - z_2)|}{(1 + t_1|z_1|)^{\frac{n}{r}}(1 + t_2|z_2|)^{\frac{n}{r}}} + \frac{1}{t_2} \frac{|\nabla_y u(x - z_1, y - z_2)|}{(1 + t_1|z_1|)^{\frac{n}{r}}(1 + t_2|z_2|)^{\frac{n}{r}}} \right\} \\ & \leq C_1 \sup_{z_1 \in \mathbb{R}^n, z_2 \in \mathbb{R}^n} \frac{|u(x - z_1, y - z_2)|}{(1 + t_1|z_1|)^{\frac{n}{r}}(1 + t_2|z_2|)^{\frac{n}{r}}}. \end{aligned}$$

Select a Schwartz function ψ on $\mathbb{R}^n$ whose Fourier transform is supported in $\{\xi : |\xi| \leq 2c\}$ and is equal to 1 on $\{\xi : |\xi| \leq c\}$. Then $\widehat{\psi}(\frac{\xi}{t_1})\widehat{\psi}(\frac{\eta}{t_2})$ is equal to 1 on the support of the Fourier transform of $u(\xi, \eta)$ and we can write

$$\begin{aligned} u(x - z_1, y - z_2) &= \left(\widehat{u}(\cdot, \cdot) \widehat{\psi}\left(\frac{\cdot}{t_1}\right) \widehat{\psi}\left(\frac{\cdot}{t_2}\right) \right)^\vee (x - z_1, y - z_2) \\ &= \int_{\mathbb{R}^{2n}} t_1^n \psi(t_1(x - z_1 - \xi)) t_2^n \psi(t_2(y - z_2 - \eta)) u(\xi, \eta) d\xi d\eta. \end{aligned}$$

We use the partial derivative to obtain

$$\begin{aligned} &|\nabla_x u(x - z_1, y - z_2)| \\ &\leq \int_{\mathbb{R}^{2n}} t_1^{n+1} |(\nabla_x \psi)(t_1(x - z_1 - \xi))| |t_2^n \psi(t_2(y - z_2 - \eta)) u(\xi, \eta)| d\xi d\eta \\ &\leq C_N \int_{\mathbb{R}^{2n}} t_1^{n+1} (1 + t_1|x - z_1 - \xi|)^{-N} t_2^n (1 + t_2|y - z_2 - \eta|)^{-N} |u(\xi, \eta)| d\xi d\eta. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{1}{t_1} \frac{|\nabla_x u(x - z_1, y - z_2)|}{(1 + t_1|z_1|)^{\frac{n}{r}} (1 + t_2|z_2|)^{\frac{n}{r}}} &\lesssim \int_{\mathbb{R}^{2n}} t_1^n t_2^n (1 + t_1|x - z_1 - \xi|)^{-N} (1 + t_2|y - z_2 - \eta|)^{-N} \\ &\quad \times \frac{|u(\xi, \eta)|}{(1 + t_1|z_1|)^{\frac{n}{r}} (1 + t_2|z_2|)^{\frac{n}{r}}} d\xi d\eta, \end{aligned}$$

where N is an arbitrarily large positive integer and C_N is a constant which depends only on N .

Since

$$1 \leq (1 + t_1|x - z_1 - \xi|)^{\frac{n}{r}} \frac{(1 + t_1|z_1|)^{\frac{n}{r}}}{(1 + t_1|x - \xi|)^{\frac{n}{r}}}$$

and

$$1 \leq (1 + t_2|y - z_2 - \eta|)^{\frac{n}{r}} \frac{(1 + t_2|z_2|)^{\frac{n}{r}}}{(1 + t_2|y - \eta|)^{\frac{n}{r}}},$$

we obtain that

$$\begin{aligned} (2.2) \quad \frac{1}{t_1} \frac{|\nabla_x u(x - z_1, y - z_2)|}{(1 + t_1|z_1|)^{\frac{n}{r}} (1 + t_2|z_2|)^{\frac{n}{r}}} &\leq C_N \int_{\mathbb{R}^{2n}} \frac{|u(\xi, \eta)|}{(1 + t_1|x - \xi|)^{\frac{n}{r}} (1 + t_2|y - \eta|)^{\frac{n}{r}}} \\ &\quad \times \frac{t_1^n}{(1 + t_1|x - z_1 - \xi|)^{N - \frac{n}{r}}} \frac{t_2^n}{(1 + t_2|y - z_2 - \eta|)^{N - \frac{n}{r}}} d\xi d\eta \\ &\leq C_1 \sup_{\xi \in \mathbb{R}^n, \eta \in \mathbb{R}^n} \frac{|u(x - \xi, y - \eta)|}{(1 + t_1|\xi|)^{\frac{n}{r}} (1 + t_2|\eta|)^{\frac{n}{r}}}. \end{aligned}$$

Similarly, we also obtain that

$$(2.3) \quad \frac{1}{t_2} \frac{|\nabla_y u(x - z_1, y - z_2)|}{(1 + t_1|z_1|)^{\frac{n}{r}} (1 + t_2|z_2|)^{\frac{n}{r}}} \leq C_1 \sup_{\xi \in \mathbb{R}^n, \eta \in \mathbb{R}^n} \frac{|u(x - \xi, y - \eta)|}{(1 + t_1|\xi|)^{\frac{n}{r}} (1 + t_2|\eta|)^{\frac{n}{r}}}.$$

Combining (2.2) and (2.3), we conclude the proof of Part 1.

