Sharp Hardy-Littlewood-Sobolev Inequalities on Quaternionic Heisenberg Groups *

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Abstract

In this paper, we get several sharp Hardy-Littlewood-Sobolev-type inequalities on quaternionic Heisenberg groups, using the symmetrization-free method of Frank and Lieb, who considered the analogues on the Heisenberg group. First, we give the sharp Hardy-Littlewood-Sobolev inequality on the quaternionic Heisenberg group and its equivalent on the sphere, for singular exponent of partial range $\lambda \geq 4$. The extremal function, as we guess, is “almost” uniquely *constant* function on the sphere. Then their dual form, a sharp conformally-invariant Sobolev-type inequality involving a (fractional) intertwining operator, and the right endpoint case, a Log-Sobolev-type inequality, are also obtained. Higher dimensional center brings extra difficulty. The conformal symmetry of the inequalities, zero center-mass technique and estimates involving meticulous computation of eigenvalues of singular kernels play a critical role in the argument.

1 Introduction

Sharp constants and extremal functions for important inequalities, especially Sobolev-type, have been studied since many years ago. It has been a general hot topic in analysis, geometry, probability, PDE and quantum field theory. They play an important role because they almost always contain or reveal profound geometric and probabilistic information of the underlying space, manifold or group. Vast beautiful literature has been done on this subject. Despite rich profound results in the framework of Riemannian geometry, the problem in the sub-Riemannian world is more interesting, but far away from being absolutely understood while some conclusive results have been obtained recently. Among so many geometric inequalities, Sobolev-type inequalities exceptionally attract attention of analysts. In this paper, we will discuss some Hardy-Littlewood-Sobolev-type (HLS) inequalities. We use notations “ $\sim$ ” and “ $\lesssim$ ” respectively for “ $=$ ” and “ $<$ ” through this paper, modulo a constant we don’t care in the present context.

The classical HLS inequality on $\mathbb{R}^n$ states that for $0 < s < n$, the Riesz potential denoted by $(-\Delta)^{-\frac{s}{2}}$, a convolution operator with kernel $|x|^{s-n}$ (or called fractional integral operator), is a linear operator of strong-type (p, q) for particular exponents, i.e., it’s bounded from Lebesgue spaces L^p to L^q . This inequality can be viewed as the *weak* Young inequality, considering the fact that function $|x|^{-\lambda}$ is indeed contained in weak Lebesgue space $L_{w}^{\frac{n}{\lambda}}$, although not in any L^p . For symmetry, we can write it in a sesquilinear form

$$\left| \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\overline{f(x)}g(y)}{|x-y|^\lambda} dx dy \right| \lesssim \|f\|_p \|g\|_q$$

with $0 < \lambda < n$ and $1 < p, q < \infty$ satisfying $\frac{1}{p} + \frac{1}{q} + \frac{\lambda}{n} = 2$. It was found by Hardy, Littlewood and Sobolev almost a century ago [26, 27, 43]. The existence of extremal functions, called *extremizers*, was first proved by Lieb in [39], combining the Riesz rearrangement inequality, refined Fatou lemma and compactness argument. Another proof of existence was given by Lions in [40], using the famous *concentration compactness* principle. By exploiting the intrinsic conformal symmetries and using symmetric decreasing rearrangement, Lieb [39] also proved for symmetric exponents $p = q = \frac{2n}{2n-\lambda}$ that function $(1+|x|^2)^{-\frac{2n-\lambda}{2}}$ is the “almost unique” extremizer, ignoring constant multiples, translations and dilations, and the *sharp*

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constant, the infima for the inequality to hold, was also given. Actually, there exists for the inequality a large conformal symmetry (invariance) group, consisting of not only translations, dilations, rotations, but also an interesting inversion $f(x) \rightarrow |x|^{-(2n-\lambda)} f(x|x|^{-2})$. This large symmetry group is the main difficulty for the existence proof of extremizers, which tells non-uniqueness and easily vanishes the weak limit of any *extremizing sequence*. However, it also gives some convenience to find some special extremizer and a unified competing symmetry method for the existence and calculation of extremizers was given by Carlen and Loss in [7]. They constructed a special strong limit using alternatively the conformal action and rearrangement to any positive L^p function, which ingeniously balanced between the “bad” and “good” roles of the symmetry group. Other symmetric rearrangement-free methods of finding the sharp constant and extremizers can be found in the work of Frank and Lieb [17, 18]. The first reference used inversion-positivity to get a result for partial exponent $\lambda \geq n-2$, while the second one demonstrate on $\mathbb{R}^n$ the method used for the Heisenberg group in a seminal paper [19].

