BOUNDARY LAYERS FOR THE 3D PRIMITIVE EQUATIONS IN A CUBE: THE SUPERCRITICAL MODES

MAKRAM HAMOUDA^{1,2}, CHANG-YEOL JUNG^{*,3} AND ROGER TEMAM¹

ABSTRACT. In this article we study the boundary layers for the viscous Linearized Primitive Equations (LPEs) when the viscosity is small. The LPEs are considered here in a cube. Besides the usual boundary layers that we analyze here too, corner layers due to the interaction between the different boundary layers are also studied.

1. INTRODUCTION

Various local areas models (LAM) are used for short term (5-7 days) weather predictions. When the Primitive Equations (PEs) are used, it is customary to use the PEs without viscosity since, usually, the effects of viscosity are felt after a period of about one week. However, the use of the inviscid Primitive Equations is confronted to the challenging problem of finding suitable boundary conditions for these equations. The issue is explained, from the practitioner point of view in the tutorial article [51]. From the mathematical point of view the issue is to find suitable boundary conditions which make these equations well-posed. In particular it is known since the work of

Date: November 14, 2015.

* Corresponding author.

Oliger and Sundstrom [33] (see also [48]) that the inviscid Primitive Equations are not well-posed for any set of local boundary equations unlike the related Euler equations of inviscid flows.

Traditionally the issue of the boundary equations for the inviscid Primitive Equations in a cube, is solved, in the linear case by considering a vertical expansion of these equations, which we recall and use below. Then suitable boundary conditions can be found for each of the modes of this expansion. It is explained in [16] that each mode of this expansion is identical to an inviscid shallow water equation and [16] contains a systematic study of the boundary conditions which are suitable for the inviscid linear shallow water equations; this work is being extended to the nonlinear case in e.g. [17], [18].

In a certain sense works like [16] give a sufficient answer to the LAM boundary condition problem, although a systematic implementation remains to be done. Assuming that the limited domain is embedded in a larger one which is computed with a coarser mesh, we know, which boundary data to extract from the coarse grid simulation to be used in the limited domain.

In relation with these practical issues we address in this article another delicate problem: namely to determine the boundary conditions for the inviscid Primitive Equations which result, from the viscous case, by letting the viscosity go to zero. This cannot be done directly, as far as we know, and the route that we follow here is to determine the different boundary layers generated, at small viscosity, by the viscous Primitive Equations and then we can obtain the desired convergences. More precisely, we surmise the boundary conditions for the inviscid PEs and then confirm this result by a convergence theorem.

Since the works of Lions, Temam and Wang on the local and global existence and regularity of solutions of the Primitive Equations (PEs) (see [27] and [28]), many advances have been done in the understanding of these equations which constitute a simplified model of the Navier-Stokes equations in the compressible case for the atmosphere, in the incompressible case for the ocean. Despite these simplifications, appearing after certain approximations as e.g. the Boussinesq or the hydrostatic approximation, the PEs continue to be the object of many theoretical and numerical mathematical studies. In this article, we continue the analysis of the boundary layers of the Linearized Primitive Equations (LPEs) as introduced in our previous works in dimension 2 in [11] and dimension 3 with periodicity in one or two directions in [12]). In the present work we deal with less regular domains. More precisely, we consider domains which contain corners restricting thus ourselves, in a first step, to the cube. Besides its concordance with many physical situations, the cube domain is numerically advantageous to handle computations for several applications like weather predictions among many others of course. Nevertheless, the particularity of the domain (here the cube) generates a new type of boundary layers called *corner layers* in addition to the boundary layers of classical types. The mathematical study of corner layers is very delicate in general; see e.g. the pioneering work [45]. Furthermore, these studies were mainly considered in the case of stationary linear problems. In order to understand the dynamics of the viscous PEs solution, when the viscosity goes to zero and up to the boundary (including the corners), we use the method of correctors that we construct in an explicit manner. Hence, the convergence results, valid up to the boundary, are then proven in suitable spaces that we will specify later on in this work.

We start the article by setting the viscous primitive equations and their expansion in the vertical direction mode by mode (see Section 2). Then, in Section 3, we formally derive the inviscid limit problem for the viscous mode solutions when the viscosity is set to be 0. In Section 4, we construct the correctors and in Section 5 we prove some estimates for these correctors. Finally, in Section 6, we state and prove the main result which shows the convergence between the viscous and inviscid LPEs solutions confirming thus the validity of our somehow heuristical choices for the correctors. This study concerns all the vertical modes $n > n_c$, i.e. the supercritical modes (see (2.10) below). The vertical modes $0 \leq n \leq n_c$ which necessitate different methods will be studied in a separate work. The mode zero $n = 0$ of the LPEs, (2.11) and (3.2) below, seems similar to the linearized Euler equations but the solutions behave differently in some respect (see e.g. [3] for the inviscid LPEs, mode zero). The subcritical modes $1 \leq n \leq n_c$ of the LPEs, see (2.12) and (3.1) below, are quite similar to the supercritical modes. However, the locations of some boundary layers and some inflows are different. The additional difficulties related to the subcritical boundary layers will be explained in more details in the Remark 6.1 below.

Of course this work makes systematic use of a number of tools and developments in boundary layer theory and in the study of the inviscid Primitive Equations. Important works in the mathematical analysis of boundary layers include e.g. the works of Eckhaus and de Jager [5], Grassman [9], Levinson [25], Masmoudi [29], Oleinik and Samokhin [34], O'Malley [31], [32], Verhulst [49], Visik and Ljusternik [50]. Beside these basic references we recall: on the engineering side the beautiful experimental book of Van Dyke [4], the works of Prandtl [36]; on the computational side, the works of Kellogg and Han [14], Stynes [47] and that of Roos, Stynes and Tobiska [41]. More recent references are recalled in the review article [7], including [8], [10], [15], [23] and [30].

2. THE SETTING OF THE PROBLEM AND MODAL DECOMPOSITION.

We consider the linearized primitive equations (LPEs) in $\mathcal{M} = \mathcal{M}' \times (-L_3, 0)$, $\mathcal{M}' = (0, L_1) \times (0, L_2)$: for $(u, v, \psi, \phi, w) = (u^\varepsilon, v^\varepsilon, \psi^\varepsilon, \phi^\varepsilon, w^\varepsilon)$, these equations read

$$(2.1) \quad \begin{cases} u_t + \bar{U}_0 u_x - f v + \phi_x - \varepsilon \Delta_3 u = F_u, \\ v_t + \bar{U}_0 v_x + f u + \phi_y - \varepsilon \Delta_3 v = F_v, \\ \psi_t + \bar{U}_0 \psi_x + N^2 w - \varepsilon \Delta_3 \psi = F_\psi, \\ \phi_z - \psi = 0, \\ u_x + v_y + w_z = 0, \end{cases}$$

where $0 < \varepsilon \ll 1$, $\bar{U}_0 > 0$, $\Delta_3 = \Delta + \partial^2 / \partial z^2$ and $\Delta = \partial^2 / \partial x^2 + \partial^2 / \partial y^2$; the Coriolis parameter f is a function of y , that is $f = f_0(1 + \beta y)$, $f_0, \beta > 0$ constants. In (2.1), (u, v, w) represents the velocity of the fluid, ψ is proportional to the temperature and the density and ϕ is related to the pressure of the fluid (see e.g. [27], [28] and [37]). Equations (2.1) correspond to the linearization of the primitive equations around the flow $\bar{U}_0$ in the direction Ox , $\bar{U}_0$ constant > 0 , and $\bar{v}_0, \bar{\phi}_0, \bar{\psi}_0 = 0$. There is also an underlying background stratification of the fluid temperature $\bar{\psi}$ which is linear in z and only appears through the constant coefficient N proportional to $d\bar{\psi}/dz$; N is called the buoyancy or Brunt-Väisälä frequency; see e.g. [35].

The boundary and initial conditions are chosen as follows, see [27] and [37]:

$$(2.2) \quad (u, v, \psi) = 0 \text{ on } \partial\mathcal{M}' \times (-L_3, 0),$$

$$(2.3) \quad \left(\frac{\partial u}{\partial z}, \frac{\partial v}{\partial z}, \psi, w\right) = 0 \text{ at } z = 0, -L_3,$$

$$(2.4) \quad (u, v, \psi) = (\tilde{u}, \tilde{v}, \tilde{\psi})(x, y, z) \text{ at } t = 0,$$

and we assume the compatibility conditions:

$$(2.5) \quad (\tilde{u}, \tilde{v}, \tilde{\psi}) \text{ satisfy (2.2).}$$

In [12] we considered simpler boundary conditions where the solutions are periodic in one direction so that we can avoid the occurrence of corner boundary layers and then the boundary layers appear only in the non-periodic direction. In this article we consider the more intriguing and interesting case where boundary layers appear on all of the boundary and this includes ordinary, parabolic and corner boundary layers.

Here and for later analysis, we introduce a suitable scalar product for the space $\mathbf{L}^2(\mathcal{M}) = L^2(\mathcal{M})^3$ as follows:

$$(2.6) \quad (U, U^*) = \int_{\mathcal{M}} (uu^* + vv^* + \frac{1}{N^2}\psi\psi^*)d\mathcal{M}, \quad |U|_{\mathbf{L}^2} = (U, U)^{\frac{1}{2}},$$

where $U = (u, v, \psi)$, $U^* = (u^*, v^*, \psi^*)$. The existence and uniqueness of a solution $U^\varepsilon = (u^\varepsilon, v^\varepsilon, \psi^\varepsilon)$ to the system (2.1)-(2.4) is then classical thanks to the standard existence results for linear parabolic problems (see e.g. [12]).

We aim to investigate the limit of the solutions $U^\varepsilon = (u^\varepsilon, v^\varepsilon, \psi^\varepsilon)$ as $\varepsilon \rightarrow 0$, in $\mathbf{L}^2(\mathcal{M})$ in space and uniformly in time, and to show that they converge to the limit solution given below. Due to the loss of the higher order terms (i.e. Laplacian) in the limit problem (i.e. when $\varepsilon = 0$), the limit U^0 of U^ε may not satisfy the boundary conditions (2.2) and (2.3). The equations for the limit solution, denoted by $U^0 = (u^0, v^0, \psi^0)$, are obtained by formally setting $\varepsilon = 0$ in (2.1); $(u, v, \psi, \phi, w) = (u^0, v^0, \psi^0, \phi^0, w^0)$ satisfy:

$$(2.7) \quad \begin{cases} u_t + \bar{U}_0 u_x - f v + \phi_x = F_u, \\ v_t + \bar{U}_0 v_x + f u + \phi_y = F_v, \\ \psi_t + \bar{U}_0 \psi_x + N^2 w = F_\psi, \\ \phi_z - \psi = 0, \\ u_x + v_y + w_z = 0. \end{cases}$$

However, the boundary conditions for U^0 are less easy. As mentioned above many choices of nonlocal boundary conditions can be considered which, associated with (2.7), produce the well-posed problem; see related studies in [37], [42]. However, one of our aims here is to find the boundary conditions satisfied by the limit U^0 of U^ε and, in fact to prove the convergence of U^ε to U^0 we need to study the boundary layers as shown below. In what follows, we will first formally derive the boundary conditions for U^0 and then confirm rigourously this choice by our error analysis and convergence theorems.

For that purpose, as in [11], [12] and [42], we need to expand the solutions U^ε and U^0 with respect to the following normal modal decompositions:

$$(2.8a) \quad (u, v, \phi) = \sum_{n \geq 0} \mathcal{U}_n(z)(u_n, v_n, \phi_n)(x, y, t),$$

$$(2.8b) \quad (w, \psi) = \sum_{n \geq 1} \mathcal{W}_n(z)(w_n, \psi_n)(x, y, t); \text{ note } w_0 = \psi_0 = 0.$$

Here $\mathcal{U}_0 = \frac{1}{\sqrt{L_3}}$, $\mathcal{U}_n = \sqrt{\frac{2}{L_3}} \cos(\lambda_n z)$ and $\mathcal{W}_n = \sqrt{\frac{2}{L_3}} \sin(\lambda_n z)$, and the frequency λ_n is given by:

$$(2.9) \quad \lambda_n = \frac{n\pi}{L_3}, n \geq 0, \text{ integer}.$$

As in [11] and [42], we consider three types of modes, namely the zero, sub- and super-critical modes. Let n_c be such that

$$(2.10) \quad \frac{n_c \pi}{L_3} = \lambda_{n_c} < \frac{N}{\bar{U}_0} < \lambda_{n_c+1} = \frac{(n_c + 1)\pi}{L_3}.$$

The mode $n = 0$ is called the *zero mode*, the modes $1 \leq n \leq n_c$ are called *subcritical*, and the modes $n > n_c$ are called *supercritical*. We do not consider the non-generic case where $L_3 N / \pi \bar{U}_0$ is an integer.

Boundary conditions for the modes: viscous case.

The modal equations for $(u_n, v_n, \psi_n, \phi_n, w_n) = (u_n^\varepsilon, v_n^\varepsilon, \psi_n^\varepsilon, \phi_n^\varepsilon, w_n^\varepsilon)$ are then: in $\mathcal{M}'$, for $n = 0$,

$$(2.11) \quad \begin{cases} u_{0t}^\varepsilon + \bar{U}_0 u_{0x}^\varepsilon - f v_0^\varepsilon + \phi_{0x}^\varepsilon - \varepsilon \Delta u_0^\varepsilon = F_{u0}, \\ v_{0t}^\varepsilon + \bar{U}_0 v_{0x}^\varepsilon + f u_0^\varepsilon + \phi_{0y}^\varepsilon - \varepsilon \Delta v_0^\varepsilon = F_{v0}, \\ u_{0x}^\varepsilon + v_{0y}^\varepsilon = 0, \end{cases}$$

and for $n \geq 1$,

$$(2.12) \quad \begin{cases} u_{nt}^\varepsilon + \bar{U}_0 u_{nx}^\varepsilon - f v_n^\varepsilon + \phi_{nx}^\varepsilon + \varepsilon \lambda_n^2 u_n^\varepsilon - \varepsilon \Delta u_n^\varepsilon = F_{un}, \\ v_{nt}^\varepsilon + \bar{U}_0 v_{nx}^\varepsilon + f u_n^\varepsilon + \phi_{ny}^\varepsilon + \varepsilon \lambda_n^2 v_n^\varepsilon - \varepsilon \Delta v_n^\varepsilon = F_{vn}, \\ \psi_{nt}^\varepsilon + \bar{U}_0 \psi_{nx}^\varepsilon + N^2 w_n^\varepsilon + \varepsilon \lambda_n^2 \psi_n^\varepsilon - \varepsilon \Delta \psi_n^\varepsilon = F_{\psi n}, \\ \phi_n^\varepsilon = -\lambda_n^{-1} \psi_n^\varepsilon, \\ w_n^\varepsilon = -\lambda_n^{-1} (u_{nx}^\varepsilon + v_{ny}^\varepsilon), \end{cases}$$

where $\lambda_n = n\pi/L_3$, as in (2.9). The boundary and initial conditions are chosen as follows: for $n = 0$,

$$(2.13) \quad (u_0, v_0) = 0 \text{ on } \partial \mathcal{M}',$$

$$(2.14) \quad (u_0, v_0) = (\tilde{u}_0, \tilde{v}_0)(x, y) \text{ at } t = 0,$$

and for $n \geq 1$,

$$(2.15) \quad (u_n, v_n, \psi_n) = 0 \text{ on } \partial \mathcal{M}',$$

$$(2.16) \quad (u_n, v_n, \psi_n) = (\tilde{u}_n, \tilde{v}_n, \tilde{\psi}_n)(x, y) \text{ at } t = 0.$$

Here we note that the following compatibility conditions are inherited from (2.5): for $n \geq 0$,

$$(2.17) \quad (\tilde{u}_n, \tilde{v}_n, \tilde{\psi}_n) = 0 \text{ on } \partial \mathcal{M}'.$$

Here we note that $\tilde{\psi}_0 \equiv 0$.

Remark 2.1. In [12] we considered a periodic boundary in x and we have rather simpler boundary layers. However, in the non-periodic domain, as we will discuss later, various boundary layers occur at all non-periodic boundaries including some corner boundary layers (see e.g. [45] for more on corner boundary layers).

For simplicity, we only consider in this article the supercritical modes. We simply set $F_{u0} = F_{v0} = \tilde{u}_0 = \tilde{v}_0 = 0$ and $F_{un} = F_{vn} = F_{\psi n} = \tilde{u}_n = \tilde{v}_n = \tilde{\psi}_n = 0$, $1 \leq n \leq n_c$, so that the zero and subcritical mode solutions (u_n, v_n, ψ_n) will not appear in the (LPEs) system. The zeroth and subcritical modes will be studied elsewhere as well as its coupling with the supercritical modes.

3. BOUNDARY CONDITIONS FOR THE MODES: INVISCID CASE

When submitted to the same expansions as (2.8), the inviscid solution $U^0 = (u^0, v^0, \psi^0, \phi^0, w^0)$ leads to the modes $(u_n^0, v_n^0, \psi_n^0, \phi_n^0, w_n^0)$, that satisfy for $n \geq 1$, the equations (2.12) with $\varepsilon = 0$, that is

$$(3.1) \quad \begin{cases} u_{nt}^0 + \bar{U}_0 u_{nx}^0 - f v_n^0 + \phi_{nx}^0 = F_{un}, \\ v_{nt}^0 + \bar{U}_0 v_{nx}^0 + f u_n^0 + \phi_{ny}^0 = F_{vn}, \\ \psi_{nt}^0 + \bar{U}_0 \psi_{nx}^0 + N^2 w_n^0 = F_{\psi n}, \\ \phi_n^0 = -\lambda_n^{-1} \psi_n^0, \\ w_n^0 = -\lambda_n^{-1} (u_{nx}^0 + v_{ny}^0). \end{cases}$$

We recall also the equation satisfied by the mode $n = 0$ although, as we said, we will not study this mode in this article:

$$(3.2) \quad \begin{cases} u_{0t}^0 + \bar{U}_0 u_{0x}^0 - f v_0^0 + \phi_{0x}^0 = F_{u0}, \\ v_{0t}^0 + \bar{U}_0 v_{0x}^0 + f u_0^0 + \phi_{0y}^0 = F_{v0}, \\ u_{0x}^0 + v_{0y}^0 = 0. \end{cases}$$

The inviscid problems (3.1) and (3.2) are supplemented with the same boundary and initial conditions, (2.13) - (2.17).

To regularize the parabolic boundary layers $\bar{\varphi}_{un}^1, \bar{\varphi}_{un}^2$ (see Lemma 5.1 below), we will need the compatibility conditions,

$$(3.3) \quad F_{un} = F_{\psi n} = 0 \text{ at } (x, y) = (0, 0), (0, L_2).$$

Throughout the paper, we will assume that (3.3) holds.

For simplicity, by dropping the subscript n (we will reintroduce the dependence on n when it is necessary), we can rewrite the system (2.12), in $\mathcal{M}'$, as follows:

$$(3.4) \quad \begin{cases} u_t^\varepsilon + \bar{U}_0 u_x^\varepsilon - f v^\varepsilon - \frac{1}{\lambda} \psi_x^\varepsilon + \varepsilon \lambda^2 u^\varepsilon - \varepsilon \Delta u^\varepsilon = F_u, \\ v_t^\varepsilon + \bar{U}_0 v_x^\varepsilon + f u^\varepsilon - \frac{1}{\lambda} \psi_y^\varepsilon + \varepsilon \lambda^2 v^\varepsilon - \varepsilon \Delta v^\varepsilon = F_v, \\ \psi_t^\varepsilon + \bar{U}_0 \psi_x^\varepsilon - \frac{N^2}{\lambda} (u_x^\varepsilon + v_y^\varepsilon) + \varepsilon \lambda^2 \psi^\varepsilon - \varepsilon \Delta \psi^\varepsilon = F_\psi. \end{cases}$$

The boundary and initial conditions associated with (3.4) are as in (2.15) and (2.16).