Part 2. We show that

$$(2.4) \quad \sup_{z_1 \in \mathbb{R}^n, z_2 \in \mathbb{R}^n} \frac{|u(x - z_1, y - z_2)|}{(1 + t_1|z_1|)^{\frac{n}{r}}(1 + t_2|z_2|)^{\frac{n}{r}}} \leq CM_s(|u|^r)(x, y)^{\frac{1}{r}}.$$

Let $|\xi| \leq \delta_1$ and $|\eta| \leq \delta_2$ for some $\delta_1, \delta_2 > 0$ to be chosen later. We use the mean value theorem to obtain the estimate

$$\begin{aligned} & u(x - z_1, y - z_2) - u(x - z_1 - \xi, y - z_2 - \eta) \\ &= (\nabla_x u)(x - z_1 - \theta\xi, y - z_2 - \theta\eta) \cdot \xi \\ &+ (\nabla_y u)(x - z_1 - \theta\xi, y - z_2 - \theta\eta) \cdot \eta \quad (0 < \theta < 1) \end{aligned}$$

for all $z_1, z_2 \in \mathbb{R}^n$. Therefore

$$\begin{aligned} & |u(x - z_1, y - z_2)| \\ & \leq \sup_{\substack{|w_1| \leq |z_1| + \delta_1 \\ |w_2| \leq |z_2| + \delta_2}} (|(\nabla_x u)(x - w_1, y - w_2)|\delta_1 + |(\nabla_y u)(x - w_1, y - w_2)|\delta_2) \\ & + |u(x - z_1 - \xi, y - z_2 - \eta)|, \end{aligned}$$

where $w_1 = z_1 + \theta\xi$ and $w_2 = z_2 + \theta\eta$. Let Q_1 be a ball with radius equal to δ_1 and z_1 being the center, and Q_2 be a ball with radius equal to δ_2 and z_2 being the center. By raising to the power of r , averaging over $Q_1 \times Q_2$ and raising to the power $\frac{1}{r}$, we derive

$$\begin{aligned} & |u(x - z_1, y - z_2)| \\ & \leq C_r^2 \sup_{\substack{|w_1| \leq |z_1| + \delta_1 \\ |w_2| \leq |z_2| + \delta_2}} |\nabla_x u(x - w_1, y - w_2)|\delta_1 + |\nabla_y u(x - w_1, y - w_2)|\delta_2 \\ & + \left(\frac{1}{v_n^2 \delta_1^n \delta_2^n} \int_{Q_1} \int_{Q_2} |u(x - z_1 - \xi, y - z_2 - \eta)|^r d\xi d\eta \right)^{\frac{1}{r}}, \end{aligned}$$

where $C_r = \max\{2^r, 2^{\frac{1}{r}}\}$ and v_n is the volume of the unit ball in $\mathbb{R}^n$.

Set $\delta_1 = \frac{\varepsilon}{t_1}$ and $\delta_2 = \frac{\varepsilon}{t_2}$ for some $\varepsilon \leq 1$. Then we have $|w_1| \leq |z_1| + \frac{\varepsilon}{t_1}, |w_2| \leq |z_2| + \frac{\varepsilon}{t_2}, \frac{1}{1+t_1|z_1|} \leq \frac{2}{1+t_1|w_1|}$ and $\frac{1}{1+t_2|z_2|} \leq \frac{2}{1+t_2|w_2|}$. Therefore we can write

$$\begin{aligned} & \frac{|u(x - z_1, y - z_2)|}{(1 + t_1|z_1|)^{\frac{n}{r}}(1 + t_2|z_2|)^{\frac{n}{r}}} \\ & \leq C_{r,n} \left(\sup_{|w_i| \leq |z_i| + \frac{\varepsilon}{t_i}} \frac{|\nabla_x u(x - w_1, y - w_2)|\varepsilon}{t_1(1 + t_1|w_1|)^{\frac{n}{r}}(1 + t_2|w_2|)^{\frac{n}{r}}} + \frac{|\nabla_y u(x - w_1, y - w_2)|\varepsilon}{t_2(1 + t_1|w_1|)^{\frac{n}{r}}(1 + t_2|w_2|)^{\frac{n}{r}}} \right. \\ & \quad \left. + \frac{(\frac{t_1^n t_2^n}{v_n^2 \varepsilon^{2n}} \int_{|\xi| \leq \frac{1}{t_1} + |z_1|} \int_{|\eta| \leq \frac{1}{t_2} + |z_2|} |u(x - z_1 - \xi, y - z_2 - \eta)|^r d\xi d\eta)^{\frac{1}{r}}}{(1 + t_1|z_1|)^{\frac{n}{r}}(1 + t_2|z_2|)^{\frac{n}{r}}} \right) \\ & \leq C_{r,n} \left(\sup_{w_i \in \mathbb{R}^n} \frac{|\nabla_x u(x - w_1, y - w_2)|\varepsilon}{t_1(1 + t_1|w_1|)^{\frac{n}{r}}(1 + t_2|w_2|)^{\frac{n}{r}}} + \frac{|\nabla_y u(x - w_1, y - w_2)|\varepsilon}{t_2(1 + t_1|w_1|)^{\frac{n}{r}}(1 + t_2|w_2|)^{\frac{n}{r}}} \right. \\ & \quad \left. + \varepsilon^{-\frac{2n}{r}} M_s(|u|^r)(x, y)^{\frac{1}{r}} \right). \end{aligned}$$