The analogous HLS inequality on the Heisenberg group $\mathbb{H}^n$ parameterized by $\mathbb{C}^n \times \mathbb{R}$ originates from Folland and Stein [16], which states that

$$\left| \iint_{\mathbb{H}^n \times \mathbb{H}^n} \frac{\overline{f(u)}g(v)}{|u^{-1}v|^\lambda} du dv \right| \lesssim \|f\|_p \|g\|_q$$

with $0 < \lambda < Q$ and $1 < p, q < \infty$ satisfying $\frac{1}{p} + \frac{1}{q} + \frac{\lambda}{Q} = 2$, where $u^{-1}v$, du , Q , $|\cdot|$ are respectively the left translation, Haar measure, homogeneous dimension and norm associated to dilation $\delta : u = (z, t) \mapsto \delta u = (\delta z, \delta^2 t)$. For the sharp problem, existence of extremizers for general exponents was recently proved by Han [24] using Lions’s concentration compactness lemmas. For calculation of extremizers for *symmetric* exponents $p = q$, first important progress was due to Jerison and Lee [32] for $\lambda = Q-2$, whose extremizer was proved to be “uniquely”

$$f \sim g \sim ((1 + |z|^2)^2 + |t|^2)^{-\frac{Q+2}{4}},$$

modulo a conformal symmetry group and they used the extremizers to study Cauchy-Riemann Yamabe problem about solutions to related Euler-Lagrange equation. Actually the authors gave a dual sharp Sobolev embedding inequality of order 2, and general Sobolev-embedding states that $\|f\|_{p^*} \lesssim \|Xf\|_p$, with $1 < p < Q$ and $p^* = \frac{pQ}{Q-p}$, where X is the horizontal gradient. Recently, Branson, Fontana and Morpurgo gave in [5] the endpoint case $\lambda \rightarrow 0$, a sharp Log-HLS inequality, which states that for any two normalized nonnegative functions $f, g \in L \log L \cap L \log(1 + |u|^2)$,

$$\iint_{\mathbb{H}^n \times \mathbb{H}^n} f(u) \log \frac{2}{|u^{-1}v|^2} g(v) du dv \leq \frac{|\mathbb{S}^{2n+1}|}{Q} \left(\int_{\mathbb{H}^n} f \log f + \int_{\mathbb{H}^n} g \log g + 2 \log 2 |\mathbb{S}^{2n+1}| \right),$$

with almost unique extremizer

$$f \sim g \sim ((1 + |z|^2)^2 + |t|^2)^{-\frac{Q}{2}},$$

from which it’s also conjectured that function

$$((1 + |z|^2)^2 + |t|^2)^{-\frac{2Q-\lambda}{4}} \tag{1.1}$$

be the almost unique extremizer for general λ . Inspired by the former two special cases, Frank and Lieb [19] recently killed this conjecture completely, by proving that a cleverly chosen extremizer is just the function above (1.1). It’s easy to note from the explicit formula (1.1) that this extremizer is similar to, but very different from that on $\mathbb{R}^n$. The level set of the Euclidean extremizer is just the Euclidean sphere, however, because of the different degrees of the coordinates in (1.1), the level set of this extremizer is neither that of the homogeneous norm, nor the isoperimetric surface (w.r.t the perimetric measure [6]). So, the symmetric rearrangement method for the Euclidean case *won’t* work again for the Heisenberg case. However, Frank and Lieb cutely choose a nice direction for the second variation around a special extremizer, which is fortunately proved to be nonnegative¹ with equality reached only by the right function. This has similar flavor to an idea of Chang and Yang [8], originating from [28]. They also used an alternative ingenious compactness argument to prove the existence of extremizers. A related doubly-weighted version of the HLS (Stein-Weiss inequalities) on the Heisenberg group was recently obtained by Han-Lu-Zhu [25] where, however, the sharp constants still remain open. The asymptotic behavior for the possible extremizers was also considered there. Some other recent, important related

¹The second variation around an extremizer should be non-positive.

results for the so-called Moser-Trudinger inequalities, an exponential class embedding at the endpoint, are Cohn-Lu [10, 11], Lam-Lu [38].

Now, as conjectured in [19], we want to extend the results onto more general 2-step nilpotent Lie groups of Heisenberg type (H-type) introduced by Kaplan [34]. We first consider a subclass—groups of Iwasawa type (I-type) [13]², the nilpotent part of the Iwasawa decomposition of a semisimple Lie group of rank one, which can be viewed as the isometry group of associated noncompact symmetric space³. It's known from the Clifford algebra characterization that there are *only* four cases for groups of I-type, including Euclidean spaces, classical Heisenberg groups, quaternionic Heisenberg groups and a 15-dimensional octonionic Heisenberg group. This paper is focused on the quaternionic Heisenberg group G with 3-dimensional center, parameterized by its Lie algebra $\mathbb{H}^n \times \text{Im } \mathbb{H}$ ($\mathbb{H}$ is the quaternions⁴) and we get a parallel result to that on the Heisenberg group. Accurately, we proved that, for $Q > \lambda \geq 4$ and *symmetric* exponent $p = \frac{2Q}{2Q-\lambda}$, extremizer exists for the sharp HLS

$$\iint_{G \times G} \frac{\overline{f(u)}g(v)}{|u^{-1}v|^\lambda} \lesssim \|f\|_p \|g\|_p, \quad (1.2)$$

and is almost uniquely

$$f \sim g \sim ((1 + |q|^2)^2 + |w|^2)^{-\frac{2Q-\lambda}{4}}, \quad (1.3)$$

with group element $u = (q, w)$ and $Q, |\cdot|$ is the homogeneous dimension and norm. Already known results by us are for $\lambda = Q - 2$ in a series of papers [21, 29, 30]. The authors considered a dual sharp embedding inequality $\|f\|_{\frac{2Q}{Q-2}} \lesssim \|Xf\|_2$ when studying the sub-Riemannian Yamabe problem, where X is the horizontal gradient. The authors in [21] proved in the unifying I-type framework that extremizer satisfying a partial symmetry for above embedding inequality is almost uniquely

$$((1 + |q|^2)^2 + |w|^2)^{-\frac{Q-2}{4}},$$

which is also used there to give all solutions with additional symmetry to the Yamabe equation. [29] gave all extremizers on the seven dimensional sphere, while [30] extended the result to all dimensions. Both the latter two papers got rid of the symmetric assumption. So, our result also extends [30].