Now, in a matrix form, we write

$$(3.5) \quad U_t^\varepsilon + \mathcal{B}U_x^\varepsilon + \mathcal{C}U_y^\varepsilon + \mathcal{D}^\varepsilon U^\varepsilon - \varepsilon \Delta U^\varepsilon = F,$$

where $U^\varepsilon = (u^\varepsilon, v^\varepsilon, \psi^\varepsilon)$, $F = (F_u, F_v, F_\psi)$ and

$$(3.6) \quad \mathcal{B} = \begin{pmatrix} \bar{U}_0 & 0 & -\frac{1}{\lambda} \\ 0 & \bar{U}_0 & 0 \\ -\frac{N^2}{\lambda} & 0 & \bar{U}_0 \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\lambda} \\ 0 & -\frac{N^2}{\lambda} & 0 \end{pmatrix}, \quad \mathcal{D}^\varepsilon = \begin{pmatrix} \varepsilon \lambda^2 & -f & 0 \\ f & \varepsilon \lambda^2 & 0 \\ 0 & 0 & \varepsilon \lambda^2 \end{pmatrix}.$$

The limit problem (i.e. $\varepsilon = 0$) of (3.5) is then:

$$(3.7) \quad U_t^0 + \mathcal{B}U_x^0 + \mathcal{C}U_y^0 + \mathcal{D}^0 U^0 = F,$$

where $U^0 = (u^0, v^0, \psi^0)$ and $\mathcal{D}^0$ is the matrix $\mathcal{D}^\varepsilon$ with $\varepsilon = 0$. This is a symmetrizable Friedrichs hyperbolic system [6].

Note that we assume all along the article that the compatibility conditions (2.17), related to the initial condition (2.16), hold for the solutions of (3.5). The boundary conditions for such a hyperbolic system are well-known, see e.g. [1]. However for the convenience of the reader and to prepare the notations - due also to the non-smooth specific geometry - we indicate hereafter the formal derivation of the boundary conditions for (3.5), corresponding to the modes $n \geq 1$. This is just a conjecture to be confirmed by the main convergence theorem below, Theorem 3.1. We proceed as follows: in each directions Ox, Oy , we diagonalize the matrices $\mathcal{B}, \mathcal{C}$ so that, for the dominant terms (that is not counting $\mathcal{D}^0$), the system (3.7) is diagonal, and we prescribe the boundary condition on the parts of the boundary where the characteristics enter the domain. This is the same as requiring that $\mathcal{B} \geq 0, \mathcal{C} \geq 0$.

Hence in the y direction the normal modes are

$$(3.8) \quad \zeta^0 = (v^0 + N^{-1}\psi^0) \quad \text{and} \quad \chi^0 = (v^0 - N^{-1}\psi^0),$$

and we propose the boundary conditions, for $0 \leq x \leq L_1$,

$$(3.9) \quad \begin{cases} \zeta^0(x, L_2, t) = (v^0 + N^{-1}\psi^0)(x, L_2, t) = 0, \\ \chi^0(x, 0, t) = (v^0 - N^{-1}\psi^0)(x, 0, t) = 0, \end{cases}$$

In the x direction, similarly to [11], we use the combinations

$$(3.10) \quad \xi^0 = (u^0 + N^{-1}\psi^0) \quad \text{and} \quad \eta^0 = (u^0 - N^{-1}\psi^0).$$

The boundary conditions are not the same depending on whether the mode is sub- or supercritical, $n < n_c$ or $n > n_c$. For the subcritical modes ($1 \leq n < n_c$), we choose

$$(3.11) \quad \begin{cases} \xi^0(L_1, y, t) = (u^0 + N^{-1}\psi^0)(L_1, y, t) = 0, \\ \eta^0(0, y, t) = (u^0 - N^{-1}\psi^0)(0, y, t) = 0, \\ v^0(0, y, t) = 0, \end{cases}$$

and for the supercritical modes ($n > n_c$), we keep (3.11)_{2,3} and just replace (3.11)₁ by

$$(3.12) \quad \xi^0(0, y, t) = (u^0 + N^{-1}\psi^0)(0, y, t) = 0,$$

and thus, for $0 \leq y \leq L_2$,

$$(3.13) \quad u^0(0, y, t) = v^0(0, y, t) = \psi^0(0, y, t) = 0.$$

Finally, the boundary conditions for the limit problem (3.7) are given by (3.9), (3.11) and (3.13). Our choice for these boundary conditions will be justified below. However, the study of the existence of the solutions to the system (3.7) associated with the boundary conditions mentioned above will be the subject of a subsequent work [13].

3.1. Positivity of the matrices $\mathcal{B}, \mathcal{C}$. The boundary conditions for the inviscid system are formally determined by the direction of the characteristics: we diagonalize the matrices $\mathcal{B}$ and $\mathcal{C}$ and, for the diagonalized system, we prescribe the boundary conditions where the characteristics enter the domain. This is the same as requiring that $\mathcal{B} \geq 0, \mathcal{C} \geq 0$.

We denote by A the operator:

$$(3.14) \quad U \in L^2(\mathcal{M}')^3 \longrightarrow AU = \mathcal{A}U := \mathcal{B}U_x + \mathcal{C}U_y,$$

with domain

$$(3.15) \quad D(A) = \{U \in L^2(\mathcal{M}')^3, \mathcal{A}U \in L^2(\mathcal{M}')^3, U \text{ satisfies} \\ \text{the boundary conditions (3.9), (3.11) and (3.13)}\}.$$

The fact that the traces in (3.9), (3.11), (3.13) make sense when $U \in L^2(\mathcal{M}')^3$ and $\mathcal{A}U \in L^2(\mathcal{M}')^3$ is well-known, see [1, 2]; the traces belong to $H_{loc}^{-\frac{1}{2}}$ of each side of $\mathcal{M}'$.

First, we state and prove the positivity of the operator $\mathcal{B}$ which is defined by (3.6).

Lemma 3.1. *Under the boundary conditions (3.9), (3.11) and (3.13), the operator $\mathcal{B}$, defined above, is positive, i.e.,*

$$(\mathcal{B}U, U) \geq 0, \text{ for all } U \in D(A).$$

Proof. Let $U = (u, v, \psi) \in D(A)$ and recall (2.6). We first write that

$$(3.16) \quad (\mathcal{B}U, U) = \int_{\mathcal{M}'} \bar{U}_0(u_x u + v_x v + N^{-2} \psi_x \psi) dx dy - \frac{1}{\lambda} \int_{\mathcal{M}'} ((\psi u)_x + (\psi v)_y) dx dy \\ := I_1 + I_2 + I_3,$$

where, by integration by parts in x and y , we have

$$(3.17) \quad I_1 = \int_0^{L_2} \left\{ \frac{\bar{U}_0}{2} (u^2 + v^2 + N^{-2} \psi^2)(L_1, y) - \frac{1}{\lambda} (\psi u)(L_1, y) \right\} dy \\ \geq \int_0^{L_2} \left\{ \frac{\bar{U}_0}{2} (u^2 + N^{-2} \psi^2)(L_1, y) - \frac{1}{\lambda} (\psi u)(L_1, y) \right\} dy,$$

$$(3.18) \quad I_2 = \frac{1}{\lambda} \int_0^{L_1} [(\psi v)(x, 0) - (\psi v)(x, L_2)] dx,$$

$$(3.19) \quad I_3 = \frac{1}{\lambda} \int_0^{L_2} (\psi u)(0, y) dy - \frac{\bar{U}_0}{2} \int_0^{L_2} (u^2 + v^2 + N^{-2} \psi^2)(0, y) dy.$$

Then, we start by proving the positivity of the integral I_2 which is dependent of the boundary conditions at $y = 0, L_2$ valid for all the modes. Thus, using (3.8) and the boundary conditions

(3.9), we infer that

$$-\psi v(x, L_2) = -\frac{N}{4}(\zeta^2 - \chi^2)(x, L_2) = \frac{N}{4}\chi^2(x, L_2) \geq 0,$$

and

$$\psi v(x, 0) = \frac{N}{4}(\zeta^2 - \chi^2)(x, 0) = \frac{N}{4}\zeta^2(x, 0) \geq 0.$$

This yields the positivity of I_2 .

Now, we consecutively prove the positivity of the integrals I_1 and I_3 for which we will treat the sub- and super-critical modes separately. Indeed, for I_1 , the positivity is straightforward for the supercritical modes ($n > n_c$); it suffices to observe that

$$\frac{\bar{U}_0}{2}(u^2 + N^{-2}\psi^2) - \frac{1}{\lambda}(\psi u) \geq \frac{\bar{U}_0}{2}\left(u - \frac{1}{\bar{U}_0\lambda}\psi\right)^2 + \frac{\bar{U}_0}{2}\left(N^{-2} - \frac{1}{\bar{U}_0^2\lambda^2}\right)\psi^2 \geq 0.$$

For the sub-critical modes ($1 \leq n \leq n_c$), we recall the definition of the combinations $\xi = u + N^{-1}\psi$ and $\eta = u - N^{-1}\psi$, given by (3.11), and we use the boundary conditions there. Hence, replacing in (3.17) u and ψ by their expressions in terms of ξ and η , we obtain

$$\begin{aligned} (3.20) \quad I_1 &\geq \int_0^{L_2} \left\{ \frac{\bar{U}_0}{8}((\xi + \eta)^2 + (\xi - \eta)^2)(L_1, y) - \frac{N}{4\lambda}(\xi^2 - \eta^2)(L_1, y) \right\} dy \\ &\geq \frac{1}{4} \int_0^{L_2} \left[\left(\bar{U}_0 - \frac{N}{\lambda} \right) \xi^2 + \left(\bar{U}_0 + \frac{N}{\lambda} \right) \eta^2 \right] (L_1, y) dy \\ &\geq (\text{thanks to (3.11)}_1) \\ &\geq \frac{1}{4} \int_0^{L_2} \left(\bar{U}_0 + \frac{N}{\lambda} \right) \eta^2(L_1, y) dy. \end{aligned}$$

Therefore, the integral I_1 is nonnegative for all the modes.

Finally, note that from (3.11)₃ and (3.13) we have $v(0, y) = 0$ for all the modes. Now, using again the combinations $\xi = u + N^{-1}\psi$, $\eta = u - N^{-1}\psi$ and thanks to (3.11)₂, as for I_1 , we similarly find that $I_3 \geq 0$ and thus the positivity of $\mathcal{B}$ is verified. $\square$

In what follows, we prove the positivity of the operator $\mathcal{C}$.

Lemma 3.2. *Under the boundary conditions (3.9), (3.11) and (3.13), the operator $\mathcal{C}$, given by (3.6), is positive, i.e.,*

$$(\mathcal{C}U, U) \geq 0, \text{ for all } U \in D(A).$$

Proof. Let $U = (u, v, \psi) \in D(A)$ and recalling (2.6), we easily see that

$$(\mathcal{C}U, U) = -\frac{1}{\lambda} \int_0^{L_1} [(\psi v)(x, L_2) - (\psi v)(x, 0)] dx.$$

We use the combinations introduced in (3.8); $\zeta = v + N^{-1}\psi$, $\chi = v - N^{-1}\psi$, and the boundary conditions (3.9). Hence, we deduce that

$$(\mathcal{C}U, U) = \frac{N}{4\lambda} \int_0^{L_1} [\chi^2(x, L_2) + \zeta^2(x, 0)] dx.$$

This gives the positivity of $\mathcal{C}$ and thus achieves the proof of the lemma. $\square$

Remark 3.1. Note that for the operator $\mathcal{D}^0$, we have $(\mathcal{D}^0 U, U) = 0$ for all $U \in D(A)$.

Remark 3.2. Proving that the inviscid linear system is well-posed when supplemented with the boundary conditions (3.9), (3.11) and (3.13) is a task different from our objectives in this article. The proof of existence and uniqueness in L^2 has been established in [16] using the linear semigroup methods. However for the proofs below, we need further regularity for U^0 , namely we will assume that

$$(3.21) \quad U^0 \in W^{1,\infty}(0, T; \mathbf{H}^3(\mathcal{M})) \cap L^\infty(0, T; \mathbf{H}^4(\mathcal{M})).$$

We plan, in a separate work, to address the issue of regularity of U^0 which will require, we believe, several compatibility conditions on the data.

3.2. Main result. We now state the convergence results which constitute the main theorem in this article. The proof of the theorem below is addressed in Subsection 6.1.

Theorem 3.1. *Assume that the compatibility conditions (2.5) and (3.3) hold. Let $U^\varepsilon = (u^\varepsilon, v^\varepsilon, \psi^\varepsilon)$ be the solution of Eq. (2.1) under the boundary and initial conditions (2.2) - (2.5) and let $U^0 = (u^0, v^0, \psi^0)$ be the solution of Eq. (2.7) supplemented with the boundary conditions (2.13), (2.15), and the initial conditions (2.14), (2.16) with (2.17), prescribed mode by mode. We assume the existence of U^0 satisfying (3.21). We also assume in (2.11), (2.12), (2.14), and (2.16) that both flows U^ε, U^0 are supercritical; that is, we assume that $(F_{u0}, F_{v0}) = 0$, $(\tilde{u}_0, \tilde{v}_0) = 0$ and $(F_{un}, F_{vn}, F_{\psi n}) = 0$, $(\tilde{u}_n, \tilde{v}_n, \tilde{\psi}_n) = 0$, $1 \leq n \leq n_c$.*

Then there exists a constant $\kappa > 0$ independent of ε such that

$$(3.22) \quad |U^\varepsilon - U^0 - \Theta|_{L^\infty(0, T; \mathbf{L}^2(\mathcal{M}))} + \sqrt{\varepsilon} |U^\varepsilon - U^0 - \Theta|_{L^2(0, T; \mathbf{H}^1(\mathcal{M}))} \leq \kappa K(U^0) \varepsilon^{\frac{3}{4}},$$

where Θ is given in (6.2) below and

$$(3.23) \quad K(U^0) = \|U^0\|_{W^{1,\infty}(0, T; \mathbf{H}^3(\mathcal{M}))} + \|U_{xy}^0\|_{L^\infty(0, T; \mathbf{H}^2(\mathcal{M}))}.$$

4. CONSTRUCTION OF THE BOUNDARY LAYERS

We now construct the boundary layer functions which correct the differences in the boundary values between U^ε and U^0 on $\partial\mathcal{M}'$ for the modes $n > n_c$. We first consider the boundary layer solutions of the dominating differential operators near the boundary and we then consider their approximate forms which can be explicitly expressed. Finally, we can set up the corner layers in the form of the intersection of the boundary layers occurring in any two adjacent edges. Depending on the nature of each boundary (characteristic or non-characteristic), the dynamics inside the boundary layer region may dramatically change. Indeed, this is merely related to the balance between the leading terms when we write the viscous equations in the fast (stretched) variables. In the following, we construct the boundary layer correctors and we distinguish them into two types, parabolic and ordinary boundary layers.

4.1. Parabolic Boundary Layers (PBL) for u^ε at $y = 0, L_2$. In the following, we will construct the correctors which solve the difference in the boundary values between the viscous solution u^ε and the limit solution u^0 at the boundaries $y = 0, L_2$. Hence, in addition to the ordinary boundary

FIGURE 1. The occurrences of boundary and corner layers on $\mathcal{M}'$ for the supercritical modes ($n > n_c$). The corner boundary layers in the gray areas; see the construction of corner boundary layers in Sec. 4.

layers studied in Sec. 4.2 and as suggested in [12], we construct a corrector φ_u solution of:

$$(4.1) \quad \varphi_{ut} + \bar{U}_0 \varphi_{ux} + \varepsilon \lambda^2 \varphi_u - \varepsilon \varphi_{uyy} = 0.$$

The associated boundary and initial conditions to (4.1) are

$$(4.2) \quad \varphi_u = 0 \text{ at } x = 0, \quad \varphi_u = -u^0 \text{ at } y = 0, L_2, \quad \varphi_u = 0 \text{ at } t = 0.$$

We then approximate φ_u by the following two PBLs, i.e. $\varphi_u \sim \bar{\varphi}_u = \bar{\varphi}_u^1 + \bar{\varphi}_u^2$ where, for $0 < x < L_1$, $t > 0$,

$$(4.3) \quad \begin{cases} \bar{\varphi}_{ut}^1 + \bar{U}_0 \bar{\varphi}_{ux}^1 + \varepsilon \lambda^2 \bar{\varphi}_u^1 - \varepsilon \bar{\varphi}_{uyy}^1 = 0, \\ \bar{\varphi}_u^1 = -u^0(x, 0, t) \text{ at } y = 0, \\ \bar{\varphi}_u^1 \rightarrow 0 \text{ as } y/\sqrt{\varepsilon} \rightarrow \infty, \\ \bar{\varphi}_u^1 = 0 \text{ at } x = 0, \\ \bar{\varphi}_u^1 = 0 \text{ at } t = 0, \end{cases}$$

and

$$(4.4) \quad \begin{cases} \bar{\varphi}_{ut}^2 + \bar{U}_0 \bar{\varphi}_{ux}^2 + \varepsilon \lambda^2 \bar{\varphi}_u^2 - \varepsilon \bar{\varphi}_{uyy}^2 = 0, \\ \bar{\varphi}_u^2 = -u^0(x, L_2, t) \text{ at } y = L_2, \\ \bar{\varphi}_u^2 \rightarrow 0 \text{ as } (L_2 - y)/\sqrt{\varepsilon} \rightarrow \infty, \\ \bar{\varphi}_u^2 = 0 \text{ at } x = 0, \\ \bar{\varphi}_u^2 = 0 \text{ at } t = 0. \end{cases}$$

Note that here the Dirichlet boundary conditions $\bar{\varphi}_u^1 = \bar{\varphi}_u^2 = 0$ are assigned at $x = 0$ whereas periodic boundary conditions are imposed in [12].

The parabolic boundary layers $\bar{\varphi}_u^1, \bar{\varphi}_u^2$ correct, respectively, the boundary values at $y = 0$ and $y = L_2$. For the supercritical modes from (3.13) and the compatibility conditions (3.3), we note

that

$$(4.5) \quad u^0(0, 0, t) = u^0(0, L_2, t) = u_x^0(0, 0, t) = u_x^0(0, L_2, t) = 0,$$

and from (2.16) and (2.17) we have in addition

$$(4.6) \quad u^0(x, 0, 0) = u^0(x, L_2, 0) = u^0(0, y, 0) = u^0(L_1, y, 0) = 0.$$

We also note that (4.5) implies

$$(4.7) \quad \bar{\varphi}_u^1 = \bar{\varphi}_{ux}^1 = 0 \text{ at } (x, y) = (0, 0), \quad \bar{\varphi}_u^2 = \bar{\varphi}_{ux}^2 = 0 \text{ at } (x, y) = (0, L_2).$$

Now, we aim to obtain explicit forms for the solutions of (4.3) and (4.4) with the compatibility conditions (4.5) and (4.6). Accordingly, we write the following generic problem for $\varphi = \varphi(x, y, t)$ as

$$(4.8) \quad \begin{cases} \varphi_t + \bar{U}_0 \varphi_x + \varepsilon \lambda^2 \varphi - \varepsilon \varphi_{yy} = 0, \\ \varphi = g_u(x, t) \text{ at } y = 0, \\ \varphi \rightarrow 0 \text{ as } y/\sqrt{\varepsilon} \rightarrow \infty, \\ \varphi = 0 \text{ at } x = 0, \\ \varphi = 0 \text{ at } t = 0, \end{cases}$$

where $g_u(x, t) = -u^0(x, 0, t) = -\gamma_0(u^0)(x, t)$, and γ_0 is the trace operator at $y = 0$. Thanks to the conditions (4.5) and (4.6), we have

$$(4.9) \quad g_u(0, t) = g_{ux}(0, t) = g_u(x, 0) = 0.$$

Indeed, assuming that (3.21) holds from (3.1) and (3.13) we find that at $x = 0$, $\bar{U}_0 u_{nx}^0 - \lambda_n^{-1} \psi_{nx}^0 = F_{un}$ and $\bar{U}_0 \psi_{nx}^0 - \lambda_n^{-1} N^2 u_{nx}^0 = F_{\psi n}$ and hence $g_{ux} = -u_x^0 = 0$ at $(x, y) = (0, 0)$. We now consider an odd smooth extension of $g(x, t)$ in x as we will see below.