Using (2.1) and setting $\varepsilon = \frac{1}{2C_{r,n}C_1} \leq 1$ with $C_1 = 2\varepsilon^{-\frac{n}{r}}$, we can achieve (2.4), where we used the hypothesis

$$\sup_{\substack{|z_1| \in \mathbb{R}^n \\ |z_2| \in \mathbb{R}^n}} \frac{|u(x - z_1, y - z_2)|}{(1 + t_1|z_1|)^{\frac{n}{r}}(1 + t_2|z_2|)^{\frac{n}{r}}} \leq \sup_{\substack{|z_1| \in \mathbb{R}^n \\ |z_2| \in \mathbb{R}^n}} \frac{B(1 + |x| + |z_1|)^{\frac{n}{r}}(1 + |y| + |z_2|)^{\frac{n}{r}}}{(1 + t_1|z_1|)^{\frac{n}{r}}(1 + t_2|z_2|)^{\frac{n}{r}}} < \infty.$$

Combing (2.1) and (2.4), we conclude the proof of Lemma 2.2.

Remark 2.3. If $\widehat{u}$ is an integrable function, then u will be a bounded function, and the condition $|u(x, y)| \leq B(1 + |x|)^{\frac{n}{r}}(1 + |y|)^{\frac{n}{r}}$ in Lemma 2.2 will not be needed.

Proof of Theorem 1.8. The proof is divided into two parts.

Part 1. We prove that

$$(2.5) \quad \|T(f)\|_{\dot{B}_p^{\rho,q}} \leq C\|f\|_{\dot{B}_p^{\rho,q}}.$$

Let $\psi(\xi)$ be a Schwartz function whose Fourier transform is supported in $\{\frac{1}{2} \leq |\xi| \leq 2\}$ and satisfies $\sum_{k \in \mathbb{Z}} |\widehat{\psi}(2^{-k}\xi)| = 1$ for all $\xi \in \{\mathbb{R}^n \setminus (0)\}$. Denote $\psi_{j,k}(\xi, \eta) = \psi_{2^{-j}}(\xi)\psi_{2^{-k}}(\eta)$, $\Delta_{j,k}(f) = \psi_{j,k} * f$ and $m_{j,k}(\xi, \eta) = \widehat{\psi}(2^{-j}\xi)\widehat{\psi}(2^{-k}\eta)m(\xi, \eta)$. It is easy to see that $m = \sum_{j,k} m_{j,k}$. By the orthogonality estimate, we have

$$\begin{aligned} \Delta_{j,k}(T(f))(x, y) &= \psi_{j,k} * T(f)(x, y) \\ &= \psi_{j,k} * (m\widehat{f})^\vee(x, y) \\ &= \left(\widehat{\psi}(2^{-j}\xi)\widehat{\psi}(2^{-k}\eta) \sum_{j',k'} \widehat{\psi}(2^{-j'}\xi)\widehat{\psi}(2^{-k'}\eta)m(\xi, \eta)\widehat{f}(\xi, \eta) \right)^\vee(x, y) \\ &= \left(\widehat{\psi}(2^{-j}\xi)\widehat{\psi}(2^{-k}\eta) \sum_{i_1, i_2} \widehat{\psi}(2^{-(j-i_1)}\xi)\widehat{\psi}(2^{-(k-i_2)}\eta)m(\xi, \eta)\widehat{f}(\xi, \eta) \right)^\vee(x, y) \\ &= \left(\widehat{\psi}(2^{-j}\xi)\widehat{\psi}(2^{-k}\eta) \sum_{i_1, i_2} m_{j-i_1, k-i_2}(\xi, \eta)\widehat{f}(\xi, \eta) \right)^\vee(x, y), \end{aligned}$$

where $(i_1, i_2 \in \{-1, 0, 1\})$. We only estimate that

$$\left(\widehat{\psi}(2^{-j}\xi)\widehat{\psi}(2^{-k}\eta)m_{j,k}(\xi, \eta)\widehat{f}(\xi, \eta) \right)^\vee(x, y) \lesssim (M_s(|\Delta_{j,k}(f)|)^r(x, y))^{\frac{1}{r}}.$$