The conformal symmetry group is again the core in our analysis. We use the *strategy* of Frank and Lieb in our proof with extra difficulty from higher dimension of center. The existence was proved in a dual form as the distance power kernel is positive definite, combining a *refined HLS*, Brezis-Lieb lemma and a Rellich-Kondrashov-type lemma. As before on $\mathbb{R}^n$ and $\mathbb{H}^n$, there is a large conformal symmetry group for (1.2), which makes it very easy for the weak limit of an extremizing sequence to vanish. However, it's proved that left translations and dilations are the *only* ways of losing compactness, which means, through them we can recover a strong limit by pulling the extremizing sequence back and the refined HLS is a beautiful tool to realize this idea. An alternative method for general non-symmetric exponents is Lions's concentration compactness argument. In finding the explicit extremizers, boundary extension of the Cayley transform, which is an isomorphism between two models of related hyperbolic space, Siegel upper domain D and unit ball B , is a basic tool to shift the inequality from the group onto the sphere. First, it's easier to deal on the sphere. Second, it gives a more clear view about how to choose the special extremizer that we finally proved to be constant, which just corresponds to the function (1.3). The critical step to filter the extremizer is utilizing the method from Herch, Chang and Yang to prove a purported *inverse second-variation* (ISV) inequality, which reaches equality only by constant functions. Actually, it's a natural idea to break the huge conformal symmetry group by restricting extremizers to functions satisfying certain *zero center-mass* condition as in above references. Fortunately, by checking the eigenvalues of the quadratic forms in the ISV inequality, any extremizer with zero center-mass property can only be constant function and finally, by turning back, we figure out all extremizers. However, a bit pity is that the special extremizer we choose is proved to be constant function only for partial exponent $\lambda \geq 4$. Note the special case $\lambda = Q - 2 = 4n + 4 \geq 4$ is included. For $\lambda < 4$, it seems that the chosen second variation and zero center-mass condition is not enough to filter the extremizers to be constant function. So, maybe more information of the conformal symmetry group should be excavated, while Euler-Lagrange equation can also do some favor. Several examples were computed, which guide us to a positive answer. We *anticipate* that constant should still be the “unique” extremizer for λ in this gap,

²This work gave a sufficient and necessary J_2 -condition for a H-type group to be of I-type, which allows to study rank one non-compact symmetric spaces avoiding classification and the semisimple Lie group theory.

³Harmonic extension of the I-type group.

⁴It's easy to distinguish from the notation $\mathbb{H}^n$ for the n order Heisenberg group.

which will be studied in our forthcoming paper. However, we remark here a very weak “local” extreme that the second variation around 1 is strictly negative.

The dual form of the sharp HLS, a sharp conformally-invariant Sobolev-type (CIS) inequality involving intertwining operators associated to the complementary series representation of Lorentz group $Sp(n, 1)$, and the right endpoint case, a sharp Log-Sobolev-type (LogS) inequality, are also obtained in this note. In [9], we worked out a similar result about the sharp HLS on the octonionic Heisenberg group, which together with [39, 19] and this paper, give a (partial) answer to the conjecture on I-type groups. Related stability on remainder terms was obtained in [41]. For general H-type or Siegel-type (high rank) groups, the higher dimensional center and lack of good symmetry structure as I-type both give more involved difficulty. This problem and a multilinear HLS will be considered in our forthcoming works.

Arrangement of this paper: In section 2, we will give some notations, definitions and basic formulas. In section 3, we give our main theorems. In section 4, we proved the existence of extremizers for HLS. In section 5, we proved an almost uniqueness and give characterization of all extremizers. Subsection 5.1 breaks the conformal symmetry by adding the zero center-mass condition. Then we compute the second variation and reduce the problem to a **claim** about an quadratic ISV inequality in subsection 5.2. The last but critical step is to prove the **claim** using Funk-Hecke formula in subsection 5.4 and 5.5. The sharp constants and explicit formulas of extremizers were proved in subsection 5.3 and Appendix B.

2 Preliminary

Quaternionic Heisenberg group. The quaternions

$$\mathbb{H} = \{q = a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}; i^2 = j^2 = k^2 = ijk = -1\},$$

endowed with multiplication

$$qq' = (aa' - bb' - cc' - dd') + (ab' + a'b + cd' - dc')i + (ac' + a'c + db' - bd')j + (ad' + a'd + bc' - cb')k,$$

form a real division algebra. The real and imaginary parts, conjugate and norm of q are

$$\operatorname{Re} q = a, \quad \operatorname{Im} q = bi + cj + dk, \quad \bar{q} = a - bi - cj - dk, \quad |q|^2 = a^2 + b^2 + c^2 + d^2$$

satisfying

$$\overline{qq'} = \bar{q}'\bar{q}, \quad |q|^2 = q\bar{q}, \quad |qq'| = |q||q'|.$$