First and in order to solve Eq. (4.8), we extend φ to $x < 0$, as an odd function; that is we set

$$(4.10) \quad \begin{cases} \tilde{\varphi}(x, y, t) = \varphi(x, y, t), \text{ for } x > 0, \\ \tilde{\varphi}(x, y, t) = -\varphi(-x, y, t), \text{ for } x < 0. \end{cases}$$

The function $\tilde{\varphi}$ then satisfies, for $x \in \mathbb{R}$, $t > 0$, $y > 0$:

$$(4.11) \quad \begin{cases} \tilde{\varphi}_t + \bar{U}_0 \tilde{\varphi}_x + \varepsilon \lambda^2 \tilde{\varphi} - \varepsilon \tilde{\varphi}_{yy} = 0, \\ \tilde{\varphi} = \tilde{g}_u(x, t) \text{ at } y = 0, \\ \tilde{\varphi} \rightarrow 0 \text{ as } y/\sqrt{\varepsilon} \rightarrow \infty, \\ \tilde{\varphi} = 0 \text{ at } t = 0. \end{cases}$$

Here, $\tilde{g}_u(x, t)$ is the smooth odd extension of $\bar{g}_u(x, t)$ in x onto $\mathbb{R}$ (see (4.10)) where $\bar{g}_u$ is a smooth extension of g_u such that $\bar{g}_u(x, t) = -u^0(x, 0, t)$ for $x \in [0, L_1]$ and zero for $x \in [2L_1, \infty)$. Thus, thanks to (4.9), the odd extension $\tilde{g}_u(x, t)$ and its derivative in x is continuous at $x = 0$ with (3.21) and hence $\partial^i \tilde{g}_u(x, t)/\partial t^i \in H_x^2(\mathbb{R})$ and

$$(4.12) \quad \left\| \frac{\partial^i \tilde{g}_u}{\partial t^i}(x, t) \right\|_{H_x^l(\mathbb{R})} \leq \kappa \left\| \frac{\partial^i g_u}{\partial t^i}(x, t) \right\|_{H_x^l(0, L_1)}, \quad l = 0, 1, 2.$$

Denoting by $\hat{\varphi} = \hat{\varphi}(\omega, y, t)$ the Fourier transform in x of $\tilde{\varphi}$, i.e., $\hat{\varphi} = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \tilde{\varphi} e^{-i\omega x} dx$ and writing $\bar{y} = y/\sqrt{\varepsilon}$, we find that $\hat{\varphi}_t + (\bar{U}_0 i\omega + \varepsilon\lambda^2)\hat{\varphi} - \hat{\varphi}_{\bar{y}\bar{y}} = 0$, and we have

$$(4.13) \quad \begin{cases} (e^{(\bar{U}_0 i\omega + \varepsilon\lambda^2)t} \hat{\varphi})_t - (e^{(\bar{U}_0 i\omega + \varepsilon\lambda^2)t} \hat{\varphi})_{\bar{y}\bar{y}} = 0, & t > 0, \bar{y} > 0, \omega \in \mathbb{R}, \\ e^{(\bar{U}_0 i\omega + \varepsilon\lambda^2)t} \hat{\varphi} = e^{(\bar{U}_0 i\omega + \varepsilon\lambda^2)t} \hat{g}_u(\omega, t) & \text{at } y = 0 \text{ } (\hat{g}_u \text{ defined as } \hat{\varphi}), \\ e^{(\bar{U}_0 i\omega + \varepsilon\lambda^2)t} \hat{\varphi} \rightarrow 0 & \text{as } \bar{y} \rightarrow \infty, \\ e^{(\bar{U}_0 i\omega + \varepsilon\lambda^2)t} \hat{\varphi} = 0 & \text{at } t = 0. \end{cases}$$

The system (4.13) is the heat equation for $e^{(\bar{U}_0 i\omega + \varepsilon\lambda^2)t} \hat{\varphi}$ on a quarter plane ($\bar{y} > 0, t > 0$) (see [45], [20], [21], [22]) and the explicit expression of $\hat{\varphi}$ is found

$$(4.14) \quad \hat{\varphi} = \sqrt{\frac{2}{\pi}} \int_{\frac{y}{\sqrt{2\varepsilon t}}}^{\infty} \exp\left(-\frac{s^2}{2} - \lambda_n^2 \frac{y^2}{2s^2} - \bar{U}_0 i\omega \frac{y^2}{2\varepsilon s^2}\right) \hat{g}_u\left(\omega, t - \frac{y^2}{2\varepsilon s^2}\right) ds.$$

Hence, we deduce the expression of $\bar{\varphi}_u^1$, the solution of (4.3), as follows:

$$(4.15) \quad \bar{\varphi}_u^1 = \tilde{\varphi} \chi_{\{0 < x < L_1\}},$$

where $\tilde{\varphi}$ is the inverse Fourier transform of $\hat{\varphi}$ whose expression is given by (4.14). Here, we recall that $\tilde{\varphi} = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \hat{\varphi} e^{i\omega x} d\omega$ and $\hat{\varphi} = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \tilde{\varphi} e^{-i\omega x} dx$.

Similarly, we obtain an analogous expression for $\bar{\varphi}_u^2$, the solution of (4.4).

Here and later it is useful to estimate the trace of $\bar{\varphi}_u^1$ at $y = L_2$ (similarly, of $\bar{\varphi}_u^2$ at $y = 0$). From (4.15); thanks to the Plancherel theorem, we obtain

$$(4.16) \quad \begin{aligned} |\bar{\varphi}_u^1(x, L_2, t)|_{L_x^2(0, L_1)} &\leq \kappa |\tilde{\varphi}(x, L_2, t)|_{L_x^2(\mathbb{R})} = \kappa |\hat{\varphi}(\omega, L_2, t)|_{L_\omega^2(\mathbb{R})} \\ &\leq \kappa \sup_{t \in [0, T]} |\hat{g}_u(\omega, t)|_{L_\omega^2(\mathbb{R})} \int_{\frac{L_2}{\sqrt{2\varepsilon t}}}^{\infty} \exp\left(-\frac{s^2}{2} - \lambda^2 \frac{L_2^2}{2s^2}\right) ds \\ &\leq \kappa \sup_{t \in [0, T]} |\tilde{g}_u(x, t)|_{L_x^2(\mathbb{R})} \exp\left(-\frac{L_2^2}{8\varepsilon T}\right) \int_{\frac{L_2}{\sqrt{2\varepsilon t}}}^{\infty} \exp\left(-\frac{s^2}{4} - \lambda^2 \frac{L_2^2}{2s^2}\right) ds \\ &\leq \kappa \sup_{t \in [0, T]} |\gamma(u^0)|_{L_x^2(0, L_1)} \exp\left(-\frac{L_2^2}{8\varepsilon T}\right), \end{aligned}$$

where γ is the trace operator at $y = 0, L_2$. Similarly, we can verify $|\bar{\varphi}_u^2(\cdot, 0, t)|_{L^2(0, L_1)} \leq \kappa \sup_{t \in [0, T]} |\gamma(u^0)|_{L_x^2(0, L_1)}$ hence these trace norms are exponentially small. However, observing that $\bar{\varphi}_u^1 = e.s.t.$ ¹ at $y = L_2$ to handle the *e.s.t.* easily in the analysis below, we introduce a smooth cut-off function $\sigma = \sigma(r)$ such that for $L = \min\{L_1, L_2\}$,

$$(4.17) \quad \sigma = \begin{cases} 1 & \text{for } 0 \leq r \leq \frac{L}{2}, \\ 0 & \text{for } r \geq \frac{2L}{3}. \end{cases}$$

We then define

$$(4.18) \quad \varphi_u^1 = \bar{\varphi}_u^1 \sigma(y), \quad \varphi_u^2 = \bar{\varphi}_u^2 \sigma(L_2 - y), \quad \varphi_u = \varphi_u^1 + \varphi_u^2.$$

More estimates on $\bar{\varphi}_u^1$, $\bar{\varphi}_u^2$ and thus $\bar{\varphi}_u$ will be derived in Sec. 5 below.

¹That is exponentially small terms, i.e. terms exponentially small in all the classical norms, C^r, H^s .

4.2. Ordinary Boundary Layers (OBL) for $v^\varepsilon, \psi^\varepsilon$ at $y = 0, L_2$. As suggested by [12], the ordinary boundary layers $\theta_\zeta, \theta_\chi$ resolve the boundary values of $\zeta^0 = v^0 + N^{-1}\psi^0, \chi^0 = v^0 - N^{-1}\psi^0$ (thus v^0, ψ^0) at $y = 0, L_2$. As in [12], the equations of the correctors θ_ζ and θ_χ , which correct the boundary values of ζ^0 and χ^0 , respectively, at $y = 0$ and $y = L_2$, are as follows:

$$(4.19) \quad \begin{cases} -\frac{N}{\lambda}\theta_{\zeta y} + \varepsilon\lambda^2\theta_\zeta - \varepsilon\theta_{\zeta yy} = -f\bar{\varphi}_u + \frac{N}{\lambda}\bar{\varphi}_{ux} := h, \\ \frac{N}{\lambda}\theta_{\chi y} + \varepsilon\lambda^2\theta_\chi - \varepsilon\theta_{\chi yy} = -f\bar{\varphi}_u - \frac{N}{\lambda}\bar{\varphi}_{ux} := g, \end{cases}$$

where $\bar{\varphi}_u = \bar{\varphi}_u^1 + \bar{\varphi}_u^2$ is the corrector defined by (4.3)-(4.4); (4.19) is supplemented with the boundary conditions, thanks to (3.9),

$$(4.20) \quad \begin{cases} \theta_\zeta(x, 0, t) = -(v^0 + N^{-1}\psi^0)(x, 0, t) \\ \quad = -2v^0(x, 0, t) = -2N^{-1}\psi^0(x, 0, t), \\ \theta_\zeta(x, L_2, t) = 0, \\ \theta_\chi(x, 0, t) = 0, \\ \theta_\chi(x, L_2, t) = -(v^0 - N^{-1}\psi^0)(x, L_2, t) \\ \quad = -2v^0(x, L_2, t) = 2N^{-1}\psi^0(x, L_2, t). \end{cases}$$

We then obtain that the approximate forms $\bar{\theta}_\zeta, \bar{\theta}_\chi$ of $\theta_\zeta, \theta_\chi$, which also satisfy Eq. (4.19), and are respectively

$$(4.21) \quad \bar{\theta}_\zeta = \mathcal{H}_\zeta + \mathcal{S}_\zeta + \mathcal{Z}_\zeta, \quad \bar{\theta}_\chi = \mathcal{H}_\chi + \mathcal{S}_\chi + \mathcal{Z}_\chi,$$

where

$$(4.22) \quad \mathcal{H}_\zeta = -(v^0 + N^{-1}\psi^0)(x, 0, t)e^{ry},$$

$$(4.23) \quad \mathcal{H}_\chi = -(v^0 - N^{-1}\psi^0)(x, L_2, t)e^{r(L_2-y)},$$

$$(4.24) \quad \mathcal{S}_\zeta = -\frac{1}{\varepsilon}e^{-(r+\frac{N}{\lambda\varepsilon})y} \int_{L_2}^y \left(e^{(2r+\frac{N}{\lambda\varepsilon})\eta} \int_0^\eta e^{-r\alpha_y} h(\alpha_y) d\alpha_y \right) d\eta,$$

$$(4.25) \quad \mathcal{S}_\chi = -\frac{1}{\varepsilon}e^{-(r+\frac{N}{\lambda\varepsilon})(L_2-y)} \int_0^y \left(e^{(2r+\frac{N}{\lambda\varepsilon})(L_2-\eta)} \int_{L_2}^\eta e^{-r(L_2-\alpha_y)} g(\alpha_y) d\alpha_y \right) d\eta,$$

$$(4.26) \quad \mathcal{Z}_\zeta = -\mathcal{S}_\zeta(x, 0, t)e^{ry}, \quad \mathcal{Z}_\chi = -\mathcal{S}_\chi(x, L_2, t)e^{r(L_2-y)},$$

and

$$(4.27) \quad r = r_n = -\frac{N}{2\lambda\varepsilon} - \frac{1}{2}\sqrt{\left(\frac{N}{\lambda\varepsilon}\right)^2 + 4\lambda^2}, \quad \lambda = \lambda_n = \frac{n\pi}{L_3}.$$

Here, we note that $\mathcal{H}_\zeta, \mathcal{H}_\chi, \mathcal{Z}_\zeta$ and $\mathcal{Z}_\chi$ are the homogeneous solutions of (4.19) and $\mathcal{S}_\zeta, \mathcal{S}_\chi$ are the particular solutions corresponding to h, g . Note that $\mathcal{Z}_\zeta + \mathcal{S}_\zeta$ and $\mathcal{Z}_\chi + \mathcal{S}_\chi$ satisfy the zero boundary condition.

As in [12], we define the correctors, at $y = 0, L_2$, $\bar{\theta}_v^y$ and $\bar{\theta}_\psi^y$, for v^ε and ψ^ε respectively,

$$(4.28) \quad \bar{\theta}_v^y = \frac{\bar{\theta}_\zeta + \bar{\theta}_\chi}{2}, \quad \bar{\theta}_\psi^y = \frac{(\bar{\theta}_\zeta - \bar{\theta}_\chi)N}{2}.$$

Using the cut-off function σ as in (4.17), we define

$$(4.29a) \quad \theta_\zeta = \bar{\theta}_\zeta \sigma(y), \quad \theta_\chi = \bar{\theta}_\chi \sigma(L_2 - y),$$

and the correctors θ_v^y and θ_ψ^y ,

$$(4.29b) \quad \theta_v^y = \frac{\theta_\zeta + \theta_\chi}{2}, \quad \theta_\psi^y = \frac{(\theta_\zeta - \theta_\chi)N}{2}.$$

Note that, at $y = 0, L_2$, the correctors $\theta_{v_n}^y$ and θ_ψ^y resolve the discrepancy in the boundary values of $v^\varepsilon - v^0$ and $\psi^\varepsilon - \psi^0$ respectively. At $x = 0$, thanks to (4.7), we note that $\mathcal{S}_\zeta = \mathcal{Z}_\zeta = \mathcal{S}_\chi = \mathcal{Z}_\chi = 0$ and thanks to (3.13), $\mathcal{H}_\zeta = \mathcal{H}_\chi = 0$. Thus, $\bar{\theta}_v^y = \bar{\theta}_\psi^y = \theta_v^y = \theta_\psi^y = 0$ at $x = 0$.

By our construction of the boundary layers, we thus corrected the discrepancies between the boundary values of U^ε and U^0 at $y = 0, L_2$. More precisely, thanks to the estimates (4.16), we note that $U^\varepsilon - U^0 - \Theta^y = 0$ at $y = 0, L_2$, $x = 0$, where $\Theta^y = (\varphi_u, \theta_v^y, \theta_\psi^y)$. However, we need to resolve the boundary values of $U^\varepsilon - U^0 - \Theta^y$ at $x = L_1$. To this, we introduce the ordinary boundary layers. Since the boundary conditions are different depending on the modes as in (3.11) and (3.13), we first consider the construction of the boundary layers θ_v^x for the modes $n > n_c$.

To construct θ_v^x , θ_u^x and θ_ψ^x , as in [11], we can identify the leading terms in (3.4) at the boundaries $x = 0, L_1$:

$$(4.30) \quad \begin{cases} \bar{U}_0 u_x^\varepsilon - \frac{1}{\lambda} \psi_x^\varepsilon + \varepsilon \lambda^2 u^\varepsilon - \varepsilon u_{xx}^\varepsilon = 0, \\ \bar{U}_0 v_x^\varepsilon + \varepsilon \lambda^2 v^\varepsilon - \varepsilon v_{xx}^\varepsilon = 0, \\ \bar{U}_0 \psi_x^\varepsilon - \frac{N^2}{\lambda} u_x^\varepsilon + \varepsilon \lambda^2 \psi^\varepsilon - \varepsilon \psi_{xx}^\varepsilon = 0. \end{cases}$$

Using the combinations $\xi^\varepsilon = u^\varepsilon + N^{-1} \psi^\varepsilon$, $\eta^\varepsilon = u^\varepsilon - N^{-1} \psi^\varepsilon$, and $\xi^0 = u^0 + N^{-1} \psi^0$, $\eta^0 = u^0 - N^{-1} \psi^0$ we rewrite (4.30):

$$(4.31) \quad \begin{cases} (\bar{U}_0 - \frac{N}{\lambda}) \xi_x^\varepsilon + \varepsilon \lambda^2 \xi^\varepsilon - \varepsilon \xi_{xx}^\varepsilon = 0, \\ \bar{U}_0 v_x^\varepsilon + \varepsilon \lambda^2 v^\varepsilon - \varepsilon v_{xx}^\varepsilon = 0, \\ (\bar{U}_0 + \frac{N}{\lambda}) \eta_x^\varepsilon + \varepsilon \lambda^2 \eta^\varepsilon - \varepsilon \eta_{xx}^\varepsilon = 0. \end{cases}$$

Accordingly we write the principal corrector equation for $\theta_\xi = \theta_\xi^x, \theta_v = \theta_v^x, \theta_\eta = \theta_\eta^x$ which can correct the boundary values of ξ^0, v^0, η^0 :

$$(4.32) \quad \begin{cases} (\bar{U}_0 - \frac{N}{\lambda}) \theta_{\xi x} + \varepsilon \lambda^2 \theta_\xi - \varepsilon \theta_{\xi xx} = 0, \\ \bar{U}_0 \theta_{vx} + \varepsilon \lambda^2 \theta_v - \varepsilon \theta_{vxx} = 0, \\ (\bar{U}_0 + \frac{N}{\lambda}) \theta_{\eta x} + \varepsilon \lambda^2 \theta_\eta - \varepsilon \theta_{\eta xx} = 0. \end{cases}$$

To impose the boundary conditions for (4.32), we first obtain the general solutions of (4.32) which are

$$(4.33) \quad \theta_\xi = c_5 \exp(r_3^- x) + c_6 \exp(-r_3^+(L_1 - x)),$$

$$(4.34) \quad \theta_\eta = c_7 \exp(r_4^- x) + c_8 \exp(-r_4^+(L_1 - x)),$$

$$(4.35) \quad \theta_v = c_9 \exp(r_5^- x) + c_{10} \exp(-r_5^+(L_1 - x)),$$

where

$$(4.36) \quad r_3^\pm = \frac{1}{2\varepsilon} \left((\bar{U}_0 - \frac{N}{\lambda}) \pm \sqrt{(\bar{U}_0 - \frac{N}{\lambda})^2 + 4\varepsilon^2 \lambda^2} \right),$$

$$(4.37) \quad r_4^\pm = \frac{1}{2\varepsilon} \left((\bar{U}_0 + \frac{N}{\lambda}) \pm \sqrt{(\bar{U}_0 + \frac{N}{\lambda})^2 + 4\varepsilon^2 \lambda^2} \right),$$

$$(4.38) \quad r_5^\pm = \frac{1}{2\varepsilon} \left(\bar{U}_0 \pm \sqrt{\bar{U}_0^2 + 4\varepsilon^2 \lambda^2} \right).$$

Here we note that

$$(4.39) \quad r_4^-, r_5^- = \mathcal{O}(\varepsilon \lambda^2).$$

However, for $r_3^\pm$, depending on the sign of $\bar{U}_0 - N/\lambda$ we note that

$$(4.40) \quad r_3^- = \mathcal{O}(\varepsilon \lambda^2) \text{ if } \bar{U}_0 - \frac{N}{\lambda} > 0, \quad r_3^+ = \mathcal{O}(\varepsilon \lambda^2) \text{ if } \bar{U}_0 - \frac{N}{\lambda} < 0,$$

where $\lambda = \lambda_n = n\pi/L_3$. For supercritical modes, i.e. $\bar{U}_0 - \frac{N}{\lambda} > 0$, we observe that, for all $\lambda = \lambda_n > 0$, there exist some $c_1, c_2 > 0$,

$$(4.41) \quad c_1 \left(\lambda_n + \frac{1}{\varepsilon} \right) \leq r_3^+ < r_5^+ < r_4^+ \leq c_2 \left(\lambda_n + \frac{1}{\varepsilon} \right).$$

Hence, we propose the following approximate forms $\bar{\theta}_\xi, \bar{\theta}_\eta, \bar{\theta}_v$ of $\theta_\xi, \theta_\eta, \theta_v$ as

$$(4.42) \quad \bar{\theta}_\xi = c_6 \exp(-r_3^+(L_1 - x)),$$

$$(4.43) \quad \bar{\theta}_\eta = c_8 \exp(-r_4^+(L_1 - x)),$$

$$(4.44) \quad \bar{\theta}_v = c_{10} \exp(-r_5^+(L_1 - x)),$$

where the constants c_6, c_8, c_{10} will be specified below.