Pick $r < \min(p, q)$ such that $N > \frac{n}{2} + \frac{n}{r}$. By Lemma 2.2, we have

$$\begin{aligned}
& \left| \left(\widehat{\psi}(2^{-j}\xi) \widehat{\psi}(2^{-k}\xi) m_{j,k}(\xi, \eta) \widehat{f}(\xi, \eta) \right)^\vee(x, y) \right| \\
&= \left| \int_{\mathbb{R}^{2n}} m_{j,k}^\vee(\xi, \eta) \Delta_{j,k}(f)(x - \xi, y - \eta) d\xi d\eta \right| \\
&\leq \sup_{z_1, z_2 \in \mathbb{R}^n} \frac{|\Delta_{j,k}(f)(x - z_1, y - z_2)|}{(1 + 2^j|z_1|)^{\frac{n}{r}} (1 + 2^k|z_2|)^{\frac{n}{r}}} \int_{\mathbb{R}^{2n}} |m_{j,k}^\vee(\xi, \eta)| (1 + 2^j|\xi|)^{\frac{n}{r}} (1 + 2^k|\eta|)^{\frac{n}{r}} d\xi d\eta \\
&\leq C_2 M_s(|\Delta_{j,k}(f)|^r(x, y))^{\frac{1}{r}} \int_{\mathbb{R}^{2n}} |m_{j,k}^\vee(\xi, \eta)| (1 + 2^j|\xi|)^{\frac{n}{r}} (1 + 2^k|\eta|)^{\frac{n}{r}} d\xi d\eta.
\end{aligned}$$

Next it suffices to estimate

$$\int_{\mathbb{R}^{2n}} |m_{j,k}^\vee(\xi, \eta)| (1 + 2^j|\xi|)^{\frac{n}{r}} (1 + 2^k|\eta|)^{\frac{n}{r}} d\xi d\eta.$$

We can use the Hölder's inequality to write

$$\begin{aligned}
& \int_{\mathbb{R}^{2n}} |m_{j,k}^\vee(\xi, \eta)| (1 + 2^j|\xi|)^N (1 + 2^k|\eta|)^N \frac{1}{(1 + 2^j|\xi|)^{N - \frac{n}{r}}} \frac{1}{(1 + 2^k|\eta|)^{N - \frac{n}{r}}} d\xi d\eta \\
&\leq \|m_{j,k}^\vee(\xi, \eta) (1 + 2^j|\xi|)^N (1 + 2^k|\eta|)^N\|_{L^2} \left\| \frac{1}{(1 + 2^j|\xi|)^{N - \frac{n}{r}}} \frac{1}{(1 + 2^k|\eta|)^{N - \frac{n}{r}}} \right\|_{L^2}.
\end{aligned}$$

It is easy to see that

$$\left\| \frac{1}{(1 + 2^j|\xi|)^{N - \frac{n}{r}}} \frac{1}{(1 + 2^k|\eta|)^{N - \frac{n}{r}}} \right\|_{L^2} = C 2^{-\frac{jn}{2}} 2^{-\frac{kn}{2}}.$$

We only estimate $\|m_{j,k}^\vee(\xi, \eta) (1 + 2^j|\xi|)^N (1 + 2^k|\eta|)^N\|_{L^2}$.

We use the obvious fact that

$$(1 + 2^j|\xi|)^N \leq C(n) \sum_{|\alpha| \leq N} |(2^j\xi)^\alpha| \quad \text{and} \quad (1 + 2^k|\eta|)^N \leq C(n) \sum_{|\beta| \leq N} |(2^k\eta)^\beta|$$

to obtain

$$\begin{aligned}
& \|m_{j,k}^\vee(\xi, \eta)(1 + 2^j|\xi|)^N(1 + 2^k|\eta|)^N\|_{L^2} \\
&= \left(\int_{\mathbb{R}^{2n}} |(m_{j,k}^\vee(\xi, \eta))|^2 (1 + 2^j|\xi|)^{2N} (1 + 2^k|\eta|)^{2N} d\xi d\eta \right)^{\frac{1}{2}} \\
&\leq C \sum_{|\alpha| \leq N, |\beta| \leq N} \left(\int_{\mathbb{R}^{2n}} |m_{j,k}^\vee(\xi, \eta)|^2 |\xi^\alpha|^2 |\eta^\beta|^2 2^{2j|\alpha|} 2^{2k|\beta|} d\xi d\eta \right)^{\frac{1}{2}} \\
&= C \sum_{|\alpha| \leq N, |\beta| \leq N} \left(\int_{\mathbb{R}^{2n}} |(\partial_\xi^\alpha \partial_\eta^\beta m_{j,k})^\vee(\xi, \eta)|^2 2^{2j|\alpha|} 2^{2k|\beta|} d\xi d\eta \right)^{\frac{1}{2}} \\
&= C \sum_{|\alpha| \leq N, |\beta| \leq N} 2^{j|\alpha|} 2^{k|\beta|} \left(\int_{\mathbb{R}^{2n}} |(\partial_\xi^\alpha \partial_\eta^\beta m_{j,k})^\vee(\xi, \eta)|^2 d\xi d\eta \right)^{\frac{1}{2}} \\
&= C \sum_{|\alpha| \leq N, |\beta| \leq N} 2^{j|\alpha|} 2^{k|\beta|} \left(\int_{\mathbb{R}^{2n}} |\partial_\xi^\alpha \partial_\eta^\beta (m_{j,k})(\xi, \eta)|^2 d\xi d\eta \right)^{\frac{1}{2}}.
\end{aligned}$$