The quaternions $\mathbb{H} \simeq \mathbb{C}^2$ as $q = (a + bi) + (c + di)j$, which can be represented by 2-order complex matrix

$$A_q = \begin{pmatrix} a + bi & c + di \\ -(c - di) & a - bi \end{pmatrix},$$

preserving the multiplications, i.e. $A_{qq'} = A_q A_{q'}$. So, if we use notation $q = (\zeta^1, \zeta^2)(\zeta^1, \zeta^2 \in \mathbb{C})$ for $q = \zeta^1 + \zeta^2 j$ and take $q' = (\eta^1, \eta^2)$, then $\bar{q} = (\bar{\zeta}^1, -\bar{\zeta}^2)$ and

$$\begin{aligned} qq' &= (\zeta^1 \eta^1 - \zeta^2 \bar{\eta}^2, \zeta^1 \eta^2 + \zeta^2 \bar{\eta}^1), \\ q\bar{q}' &= (\zeta^1 \bar{\eta}^1 + \zeta^2 \bar{\eta}^2, \zeta^2 \eta^1 - \zeta^1 \eta^2). \end{aligned}$$

For $q = (q_1, \dots, q_n) \in \mathbb{H}^n$, we use scalar product $q \cdot \bar{q}' = \sum_{j=1}^n q_j \bar{q}'_j$, then quaternionic unitary group, also called compact symplectic group,

$$Sp(n) = \{A \in M_n(\mathbb{H}) : \bar{A}^T A = A \bar{A}^T = I\}$$

preserves this product by right action on $\mathbb{H}^n$.

The quaternionic Heisenberg group is a 2-step nilpotent Lie group identified with its Lie algebra $G = \mathbb{H}^n \times \operatorname{Im} \mathbb{H}$ with group multiplication law⁵

$$uu' = (q, w)(q', w') = (q + q', w + w' + 2\operatorname{Im} q \cdot \bar{q}'),$$

where $u = (q, w), u' = (q', w')$ are two group elements. Here “ $\cdot$ ” is the scalar product in $\mathbb{H}^n$. This group can be viewed as the nilpotent part of Iwasawa decomposition of Lorentz group $Sp(n + 1, 1)$, which is

⁵Note here that we need take a little care to choose a proper inner product on the Lie algebra to make G with this law to be a group of H-type.

the isometry group of associated rank one hyperbolic symmetric space over quaternions, identified with homogeneous space $Sp(n+1, 1)/(Sp(n+1)Sp(1))$. We denote the homogeneous dimension and norm by

$$Q = 4n + 6, \quad |u| = |(q, w)| = (|q|^4 + |w|^2)^{\frac{1}{4}},$$

associated to the group dilation $\delta u = (\delta q, \delta^2 w)$ for any $\delta > 0$ and use $du = dqdw$, the Lebesgue measure, for the invariant Haar measure. Then the group distance of two elements $u = (q, w), v = (q', w')$ is defined naturally by

$$d_G(u, v) = |v^{-1}u| = (|q - q'|^4 + |w - w' + 2\operatorname{Im} q \cdot \bar{q}'|^2)^{\frac{1}{4}}.$$

The left-invariant vector fields corresponding to one-parameter real coordinates subgroups are given by

$$\begin{aligned} X_j^0 &= \frac{\partial}{\partial q_j^0} + 2q_j^1 \frac{\partial}{\partial w_1} + 2q_j^2 \frac{\partial}{\partial w_2} + 2q_j^3 \frac{\partial}{\partial w_3}, \\ X_j^1 &= \frac{\partial}{\partial q_j^1} - 2q_j^0 \frac{\partial}{\partial w_1} - 2q_j^3 \frac{\partial}{\partial w_2} + 2q_j^2 \frac{\partial}{\partial w_3}, \\ X_j^2 &= \frac{\partial}{\partial q_j^2} + 2q_j^3 \frac{\partial}{\partial w_1} - 2q_j^0 \frac{\partial}{\partial w_2} - 2q_j^1 \frac{\partial}{\partial w_3}, \\ X_j^3 &= \frac{\partial}{\partial q_j^3} - 2q_j^2 \frac{\partial}{\partial w_1} + 2q_j^1 \frac{\partial}{\partial w_2} - 2q_j^0 \frac{\partial}{\partial w_3}, \\ T^k &= \frac{\partial}{\partial w_k}, \quad 1 \leq j \leq n, \quad 1 \leq k \leq 3, \end{aligned}$$

all of which form together a basis of the Lie algebra and a second-order left-invariant differential operator, the *sublaplacian*⁶, is defined by

$$\mathcal{L} = -\frac{1}{4} \sum_{1 \leq j \leq n, 0 \leq k \leq 3} (X_j^k)^2,$$

which is independent of the choice of basis and hypoelliptic from a famous theorem of Hörmander. The fundamental solution of $\mathcal{L}$ was proved, by analogy with Folland's argument⁷, to be

$$\mathcal{L}^{-1}(u, v) = \frac{2^{\frac{Q}{2}-5} \Gamma(\frac{Q-2}{4}) \Gamma(\frac{Q-6}{4})}{\pi^{\frac{Q}{2}-1}} d_G^{2-Q}(u, v). \quad (2.1)$$

An analogue was proved in [34] for general H-type, heuristically believed to be the biggest group class with this property.