4.3. OBL for v^ε at $x = L_1$. We have constructed θ_v^y so that $v^\varepsilon - v^0 - \theta_v^y = 0$ at $y = 0, L_2$. Thanks to the conditions (3.9), (3.11) and (3.13), we also note that $v^\varepsilon - v^0 - \theta_v^y = 0$ at $x = 0$. Hence we just correct the discrepant boundary values of it at $x = L_1$. For this, we first correct $-v^0(L_1, y, t)$ at $x = L_1$ by the so-called ordinary boundary layer $\theta_v^x = \theta_v$ as in (4.35) and from (4.44) we thus find the corrector $\bar{\theta}_v^x = \bar{\theta}_v$,

$$(4.45) \quad \bar{\theta}_v^x = -v^0(L_1, y, t) e^{-r_5^+(L_1 - x)},$$

and define the corrector θ_v^x

$$(4.46) \quad \theta_v^x = \bar{\theta}_v^x \sigma(L_1 - x).$$

4.4. **OBL for $u^\varepsilon, \psi^\varepsilon$ at $x = L_1$.** Before we proceed, as we did in (4.29) and (4.28), using (4.42) and (4.43) we define the correctors, at $x = 0, L_1$, $\bar{\theta}_u^x$ and $\bar{\theta}_\psi^x$,

$$(4.47) \quad \bar{\theta}_u^x = \frac{\bar{\theta}_\xi + \bar{\theta}_\eta}{2}, \quad \bar{\theta}_\psi^x = \frac{(\bar{\theta}_\xi - \bar{\theta}_\eta)N}{2}.$$

We determine the coefficients c_6, c_8 in $\bar{\theta}_u^x, \bar{\theta}_\psi^x$ to correct $u^\varepsilon - u^0 - \varphi_u$ at $x = 0, L_1$. Note that this quantity is zero at $x = 0, y = 0, L_2$. Hence we just need to correct the boundary value at $x = L_1$. From the explicit expressions of the correctors in (4.42) and (4.43), we obtain the approximate form $\bar{\theta}_u^x$ of θ_u by

$$(4.48) \quad \bar{\theta}_u^x = \frac{1}{2}(\bar{\theta}_\xi + \bar{\theta}_\eta) = \frac{1}{2}(c_6 e^{-r_3^+(L_1-x)} + c_8 e^{-r_4^+(L_1-x)}).$$

We first correct the boundary value of $u^\varepsilon - u^0$ at $x = L_1$ by setting

$$(4.49) \quad c_6 + c_8 = -2u^0(L_1, y, t).$$

Similarly, to correct $\psi^\varepsilon - \psi^0 - \bar{\theta}_\psi^y$ at $x = L_1$, the approximate boundary layers $\bar{\theta}_\psi^x$ of θ_ψ is also found

$$(4.50) \quad \bar{\theta}_\psi^x = \frac{N}{2}(\bar{\theta}_\xi - \bar{\theta}_\eta) = \frac{N}{2}(c_6 e^{-r_3^+(L_1-x)} - c_8 e^{-r_4^+(L_1-x)}).$$

To correct the boundary values of $\psi^\varepsilon - \psi^0$ at $x = L_1$, we set

$$(4.51) \quad c_6 - c_8 = -2N^{-1}\psi^0(L_1, y, t).$$

From (4.49), (4.51) we then easily find c_6, c_8 so that

$$(4.52) \quad \begin{aligned} \bar{\theta}_\xi &= (-u^0 - N^{-1}\psi^0)(L_1, y, t) \exp(-r_3^+(L_1 - x)), \\ \bar{\theta}_\eta &= (-u^0 + N^{-1}\psi^0)(L_1, y, t) \exp(-r_4^+(L_1 - x)). \end{aligned}$$

Writing

$$(4.53a) \quad \theta_\xi = \bar{\theta}_\xi \sigma(L_1 - x), \quad \theta_\eta = \bar{\theta}_\eta \sigma(L_1 - x),$$

we thus define θ_u^x and θ_ψ^x , respectively

$$(4.53b) \quad \theta_u^x = \frac{\theta_\xi + \theta_\eta}{2}, \quad \theta_\psi^x = \frac{(\theta_\xi - \theta_\eta)N}{2}.$$

We have constructed boundary layers to resolve the discrepancies due to the outer solution U^0 along the boundary $\partial\mathcal{M}'$. However, there are further discrepancies due to the boundary layers at the corners which will be discussed in Section 4.5 below. At this point, it is noteworthy to investigate the remaining boundary values as in Table 1.

TABLE 1. Discrepancies along the boundaries $\partial\mathcal{M}'$.

	$x = 0$	$y = 0, L_2$	$x = L_1$
$v^\varepsilon - v^0 - \theta_v^y - \theta_v^x$	0	$-\theta_v^x$	$-\theta_v^y = -\frac{1}{2}(\theta_\zeta + \theta_\chi)$
$u^\varepsilon - u^0 - \varphi_u - \theta_u^x$	0	$-\theta_u^x$	$-\varphi_u$
$\psi^\varepsilon - \psi^0 - \theta_\psi^y - \theta_\psi^x$	0	$-\theta_\psi^x$	$-\theta_\psi^y = -\frac{N}{2}(\theta_\zeta - \theta_\chi)$

Here, no further discrepancies near $(0, 0)$, $(0, L_2)$ arise and hence the corner boundary layers are suppressed at those two corners. This is because $u^0 = v^0 = \psi^0 = 0$ at $(0, 0)$, $(0, L_2)$ indicated from (3.13) for supercritical modes. However, we need to resolve the discrepancies at $(L_1, 0)$, (L_1, L_2) by the so-called *corner boundary layers* which we now introduce.

4.5. Corner Boundary Layers (CBL) for $u^\varepsilon, v^\varepsilon, \psi^\varepsilon$ at $(L_1, 0)$, (L_1, L_2) . As indicated in Table 1 we need to resolve the discrepancies at the corners (see Figure 1) of thickness $\mathcal{O}(\sqrt{\varepsilon})$ and $\mathcal{O}(\varepsilon)$. To resolve all the discrepancies along $\partial\mathcal{M}'$, using in (3.4) the stretched variables with the thickness of $\mathcal{O}(\varepsilon)$ and adding the stiff terms involving $\varepsilon\lambda^2$ we introduce the corner boundary layers $(\rho_u^1, \rho_v^1, \rho_\psi^1)$; they are given as solutions of

$$(4.54) \quad \begin{cases} \bar{U}_0 \rho_{ux}^1 - \frac{1}{\lambda} \rho_{\psi x}^1 + \varepsilon \lambda^2 \rho_u^1 - \varepsilon \rho_{uux}^1 - \varepsilon \rho_{uyy}^1 = -\bar{\rho}_{ut}^1, \\ \bar{U}_0 \rho_{vx}^1 - \frac{1}{\lambda} \rho_{\psi y}^1 + \varepsilon \lambda^2 \rho_v^1 - \varepsilon \rho_{vxx}^1 - \varepsilon \rho_{vyy}^1 = 0, \\ \bar{U}_0 \rho_{\psi x}^1 - \frac{N^2}{\lambda} (\rho_{ux}^1 + \rho_{vy}^1) + \varepsilon \lambda^2 \rho_\psi^1 - \varepsilon \rho_{\psi xx}^1 - \varepsilon \rho_{\psi yy}^1 = 0. \end{cases}$$

Here, $\bar{\rho}_u^1$ is defined in (4.56) below. This term is introduced to regularize Eq. (4.54) and it will be justified in Remark 5.3 after the proof of Lemma 5.4.

The boundary conditions for (4.54) are supplemented below. Then the well-posedness is standard. The correctors $(\rho_u^1, \rho_v^1, \rho_\psi^1)$ will be justified in the convergence error analysis below.

The following boundary conditions for (4.54) are then supplemented

$$(4.55) \quad \rho_u^1|_{\partial\mathcal{M}'} = \bar{\rho}_u^1|_{\partial\mathcal{M}'}, \quad \rho_v^1|_{\partial\mathcal{M}'} = \bar{\rho}_v^1|_{\partial\mathcal{M}'}, \quad \rho_\psi^1|_{\partial\mathcal{M}'} = \bar{\rho}_\psi^1|_{\partial\mathcal{M}'},$$

where

$$(4.56a) \quad \bar{\rho}_u^1 = \bar{\rho}_u^1 \sigma(L_1 - x) \sigma(y),$$

$$(4.56b) \quad \bar{\rho}_v^1 = \bar{\rho}_v^1 \sigma(L_1 - x) \sigma(y),$$

$$(4.56c) \quad \bar{\rho}_\psi^1 = \bar{\rho}_\psi^1 \sigma(L_1 - x) \sigma(y),$$

and

$$(4.56d) \quad \bar{\rho}_u^1 = -e^{-r_3^+(L_1-x)} \bar{\varphi}_u^{+,1}(L_1, y, t) - e^{-r_4^+(L_1-x)} \bar{\varphi}_u^{-,1}(L_1, y, t),$$

$$(4.56e) \quad \bar{\rho}_v^1 = v^0(L_1, 0, t) e^{-r_5^+(L_1-x)} e^{ry} - \frac{1}{2} (\mathcal{S}_\zeta + \mathcal{Z}_\zeta)(L_1, y, t) e^{-\frac{\bar{U}_0}{\varepsilon}(L_1-x)},$$

$$(4.56f) \quad \begin{aligned} \bar{\rho}_\psi^1 &= \frac{N}{2} (u^0 + N^{-1} \psi^0)(L_1, 0, t) e^{-r_3^+(L_1-x)} e^{ry} \\ &\quad - \frac{N}{2} (u^0 - N^{-1} \psi^0)(L_1, 0, t) e^{-r_4^+(L_1-x)} e^{ry} \\ &\quad - \frac{N}{2} (\mathcal{S}_\zeta + \mathcal{Z}_\zeta)(L_1, y, t) e^{-\frac{\bar{U}_0}{\varepsilon}(L_1-x)}. \end{aligned}$$

Here, $\bar{\varphi}_u^{+,1}$ and $\bar{\varphi}_u^{-,1}$ are the parabolic boundary layers given in (4.3) with the boundary conditions at $y = 0$ being replaced by $-\frac{1}{2}(u^0 + N^{-1}\psi^0)(x, 0, t)$ and $-\frac{1}{2}(u^0 - N^{-1}\psi^0)(x, 0, t)$, respectively. Thus, thanks to the superposition of solutions, we note that $\bar{\varphi}_u^{+,1} + \bar{\varphi}_u^{-,1} = \bar{\varphi}_u^1$.

Similarly, we construct the corner boundary layers $(\rho_u^2, \rho_v^2, \rho_\psi^2)$ at (L_1, L_2) as well as $(\bar{\rho}_u^2, \bar{\rho}_v^2, \bar{\rho}_\psi^2)$ and $(\bar{\bar{\rho}}_u^2, \bar{\bar{\rho}}_v^2, \bar{\bar{\rho}}_\psi^2)$. Then we define

$$(4.57) \quad (\rho_u, \rho_v, \rho_\psi) = (\rho_u^1 + \rho_u^2, \rho_v^1 + \rho_v^2, \rho_\psi^1 + \rho_\psi^2).$$

From (4.56) we note the boundary values of the CBLs given in Table 2 where we used the fact that from (3.9) $(v^0 + N^{-1}\psi^0)(L_1, L_2, t) = (v^0 - N^{-1}\psi^0)(L_1, 0, t) = 0$ and that from (4.24)-(4.26) $(\mathcal{S}_\chi + \mathcal{Z}_\chi)(L_1, L_2, t) = (\mathcal{S}_\zeta + \mathcal{Z}_\zeta)(L_1, 0, t) = 0$, and hence, all discrepancies indicated in Table 1 are thus resolved at $y = 0, L_2$ and no further ones occur.

TABLE 2. The boundary values of CBLs $(\rho_u^1, \rho_v^1, \rho_\psi^1)$ along $\partial\mathcal{M}'$.

	$x = 0, y = L_2$	$y = 0$	$x = L_1$
ρ_v^1	0	$\bar{\rho}_v^1(x, 0, t) = -\theta_v^x$	$\bar{\rho}_v^1(L_1, y, t) = -\frac{1}{2}\theta_\zeta$
ρ_u^1	0	$\bar{\rho}_u^1(x, 0, t) = -\theta_u^x$	$(-\bar{\varphi}_u^{+,1} - \bar{\varphi}_u^{-,1})\sigma(y) = -\varphi_u^1$
ρ_ψ^1	0	$\bar{\rho}_\psi^1(x, 0, t) = -\theta_\psi^x$	$\bar{\rho}_\psi^1(L_1, y, t) = -\frac{N}{2}\theta_\zeta$

5. ESTIMATES ON THE BOUNDARY LAYERS

Thanks to the explicit form of the approximate correctors, we easily estimate the correctors in the Sobolev spaces. We reintroduce here the dependence on n , i.e. $\lambda = \lambda_n = n\pi/L_3$. Let us recall that we assume the compatibility conditions (3.3).

We first estimate the parabolic boundary layers $\bar{\varphi}_{un} = \bar{\varphi}_{un}^1 + \bar{\varphi}_{un}^2$.

Lemma 5.1. *Assume that $i, l, m \geq 0$, $0 \leq 2\max\{i, l-1\} + m \leq 2$, $n > n_c$ and $0 \leq t \leq T$. Then there exist constants $\kappa_j, c > 0$, independent of ε and n such that*

$$(5.1) \quad \left| y^j (L_2 - y)^j \frac{\partial^{i+l+m}(\bar{\varphi}_{un}^1, \bar{\varphi}_{un}^2, \bar{\varphi}_{un})}{\partial t^i \partial x^l \partial y^m} \right|_{L^\infty(0,T;\mathbf{L}^2(\mathcal{M}'))} \leq \kappa_j K_{i,l,m}^2(U_n^0) \varepsilon^{-\frac{m}{2}} \left(\lambda_n + \frac{1}{\sqrt{\varepsilon}} \right)^{-j-\frac{1}{2}},$$

where

$$(5.2) \quad \begin{aligned} K_{i,l,m}^2(U_n^0) &= \|\gamma u_n^0\|_{W^{i,\infty}(0,T;H_x^l(0,L_1))} \text{ if } m = 0, \\ K_{i,l,m}^2(U_n^0) &= \sum_{p+q+r=1, p,q,r \geq 0} (\varepsilon \lambda_n^2)^p \|\gamma u_n^0\|_{W^{q,\infty}(0,T;H_x^{l+r}(0,L_1))} \text{ if } m = 1, 2. \end{aligned}$$

Proof. Before we proceed, we recall that the solution $\bar{\varphi}_{un}^1 = \tilde{\varphi}$ is the inverse Fourier transform of $\hat{\varphi}$:

$$(5.3) \quad \hat{\varphi} = \sqrt{\frac{2}{\pi}} \int_{\frac{y}{\sqrt{2\varepsilon t}}}^{\infty} \exp\left(-\frac{s^2}{2} - \lambda_n^2 \frac{y^2}{2s^2} - \bar{U}_0 i \omega \frac{y^2}{2\varepsilon s^2}\right) \hat{g}_u\left(\omega, t - \frac{y^2}{2\varepsilon s^2}\right) ds,$$

where $g_u(x, t) = -u^0(x, 0, t)$, $0 \leq x \leq L_1$ is given. Thanks to the compatibility conditions (4.9), we first note that $\hat{g}_u(\omega, 0) = 0$. Hence differentiating in t and x , we obtain that, for $i = 0, 1$,

$$\begin{aligned}
(5.4) \quad & \left| \frac{\partial^i \bar{\varphi}_{un}^1(\cdot, y, t)}{\partial t^i} \right|_{H_x^l(0, L_1)} \leq \kappa \left| \frac{\partial^i \tilde{\varphi}}{\partial t^i}(\cdot, y, t) \right|_{H_x^l(\mathbb{R})} = \kappa \left| (i\omega)^l \frac{\partial^i \hat{\varphi}}{\partial t^i}(\cdot, y, t) \right|_{H_\omega^l(\mathbb{R})} \\
& \leq \left(-\frac{s^2}{2} - \lambda_n^2 \frac{y^2}{2s^2} = -\frac{3s^2}{8} - \frac{s^2}{8} - \lambda_n^2 \frac{y^2}{2s^2} \leq -\frac{3s^2}{8} - \frac{\lambda_n y}{2} \right) \\
& \leq \kappa \sup_{t \in [0, T]} \left| (i\omega)^l \frac{\partial^i \hat{g}_u}{\partial t^i}(\omega, t) \right|_{L_\omega^2(\mathbb{R})} \exp\left(-\frac{\lambda_n y}{2}\right) \int_{\frac{y}{\sqrt{2\varepsilon}t}}^\infty \exp\left(-\frac{3s^2}{8}\right) ds \\
& \leq (\exp(-3s^2/8) \leq \kappa \exp(-cs), c > 0 \text{ constant}) \\
& \leq \kappa \sup_{t \in [0, T]} \left| (i\omega)^l \frac{\partial^i \hat{g}_u}{\partial t^i}(\omega, t) \right|_{L_\omega^2(\mathbb{R})} \exp\left(-c\left(\lambda_n + \frac{1}{\sqrt{\varepsilon}}\right)y\right) \\
& \leq \kappa \sup_{t \in [0, T]} \left| \frac{\partial^i(\gamma u_n^0)}{\partial t^i}(\cdot, t) \right|_{H_x^l(0, L_1)} \exp\left(-c\left(\lambda_n + \frac{1}{\sqrt{\varepsilon}}\right)y\right),
\end{aligned}$$

which verifies (5.1) for $i = 0, 1$ and $m = 0$ and from (4.12) it holds for $l = 0, 1, 2$.