For multi-indices $\delta = (\delta_1, \delta_2, \dots, \delta_n)$ and $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$, we denote $\delta \leq \alpha$ to mean $\delta_i \leq \alpha_i$ and $\gamma \leq \beta$ to mean $\gamma_i \leq \beta_i$ for all $i = 1, 2, \dots, n$. Considering that $\widehat{\psi}(\xi)$ is supported on $\{\xi \mid \frac{1}{2} \leq |\xi| \leq 2\}$, we can use Leibniz's rule to obtain

$$\begin{aligned}
& \int_{\mathbb{R}^{2n}} |\partial_\xi^\alpha \partial_\eta^\beta (m_{j,k})(\xi, \eta)|^2 d\xi d\eta \\
&\leq \sum_{\substack{\delta \leq \alpha \\ \gamma \leq \beta}} C_{\delta, \gamma} \int_{\mathbb{R}^{2n}} |2^{-j|\alpha-\delta|-k|\beta-\gamma|} (\partial_\xi^{\alpha-\delta}(\widehat{\psi})) (2^{-j}\xi) (\partial_\eta^{\beta-\gamma}(\widehat{\psi})) (2^{-k}\eta) \partial_\xi^\delta \partial_\eta^\gamma m(\xi, \eta)|^2 d\xi d\eta \\
&\leq \sum_{\substack{\delta \leq \alpha \\ \gamma \leq \beta}} C_{\delta, \gamma} 2^{-2j|\alpha|} 2^{2j|\delta|} 2^{-2k|\beta|} 2^{2k|\gamma|} \int_{\substack{2^{j-1} \leq |\xi| \leq 2^{j+1} \\ 2^{k-1} \leq |\eta| \leq 2^{k+1}}} |\partial_\xi^\delta \partial_\eta^\gamma m(\xi, \eta)|^2 d\xi d\eta \\
&\leq \sum_{\substack{\delta \leq \alpha \\ \gamma \leq \beta}} C_{\delta, \gamma} 2^{-2j|\alpha|} 2^{2j|\delta|} 2^{-2k|\beta|} 2^{2k|\gamma|} 2^{jn} 2^{kn} 2^{-2j|\delta|} 2^{-2k|\gamma|}.
\end{aligned}$$

Then

$$\begin{aligned}
& \|m_{j,k}^\vee(\xi, \eta)(1 + 2^j|\xi|)^N(1 + 2^k|\eta|)^N\|_{L^2} \left\| \frac{1}{(1 + 2^j|\xi|)^{N-\frac{n}{r}}} \frac{1}{(1 + 2^k|\eta|)^{N-\frac{n}{r}}} \right\|_{L^2} \\
&= C 2^{-\frac{jn}{2}} 2^{-\frac{kn}{2}} \sum_{|\alpha| \leq N, |\beta| \leq N} C_{\alpha, \beta} 2^{j|\alpha|} 2^{k|\beta|} \left(\int_{\mathbb{R}^{2n}} |\partial_\xi^\alpha \partial_\eta^\beta (m_{j,k})(\xi, \eta)|^2 d\xi d\eta \right)^{\frac{1}{2}} \\
&\leq C 2^{-\frac{jn}{2}} 2^{-\frac{kn}{2}} \sum_{|\alpha| \leq N, |\beta| \leq N} C_{\alpha, \beta} 2^{j|\alpha|} 2^{k|\beta|} \left(\sum_{\delta \leq \alpha, \gamma \leq \beta} C_{\delta, \gamma} 2^{jn} 2^{kn} 2^{-2j|\delta|} 2^{-2k|\gamma|} \right)^{\frac{1}{2}} \\
&< \infty,
\end{aligned}$$

where C depends only on n, N, A, r . Therefore

$$\psi_{j,k} * (m_{j,k} \widehat{f})^\vee \lesssim (M_s(|\Delta_{j,k}(f)|)^r(x, y))^{\frac{1}{r}}.$$

Similarly we obtain

$$\psi_{j,k} * (m_{j-i_1, k-i_2} \widehat{f})^\vee \lesssim (M_s(|\Delta_{j,k}(f)|)^r(x, y))^{\frac{1}{r}} \quad (i \in \{1, 0, -1\}).$$

Then

$$\psi_{j,k} * (m \widehat{f})^\vee \lesssim (M_s(|\Delta_{j,k}(f)|)^r(x, y))^{\frac{1}{r}}.$$

Therefore we conclude that

$$\begin{aligned} \|T(f)\|_{\dot{B}_p^{\rho,q}} &= \|(m \widehat{f})^\vee\|_{\dot{B}_p^{\rho,q}} \\ &\leq C \left(\sum_{j,k} \left(2^{j\rho_1+k\rho_2} \|(M_s(|\Delta_{j,k}(f)|)^r)^{\frac{1}{r}}\|_{L^p} \right)^q \right)^{\frac{1}{q}} \\ &\leq C \left(\sum_{j,k} \left(2^{j\rho_1+k\rho_2} \|(|\Delta_{j,k}(f)|^r)^{\frac{1}{r}}\|_{L^{\frac{p}{r}}} \right)^q \right)^{\frac{1}{q}} \\ &= C \left(\sum_{j,k} \left(2^{j\rho_1+k\rho_2} \|\Delta_{j,k}(f)\|_{L^p} \right)^q \right)^{\frac{1}{q}} \\ &= C \|f\|_{\dot{B}_p^{\rho,q}} \end{aligned}$$

for all $f \in \mathcal{S}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Since $\mathcal{S}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ are dense in $\dot{B}_p^{\rho,q}$, the operator T can be extended to a bounded operator on $\dot{B}_p^{\rho,q}$. Then the proof of Part 1 is accomplished.