The Cayley transform and Peter-Weyl decomposition. We denote the quaternionic sphere by

$$S = \{\zeta = (\zeta', \zeta_{n+1}) \in \mathbb{H}^n \times \mathbb{H} : |\zeta'|^2 + |\zeta_{n+1}|^2 = 1\} \cong \mathbb{S}^{4n+3}$$

endowed with the Lebesgue surface measure $d\zeta$. The quaternionic Heisenberg group G is then equivalent to the punctured sphere $S \setminus \{o\}$ with south pole $o = (0, \dots, 0, -1)$, through the (boundary) Cayley transform, which and its inversion are defined by⁸

$$\begin{aligned} \mathcal{C} : \quad G &\longrightarrow S \setminus \{o\} \\ u = (q, w) &\longmapsto \zeta = (\zeta', \zeta_{n+1}) = \left(\frac{2q}{1 + |q|^2 - w}, \frac{1 - |q|^2 + w}{1 + |q|^2 - w} \right), \\ \mathcal{C}^{-1} : \quad S \setminus \{o\} &\longrightarrow G \\ \zeta = (\zeta', \zeta_{n+1}) &\longmapsto u = (q, w) = \left(\frac{\zeta'}{1 + \zeta_{n+1}}, -\operatorname{Im} \frac{1 - \zeta_{n+1}}{1 + \zeta_{n+1}} \right), \end{aligned} \quad (2.2)$$

with Jacobian determinant

$$\begin{aligned} |J_{\mathcal{C}}(u)| &= 2^{Q-3} ((1 + |q|^2)^2 + |w|^2)^{-\frac{Q}{2}} \\ &= 2^{-3} |1 + \zeta_{n+1}|^Q. \end{aligned} \quad (2.3)$$

⁶A counterpart of the Euclidean Laplacian $-\Delta$.

⁷In [15], Folland proved a similar formula on the Heisenberg group.

⁸We use the left quotient $\frac{q}{q'} = (q')^{-1}q$.

This transform is a generalization of the stereographic projection on $\mathbb{R}^n$ and is actually conformal. We define a distance on S by

$$d_S(\zeta, \eta) = 2^{-\frac{1}{2}} |1 - \zeta \cdot \bar{\eta}|^{\frac{1}{2}}. \quad (2.4)$$

It's interesting to note that there exists the following relation between distances on the sphere S and group G ,

$$\begin{aligned} d_S(\zeta, \eta) &= ((1 + |q|^2)^2 + |w|^2)^{-\frac{1}{4}} ((1 + |q'|^2)^2 + |w'|^2)^{-\frac{1}{4}} d_G(u, v) \\ &= 2^{\frac{3}{2}}^{-1} |J_C(u)|^{\frac{1}{2Q}} |J_C(v)|^{\frac{1}{2Q}} d_G(u, v). \end{aligned} \quad (2.5)$$

The Cayley transform and sphere distance both have generalized counterparts on H-type groups, see [4, 2]⁹, and [31] is also a nice book for reference.

If we take complex coordinates

$$\zeta = (\zeta_1^1, \zeta_1^2; \dots; \zeta_n^1, \zeta_n^2; \zeta_{n+1}^1, \zeta_{n+1}^2) \in \mathbb{C}^{2n+2} \quad \text{with } \zeta_j^1, \zeta_j^2 \in \mathbb{C} \text{ for } 1 \leq j \leq n+1,$$

then we can write the sphere distance (2.4) in the following complex form

$$d_S(\zeta, \eta) = 2^{-\frac{1}{2}} (|1 - \zeta \cdot_{\mathbb{C}} (\bar{\eta}^1, \bar{\eta}^2)|^2 + |\zeta \cdot_{\mathbb{C}} (-\eta^2, \eta^1)|^2)^{\frac{1}{4}},$$

where we use the complex product “ $\cdot_{\mathbb{C}}$ ”. Note that the distance d_S is not invariant under the action of complex unitary group $U(2n+2)$, so the associated integral operator with kernel $d_S(\zeta, \eta)$ is not diagonal with respect to the irreducible complex bigraded spherical harmonics decomposition of $L^2(S)$. However, this is not strange as the quaternion unitary group $Sp(n+1)$ is much smaller than $U(2n+2)$. Actually, the maximal compact subgroup (stabilize the origin point) of the isometry group $Sp(n+1, 1)$ of the quaternionic (ball-model) symmetric space is $Sp(n+1)Sp(1)$, which leaves invariant and acts transitively on the boundary sphere and its subgroup leaving the north pole fixed is isomorphic to $Sp(n)Sp(1)$, then we can realize the spherical principle series representation of $Sp(n+1)Sp(1)$ on $L^2(S) \cong L^2(Sp(n+1)Sp(1)/(Sp(n)Sp(1)))$. We need to decompose further the real spherical harmonic subspace into more $Sp(n+1)Sp(1)$ -irreducible subspaces as the action of $Sp(n+1)Sp(1)$ is contained in $O(4n+4)$. Going to details, we first have (real) $O(4n+4)$ -irreducible decomposition

$$L^2(S) = \bigoplus_{k \geq 0} \mathcal{H}_k,$$

with $\mathcal{H}_k$ being the space of k -homogeneous harmonic polynomials in real variables, and for quaternionic case, as $\mathcal{H}_k$ is also invariant under the action of $Sp(n+1)Sp(1)$, then we can decompose $\mathcal{H}_k$ further into $Sp(n+1)Sp(1)$ -irreducible parts, and finally the $Sp(n+1)Sp(1)$ -irreducible decomposition is given by

$$L^2(S) = \bigoplus_{j \geq k \geq 0} V_{j,k}, \quad (2.6)$$

where $V_{j,k} \subset \mathcal{H}_{j+k}$ is called (quaternionic) “ (j, k) -bispherical harmonic” subspace, which is generated by the action of $Sp(n+1)Sp(1)$ on the zonal spherical harmonic¹⁰ given by (Theorem 3.1 (4) in [33])