For $i = 0, m = 2$, we see with (4.3)₁ that $\varepsilon |\bar{\varphi}_{uyy}^1|_{H^1(0, L_1)} \leq \kappa |\bar{\varphi}_u^1|_{H^{l+1}(0, L_1)} + \kappa |\bar{\varphi}_{ut}^1|_{H^1(0, L_1)} + \kappa \varepsilon \lambda_n^2 |\varphi_u^1|_{H^1(0, L_1)}$ and it holds for $l = 0, 1$. From (5.4) we immediately deduce that

$$(5.5) \quad \varepsilon \left| \frac{\partial^2 \bar{\varphi}_{un}^1(\cdot, y, t)}{\partial y^2} \right|_{H_x^l(0, L_1)} \leq \kappa K_{i,l,2}^2(U_n^0) \exp\left(-c\left(\lambda_n + \frac{1}{\sqrt{\varepsilon}}\right)y\right).$$

For $i = 0, m = 1$, since $\hat{g}_u(\omega, 0) = 0$, differentiating (5.3) in y we obtain

$$\begin{aligned}
(5.6) \quad & \frac{\partial \hat{\varphi}}{\partial y} = \sqrt{\frac{2}{\pi}} \int_{\frac{y}{\sqrt{2\varepsilon}t}}^\infty \exp\left(-\frac{s^2}{2} + (-\varepsilon \lambda_n^2 - \bar{U}_0 i\omega) \frac{y^2}{2\varepsilon s^2}\right) \\
& \cdot \left\{ -\varepsilon \lambda_n^2 - \bar{U}_0 i\omega - \frac{\partial}{\partial t} \right\} \hat{g}_u\left(\omega, t - \frac{y^2}{2\varepsilon s^2}\right) \frac{y}{\varepsilon s^2} ds;
\end{aligned}$$

then

$$\begin{aligned}
(5.7) \quad & \left| \frac{\partial \bar{\varphi}_{un}^1(\cdot, y, t)}{\partial y} \right|_{H_x^l(0, L_1)} \leq \kappa \left| \frac{\partial \tilde{\varphi}}{\partial y}(\cdot, y, t) \right|_{H_x^l(\mathbb{R})} = \kappa \left| (i\omega)^l \frac{\partial \hat{\varphi}}{\partial y}(\cdot, y, t) \right|_{H_\omega^l(\mathbb{R})} \\
& \leq \kappa \sup_{t \in [0, T]} \left| \left\{ -\varepsilon \lambda_n^2 - \bar{U}_0 i\omega - \frac{\partial}{\partial t} \right\} \hat{g}_u(\omega, t) \right|_{H_\omega^l(\mathbb{R})} \\
& \quad \times \varepsilon^{-1} \int_{\frac{y}{\sqrt{2\varepsilon}t}}^\infty \exp\left(-\frac{s^2}{2} - \lambda_n^2 \frac{y^2}{2s^2}\right) \frac{y}{s^2} ds \\
& \leq \kappa K_{i,l,1}^2(U_n^0) \exp(-c\lambda_n y) \varepsilon^{-1} \int_{\frac{y}{\sqrt{2\varepsilon}t}}^\infty \exp\left(-\frac{3s^2}{8}\right) \frac{y}{s^2} ds \\
& \leq \kappa K_{i,l,1}^2(U_n^0) \varepsilon^{-\frac{1}{2}} \exp(-c\lambda_n y) \exp\left(-c\frac{y}{\sqrt{\varepsilon}}\right).
\end{aligned}$$

This is valid for $l = 0, 1$. Here we used the following observation: integrating by parts

$$\begin{aligned}
 (5.8) \quad & \int_{\frac{y}{\sqrt{2\varepsilon t}}}^{\infty} \exp\left(-\frac{3s^2}{8}\right) \frac{y}{s^2} ds = -y \int_{\frac{y}{\sqrt{2\varepsilon t}}}^{\infty} \exp\left(-\frac{3s^2}{8}\right) d(s^{-1}) \\
 & = \sqrt{2\varepsilon t} \exp\left(-\frac{3y^2}{16\varepsilon t}\right) - \frac{3y}{4} \int_{\frac{y}{\sqrt{2\varepsilon t}}}^{\infty} \exp\left(-\frac{3s^2}{8}\right) ds \\
 & \leq \kappa\sqrt{\varepsilon} \exp(-c\frac{y}{\sqrt{\varepsilon}}) + \kappa y \exp(-c\frac{y}{\sqrt{\varepsilon}}) \leq \kappa\sqrt{\varepsilon} \exp\left(-c\frac{y}{\sqrt{\varepsilon}}\right).
 \end{aligned}$$

Hence, (5.1) follows using the fact that $|y^j \exp(-\alpha\frac{y}{\varepsilon})|_{L^p(0, L_2)} \leq \kappa(\frac{\varepsilon}{\alpha})^{j+\frac{1}{p}}$ for $\alpha > 0$, $1 \leq p \leq \infty$ (see e.g [11]). $\square$

We next estimate the ordinary boundary layers $\bar{\theta}_{\psi n}^y = (\bar{\theta}_{\zeta n} + \bar{\theta}_{\chi n})/2$, $\bar{\theta}_{\psi n}^y = (\bar{\theta}_{\zeta n} - \bar{\theta}_{\chi n})N/2$. Here, we recall that $\bar{\theta}_{\zeta n} = \mathcal{H}_{\zeta} + \mathcal{S}_{\zeta} + \mathcal{Z}_{\zeta}$, $\bar{\theta}_{\chi n} = \mathcal{H}_{\chi} + \mathcal{S}_{\chi} + \mathcal{Z}_{\chi}$; see (4.20)-(4.27).

Lemma 5.2. *Assume that $i, j, l \geq 0$, $n > n_c$. Then there exist constants $\kappa_j > 0$, independent of ε , n such that for $m \geq 0$,*

$$(5.9) \quad \left| y^j (L_2 - y)^j \frac{\partial^{i+l+m}(\mathcal{H}_{\zeta}, \mathcal{H}_{\chi})}{\partial t^i \partial x^l \partial y^m} \right|_{L^\infty(0, T; \mathbf{L}^2(\mathcal{M}'))} \leq \kappa_j K_{i,l}^1(U_n^0) (-r_n)^{-j+m-\frac{1}{2}},$$

and for $m = 0, 1$, $l = 0, 1$,

$$(5.10) \quad \left| y^j (L_2 - y)^j \frac{\partial^{i+l+m}(\mathcal{S}_{\zeta}, \mathcal{S}_{\chi})}{\partial t^i \partial x^l \partial y^m} \right|_{L^\infty(0, T; \mathbf{L}^2(\mathcal{M}'))} \leq \kappa_j K_{i,l+1}^1(U_n^0) \lambda_n \left(\lambda_n + \frac{1}{\sqrt{\varepsilon}} \right)^{-j+m-\frac{3}{2}},$$

$$\begin{aligned}
 (5.11) \quad & \left| y^j (L_2 - y)^j \frac{\partial^{i+l+m}(\mathcal{Z}_{\zeta}, \mathcal{Z}_{\chi})}{\partial t^i \partial x^l \partial y^m} \right|_{L^\infty(0, T; \mathbf{L}^2(\mathcal{M}'))} \\
 & \leq \kappa_j K_{i,l+1}^1(U_n^0) \lambda_n \left(\lambda_n + \frac{1}{\sqrt{\varepsilon}} \right)^{-1} (-r_n)^{-j+m-\frac{1}{2}},
 \end{aligned}$$

where $\mathcal{H}_{\zeta}, \mathcal{H}_{\chi}$ are as in (4.22), $\mathcal{Z}_{\zeta}, \mathcal{Z}_{\chi}$ as in (4.26), $\mathcal{S}_{\zeta}, \mathcal{S}_{\chi}$ are as in (4.24)-(4.25), and

$$(5.12) \quad K_{i,l}^1(U_n^0) = \|\gamma U_n^0\|_{W^{i,\infty}(0, T; \mathbf{H}_x^l(0, L_1))},$$

$$(5.13) \quad r_n = -\frac{N}{2\lambda_n \varepsilon} - \frac{1}{2} \sqrt{\left(\frac{N}{\lambda_n \varepsilon}\right)^2 + 4\lambda_n^2}.$$

Remark 5.1. Lemmas 5.2 and 5.3 are valid for all $i \geq 0$, but we will need only $i = 0, 1$ in accordance with (3.21).

Proof. Recalling (4.22)-(4.27) it suffices, by symmetry, to estimate the following terms:

$$(5.14) \quad \mathcal{H}_{\zeta} = -(v_n^0 + N^{-1}\psi_n^0)(x, 0, t)e^{r_n y},$$

$$(5.15) \quad \mathcal{S}_{\zeta} = -\frac{1}{\varepsilon} e^{-(r_n + \frac{N}{\lambda_n \varepsilon})y} \int_{L_2}^y \left(e^{(2r_n + \frac{N}{\lambda_n \varepsilon})\eta} \int_0^\eta e^{-r_n \alpha_y} h(\alpha_y) d\alpha_y \right) d\eta,$$

$$(5.16) \quad \mathcal{Z}_{\zeta} = -\mathcal{S}_{\zeta}(x, 0, t)e^{r_n y}.$$

Using $2ab \leq a^2 + b^2$, $a, b \geq 0$, we first note that $\left(\frac{N}{\lambda_n \varepsilon}\right)^2 + 4\lambda_n^2 \geq \frac{2N}{\varepsilon} + 3\lambda_n^2$, and hence

$$(5.17) \quad -r_n = \frac{N}{2\lambda_n \varepsilon} + \frac{1}{2} \sqrt{\left(\frac{N}{\lambda_n \varepsilon}\right)^2 + 4\lambda_n^2} \geq \frac{N}{2\lambda_n \varepsilon} + c_0 \left(\lambda_n + \frac{1}{\sqrt{\varepsilon}}\right),$$

where

$$(5.18) \quad c_0 = (\sqrt{2N} + \sqrt{3})/4.$$

We begin by estimating the first term $\mathcal{H}_\zeta$. We infer from (5.14) that

$$(5.19) \quad \begin{aligned} \left| (y(L_2 - y))^j \frac{\partial^{i+m} \mathcal{H}_\zeta}{\partial t^i \partial y^m} \right|_{H_x^l(0, L_1)} &\leq \kappa K_{i,l}^1(U_n^0) (-r_n)^m (y(L_2 - y))^j \exp(r_n y) \\ &\leq \kappa K_{i,l}^1(U_n^0) (-r_n)^{m-j} (-r_n y)^j \exp(r_n y) \leq \kappa K_{i,l}^1(U_n^0) (-r_n)^{m-j} \exp(cr_n y). \end{aligned}$$

Integrating over y , the estimate for $\mathcal{H}_\zeta$ follows.

Then, we estimate $\mathcal{S}_\zeta$. We first estimate $h(\alpha_y) = h(x, \alpha_y, t)$ and, using (5.4) we find: for $l = 0, 1$,

$$(5.20) \quad \begin{aligned} \left| \frac{\partial^i}{\partial t^i} h(\alpha_y) \right|_{H_x^l(0, L_1)} &= \left| \frac{\partial^i}{\partial t^i} \left(-f \bar{\varphi}_u + \frac{N}{\lambda_n} \bar{\varphi}_{ux} \right) \right|_{H_x^l(0, L_1)} \\ &\leq \kappa K_{i,l+1}^1(U_n^0) e^{-c(\lambda_n + \frac{1}{\sqrt{\varepsilon}})\alpha_y}, \end{aligned}$$

and then estimate $I(\eta) = \int_0^\eta e^{-r_n \alpha_y} h(\alpha_y) d\alpha_y$, $\eta \geq 0$. Hence, we find

$$(5.21) \quad \begin{aligned} \left| \frac{\partial^i}{\partial t^i} I(\eta) \right|_{H_x^l(0, L_1)} &\leq \int_0^\eta e^{-r_n \alpha_y} \left| \frac{\partial^i}{\partial t^i} h(\alpha_y) \right|_{H_x^l(0, L_1)} d\alpha_y \\ &\leq \kappa K_{i,l+1}^1(U_n^0) \int_0^\eta e^{-r_n \alpha_y - c(\lambda_n + \frac{1}{\sqrt{\varepsilon}})\alpha_y} d\alpha_y, \end{aligned}$$

choosing $0 < c \leq c_0$, c_0 as in (5.17), (5.18) so that $-r_n - c(\lambda_n + \frac{1}{\sqrt{\varepsilon}}) \geq \frac{N}{2\lambda_n \varepsilon}$ we obtain

$$(5.22) \quad \left| \frac{\partial^i}{\partial t^i} I(\eta) \right|_{H_x^l(0, L_1)} \leq \kappa K_{i,l+1}^1(U_n^0) \lambda_n \varepsilon e^{-r_n \eta - c(\lambda_n + \frac{1}{\sqrt{\varepsilon}})\eta}.$$

Hence, we find that

$$(5.23) \quad \begin{aligned} \left| \frac{\partial^i}{\partial t^i} \mathcal{S}_\zeta \right|_{H_x^l(0, L_1)} &\leq \frac{1}{\varepsilon} e^{-(r_n + \frac{N}{\lambda_n \varepsilon})y} \int_y^{L_2} e^{(2r_n + \frac{N}{\lambda_n \varepsilon})\eta} \left| \frac{\partial^i}{\partial t^i} I(\eta) \right|_{H_x^l(0, L_1)} d\eta \\ &\leq \kappa K_{i,l+1}^1(U_n^0) \lambda_n \int_y^{L_2} e^{(r_n + \frac{N}{\lambda_n \varepsilon})(\eta - y)} e^{-c(\lambda_n + \frac{1}{\sqrt{\varepsilon}})\eta} d\eta \\ &\leq (\text{from (5.13)}) \leq \kappa K_{i,l+1}^1(U_n^0) \lambda_n \int_y^{L_2} e^{-c(\lambda_n + \frac{1}{\sqrt{\varepsilon}})\eta} d\eta \\ &\leq \kappa K_{i,l+1}^1(U_n^0) \lambda_n \left(\lambda_n + \frac{1}{\sqrt{\varepsilon}} \right)^{-1} e^{-c(\lambda_n + \frac{1}{\sqrt{\varepsilon}})y}, \end{aligned}$$

and thus

$$(5.24) \quad \begin{aligned} & \left| y^j (L_2 - y)^j \frac{\partial^i}{\partial t^i} \mathcal{S}_\zeta \right|_{H_x^l(0, L_1)} \\ & \leq J_1 := \kappa K_{i, l+1}^1(U_n^0) \lambda_n \left(\lambda_n + \frac{1}{\sqrt{\varepsilon}} \right)^{-j-1} e^{-c(\lambda_n + \frac{1}{\sqrt{\varepsilon}})y}. \end{aligned}$$

Differentiating (5.15) in y , we find that

$$(5.25) \quad \mathcal{S}_{\zeta y} = \left(-r_n - \frac{N}{\lambda_n \varepsilon} \right) \mathcal{S}_\zeta - \frac{1}{\varepsilon} e^{r_n y} I(y).$$

Since $0 \leq -r_n - \frac{N}{\lambda_n \varepsilon} = -\frac{N}{2\lambda_n \varepsilon} + \frac{1}{2} \sqrt{\left(\frac{N}{\lambda_n \varepsilon} \right)^2 + 4\lambda_n^2} = \lambda_n^2 \left(\frac{N}{2\lambda_n \varepsilon} + \frac{1}{2} \sqrt{\left(\frac{N}{\lambda_n \varepsilon} \right)^2 + 4\lambda_n^2} \right)^{-1} \leq \lambda_n$, using (5.17), (5.22) - (5.25), we deduce that

$$(5.26) \quad \begin{aligned} & \left| y^j (L_2 - y)^j \frac{\partial^{i+1}}{\partial t^i \partial y} \mathcal{S}_\zeta \right|_{H_x^l(0, L_1)} \\ & \leq \kappa \lambda_n J_1 + \kappa K_{i, l+1}^1(U_n^0) \lambda_n \left(\lambda_n + \frac{1}{\sqrt{\varepsilon}} \right)^{-j} e^{-c(\lambda_n + \frac{1}{\sqrt{\varepsilon}})y} \\ & \leq \kappa K_{i, l+1}^1(U_n^0) \lambda_n \left(\lambda_n + \frac{1}{\sqrt{\varepsilon}} \right)^{-j} e^{-c(\lambda_n + \frac{1}{\sqrt{\varepsilon}})y}. \end{aligned}$$

Hence, taking the L_y^2 -norm in (5.24) and (5.26), we conclude that (5.10) holds for $\mathcal{S}_\zeta$ and $m = 0, 1$.

Using (5.16) and (5.23) with $y = 0$, as we did in (5.19) we finally find that

$$(5.27) \quad \begin{aligned} & \left| y^j (L_2 - y)^j \frac{\partial^{i+l+m} \mathcal{Z}_\zeta}{\partial t^i \partial x^l \partial y^m} \right|_{H_x^l(0, L_1)} \\ & \leq \kappa K_{i, l+1}^1(U_n^0) \lambda_n \left(\lambda_n + \frac{1}{\sqrt{\varepsilon}} \right)^{-1} (-r_n)^{m-j} \exp(cr_n y). \end{aligned}$$

This implies (5.11). The lemma thus follows. $\square$

Remark 5.2. From (5.17) we note that

$$(5.28) \quad c_1 \left(\frac{1}{\lambda_n \varepsilon} + \lambda_n + \frac{1}{\sqrt{\varepsilon}} \right) \leq -r_n \leq c_2 \left(\lambda_n + \frac{1}{\varepsilon} \right), \text{ for some } c_1, c_2 > 0.$$

If $-j + m - \frac{1}{2} \leq 0$, $m = 0, 1$, $l \geq 0$, thanks to (5.28), Lemma 5.2 implies that

$$(5.29) \quad \begin{aligned} & \left| y^j (L_2 - y)^j \frac{\partial^{i+l+m} (\bar{\theta}_v^y, \bar{\theta}_\psi^y)}{\partial t^i \partial x^l \partial y^m} \right|_{L^\infty(0, T; L^2(\mathcal{M}'))} \\ & \leq \kappa K_{i, l}^1(U_n^0) (\lambda_n \varepsilon)^{j-m+\frac{1}{2}} + \kappa K_{i, l+1}^1(U_n^0) \lambda_n \varepsilon^{\frac{j-m}{2} + \frac{3}{4}}. \end{aligned}$$

At the boundaries $x = 0, L_1$, we estimate the ordinary correctors $(\bar{\theta}_{un}^x, \bar{\theta}_{vn}^x, \bar{\theta}_{\psi n}^x)$ as follows.

Lemma 5.3. Assume that $i, l, m, k \geq 0$, $0 \leq t \leq T$, $n > n_c$. Then there exist constants $\kappa_k, c > 0$, independent of ε and n , such that

$$(5.30) \quad \left| x^k (L_1 - x)^k \frac{\partial^{i+l+m}(\bar{\theta}_{un}^x, \bar{\theta}_{vn}^x, \bar{\theta}_{\psi n}^x)}{\partial t^i \partial x^l \partial y^m} \right|_{L^\infty(0,T;\mathbf{L}^2(\mathcal{M}'))} \leq \kappa_k K_{i,m}^3(U_n^0) \left(\lambda_n + \frac{1}{\varepsilon} \right)^{-k+l-\frac{1}{2}},$$

where

$$(5.31) \quad K_{i,m}^3(U_n^0) = \|\tilde{\gamma} U_n^0\|_{W^{i,\infty}(0,T;\mathbf{H}_y^m(0,L_2))},$$

and $\tilde{\gamma}$ is the trace operator at $x = 0, L_1$.

Proof. We infer from the explicit expressions of the boundary layer correctors as in (4.45), (4.47), (4.52) that for $n > n_c$ and for $0 < c < 1$:

$$(5.32) \quad \left| x^k (L_1 - x)^k \frac{\partial^{i+l}(\bar{\theta}_{un}^x, \bar{\theta}_{vn}^x, \bar{\theta}_{\psi n}^x)}{\partial t^i \partial x^l} \right|_{\mathbf{H}_y^m(0,L_2)} \leq \kappa K_{i,m}^3(U_n^0) \left[(r_3^+)^{-k+l} e^{-cr_3^+(L_1-x)} + (r_4^+)^{-k+l} e^{-cr_4^+(L_1-x)} + (r_5^+)^{-k+l} e^{-cr_5^+(L_1-x)} \right].$$

Thanks to (4.41), the lemma follows. $\square$

Lemma 5.4. Let $\Upsilon_n = (\rho_{un}, \rho_{vn}, \rho_{\psi n})$, $n > n_c$, be the solutions of Eq. (4.54). Then there exists a constant $\kappa > 0$, independent of ε and n , such that for $i = 0, 1$,

$$(5.33) \quad |\Upsilon_n|_{W^{i,\infty}(0,T;\mathbf{L}^2(\mathcal{M}'))}, |\tilde{\Upsilon}_n|_{W^{i,\infty}(0,T;\mathbf{L}^2(\mathcal{M}'))} \leq \kappa K_i^4(U_n^0) \varepsilon^{\frac{3}{4}},$$

$$(5.34) \quad |\Upsilon_n - \tilde{\Upsilon}_n|_{W^{i,\infty}(0,T;\mathbf{H}^1(\mathcal{M}'))} \leq \kappa K_i^4(U_n^0) \varepsilon^{-\frac{1}{4}},$$

where $\tilde{\Upsilon}_n = (\bar{\rho}_{un}, \bar{\rho}_{vn}, \bar{\rho}_{\psi n})$ is given in (4.56) and

$$(5.35) \quad K_i^4(U_n^0) = \|U_n^0\|_{W^{i,\infty}(0,T;\mathbf{H}^3(\mathcal{M}'))}.$$

Proof. By symmetry, it suffices to estimate $\Upsilon_n^1 = (\rho_{un}^1, \rho_{vn}^1, \rho_{\psi n}^1)$, the solutions of Eq. (4.54). Let $\tilde{\Upsilon}_n = \tilde{\Upsilon}_n^1 = (\tilde{\rho}_{un}^1, \tilde{\rho}_{vn}^1, \tilde{\rho}_{\psi n}^1) = (\rho_{un}^1 - \bar{\rho}_{un}^1, \rho_{vn}^1 - \bar{\rho}_{vn}^1, \rho_{\psi n}^1 - \bar{\rho}_{\psi n}^1)$ where $\tilde{\Upsilon}_n^1 = (\bar{\rho}_{un}^1, \bar{\rho}_{vn}^1, \bar{\rho}_{\psi n}^1)$ is defined in (4.56). We then have

$$(5.36) \quad \left\{ \begin{array}{l} \bar{U}_0 \tilde{\rho}_{un}^1 - \frac{1}{\lambda_n} \tilde{\rho}_{\psi n}^1 + \varepsilon \lambda_n^2 \tilde{\rho}_{un}^1 - \varepsilon \tilde{\rho}_{un}^1 - \varepsilon \tilde{\rho}_{un}^1 \\ = \mathcal{R}_u(\bar{\rho}^1) := -(\bar{\rho}_{un}^1 + \bar{U}_0 \bar{\rho}_{un}^1 - \frac{1}{\lambda_n} \bar{\rho}_{\psi n}^1 + \varepsilon \lambda_n^2 \bar{\rho}_{un}^1 - \varepsilon \bar{\rho}_{un}^1 - \varepsilon \bar{\rho}_{un}^1), \\ \bar{U}_0 \tilde{\rho}_{vn}^1 - \frac{1}{\lambda_n} \tilde{\rho}_{\psi n}^1 + \varepsilon \lambda_n^2 \tilde{\rho}_{vn}^1 - \varepsilon \tilde{\rho}_{vn}^1 - \varepsilon \tilde{\rho}_{vn}^1 \\ = \mathcal{R}_v(\bar{\rho}^1) := -(\bar{U}_0 \bar{\rho}_{vn}^1 - \frac{1}{\lambda_n} \bar{\rho}_{\psi n}^1 + \varepsilon \lambda_n^2 \bar{\rho}_{vn}^1 - \varepsilon \bar{\rho}_{vn}^1 - \varepsilon \bar{\rho}_{vn}^1), \\ \bar{U}_0 \tilde{\rho}_{\psi n}^1 - \frac{N^2}{\lambda_n} (\tilde{\rho}_{un}^1 + \tilde{\rho}_{vn}^1) + \varepsilon \lambda_n^2 \tilde{\rho}_{\psi n}^1 - \varepsilon \tilde{\rho}_{\psi n}^1 - \varepsilon \tilde{\rho}_{\psi n}^1 \\ = \mathcal{R}_\psi(\bar{\rho}^1) := -(\bar{U}_0 \bar{\rho}_{\psi n}^1 - \frac{N^2}{\lambda_n} (\bar{\rho}_{un}^1 + \bar{\rho}_{vn}^1) + \varepsilon \lambda_n^2 \bar{\rho}_{\psi n}^1 - \varepsilon \bar{\rho}_{\psi n}^1 - \varepsilon \bar{\rho}_{\psi n}^1). \end{array} \right.$$

We first estimate $\tilde{\Upsilon}_n^1$. For this, we need to estimate $(\mathcal{R}_u(\bar{\rho}^1), \mathcal{R}_v(\bar{\rho}^1), \mathcal{R}_\psi(\bar{\rho}^1))$. From the relationship (4.56) we first estimate $(\mathcal{R}_u(\bar{\rho}^1), \mathcal{R}_v(\bar{\rho}^1), \mathcal{R}_\psi(\bar{\rho}^1))$ which are the right-hand side of (5.36) with $(\bar{\rho}_u^1, \bar{\rho}_v^1, \bar{\rho}_\psi^1)$ being replaced by $(\bar{\rho}_u^1, \bar{\rho}_v^1, \bar{\rho}_\psi^1)$.