Part 2. We show that

$$(2.6) \quad \|T(f)\|_{\dot{F}_p^{\rho,q}} \leq C \|f\|_{\dot{F}_p^{\rho,q}}.$$

In Part 1, we have obtained

$$\psi_{j,k} * (m \widehat{f})^\vee \lesssim (M_s(|\Delta_{j,k}(f)|)^r)^{\frac{1}{r}}$$

for all $f \in \mathcal{S}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Pick $r < \min\{p, q\}$ such that $\frac{p}{r}, \frac{q}{r} > 1$. According to

Fefferman-Stein vector-valued inequality in the bi-parameter setting, we obtain

$$\begin{aligned}
\|(T(f))\|_{\dot{F}_p^{\rho,q}} &= \|(m\hat{f})^\vee\|_{\dot{F}_p^{\rho,q}} \\
&\leq C \left\| \left(\sum_{j,k} \left(2^{j\rho_1+k\rho_2} (M_s(|\Delta_{j,k}(f)|^r))^{\frac{1}{r}} \right)^q \right)^{\frac{1}{q}} \right\|_{L^p} \\
&\leq C \left\| \left(\sum_{j,k} \left(2^{(j\rho_1+k\rho_2)r} |\Delta_{j,k}(f)|^r \right)^{\frac{q}{r}} \right)^{\frac{r}{q}} \right\|_{L^{\frac{p}{r}}}^{\frac{1}{r}} \\
&= C \left\| \left(\sum_{j,k} \left(2^{j\rho_1+k\rho_2} |\Delta_{j,k}(f)| \right)^q \right)^{\frac{1}{q}} \right\|_{L^p} \\
&= C \|f\|_{\dot{F}_p^{\rho,q}},
\end{aligned}$$

which concludes the proof of Theorem 1.8.

3 Proof of Theorem 1.10.

In this section, we prove the bi-parameter Fourier multiplier operators are bounded on weighted bi-parameter Besov-Lipschitz and Triebel-Lizorkin spaces. First we state a well-known vector-valued weighted maximal inequality.

Lemma 3.1. *Assume $1 < p, q < \infty$ and $w \in A_p(\mathbb{R}^n \times \mathbb{R}^n)$. Then there exists a constant $C > 0$ such that*

$$\left\| \left\{ \sum_{k \in \mathbb{Z}} (M_s(f_k))^q \right\}^{\frac{1}{q}} \right\|_{L^p(w)} \leq C \left\| \left\{ \sum_{k \in \mathbb{Z}} |f_k|^q \right\}^{\frac{1}{q}} \right\|_{L^p(w)}$$

for all sequences $\{f_k\}$ of locally integrable functions on $\mathbb{R}^n \times \mathbb{R}^n$.

Proof of Theorem 1.10 In the proof of Theorem 1.8, we have obtained the pointwise estimate

$$|\Delta_{j,k}(T(f))| \lesssim (M_s(|\Delta_{j,k}(f)|^r))^{\frac{1}{r}}.$$

Pick r such that $1 < \frac{p}{r}, \frac{q}{r} < \infty$ and $w \in A_{\frac{p}{r}}$, we have

$$\begin{aligned}
(3.1) \quad \left\| \Delta_{j,k} \left((m\hat{f})^\vee \right) \right\|_{L^p(w)} &\leq C \left\| (M_s(|\Delta_{j,k}(f)|^r))^{\frac{1}{r}} \right\|_{L^p(w)} \\
&\leq C \|\Delta_{j,k}(f)\|_{L^p(w)}.
\end{aligned}$$

By Lemma 3.1 and (3.1), we can derive that

$$\begin{aligned}
\|T(f)\|_{\dot{B}_p^{\rho,q}(w)} &= \|(\widehat{mf})^\vee\|_{\dot{B}_p^{\rho,q}(w)} \\
&= \left(\sum_{j,k} \left(2^{j\alpha_1+k\alpha_2} \left\| \Delta_{j,k}((\widehat{mf})^\vee) \right\|_{L^p(w)} \right)^q \right)^{\frac{1}{q}} \\
&\leq C \left(\sum_{j,k} \left(2^{j\alpha_1+k\alpha_2} \left\| (M_s(|\Delta_{j,k}(f)|^r))^{\frac{1}{r}} \right\|_{L^p(w)} \right)^q \right)^{\frac{1}{q}} \\
&\leq C \left(\sum_{j,k} \left(2^{j\alpha_1+k\alpha_2} \left\| \Delta_{j,k}(f)^r \right\|_{L^{\frac{p}{r}}(w)}^{\frac{1}{r}} \right)^q \right)^{\frac{1}{q}} \\
&= C \|f\|_{\dot{B}_p^{\rho,q}(w)}
\end{aligned}$$