$$\begin{aligned} Z_{j,k}(\zeta) &\sim \text{Proj}_{\mathcal{H}} \left(\sum_{l=k}^{\lfloor \frac{j+k}{2} \rfloor} (-1)^l \binom{j+k-l}{l} (2\text{Re}\zeta_{n+1})^{j+k-2l} |\zeta_{n+1}|^{2l} \right) \\ &\sim \frac{\sin(j-k+1)\phi}{\sin\phi} \cos^{j-k} \theta P_k^{(2n-1, j-k+1)}(\cos 2\theta), \end{aligned} \quad (2.7)$$

with $|\zeta_{n+1}| = \cos\theta$, $\text{Re}\zeta_{n+1} = \cos\theta \cos\phi$ ($\theta \in [0, \frac{\pi}{2}]$, $\phi \in [0, \pi]$), where $\text{Proj}_{\mathcal{H}}$ is the projection operator onto harmonic subspace $\mathcal{H}$ consisting of all harmonic polynomials and $P_k^{(2n-1, j-k+1)}(z)$ is a Jacobi

⁹[4] defined a general “distance” on the boundary of solvable extension in a unifying H-type setting, $d_S(\zeta, \eta) = 2^{-\frac{1}{2}} |P_{\eta}\zeta - \eta|^{\frac{1}{2}}$, with P_{η} being the orthogonal projection operator onto $T_{\eta}^{(2)} \oplus \mathbb{R}\eta$, where $T_{\eta}^{(2)}$ is the second layer of tangent space at point η . This function d_S was proved to be a distance for and *only* for I-type, when the Cayley transform is conformal with respect to the C-C metric (see also [2] for Sobolev spaces and related operators).

¹⁰We call elements in $V_{j,k}$ (bi)spherical harmonics and define a zonal harmonic $Z_{j,k}^{\eta}(\zeta)$ in $V_{j,k}$ to be the unique element, modulo a constant, that is invariant under the action of the subgroup fixing η , which is isomorphic to $Sp(n)Sp(1)$. Uniqueness is guaranteed by [37]. When $\eta = (0, \dots, 1)$, we denote it by $Z_{j,k}(\zeta)$. Besides, the projection kernel onto $V_{j,k}$ denoted by $Z_{j,k}(\zeta, \eta)$ or $Z_{j,k}(\zeta \cdot \bar{\eta})$ is nothing but a zonal harmonic modulo a constant.

polynomial¹¹. See e.g. [33] for the classical spherical harmonic realization of spherical principle series. In other words, we get the Peter-Weyl decomposition of $L^2(S)$ under the representation action of $Sp(n+1)Sp(1)$. Using the explicit formula for the reproducing kernel of bispherical harmonic subspaces, we will give a quaternionic analogue of the Funk-Hecke formula, which is useful in proving a critical quadratic ISV inequality for sharp HLS inequality by computing the associated eigenvalues.

The sublaplacian and *conformal sublaplacian* on S are defined by

$$\mathcal{L}' = -\frac{1}{4} \sum_{1 \leq j \leq n, 0 \leq k \leq 3} (|\mathcal{C}_* X_j^k|^{-1} \mathcal{C}_* X_j^k)^2, \quad \mathcal{D} = \mathcal{L}' + n(n+1),$$

where $\mathcal{C}_*$ is the induced tangent map of the Cayley transform. The operator $\mathcal{D}$ is an analogue of the Geller-type sublaplacian on the complex sphere [22]. The fundamental solution and spectrum with regard to (2.6) of $\mathcal{D}$ are given by

$$\mathcal{D}^{-1}(\zeta, \eta) = \frac{\Gamma(\frac{Q-2}{4})\Gamma(\frac{Q-6}{4})}{2^{\frac{Q}{2}}\pi^{\frac{Q}{2}-1}} d_S^{2-Q}(\zeta, \eta), \quad \lambda_{j,k}(\mathcal{D}) = \frac{\Gamma(j + \frac{Q+2}{4})\Gamma(k + \frac{Q+2}{4} - 1)}{\Gamma(j + \frac{Q-2}{4})\Gamma(k + \frac{Q-2}{4} - 1)}, \quad (2.8)$$

and from the definitions, we have the following relation between the sublaplacian $\mathcal{L}$ on G and conformal sublaplacian $\mathcal{D}$ on S ,

$$\mathcal{L} \left((2^3 |J_{\mathcal{C}}|)^{\frac{Q-2}{2Q}} (F \circ \mathcal{C}) \right) = (2^3 |J_{\mathcal{C}}|)^{\frac{Q+2}{2Q}} (\mathcal{D}F) \circ \mathcal{C}, \quad \forall F \in C^\infty(S). \quad (2.9)$$

See e.g [2] for a proof in unifying I-type setting. Note that the constant 2^3 comes from the difference between the standard volume elements we use here and those associated with the standard contact forms for G and S . The sub-(and conformal sub-) laplacians have one kind of generalization: intertwining operators.

Intertwining operators on G and S : general intertwining operators are defined for principle (complementary) series representations of semisimple Lie groups (of real rank one), here we concern the Lorentz group $Sp(n+1, 1)$. See e.g. [35, 33, 12] and that of $SU(n+1, 1)$ was also studied in [5] for Heisenberg groups in analysis language.