From (4.3) and (4.4) for $\bar{\varphi}_u^{\pm,1}$ recalling (4.56) for $\bar{\rho}_u^1$ we find that

$$(5.37) \quad \begin{aligned} \mathcal{R}_u(\bar{\rho}^1) = & -(\bar{U}_0 \bar{\rho}_{unx}^1 - \frac{1}{\lambda_n} \bar{\rho}_{\psi nx}^1 - \varepsilon \bar{\rho}_{unxx}^1) \\ & + \bar{U}_0(-e^{-r_3^+(L_1-x)} \bar{\varphi}_{unx}^{+,1}(L_1, y, t) - e^{-r_4^+(L_1-x)} \bar{\varphi}_{unx}^{-,1}(L_1, y, t)). \end{aligned}$$

We note that the estimates as in Lemma 5.1 for $\bar{\varphi}_{unx}^{\pm,1}(L_1, y, t)$ also hold. Letting then

$$(5.38) \quad \Psi(x, y) = \exp\left(-c(\lambda_n + \frac{1}{\varepsilon})(L_1 - x)\right) \exp\left(-c(\lambda_n + \frac{1}{\sqrt{\varepsilon}})y\right), \text{ some } c > 0,$$

we estimate $\mathcal{R}_u(\bar{\rho}^1)$. Thanks to (4.41), (5.28) and (5.7), we find that

$$(5.39) \quad |\partial_t^i \mathcal{R}_u(\bar{\rho}^1)| \leq \kappa K_i^4(U_n^0)(1 + \varepsilon \lambda_n)(\lambda_n + \frac{1}{\varepsilon})\Psi(x, y).$$

Thanks to the compact support of the cut-off functions $\sigma(L_1 - x), \sigma(y)$, after elementary computations, we find that

$$(5.40) \quad |\partial_t^i (\mathcal{R}_u(\bar{\rho}^1) - \mathcal{R}_u(\bar{\rho}^1)\sigma(L_1 - x)\sigma(y))| \leq \kappa K_i^4(U_n^0)(\lambda_n + \frac{1}{\varepsilon}) \exp\left(-\frac{c}{\sqrt{\varepsilon}}\right),$$

which is exponentially small, and thus, we have

$$(5.41) \quad |\partial_t^i \mathcal{R}_u(\bar{\rho}^1)| \leq \kappa K_i^4(U_n^0)(1 + \varepsilon \lambda_n)(\lambda_n + \frac{1}{\varepsilon})\Psi(x, y), \quad i = 0, 1.$$

Using (4.19) we note that $(-\frac{N}{\lambda} \partial_y + \varepsilon \lambda^2 - \varepsilon \partial_y^2)e^{ry} = 0$ and $(-\frac{N}{\lambda} \partial_y + \varepsilon \lambda^2 - \varepsilon \partial_y^2)(\mathcal{S}_\zeta + \mathcal{Z}_\zeta)(L_1, y, t) = h(L_1, y, t) = (-f \bar{\varphi}_u + \frac{N}{\lambda} \bar{\varphi}_{ux})(L_1, y, t)$. The terms with $\varepsilon \lambda_n^2$ are then absorbed. More precisely, recalling (4.56) for $\bar{\rho}_v^1$ and $\bar{\rho}_\psi^1$ we find that

$$(5.42) \quad \begin{cases} \mathcal{R}_v(\bar{\rho}^1) = -(\bar{U}_0 \bar{\rho}_{vnx}^1 - \frac{1}{\lambda_n} \bar{\rho}_{\psi ny}^1 - \varepsilon \bar{\rho}_{vnxx}^1) \\ \quad - \frac{N}{\lambda} \bar{\rho}_{vy}^1 + \frac{1}{2} h(L_1, y, t) e^{-\frac{\bar{U}_0}{\varepsilon}(L_1-x)}, \\ \mathcal{R}_\psi(\bar{\rho}^1) = -(\bar{U}_0 \bar{\rho}_{\psi nx}^1 - \frac{N^2}{\lambda_n} (\bar{\rho}_{unx}^1 + \bar{\rho}_{vny}^1) - \varepsilon \bar{\rho}_{\psi nxx}^1) \\ \quad - \frac{N}{\lambda} \bar{\rho}_{\psi y}^1 + \frac{N}{2} h(L_1, y, t) e^{-\frac{\bar{U}_0}{\varepsilon}(L_1-x)}. \end{cases}$$

Thus, using (4.41), (5.28), (5.23) and (5.27), the terms $\mathcal{R}_v(\bar{\rho}^1), \mathcal{R}_\psi(\bar{\rho}^1)$ as well as $\mathcal{R}_v(\bar{\rho}^1), \mathcal{R}_\psi(\bar{\rho}^1)$ are similarly estimated. Letting $\mathcal{R}(\bar{\rho}) = (\mathcal{R}_u(\bar{\rho}), \mathcal{R}_v(\bar{\rho}), \mathcal{R}_\psi(\bar{\rho})) = (\mathcal{R}_u(\bar{\rho}^1), \mathcal{R}_v(\bar{\rho}^1), \mathcal{R}_\psi(\bar{\rho}^1))$ we thus deduce the same bound as (5.41) for $\mathcal{R}(\bar{\rho})$.

We are then ready to estimate $\tilde{\Upsilon}_n^1 = (\tilde{\rho}_{un}^1, \tilde{\rho}_{vn}^1, \tilde{\rho}_{\psi n}^1)$, the solutions of (5.36). For simplicity in the notations, we write $(\tilde{\rho}_{un}^1, \tilde{\rho}_{vn}^1, \tilde{\rho}_{\psi n}^1) = (\rho_u, \rho_v, \rho_\psi)$ and rewrite (5.36) as

$$(5.43) \quad \begin{cases} \bar{U}_0 \rho_{ux} - \frac{1}{\lambda_n} \rho_{\psi x} + \varepsilon \lambda_n^2 \rho_u - \varepsilon \rho_{uux} - \varepsilon \rho_{uyy} = \mathcal{R}_u(\bar{\rho}), \\ \bar{U}_0 \rho_{vx} - \frac{1}{\lambda_n} \rho_{\psi y} + \varepsilon \lambda_n^2 \rho_v - \varepsilon \rho_{vxx} - \varepsilon \rho_{vyy} = \mathcal{R}_v(\bar{\rho}), \\ \bar{U}_0 \rho_{\psi x} - \frac{N^2}{\lambda_n} (\rho_{ux} + \rho_{vy}) + \varepsilon \lambda_n^2 \rho_\psi - \varepsilon \rho_{\psi xx} - \varepsilon \rho_{\psi yy} = \mathcal{R}_\psi(\bar{\rho}). \end{cases}$$

Multiplying (5.43)_{1,2,3} by $(L_1 - x)(\rho_u, \rho_v, N^{-2}\rho_\psi)$ and integrating over $\mathcal{M}'$ from (4.56) since $(\rho_u, \rho_v, \rho_\psi) = 0$ on $\partial\mathcal{M}'$ we find that

$$\begin{aligned}
 & \frac{\bar{U}_0}{2} \int_{\mathcal{M}'} (|\rho_u|^2 + |\rho_v|^2 + N^{-2}|\rho_\psi|^2) dx dy \\
 (5.44) \quad & + \varepsilon \int_{\mathcal{M}'} (L_1 - x)(|\nabla \rho_u|^2 + |\nabla \rho_v|^2 + N^{-2}|\nabla \rho_\psi|^2) dx dy \\
 & \leq \frac{1}{\lambda_n} \int_{\mathcal{M}'} \rho_u \rho_\psi dx dy + \kappa \int_{\mathcal{M}'} (L_1 - x) |\mathcal{R}(\bar{\rho})| (|\rho_u| + |\rho_v| + N^{-2}|\rho_\psi|) dx dy.
 \end{aligned}$$

Since $\frac{1}{\lambda_n} \rho_u \rho_\psi = \frac{N}{\lambda_n} \rho_u N^{-1} \rho_\psi \leq \frac{N}{2\lambda_n} (|\rho_u|^2 + N^{-2}|\rho_\psi|^2)$, writing $\Phi = (\rho_u, \rho_v, N^{-1}\rho_\psi)$ we find that

$$\begin{aligned}
 (5.45) \quad & \left(\frac{\bar{U}_0}{2} - \frac{N}{2\lambda_n} \right) \int_{\mathcal{M}'} |\Phi|^2 dx dy + \varepsilon \int_{\mathcal{M}'} (L_1 - x) |\nabla \Phi|^2 dx dy \\
 & \leq \kappa \int_{\mathcal{M}'} (L_1 - x) |\mathcal{R}(\bar{\rho})| |\Phi| dx dy.
 \end{aligned}$$

For the supercritical modes, observing that $\bar{U}_0 - N/\lambda_n > 0$, we hence obtain

$$\begin{aligned}
 (5.46) \quad & \left(\bar{U}_0 - \frac{N}{\lambda_n} \right) \int_{\mathcal{M}'} |\Phi|^2 dx dy \leq \kappa \int_{\mathcal{M}'} (L_1 - x)^2 |\mathcal{R}(\bar{\rho})|^2 dx dy \\
 & \leq \kappa (K_0^4(U_n^0)(1 + \varepsilon\lambda_n))^2 \int_{\mathcal{M}'} (L_1 - x)^2 \left(\left(\lambda_n + \frac{1}{\varepsilon} \right) \Psi(x, y) \right)^2 dx dy \\
 & \leq \kappa (K_0^4(U_n^0)(1 + \varepsilon\lambda_n))^2 \left(\lambda_n + \frac{1}{\varepsilon} \right)^{-1} \left(\lambda_n + \frac{1}{\sqrt{\varepsilon}} \right)^{-1} \\
 & \leq \kappa (K_0^4(U_n^0))^2 \varepsilon^{\frac{3}{2}}.
 \end{aligned}$$

We now estimate $\bar{\Upsilon}_n = (\bar{\rho}_{un}, \bar{\rho}_{vn}, \bar{\rho}_{\psi n})$ is given in (4.56) and thus $(\bar{\rho}_{un}, \bar{\rho}_{vn}, \bar{\rho}_{\psi n})$. For this, we just observe that

$$(5.47) \quad \left| \left(e^{-r_3^+(L_1-x)}, e^{-r_4^+(L_1-x)}, e^{-r_5^+(L_1-x)}, e^{-\frac{\bar{U}_0}{\varepsilon}(L_1-x)} \right) \right|_{L^2(0, L_1)} \leq \kappa \varepsilon^{\frac{1}{2}}, \quad |e^{ry}|_{L^2(0, L_2)} \leq \kappa \varepsilon^{\frac{1}{4}},$$

from Lemma 5.1,

$$(5.48) \quad |\partial_t^i \bar{\varphi}_u^{\pm, 1}(L_1, y, t)|_{L_y^2(0, L_2)} \leq \kappa |\partial_t^i \partial_x (\bar{\varphi}_u^{\pm, 1})|_{L^2(\mathcal{M}')} \leq \kappa K_{i, 1, 0}^2(U_n^0) \varepsilon^{\frac{1}{4}},$$

and from Lemma 5.2,

$$(5.49) \quad |\partial_t^i (\mathcal{S}_\zeta + \mathcal{Z}_\zeta)(L_1, y, t)|_{L_y^2(0, L_2)} \leq \kappa |\partial_t^i \partial_x (\mathcal{S}_\zeta + \mathcal{Z}_\zeta)|_{L^2(\mathcal{M}')} \leq \kappa K_{i, 2}^1(U_n^0) \varepsilon^{\frac{1}{4}}.$$

This implies $|\bar{\Upsilon}_n|_{W^{i, \infty}(0, T; \mathbf{L}^2(\mathcal{M}'))} \leq \kappa K_i^4(U_n^0) \varepsilon^{\frac{3}{4}}$ and hence (5.46) implies (5.33) for $i = 0$.

Multiplying (5.43)_{1,2,3} by Φ , integrating over $\mathcal{M}'$, and applying the Hardy inequality we then find that

$$\begin{aligned}
 (5.50) \quad & \varepsilon \int_{\mathcal{M}'} |\nabla \Phi|^2 dx dy \leq \kappa \int_{\mathcal{M}'} |\mathcal{R}(\bar{\rho})| |\Phi| dx dy \\
 & \leq \kappa \varepsilon^{-1} \int_{\mathcal{M}'} (L_1 - x)^2 |\mathcal{R}(\bar{\rho})|^2 dx dy + \frac{\varepsilon}{2} \int_{\mathcal{M}'} |\nabla \Phi|^2 dx dy.
 \end{aligned}$$

This implies (5.34) for $i = 0$. The estimates for the time derivative $\tilde{\Upsilon}_{nt}^1$ similarly follow; it suffices to differentiate (5.36) in time and carry out similar computations as above. $\square$

Remark 5.3. Thanks to the term $-\bar{\rho}_{unt}^1$ in (4.54), as in (5.37) we can avoid the differentiation of $\varphi^{\pm,1}$ in time. Hence, differentiating (5.36) in time its right-hand side is differentiated only once in time too, and hence we achieve the estimates for $i = 1$ as in Lemma 5.4.

6. GLOBAL CONVERGENCE ANALYSIS FOR THE MODES $n > n_c$

In this section, we will confirm our choice for the correctors previously introduced and validate the approximations of the solutions of (2.1). Using the modal expansions and the boundary conditions chosen for $U^{0,2}$ we may write the global corrector in supercritical modes as

$$(6.1) \quad \bar{\Theta} = \begin{pmatrix} \bar{\vartheta}_u \\ \bar{\vartheta}_v \\ \bar{\vartheta}_\psi \end{pmatrix} = \sum_{n=n_c+1}^{\infty} \begin{pmatrix} \bar{\vartheta}_{un}\mathcal{U}_n \\ \bar{\vartheta}_{vn}\mathcal{U}_n \\ \bar{\vartheta}_{\psi n}\mathcal{W}_n \end{pmatrix} := \sum_{n=n_c+1}^{\infty} \begin{pmatrix} (\bar{\varphi}_{un} + \bar{\theta}_{un}^x + \rho_{un})\mathcal{U}_n \\ (\bar{\theta}_{vn}^y + \bar{\theta}_{vn}^x + \rho_{vn})\mathcal{U}_n \\ (\bar{\theta}_{\psi n}^y + \bar{\theta}_{\psi n}^x + \rho_{\psi n})\mathcal{W}_n \end{pmatrix},$$

and

$$(6.2) \quad \Theta = \begin{pmatrix} \vartheta_u \\ \vartheta_v \\ \vartheta_\psi \end{pmatrix} = \sum_{n=n_c+1}^{\infty} \begin{pmatrix} \vartheta_{un}\mathcal{U}_n \\ \vartheta_{vn}\mathcal{U}_n \\ \vartheta_{\psi n}\mathcal{W}_n \end{pmatrix} := \sum_{n=n_c+1}^{\infty} \begin{pmatrix} (\varphi_{un} + \theta_{un}^x + \rho_{un})\mathcal{U}_n \\ (\theta_{vn}^y + \theta_{vn}^x + \rho_{vn})\mathcal{U}_n \\ (\theta_{\psi n}^y + \theta_{\psi n}^x + \rho_{\psi n})\mathcal{W}_n \end{pmatrix},$$

Here in (6.2), we collected all the correctors which resolve the boundary values between U^ε and U^0 on $\partial\mathcal{M}'$.