and

$$\begin{aligned}
\|T(f)\|_{\dot{F}_p^{\rho,q}(w)} &= \|(\widehat{mf})^\vee\|_{\dot{F}_p^{\rho,q}(w)} \\
&= \left\| \left(\sum_{j,k} \left(2^{j\alpha_1+k\alpha_2} |\Delta_{j,k}((\widehat{mf})^\vee)| \right)^q \right)^{\frac{1}{q}} \right\|_{L^p(w)} \\
&\leq C \left\| \left(\sum_{j,k} \left(2^{j\alpha_1+k\alpha_2} (M_s(|\Delta_{j,k}(f)|^r))^{\frac{1}{r}} \right)^q \right)^{\frac{1}{q}} \right\|_{L^p(w)} \\
&\leq C \left\| \left(\sum_{j,k} \left(2^{(j\alpha_1+k\alpha_2)r} |\Delta_{j,k}(f)|^r \right)^{\frac{q}{r}} \right)^{\frac{r}{q}} \right\|_{L^{\frac{p}{r}}(w)}^{\frac{1}{r}} \\
&= C \|f\|_{\dot{F}_p^{\rho,q}(w)},
\end{aligned}$$

which concludes the proof of Theorem 1.10.

4 Extension of the main results: Proofs of Theorems 1.11 and 1.12

In Sections 2 and 3, we have proved Theorem 1.8 and Theorem 1.10. Now we generalize the results of Theorem 1.8 and Theorem 1.10 to more general cases where the multipliers have the worse singularity at the origin.

Proof of Theorem 1.11. By the Littlewood-Paley decomposition, we can

derive that

$$\begin{aligned}\Delta_{j,k}(T(f)) &= \psi_{j,k} * T(f) \\ &= \left(\widehat{\psi}(2^{-j}\xi) \widehat{\psi}(2^{-k}\eta) \sum_{i_1, i_2} m_{j-i_1, k-i_2}(\xi, \eta) \widehat{f}(\xi, \eta) \right)^\vee(x, y),\end{aligned}$$

where $(i_1, i_2 \in \{-1, 0, 1\})$. We only verify that

$$\left(\widehat{\psi}(2^{-j}\xi) \widehat{\psi}(2^{-k}\eta) m_{j,k}(\xi, \eta) \widehat{f}(\xi, \eta) \right)^\vee(x, y) \lesssim 2^{-jm_1 - km_2} (M_s(|\Delta_{j,k}(f)|)^r(x, y))^{\frac{1}{r}}.$$

Pick $r < \min\{p, q\}$ such that $N > \frac{n}{2} + \frac{n}{r}$. We write

$$\begin{aligned}(4.1) \quad & \left| \left(\widehat{\psi}(2^{-j}\xi) \widehat{\psi}(2^{-k}\eta) m_{j,k}(\xi, \eta) \widehat{f}(\xi, \eta) \right)^\vee(x, y) \right| \\ &= \left| \int_{\mathbb{R}^{2n}} m_{j,k}^\vee(\xi, \eta) \Delta_{j,k}(f)(x - \xi, y - \eta) d\xi d\eta \right| \\ &\leq C_2 M_s(|\Delta_{j,k}(f)|^r(x, y))^{\frac{1}{r}} \int_{\mathbb{R}^{2n}} |m_{j,k}^\vee(\xi, \eta)| (1 + 2^j|\xi|)^{\frac{n}{r}} (1 + 2^k|\eta|)^{\frac{n}{r}} d\xi d\eta.\end{aligned}$$

Next we estimate

$$\int_{\mathbb{R}^{2n}} |m_{j,k}^\vee(\xi, \eta)| (1 + 2^j|\xi|)^{\frac{n}{r}} (1 + 2^k|\eta|)^{\frac{n}{r}} d\xi d\eta.$$

We use the Hölder's inequality to obtain

$$\begin{aligned}& \int_{\mathbb{R}^{2n}} |m_{j,k}^\vee(\xi, \eta)| (1 + 2^j|\xi|)^N (1 + 2^k|\eta|)^N \frac{1}{(1 + 2^j|\xi|)^{N - \frac{n}{r}}} \frac{1}{(1 + 2^k|\eta|)^{N - \frac{n}{r}}} d\xi d\eta \\ &\leq C 2^{-\frac{jn}{2}} 2^{-\frac{kn}{2}} \|m_{j,k}^\vee(\xi, \eta) (1 + 2^j|\xi|)^N (1 + 2^k|\eta|)^N\|_{L^2}.\end{aligned}$$

It suffices to estimate $\|m_{j,k}^\vee(\xi, \eta) (1 + 2^j|\xi|)^N (1 + 2^k|\eta|)^N\|_{L^2}$.