Denote $Aut(G)$ the set of all conformal transformations¹² on G , any element of which is composition of translations, rotations (on q), dilations and inversion

$$\sigma_{inv} : (q, w) \mapsto \left(-\frac{q}{|q|^2 - w}, -\frac{w}{|q|^4 + |w|^2} \right). \quad (2.10)$$

It's a celebrated theorem that $Aut(G) \cong Sp(n+1, 1)$ and like the Cayley transform, this inversion is a boundary extension of an isometry¹³ on related rank one symmetric space. For the quaternionic sphere, we denote similarly the set of all conformal transformations on S by

$$Aut(S) = \{ \tau = \mathcal{C} \circ \sigma \circ \mathcal{C}^{-1} : \sigma \in Aut(G) \}.$$

Before going on, it's necessary to define the (inhomogeneous Folland-Stein) Sobolev space of exponent $(d, 2)(d > 0)$, denoted by $W^{d,2}$ here, to be the completion of $C^\infty(S)$ (or $\mathcal{D}(G)$) w.r.t the norm $\|f\|_{W^{d,2}} = \|(I + \mathcal{L}')^{\frac{d}{2}} f\|_2$ (or $\|(I + \mathcal{L})^{\frac{d}{2}} f\|_2$), where $\mathcal{D}(G)$ means the smooth function space of compact support on G . It's obvious that $\|\cdot\|_{W^{d,2}(S)}$ is equivalent to $\|\mathcal{D}^{\frac{d}{2}} \cdot\|_2$.

Then for $d \in (0, Q)$, we define the intertwining operator $\mathcal{A}_d$ on S to be any operator satisfying

$$|J_\tau|^{\frac{Q+d}{2Q}} (\mathcal{A}_d F) \circ \tau = \mathcal{A}_d \left(|J_\tau|^{\frac{Q-d}{2Q}} (F \circ \tau) \right), \quad (2.11)$$

for all $F \in C^\infty(S)$, $\tau \in Aut(S)$ and $|J_\tau|$ is the Jacobian determinant of τ . The definition states that operator $\mathcal{A}_d$ intertwines with two principle (complementary) series representations π_d, π_{-d} of $Sp(n+1, 1)$, with $\pi_d(\tau) : F \rightarrow |J_\tau|^{\frac{Q+d}{2Q}} F \circ \tau$. It was first proved in [33] (with more representation language)

¹¹Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}_{n \in \mathbb{N}}$ is a family of orthogonal polynomials associated to weight $(1-z)^\alpha(1+z)^\beta$ on interval $[-1, 1]$.

¹²A conformal transformation is a diffeomorphism preserving the contact structure.

¹³For general H-type, we have similar inversion on the solvable extension — NA -group, which is proved to be an isometry (on the rank one symmetric space) if and *only* if the group is of I-type. For inversion on H-type and I-type group, see e.g. [36, 13].

that the intertwining operator is diagonal w.r.t to the $Sp(n+1)Sp(1)$ -irreducible bispherical harmonic decomposition (2.6), and its spectrum is, modulo a constant dependent of d , uniquely given by,

$$\lambda_{j,k}(\mathcal{A}_d|_{V_{j,k}}) = \frac{\Gamma(j + \frac{Q+d}{4}) \Gamma(k + \frac{Q+d}{4} - 1)}{\Gamma(j + \frac{Q-d}{4}) \Gamma(k + \frac{Q-d}{4} - 1)}. \quad (2.12)$$

We also simulate in Appendix A the calculus of [5] to get the equivalence between this spectral decomposition and the intertwining relation (2.11). Then the operator can be extended onto Folland-Stein-Sobolev space $W^{\frac{d}{2},2}(S)$. If we choose the d -dependent constant to be 1, then by (1) in Lemma 5.5 (corresponds to $\alpha = \frac{Q-d}{4}$), the fundamental solution of $\mathcal{A}_d$ is given by

$$\mathcal{A}_d^{-1}(\zeta, \eta) = c'_d d_S^{d-Q}(\zeta, \eta) \quad (2.13)$$

with

$$c_d'^{-1} = \frac{2^{\frac{Q-d}{2}+1} \pi^{\frac{Q}{2}-1} \Gamma(\frac{d}{2})}{\Gamma(\frac{Q-d}{4}) \Gamma(\frac{Q-d}{4} - 1)}, \quad (2.14)$$

and this result is also listed in [2] and implicitly contained in [33].

The corresponding intertwining operator $\mathcal{L}_d$ on G can be defined similarly to be any operator satisfying

$$|J_\sigma|^{\frac{Q+d}{2Q}}(\mathcal{L}_d f) \circ \sigma = \mathcal{L}_d \left(|J_\sigma|^{\frac{Q-d}{2Q}}(f \circ \sigma) \right), \quad (2.15)$$

for all $\sigma \in \text{Aut}(G)$, $f \in \mathcal{D}(G)$, or equivalently by a similar relation with $\mathcal{A}_d$ to (2.9) using the Cayley transform

$$\mathcal{L}_d \left((2^3 |J_C|)^{\frac{Q-d}{2Q}}(F \circ \mathcal{C}) \right) = (2^3 |J_C|)^{\frac{Q+d}{2Q}}(\mathcal{A}_d F) \circ \mathcal{C}, \quad (2.16)$$