From (4.32) and (4.19) we first notice that, for all $n \geq 1$,

$$(6.3) \quad \begin{cases} \bar{U}_0 \bar{\theta}_{unx}^x - \lambda_n^{-1} \bar{\theta}_{\psi nx}^x + \varepsilon \lambda_n^2 \bar{\theta}_{un}^x - \varepsilon \bar{\theta}_{unxx}^x = 0, \\ \bar{U}_0 \bar{\theta}_{\psi nx}^x - N^2 \lambda_n^{-1} \bar{\theta}_{unx}^x + \varepsilon \lambda_n^2 \bar{\theta}_{\psi n}^x - \varepsilon \bar{\theta}_{\psi nxx}^x = 0, \\ \bar{U}_0 \bar{\theta}_{vnx}^x + \varepsilon \lambda_n^2 \bar{\theta}_{vn}^x - \varepsilon \bar{\theta}_{vnxx}^x = 0, \end{cases}$$

$$(6.4) \quad \begin{cases} -\lambda_n^{-1} \bar{\theta}_{\psi ny}^y + \varepsilon \lambda_n^2 \bar{\theta}_{vn}^y - \varepsilon \bar{\theta}_{vnny}^y = -f \bar{\varphi}_{un}, \\ -N^2 \lambda_n^{-1} \bar{\theta}_{vny}^y + \varepsilon \lambda_n^2 \bar{\theta}_{\psi n}^y - \varepsilon \bar{\theta}_{\psi nny}^y = N^2 \lambda_n^{-1} \bar{\varphi}_{unx}. \end{cases}$$

We now derive the equations for the quantities $(\bar{\vartheta}_u, \bar{\vartheta}_v, \bar{\vartheta}_\psi)$, introduced in the definition of the global corrector (6.1):

$$(6.5a) \quad \left\{ \begin{aligned} & \bar{\vartheta}_{ut} + \bar{U}_0 \bar{\vartheta}_{ux} - \varepsilon \bar{\vartheta}_{uzz} - \varepsilon \bar{\vartheta}_{uyy} - \varepsilon \bar{\vartheta}_{uxx} \\ &= \sum_{n=n_c+1}^{\infty} (\bar{\varphi}_{unt} + \bar{U}_0 \bar{\varphi}_{unx} + \varepsilon \lambda_n^2 \bar{\varphi}_{un} - \varepsilon \bar{\varphi}_{unyy} - \varepsilon \bar{\varphi}_{unxx}) \mathcal{U}_n + \\ & \quad + \sum_{n=n_c+1}^{\infty} (\bar{\theta}_{unt}^x + \bar{U}_0 \bar{\theta}_{unx}^x + \varepsilon \lambda_n^2 \bar{\theta}_{un}^x - \varepsilon \bar{\theta}_{unyy}^x - \varepsilon \bar{\theta}_{unxx}^x) \mathcal{U}_n + \\ & \quad + \sum_{n=n_c+1}^{\infty} (\rho_{unt} + \bar{U}_0 \rho_{unx} + \varepsilon \lambda_n^2 \rho_{un} - \varepsilon \rho_{unyy} - \varepsilon \rho_{unxx}) \mathcal{U}_n \\ &= (\text{using (4.3), (4.4), (4.54) and (6.3)}_1) \\ &= -\varepsilon \sum_{n=n_c+1}^{\infty} \bar{\varphi}_{unxx} \mathcal{U}_n + \\ & \quad + \sum_{n=n_c+1}^{\infty} (\bar{\theta}_{unt}^x - \varepsilon \bar{\theta}_{unyy}^x + \lambda_n^{-1} \bar{\theta}_{\psi nx}^x) \mathcal{U}_n + \\ & \quad + \sum_{n=n_c+1}^{\infty} (\rho_{unt} + \lambda_n^{-1} \rho_{\psi nx} - \rho_{unt}) \mathcal{U}_n \\ &:= R_1, \end{aligned} \right.$$

²However, we still assume the existence of U^0 satisfying (3.21).

$$(6.5b) \quad \left\{ \begin{array}{l} \bar{U}_0 \bar{\vartheta}_{vx} - \varepsilon \bar{\vartheta}_{vzz} - \varepsilon \bar{\vartheta}_{vyy} - \varepsilon \bar{\vartheta}_{vxx} \\ = \sum_{n=n_c+1}^{\infty} (\bar{U}_0 \bar{\theta}_{vnx}^y + \varepsilon \lambda_n^2 \bar{\theta}_{vn}^y - \varepsilon \bar{\theta}_{vnyy}^y - \varepsilon \bar{\theta}_{vnxx}^y) \mathcal{U}_n + \\ + \sum_{n=n_c+1}^{\infty} (\bar{U}_0 \bar{\theta}_{vnx}^x + \varepsilon \lambda_n^2 \bar{\theta}_{vn}^x - \varepsilon \bar{\theta}_{vnyy}^x - \varepsilon \bar{\theta}_{vnxx}^x) \mathcal{U}_n + \\ + \sum_{n=n_c+1}^{\infty} (\bar{U}_0 \rho_{vnx} + \varepsilon \lambda_n^2 \rho_{vn} - \varepsilon \rho_{vnyy} - \varepsilon \rho_{vnxx}) \mathcal{U}_n \\ = (\text{thanks to (6.3)}_3, (4.54) \text{ and (6.4)}_1) \\ = \sum_{n=n_c+1}^{\infty} (\bar{U}_0 \bar{\theta}_{vnx}^y - f \bar{\varphi}_{un} + \lambda_n^{-1} \bar{\theta}_{\psi ny}^y - \varepsilon \bar{\theta}_{vnxx}^y) \mathcal{U}_n + \\ + \sum_{n=n_c+1}^{\infty} (-\varepsilon \bar{\theta}_{vnyy}^x) \mathcal{U}_n + \\ + \sum_{n=n_c+1}^{\infty} (\lambda_n^{-1} \rho_{\psi ny}) \mathcal{U}_n \\ := R_2, \end{array} \right.$$

$$(6.5c) \quad \left\{ \begin{array}{l} \bar{U}_0 \bar{\vartheta}_{\psi x} - \varepsilon \bar{\vartheta}_{\psi zz} - \varepsilon \bar{\vartheta}_{\psi yy} - \varepsilon \bar{\vartheta}_{\psi xx} \\ = \sum_{n=n_c+1}^{\infty} (\bar{U}_0 \bar{\theta}_{\psi nx}^y + \varepsilon \lambda_n^2 \bar{\theta}_{\psi n}^y - \varepsilon \bar{\theta}_{\psi nyy}^y - \varepsilon \bar{\theta}_{\psi nxx}^y) \mathcal{W}_n + \\ + \sum_{n=n_c+1}^{\infty} (\bar{U}_0 \bar{\theta}_{\psi nx}^x + \varepsilon \lambda_n^2 \bar{\theta}_{\psi n}^x - \varepsilon \bar{\theta}_{\psi nyy}^x - \varepsilon \bar{\theta}_{\psi nxx}^x) \mathcal{W}_n + \\ + \sum_{n=n_c+1}^{\infty} (\bar{U}_0 \rho_{\psi nx} + \varepsilon \lambda_n^2 \rho_{\psi n} - \varepsilon \rho_{\psi nyy} - \varepsilon \rho_{\psi nxx}) \mathcal{W}_n \\ = (\text{using (6.3)}_2, (4.54) \text{ and (6.4)}_2) \\ = \sum_{n=n_c+1}^{\infty} (\bar{U}_0 \bar{\theta}_{\psi nx}^y + N^2 \lambda_n^{-1} \bar{\theta}_{vny}^y + N^2 \lambda_n^{-1} \bar{\varphi}_{unx} - \varepsilon \bar{\theta}_{\psi nxx}^y) \mathcal{W}_n + \\ + \sum_{n=n_c+1}^{\infty} (-\varepsilon \bar{\theta}_{\psi nyy}^x + N^2 \lambda_n^{-1} \bar{\theta}_{unx}^x) \mathcal{W}_n + \\ + \sum_{n=n_c+1}^{\infty} (N^2 \lambda_n^{-1} (\rho_{unx} + \rho_{vny})) \mathcal{W}_n \\ := R_3. \end{array} \right.$$

6.1. Proof of the main Theorem 3.1. We estimate the asymptotic error $\bar{W} = (\bar{W}_1, \bar{W}_2, \bar{W}_3) := U^\varepsilon - U^0 - \bar{\Theta}$. For that purpose, we define the auxiliary correctors $(\bar{\vartheta}_\phi, \bar{\vartheta}_w)$,

$$(6.6) \quad \left\{ \begin{array}{l} \bar{\vartheta}_\phi = \sum_{n=n_c+1}^{\infty} (\rho_{\phi n} + \bar{\theta}_{\phi n}) \mathcal{U}_n, \text{ where } \rho_{\phi n} = -\lambda_n^{-1} \rho_{\psi n}, \text{ and} \\ \bar{\theta}_{\phi n} = -\lambda_n^{-1} (\bar{\theta}_{\psi n}^y + \bar{\theta}_{\psi n}^x) \text{ as in (6.1),} \\ \bar{\vartheta}_w = \sum_{n=n_c+1}^{\infty} (\rho_{wn} + \bar{\theta}_{wn}) \mathcal{W}_n, \text{ where } \rho_{wn} = -\lambda_n^{-1} (\rho_{unx} + \rho_{vny}), \text{ and} \\ \bar{\theta}_{wn} = -\lambda_n^{-1} ((\bar{\varphi}_{un} + \bar{\theta}_{un}^x)_x + (\bar{\theta}_{vn}^y + \bar{\theta}_{vn}^x)_y) \text{ as in (6.1).} \end{array} \right.$$

We then note that

$$(6.7) \quad \bar{\vartheta}_{\phi z} - \bar{\vartheta}_\psi = \bar{\vartheta}_{ux} + \bar{\vartheta}_{vy} + \bar{\vartheta}_{wz} = 0.$$

Using the auxiliary correctors (6.6), (6.5a)-(6.5c), we obtain from (2.1) and (2.7) the equation for $\bar{W} = (\bar{W}_1, \bar{W}_2, \bar{W}_3) = U^\varepsilon - U^0 - \bar{\Theta}$, namely:

$$(6.8) \quad \mathcal{L}(\bar{W}) = \left\{ \begin{array}{l} \bar{W}_{1t} + \bar{U}_0 \bar{W}_{1x} - f \bar{W}_2 + (\phi^\varepsilon - \phi^0 - \bar{\vartheta}_\phi)_x - \varepsilon \Delta_3 \bar{W}_1 \\ \bar{W}_{2t} + \bar{U}_0 \bar{W}_{2x} + f \bar{W}_1 + (\phi^\varepsilon - \phi^0 - \bar{\vartheta}_\phi)_y - \varepsilon \Delta_3 \bar{W}_2 \\ \bar{W}_{3t} + \bar{U}_0 \bar{W}_{3x} + N^2 (w^\varepsilon - w^0 - \bar{\vartheta}_w) - \varepsilon \Delta_3 \bar{W}_3 \\ (\phi^\varepsilon - \phi^0 - \bar{\vartheta}_\phi)_z - \bar{W}_3 \\ \bar{W}_{1x} + \bar{W}_{2y} + (w^\varepsilon - w^0 - \bar{\vartheta}_w)_z \end{array} \right. = \left(\begin{array}{l} \varepsilon \Delta_3 u^0 + RHS_1 \\ \varepsilon \Delta_3 v^0 + RHS_2 \\ \varepsilon \Delta_3 \psi^0 + RHS_3 \\ 0 \\ 0 \end{array} \right),$$

where

$$(6.9) \quad \begin{aligned} RHS_1 &= f\bar{\vartheta}_v + \sum_{n=n_c+1}^{\infty} \lambda_n^{-1} (\bar{\theta}_{\psi n}^y + \bar{\theta}_{\psi n}^x + \rho_{\psi n})_x \mathcal{U}_n - R_1 \\ RHS_2 &= -\bar{\vartheta}_{vt} - f\bar{\vartheta}_u + \sum_{n=n_c+1}^{\infty} \lambda_n^{-1} (\bar{\theta}_{\psi n}^y + \bar{\theta}_{\psi n}^x + \rho_{\psi n})_y \mathcal{U}_n - R_2 \\ RHS_3 &= -\bar{\vartheta}_{\psi t} + N^2 \sum_{n=n_c+1}^{\infty} \lambda_n^{-1} ((\bar{\varphi}_{un} + \bar{\theta}_{un}^x + \rho_{un})_x + (\bar{\theta}_{vn}^y + \bar{\theta}_{vn}^x + \rho_{vn})_y) \mathcal{W}_n - R_3, \end{aligned}$$

and R_1, R_2, R_3 are as in (6.5a)-(6.5c), respectively.

We further simplify (6.9). From (6.5) we find that

$$(6.10) \quad \begin{aligned} RHS_1 &= \sum_{n=n_c+1}^{\infty} f(\bar{\theta}_{vn}^y + \bar{\theta}_{vn}^x + \rho_{vn}) \mathcal{U}_n + \sum_{n=n_c+1}^{\infty} \lambda_n^{-1} \bar{\theta}_{\psi n}^y \mathcal{U}_n + \varepsilon \sum_{n=n_c+1}^{\infty} \bar{\varphi}_{unxx} \mathcal{U}_n - \\ &\quad - \sum_{n=n_c+1}^{\infty} (\bar{\theta}_{unt}^x - \varepsilon \bar{\theta}_{unyy}^x) \mathcal{U}_n - \sum_{n=n_c+1}^{\infty} (\rho_{unt} - \rho_{unt}) \mathcal{U}_n, \end{aligned}$$

$$(6.11) \quad \begin{aligned} RHS_2 &= -\bar{\vartheta}_{vt} - \sum_{n=n_c+1}^{\infty} f(\bar{\theta}_{un}^x + \rho_{un}) \mathcal{U}_n + \sum_{n=n_c+1}^{\infty} \lambda_n^{-1} \bar{\theta}_{\psi n}^x \mathcal{U}_n - \\ &\quad - \sum_{n=n_c+1}^{\infty} (\bar{U}_0 \bar{\theta}_{vnx}^y - \varepsilon \bar{\theta}_{vnx}^y) \mathcal{U}_n + \sum_{n=n_c+1}^{\infty} \varepsilon \bar{\theta}_{vnyy}^x \mathcal{U}_n, \end{aligned}$$

$$(6.12) \quad \begin{aligned} RHS_3 &= -\bar{\vartheta}_{\psi t} + N^2 \sum_{n=n_c+1}^{\infty} \lambda_n^{-1} \bar{\theta}_{vny}^x \mathcal{W}_n - \sum_{n=n_c+1}^{\infty} (\bar{U}_0 \bar{\theta}_{\psi nx}^y - \varepsilon \bar{\theta}_{\psi nx}^y) \mathcal{W}_n + \\ &\quad + \sum_{n=n_c+1}^{\infty} \varepsilon \bar{\theta}_{\psi nyy}^x \mathcal{W}_n. \end{aligned}$$

We take, as in (2.6), the scalar product of (6.8)_{1,2,3} over $\mathbf{L}^2(\mathcal{M})$ with $W := (W_1, W_2, W_3) = U^\varepsilon - U^0 - \Theta$. Here, we note that $W = 0$ on $\partial\mathcal{M}'$. Since

$$(6.13) \quad \mathcal{L}(W) = \varepsilon \Delta_3 U^0 + RHS + \mathcal{L}(W - \bar{W}),$$

with $RHS = (RHS_1, RHS_2, RHS_3)^T$, we find that

$$(6.14) \quad \frac{1}{2} \frac{d}{dt} |W|_{\mathbf{L}^2}^2 + \varepsilon |W|_{\mathbf{H}^1}^2 = (\varepsilon \Delta_3 U^0 + RHS, W) + (\mathcal{L}(W - \bar{W}), W).$$

From the estimates of the boundary layers derived in Sec. 5 and in particular the pointwise estimates in the proofs therein we note that the term $(\mathcal{L}(W - \bar{W}), W) = (\mathcal{L}(\bar{\Theta} - \Theta), W)$ is exponentially small; it can be estimated and then absorbed by other norms.

We now estimate the right-hand side of (6.14) term by term. We first see that

$$(6.15) \quad |(\varepsilon \Delta_3 U^0, W)| \lesssim \varepsilon |U^0|_{\mathbf{H}^2(\mathcal{M})} |W|_{\mathbf{L}^2(\mathcal{M})}.$$

Here, by $a \lesssim b$ we mean $a \leq \kappa b$ where $\kappa > 0$ is a constant independent of ε .

Now, using the Hardy inequality, Remark 5.2 and Lemmas 5.1, 5.3 and 5.4, we deduce that

$$\begin{aligned}
& |(RHS_1, W_1)| \\
& \lesssim \varepsilon \left| \sum_{n=n_c+1}^{\infty} \bar{\varphi}_{unx} \mathcal{U}_n + \sum_{n=n_c+1}^{\infty} \bar{\theta}_{uny}^x \mathcal{U}_n \right|_{L^2(\mathcal{M})} |W_1|_{H^1(\mathcal{M})} + \\
& + \left| x(L_1 - x) \sum_{n=n_c+1}^{\infty} (f\bar{\theta}_{vn}^x - \bar{\theta}_{unt}^x) \mathcal{U}_n \right|_{L^2(\mathcal{M})} \left| \frac{W_1}{x(L_1 - x)} \right|_{L^2(\mathcal{M})} + \\
& + \left| y(L_2 - y) \sum_{n=n_c+1}^{\infty} (f\bar{\theta}_{vn}^y + \lambda_n^{-1} \bar{\theta}_{\psi nx}^y) \mathcal{U}_n \right|_{L^2(\mathcal{M})} \left| \frac{W_1}{y(L_2 - y)} \right|_{L^2(\mathcal{M})} + \\
& + \left| \sum_{n=n_c+1}^{\infty} (f\rho_{vn}) \mathcal{U}_n \right|_{L^2(\mathcal{M})} |W_1|_{L^2(\mathcal{M})} \\
(6.16) \quad & \lesssim \varepsilon^{\frac{5}{4}} \left[\sum_{n=n_c+1}^{\infty} ((K_{0,1,0}^2)^2 + (K_{0,1}^3)^2)(U_n^0) \right]^{\frac{1}{2}} |W_1|_{H^1(\mathcal{M})} + \\
& + \varepsilon^{\frac{3}{2}} \left[\sum_{n=n_c+1}^{\infty} (K_{1,0}^3)^2(U_n^0) \right]^{\frac{1}{2}} |W_1|_{H^1(\mathcal{M})} + \\
& + \varepsilon^{\frac{5}{4}} \left[\sum_{n=n_c+1}^{\infty} ((K_{0,1}^1)^2 \lambda_n^3 + (K_{0,2}^1)^2 \lambda_n^2)(U_n^0) \right]^{\frac{1}{2}} |W_1|_{H^1(\mathcal{M})} + \\
& + \varepsilon^{\frac{3}{4}} \left[\sum_{n=n_c+1}^{\infty} (K_1^4)^2(U_n^0) \right]^{\frac{1}{2}} |W_1|_{L^2(\mathcal{M})}.
\end{aligned}$$

Using $\bar{\vartheta}_v = \sum_{n=n_c+1}^{\infty} (\bar{\theta}_{vn}^y + \bar{\theta}_{vn}^x + \rho_{vn}) \mathcal{U}_n$ and $\bar{\vartheta}_{\psi} = \sum_{n=n_c+1}^{\infty} (\bar{\theta}_{\psi n}^y + \bar{\theta}_{\psi n}^x + \rho_{\psi n}) \mathcal{W}_n$, we then similarly estimate the terms, RHS_2 , RHS_3 as follows:

$$\begin{aligned}
& |(RHS_2, W_2)| \\
& \lesssim \varepsilon \left| \sum_{n=n_c+1}^{\infty} \bar{\theta}_{vny}^x \mathcal{U}_n + \sum_{n=n_c+1}^{\infty} \bar{\theta}_{vnx}^y \mathcal{U}_n \right|_{L^2(\mathcal{M})} |W_2|_{H^1(\mathcal{M})} + \\
& + \left| x(L_1 - x) \sum_{n=n_c+1}^{\infty} (\bar{\theta}_{vnt}^x + f\bar{\theta}_{un}^x - \lambda_n^{-1} \bar{\theta}_{\psi ny}^x) \mathcal{U}_n \right|_{L^2(\mathcal{M})} \left| \frac{W_2}{x(L_1 - x)} \right|_{L^2(\mathcal{M})} + \\
(6.17) \quad & + \left| y(L_2 - y) \sum_{n=n_c+1}^{\infty} (\bar{\theta}_{vnt}^y + \bar{U}_0 \bar{\theta}_{vnx}^y) \mathcal{U}_n \right|_{L^2(\mathcal{M})} \left| \frac{W_2}{y(L_2 - y)} \right|_{L^2(\mathcal{M})} + \\
& + \left| \sum_{n=n_c+1}^{\infty} (\rho_{vnt} + f\rho_{un}) \mathcal{U}_n \right|_{L^2(\mathcal{M})} |W_2|_{L^2(\mathcal{M})}
\end{aligned}$$

$$\begin{aligned}
(6.18) \quad & \lesssim \varepsilon^{\frac{3}{2}} \left[\sum_{n=n_c+1}^{\infty} ((K_{0,1}^3)^2 + (K_{0,2}^1)^2 \lambda_n^2)(U_n^0) \right]^{\frac{1}{2}} |W_2|_{H^1(\mathcal{M})} + \\
& + \varepsilon^{\frac{3}{2}} \left[\sum_{n=n_c+1}^{\infty} ((K_{1,0}^3)^2 + (K_{0,1}^3)^2)(U_n^0) \right]^{\frac{1}{2}} |W_2|_{H^1(\mathcal{M})} + \\
& + \varepsilon^{\frac{5}{4}} \left[\sum_{n=n_c+1}^{\infty} ((K_{1,0}^1)^2 \lambda_n^3 + (K_{1,1}^1)^2 \lambda_n^2 + (K_{0,1}^1)^2 \lambda_n^3 + (K_{0,2}^1)^2 \lambda_n^2)(U_n^0) \right]^{\frac{1}{2}} |W_2|_{H^1(\mathcal{M})} + \\
& + \varepsilon^{\frac{3}{4}} \left[\sum_{n=n_c+1}^{\infty} (K_1^4)^2 (U_n^0) \right]^{\frac{1}{2}} |W_2|_{L^2(\mathcal{M})}.
\end{aligned}$$

and

$$\begin{aligned}
(6.19) \quad & |(RHS_3, W_3)| \\
& \lesssim \varepsilon \left| \sum_{n=n_c+1}^{\infty} (\bar{\theta}_{\psi ny}^x + \bar{\theta}_{\psi nx}^y) \mathcal{W}_n \right|_{L^2(\mathcal{M})} |W_3|_{H^1(\mathcal{M})} + \\
& + \left| x(L_1 - x) \sum_{n=n_c+1}^{\infty} (-\bar{\theta}_{\psi nt}^x + N^2 \lambda_n^{-1} \bar{\theta}_{vny}^x) \mathcal{W}_n \right|_{L^2(\mathcal{M})} \left| \frac{W_3}{x(L_1 - x)} \right|_{L^2(\mathcal{M})} + \\
& + \left| y(L_2 - y) \sum_{n=n_c+1}^{\infty} (\bar{\theta}_{\psi nt}^y + \bar{U}_0 \bar{\theta}_{\psi nx}^y) \mathcal{W}_n \right|_{L^2(\mathcal{M})} \left| \frac{W_3}{y(L_2 - y)} \right|_{L^2(\mathcal{M})} + \\
& + \left| \sum_{n=n_c+1}^{\infty} (\rho_{\psi nt}) \mathcal{W}_n \right|_{L^2(\mathcal{M})} |W_3|_{L^2(\mathcal{M})}
\end{aligned}$$

$$\begin{aligned}
(6.20) \quad & \lesssim \varepsilon^{\frac{3}{2}} \left[\sum_{n=n_c+1}^{\infty} ((K_{0,1}^3)^2 + (K_{0,2}^1)^2 \lambda_n^2)(U_n^0) \right]^{\frac{1}{2}} |W_3|_{H^1(\mathcal{M})} + \\
& + \varepsilon^{\frac{3}{2}} \left[\sum_{n=n_c+1}^{\infty} ((K_{1,0}^3)^2 + (K_{0,1}^3)^2)(U_n^0) \right]^{\frac{1}{2}} |W_3|_{H^1(\mathcal{M})} + \\
& + \varepsilon^{\frac{5}{4}} \left[\sum_{n=n_c+1}^{\infty} ((K_{1,0}^1)^2 \lambda_n^3 + (K_{1,1}^1)^2 \lambda_n^2 + (K_{0,1}^1)^2 \lambda_n^3 + (K_{0,2}^1)^2 \lambda_n^2)(U_n^0) \right]^{\frac{1}{2}} |W_3|_{H^1(\mathcal{M})} + \\
& + \varepsilon^{\frac{3}{4}} \left[\sum_{n=n_c+1}^{\infty} (K_1^4)^2 (U_n^0) \right]^{\frac{1}{2}} |W_3|_{L^2(\mathcal{M})}.
\end{aligned}$$