$$\begin{aligned}(4.2) \quad & \|m_{j,k}^\vee(\xi, \eta) (1 + 2^j|\xi|)^N (1 + 2^k|\eta|)^N\|_{L^2} \\ &\leq C \sum_{|\alpha| \leq N, |\beta| \leq N} \left(\int_{\mathbb{R}^{2n}} |m_{j,k}^\vee(\xi, \eta)|^2 |\xi^\alpha|^2 |\eta^\beta|^2 2^{2j|\alpha|} 2^{2k|\beta|} d\xi d\eta \right)^{\frac{1}{2}} \\ &= C \sum_{|\alpha| \leq N, |\beta| \leq N} C_{\alpha, \beta} 2^{j|\alpha|} 2^{k|\beta|} \left(\int_{\mathbb{R}^{2n}} |\partial_\xi^\alpha \partial_\eta^\beta (m_{j,k})(\xi, \eta)|^2 d\xi d\eta \right)^{\frac{1}{2}}.\end{aligned}$$

We use Leibniz's rule to obtain

$$\begin{aligned}
(4.3) \quad & \int_{\mathbb{R}^{2n}} |\partial_\xi^\alpha \partial_\eta^\beta (m_{j,k})(\xi, \eta)|^2 d\xi d\eta \\
& \leq \sum_{\substack{\delta \leq \alpha \\ \gamma \leq \beta}} C_{\delta, \gamma} \int_{\mathbb{R}^{2n}} |2^{-j|\alpha-\delta|} 2^{-k|\beta-\gamma|} (\partial_\xi^{\alpha-\delta}(\widehat{\psi})) (2^{-j}\xi) (\partial_\eta^{\beta-\gamma}(\widehat{\psi})) (2^{-k}\eta) \partial_\xi^\delta \partial_\beta^\gamma m(\xi, \eta)|^2 d\xi d\eta \\
& \leq \sum_{\substack{\delta \leq \alpha \\ \gamma \leq \beta}} C_{\delta, \gamma} 2^{-2j|\alpha|} 2^{2j|\delta|} 2^{-2k|\beta|} 2^{2k|\gamma|} 2^{jn} 2^{kn} 2^{-2j(|\delta|+m_1)} 2^{-2k(|\gamma|+m_2)}.
\end{aligned}$$

According to (4.2) and (4.3), we derive

$$\begin{aligned}
(4.4) \quad & \|m_{j,k}^\vee(\xi, \eta) (1 + 2^j |\xi|)^N (1 + 2^k |\eta|)^N\|_{L^2} \\
& \leq C \sum_{|\alpha| \leq N, |\beta| \leq N} C_{\alpha, \beta} 2^{j|\alpha|} 2^{k|\beta|} \left(\sum_{\substack{\delta \leq \alpha \\ \gamma \leq \beta}} C_{\delta, \gamma} 2^{jn} 2^{kn} 2^{-2j(|\alpha|+m_1)} 2^{-2k(|\beta|+m_2)} \right)^{\frac{1}{2}} \\
& = 2^{-jm_1 - km_2} C < \infty.
\end{aligned}$$

Combining (4.1) and (4.4), we have

$$\psi_{j,k} * (m_{j,k} \widehat{f})^\vee \lesssim 2^{-jm_1 - km_2} (M_s(|\Delta_{j,k}(f)|)^r(x, y))^{\frac{1}{r}}.$$

Similarly we also have

$$\psi_{j,k} * (m_{j-i_1, k-i_2} \widehat{f})^\vee \lesssim 2^{-jm_1 - km_2} (M_s(|\Delta_{j,k}(f)|)^r(x, y))^{\frac{1}{r}} \quad (i \in \{1, 0, -1\}).$$

Therefore

$$\begin{aligned}
\|T(f)\|_{\dot{B}_p^{p+m, q}} &= \|(m\widehat{f})^\vee\|_{\dot{B}_p^{p+m, q}} \\
&\leq C \left(\sum_{j,k} \left(2^{j\rho_1 + k\rho_2} \left\| (M_s(|\Delta_{j,k}(f)|)^r)^{\frac{1}{r}} \right\|_{L^p} \right)^q \right)^{\frac{1}{q}} \\
&\leq C \left(\sum_{j,k} \left(2^{j\rho_1 + k\rho_2} \left\| (|\Delta_{j,k}(f)|)^r \right\|_{L^{\frac{p}{r}}}^{\frac{1}{r}} \right)^q \right)^{\frac{1}{q}} \\
&\leq C \|f\|_{\dot{B}_p^{p, q}}
\end{aligned}$$

and

$$\begin{aligned}
\|(T(f))\|_{\dot{F}_p^{\rho+m,q}} &= \|(m\widehat{f})^\vee\|_{\dot{F}_p^{\rho+m,q}} \\
&\leq C \left\| \left(\sum_{j,k} \left(2^{j\rho_1+k\rho_2} (M_s(|\Delta_{j,k}(f)|^r))^{\frac{1}{r}} \right)^q \right)^{\frac{1}{q}} \right\|_{L^p} \\
&\leq C \left\| \left(\sum_{j,k} \left(2^{(j\rho_1+k\rho_2)r} |\Delta_{j,k}(f)|^r \right)^{\frac{q}{r}} \right)^{\frac{r}{q}} \right\|_{L^{\frac{p}{r}}}^{\frac{1}{r}} \\
&\leq C \|f\|_{\dot{F}_p^{\rho,q}},
\end{aligned}$$

which concludes the proof of Theorem 1.11.

Since the proof of Theorem 1.12 is similar to that of Theorem 1.10, we omit the details.

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