From (6.14) - (6.19) we then obtain that

$$(6.21) \quad \frac{1}{2} \frac{d}{dt} |W|_{\mathbf{L}^2}^2 + \varepsilon |W|_{\mathbf{H}^1}^2 \leq \kappa (K(U^0))^2 \varepsilon^{\frac{3}{2}} + \frac{\varepsilon}{2} |W|_{\mathbf{H}^1(\mathcal{M})}^2 + \kappa |W|_{\mathbf{L}^2(\mathcal{M})}^2,$$

where

$$\begin{aligned}
(6.22) \quad & K(U^0) \\
& \lesssim \left[\sum_{n=n_c+1}^{\infty} (K_1^4)^2(U_n^0) \right]^{\frac{1}{2}} + \left[\sum_{n=n_c+1}^{\infty} ((K_{0,1,0}^2)^2 + (K_{0,1}^3)^2 + (K_{1,0}^3)^2)(U_n^0) \right]^{\frac{1}{2}} + \\
& + \left[\sum_{n=n_c+1}^{\infty} \lambda_n^3((K_{0,1}^1)^2 + (K_{1,0}^1)^2)(U_n^0) \right]^{\frac{1}{2}} + \left[\sum_{n=n_c+1}^{\infty} \lambda_n^2((K_{0,2}^1)^2 + (K_{1,1}^1)^2)(U_n^0) \right]^{\frac{1}{2}} \\
& \lesssim \|U^0\|_{W^{1,\infty}(0,T;\mathbf{H}^3(\mathcal{M}))} + \|\gamma U_x^0\|_{L^\infty(0,T;\mathbf{L}^2(\mathcal{M}''))} + \\
& + \|\tilde{\gamma} U_y^0\| + \|\tilde{\gamma} U_t^0\|_{L^\infty(0,T;\mathbf{L}^2(\mathcal{M}'''))} + \|\gamma U_{xx}^0\| + \|\gamma U_{tzz}^0\|_{L^\infty(0,T;\mathbf{L}^2(\mathcal{M}''))} + \\
& + \|\gamma U_{xxz}^0\| + \|\gamma U_{txz}^0\|_{L^\infty(0,T;\mathbf{L}^2(\mathcal{M}''))},
\end{aligned}$$

where $\mathcal{M}'' = (0, L_1) \times (-L_3, 0)$ and γf is the trace of f at $y = 0, L_2$, and $\mathcal{M}''' = (0, L_2) \times (-L_3, 0)$ and $\tilde{\gamma} f$ is the trace of f at $x = 0, L_1$. Hence, $K(U^0)$ is bounded by the right-hand side of (3.23) and this completes the proof of Theorem 3.1.

Remark 6.1. For the subcritical modes ($1 \leq n \leq n_c$), since λ_n is bounded and $\bar{U}_0 - N/\lambda_n < 0$, i.e. $\lambda_1 \leq \lambda_n \leq \lambda_{n_c} < N/\bar{U}_0$, we have to modify the boundary layers constructed in Section 4. First, since λ_n is bounded, the parameter $\varepsilon \lambda_n^2$ is small and hence the terms with $\varepsilon \lambda_n^2$ can be essentially ignored. We would like to emphasize the parabolic and ordinary boundary layers for $u^\varepsilon, \psi^\varepsilon$ at $y = 0, L_2$ and $x = 0, L_1$ respectively because the other boundary layers can be similarly constructed. From (3.4), with $F = 0$, dropping the small terms $\varepsilon \lambda^2 U^\varepsilon$ and the Coriolis terms, and using the combinations $\xi^\varepsilon = u^\varepsilon + N^{-1}\psi^\varepsilon$ and $\eta^\varepsilon = u^\varepsilon - N^{-1}\psi^\varepsilon$ (see (4.31) above) we can write

$$(6.23) \quad \begin{cases} \xi_t^\varepsilon + (\bar{U}_0 - \frac{N}{\lambda})\xi_x^\varepsilon - \frac{N}{\lambda}v_y^\varepsilon - \varepsilon\Delta\xi^\varepsilon = 0, \\ v_t^\varepsilon + \bar{U}_0v_x^\varepsilon - \frac{1}{\lambda}\psi_y^\varepsilon - \varepsilon\Delta v^\varepsilon = 0, \\ \eta_t^\varepsilon + (\bar{U}_0 + \frac{N}{\lambda})\eta_x^\varepsilon + \frac{N}{\lambda}v_y^\varepsilon - \varepsilon\Delta\eta^\varepsilon = 0. \end{cases}$$

Then depending on the location of the inflows, we need to devise the subcritical boundary layers along $\partial\mathcal{M}'$. Since $\bar{U}_0 - N/\lambda_n < 0$, the inflows are at $x = L_1$ for ξ^ε and at $x = 0$ for η^ε .

Hence the parabolic boundary layers φ_ξ and φ_η corresponding to ξ^ε and η^ε , respectively, are supplemented with the zero condition at the inflows $x = L_1$ and $x = 0$, respectively. Thus, we have two types of subcritical parabolic boundary layers for $u^\varepsilon, \psi^\varepsilon$ at $y = 0, L_2$. One is supplemented with the zero condition at $x = L_1$ and the other at $x = 0$ whereas all supercritical parabolic boundary layers are supplemented with the zero condition only at $x = 0$ (see (4.3), (4.4)). To regularize the subcritical parabolic boundary layers, we need the compatibility conditions like (4.7) at the inflow corners.

The subcritical ordinary boundary layers for $u^\varepsilon, \psi^\varepsilon$ at $x = 0, L_1$ are formed at both $x = 0$ and $x = L_1$ whereas the same boundary layers are only at $x = L_1$ for the supercritical modes. Following (4.33) and (4.40), we take instead $\bar{\theta}_\xi = c_5 \exp(r_3^- x)$ in (4.42) which is the boundary layer at $x = 0$.

Then the subcritical corner layers also need to be constructed to resolve further discrepancies due to the subcritical boundary layers near the corners. In addition, since $\bar{U}_0 - N/\lambda_n < 0$, the analysis in (5.46) fails and some delicate analysis will be necessitated.

We will discuss all the subcritical corner and boundary layers in a forthcoming article.

ACKNOWLEDGMENTS

This work was supported in part by NSF Grants DMS 0906440 and DMS 1206438 and DMS 1510249, and by the Research Fund of Indiana University and by the Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education(2015R1D1A1A01059837).

REFERENCES

- [1] Claude Bardos and Jeffrey Rauch. Maximal positive boundary value problems as limits of singular perturbation problems. *Trans. Amer. Math. Soc.*, 270(2):377–408, 1982. 7, 8
- [2] Sylvie Benzoni-Gavage and Denis Serre. *Multidimensional hyperbolic partial differential equations*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2007. First-order systems and applications. 8
- [3] Qingshan Chen, Ming-Cheng Shiue and Roger Temam. The Barotropic mode for the Primitive equations. *J. Sci. Comput.*, 45(1-3):167–199, 2010. 3
- [4] M. Van Dyke. *An album of fluid motion*. The Parabolic Press, Stanford, California, 1982. 3
- [5] Wiktor Eckhaus. Boundary layers in linear elliptic singular perturbation problems. *SIAM Rev.*, 14:225–270, 1972. 3
- [6] K. O. Friedrichs. Symmetric hyperbolic linear differential equations. *Comm. Pure Appl. Math.*, 7:345–392, 1954. 7
- [7] Gung-Min Gie, Chang-Yeol Jung and Roger Temam. Recent progresses in boundary layer theory. *Discrete Contin. Dyn. Syst.*, 36:2521 – 2583, 2016. 3
- [8] O. Guès, G. Métivier, M. Williams, and K. Zumbrun. Boundary layer and long time stability for multidimensional viscous shocks. *Discrete Contin. Dyn. Syst.*, 11(1):131–160, 2004. 3
- [9] J. Grasman. *On the birth of boundary layers*. Mathematisch Centrum, Amsterdam, 1971. Mathematical Centre Tracts, No. 36. 3
- [10] E. Grenier. Boundary layers. In *Handbook of mathematical fluid dynamics. Vol. III*, pages 245–309. North-Holland, Amsterdam, 2004. 3
- [11] Makram Hamouda, Chang-Yeol Jung, and Roger Temam. Boundary layers for the 2D linearized primitive equations. *Commun. Pure Appl. Anal.*, 8(1):335–359, 2009. 2, 5, 7, 15, 21
- [12] Makram Hamouda, Chang-Yeol Jung, and Roger Temam. Asymptotic analysis for the 3D primitive equations in a channel. *Discrete Contin. Dyn. Syst. Ser. S*, 6(2):401–422, 2013. 2, 4, 5, 6, 11, 14, 15
- [13] Makram Hamouda, Chang-Yeol Jung, and Roger Temam. Well-posedness of the 3D linearized inviscid primitive equations in a cube. *In preparation*. 8
- [14] H. Han and R. B. Kellogg. Differentiability properties of solutions of the equation $-\epsilon^2 \Delta u + ru = f(x, y)$ in a square. *SIAM J. Math. Anal.*, 21(2):394–408, 1990. 3
- [15] Daozhi Han, Anna L. Mazzucato, Dongjuan Niu, and Xiaoming Wang. Boundary layer for a class of nonlinear pipe flow. *J. Differential Equations*, 252(12):6387–6413, 2012. 3
- [16] Aimin Huang and Roger Temam. The linearized 2D inviscid shallow water equations in a rectangle: boundary conditions and well-posedness. *Arch. Ration. Mech. Anal.*, 211(3):1027–1063, 2014. 2, 10
- [17] Aimin Huang and Roger Temam. The nonlinear 2D subcritical inviscid shallow water equations with periodicity in one direction. *Commun. Pure Appl. Anal.*, 13(5):2005 – 2038, 2014. 2

- [18] Aimin Huang, Madalina Petcu and Roger Temam. The nonlinear 2D supercritical inviscid shallow water equations in a rectangle. *Asymptotic Analysis*, 93(3):187–218, 2015. [2](#)
- [19] Aimin Huang and Roger Temam. The 2D nonlinear fully hyperbolic inviscid shallow water equations in a rectangle. *To appear*.
- [20] Chang-Yeol Jung and Roger Temam. Numerical approximation of two-dimensional convection-diffusion equations with multiple boundary layers. *Int. J. Numer. Anal. Model.*, 2(4):367–408, 2005. [13](#)
- [21] Chang-Yeol Jung and Roger Temam. Singular perturbations and boundary layer theory for convection-diffusion equations in a circle: the generic noncompatible case. *SIAM J. Math. Anal.*, 44(6):4274–4296, 2012. [13](#)
- [22] Chang-Yeol Jung and Roger Temam. Boundary layer theory for convection-diffusion equations in a circle. *Russian Math. Surveys*, 69(3):435–480, 2014. [13](#)
- [23] R. Bruce Kellogg and Martin Stynes. Corner singularities and boundary layers in a simple convection-diffusion problem. *J. Differential Equations*, 213(1):81–120, 2005. [3](#)
- [24] P. D. Lax and R. S. Phillips. Local boundary conditions for dissipative symmetric linear differential operators. *Comm. Pure Appl. Math.*, 13:427–455, 1960.
- [25] Norman Levinson. The first boundary value problem for $\varepsilon\Delta u + A(x, y)u_x + B(x, y)u_y + C(x, y)u = D(x, y)$ for small ε . *Ann. of Math. (2)*, 51:428–445, 1950. [3](#)
- [26] J.-L. Lions. *Perturbations singulières dans les problèmes aux limites et en contrôle optimal*. Lecture Notes in Mathematics, Vol. 323. Springer-Verlag, Berlin-New York, 1973.
- [27] J. L. Lions, R. Temam and S. Wang, *New formulations of the primitive equations of atmosphere and applications*. Nonlinearity 5 (1992), no. 2, 237–288. [2](#), [3](#), [4](#)
- [28] J. L. Lions, R. Temam and S. Wang, *On the equations of the large-scale ocean*. Nonlinearity 5 (1992), no. 5, 1007–1053. [2](#), [3](#)
- [29] Nader Masmoudi. The Euler limit of the Navier-Stokes equations, and rotating fluids with boundary. *Arch. Rational Mech. Anal.*, 142(4):375–394, 1998. [3](#)
- [30] Anna Mazzucato, Dongjuan Niu, and Xiaoming Wang. Boundary layer associated with a class of 3D nonlinear plane parallel channel flows. *Indiana Univ. Math. J.*, 60(4):1113–1136, 2011. [3](#)
- [31] Robert E. O’Malley, Jr. *Singular perturbation methods for ordinary differential equations*, volume 89 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1991. [3](#)
- [32] Robert E. O’Malley, Jr. Singularly perturbed linear two-point boundary value problems. *SIAM Rev.*, 50(3):459–482, 2008. [3](#)
- [33] Joseph Oliger and Arne Sundström. Theoretical and practical aspects of some initial boundary value problems in fluid dynamics. *SIAM J. Appl. Math.*, 35(3):419–446, 1978. [2](#)
- [34] O. A. Oleinik and V. N. Samokhin. *Mathematical models in boundary layer theory*, volume 15 of *Applied Mathematics and Mathematical Computation*. Chapman & Hall/CRC, Boca Raton, FL, 1999. [3](#)
- [35] Joseph Pedlosky. *Geophysical Fluid Dynamics*, Springer-Verlag, New York, 1987. [3](#)
- [36] Ludwig Prandtl. *Gesammelte Abhandlungen zur angewandten Mechanik, Hydro- und Aerodynamik*. Herausgegeben von Walter Tollmien, Hermann Schlichting, Henry Görtler. Schriftleitung: F. W. Riegels. In 3 Teilen. Springer-Verlag, Berlin, 1961. [3](#)
- [37] M. Petcu, R. Temam and M. Ziane, *Some mathematical problems in geophysical fluid dynamics*. Handbook of numerical analysis. **Vol. XIV**. Special volume: computational methods for the atmosphere and the oceans, 577–750, Handb. Numer. Anal., 14, Elsevier/North-Holland, Amsterdam, 2009. [3](#), [4](#)
- [38] Jeffrey Rauch. Symmetric positive systems with boundary characteristic of constant multiplicity. *Trans. Amer. Math. Soc.*, 291(1):167–187, 1985.
- [39] Jeffrey B. Rauch and Frank J. Massey, III. Differentiability of solutions to hyperbolic initial-boundary value problems. *Trans. Amer. Math. Soc.*, 189:303–318, 1974.
- [40] W. H. Reid. Uniform asymptotic approximations to the solutions of the Orr-Sommerfeld equation. I. II, Plane Couette flow. *Studies in Appl. Math.*, 53:91–110, 1974, 53:217–224, 1974.

- [41] H.-G. Roos, M. Stynes, and L. Tobiska. *Numerical methods for singularly perturbed differential equations*, volume 24 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1996. Convection-diffusion and flow problems. [3](#)
- [42] Antoine Rousseau, Roger M. Temam, and Joseph J. Tribbia. Boundary value problems for the inviscid primitive equations in limited domains. In *Handbook of numerical analysis. Vol. XIV. Special volume: computational methods for the atmosphere and the oceans*, volume 14 of *Handb. Numer. Anal.*, pages 481–575. Elsevier/North-Holland, Amsterdam, 2009. [4](#), [5](#)
- [43] Leonard Sarason. Differentiable solutions of symmetrizable and singular symmetric first order systems. *Arch. Rational Mech. Anal.*, 26:357–384, 1967.
- [44] Eric Simonnet, Michael Ghil, Kayo Ide, Roger Temam, and Shouhong Wang. Low-frequency variability in shallow-water models of the wind-driven ocean circulation. I. Steady-state solution. *J. Phys. Oceanogr.*, 33(4):712–728, 2003.
- [45] Shagi-Di Shih and R. Bruce Kellogg. Asymptotic analysis of a singular perturbation problem. *SIAM J. Math. Anal.*, 18(5):1467–1511, 1987. [2](#), [6](#), [13](#)
- [46] Donald R. Smith. *Singular-perturbation theory*. Cambridge University Press, Cambridge, 1985. An introduction with applications.
- [47] Martin Stynes. Steady-state convection-diffusion problems. *Acta Numer.*, 14:445–508, 2005. [3](#)
- [48] Roger Temam and Joseph Tribbia. Open boundary conditions for the primitive and Boussinesq equations. *J. Atmospheric Sci.*, 60(21):2647–2660, 2003. [2](#)
- [49] Ferdinand Verhulst. *Methods and applications of singular perturbations*, volume 50 of *Texts in Applied Mathematics*. Springer, New York, 2005. Boundary layers and multiple timescale dynamics. [3](#)
- [50] M. I. Višik and L. A. Ljusternik. Regular degeneration and boundary layer for linear differential equations with small parameter. *Amer. Math. Soc. Transl. (2)*, 20:239–364, 1962. [3](#)
- [51] T.T. Warner, R. A. Peterson and R. E. Treadon. A tutorial on lateral boundary conditions as a basic and potentially serious limitation to regional numerical weather prediction. *Bull. Am. Meteorol. Soc.*, 78(11):2599–2617, 1997. [1](#)

¹ THE INSTITUTE FOR SCIENTIFIC COMPUTING AND APPLIED MATHEMATICS,, INDIANA UNIVERSITY, 831 E. 3RD ST., RAWLES HALL,, BLOOMINGTON, IN 47405, USA

² UNIVERSITY OF CARTHAGE, FACUTLY OF SCIENCES AT BIZERTE, DEPARTMENT OF MATHEMATICS, 7021 ZARZOUNA, BIZERTE, TUNISIA.

³ DEPARTMENT OF MATHEMATICAL SCIENCES, SCHOOL OF NATURAL SCIENCE, ULSAN NATIONAL INSTITUTE OF SCIENCE AND TECHNOLOGY, UNIST-GIL 50, ULSAN 689-798, REPUBLIC OF KOREA