for all $F \in C^\infty(S)$. Take $\{\Phi_k^\lambda\}_{k \in \mathbb{N}, \lambda \in \text{Im } \mathbb{H} \cong \mathbb{R}^3}$ to be the $Sp(n)$ -spherical functions on G , which are normalized joint radial¹⁴ eigenfunctions of sublaplacian $\mathcal{L}$ and $\mathcal{T} = \left(\frac{\partial}{\partial w_1}, \frac{\partial}{\partial w_2}, \frac{\partial}{\partial w_3} \right)$, with joint spectrum $(2(k+n)|\lambda|, i\lambda)$, and their formulas are given by

$$\Phi_k^\lambda(q, w) = e^{i\lambda \cdot_{\mathbb{R}} w - |\lambda||q|^2} L_k^{2n-1}(|\lambda||q|^2), \quad (2.17)$$

where $\lambda \cdot_{\mathbb{R}} w = \sum_{j=1}^3 \lambda_j w_j$ and L_k^{2n-1} is the k -th classical Laguerre polynomial¹⁵ of order $2n-1$. Then we can define the spherical Fourier transform “ $\hat{f}$ ” on G using spherical functions (2.17), which acts on radial functions in $L^1(G)$ as

$$\hat{\cdot}: f \mapsto \hat{f}(\lambda, k) = \int_G f(u) \Phi_k^\lambda(u). \quad (2.18)$$

Generalized spherical functions and Fourier transform were studied on groups of H-type and their solvable extensions by Damek and Ricci [14]. From the intertwining relation (2.15) and using functional calculus of the spherical Fourier transform (2.18), we have that the operator $\mathcal{L}_d$ is, modulo a constant, uniquely given by

$$\widehat{\mathcal{L}_d f}(\lambda, k) = |2\lambda|^{\frac{d}{2}} \frac{\Gamma(k + \frac{Q+d}{4} - 1)}{\Gamma(k + \frac{Q-d}{4} - 1)} \hat{f}(\lambda, k), \quad \text{or} \quad \mathcal{L}_d = |2\mathcal{T}|^{\frac{d}{2}} \frac{\Gamma(|2\mathcal{T}|^{-1} \mathcal{L} + \frac{2+d}{4})}{\Gamma(|2\mathcal{T}|^{-1} \mathcal{L} + \frac{2-d}{4})}. \quad (2.19)$$

Then the operator can also be extended to the Folland-Stein-Sobolev space $W^{\frac{d}{2},2}(G)$ as before. The fundamental solution of $\mathcal{L}_d$ is given by

$$\mathcal{L}_d^{-1}(u, v) = c_d d_G^{d-Q}(u, v) \quad (2.20)$$

with

$$c_d = 2^{Q-d-3} c_d', \quad (2.21)$$

where c_d' is defined in (2.14). We can move the argument on the sphere directly onto the group to prove (2.19) and (2.20), which is also contained in [12], where similar operators on the nilpotent subgroups of all rank one semisimple Lie groups were studied.

¹⁴We use “radial” here to mean that the function only depends on $|q|$ and w .

¹⁵Laguerre polynomials $\{L_n^\alpha\}_{n \in \mathbb{N}}$ is a family of orthogonal polynomials with regards to weight $e^{-x} x^\alpha$ on $(0, \infty)$.

The intertwining operators are one kind of generalization of fractional (conformal) sublaplacians, whose second order counterparts are just the (conformal) sublaplacian defined above respectively on the sphere and group, and especially on the Euclidean space and sphere, $\mathcal{L}_d = (-\Delta)^{\frac{d}{2}}$ and $\mathcal{A}_d$ is the spherical picture obtained from $(-\Delta)^{\frac{d}{2}}$ by the stereographic projection. We can also try to consider the intertwining operator at $d = Q$, then we may get corresponding analogue of Beckner-Onofri inequality, which is not included in this paper. Above results were all discussed for the Heisenberg group (and complex sphere) in [5] and can also be derived from the theory of Knapp-Stein intertwining operators in [35], both in the compact and noncompact pictures.

3 Main Results

Return to the HLS inequality, we now can state our main sharp HLS and related HLS-type theorems. The HLS with general exponents states (in the Hermitian form) that

$$\left| \iint_{G \times G} \frac{\overline{f(u)}g(v)}{|u^{-1}v|^\lambda} dudv \right| \lesssim \|f\|_p \|g\|_q, \quad \frac{1}{p} + \frac{1}{q} + \frac{\lambda}{Q} = 2,$$

which also have a fractional integral operator form: $\|f * |u|^{-\lambda}\|_{q'} \lesssim \|f\|_p$. Define the associated functional

$$I(f, g) := \left| \iint_{G \times G} \frac{\overline{f(u)}g(v)}{|u^{-1}v|^\lambda} dudv \right| / (\|f\|_p \|g\|_q),$$

then our extremal problem is to ask whether there is a non-vanishing function pair (f_*, g_*) , s.t.,

$$I(f_*, g_*) = \sup_{f \in L^p, g \in L^q, f, g \neq 0} I(f, g),$$

and if exists, is it unique, in what sense and furthermore can we figure out all of these function pairs using elementary functions? We call this *a priori* function pair an *extremizer pair* and for the quadratic functional $I(f) := I(f, f)$, we call f_* an (quadratic) *extremizer* if $I(f_*) = \sup_{0 \neq f} I(f)$. Any sequence $\{f_j\}$ s.t. $I(f_j)$ converges to the supremum is called an *extremizing sequence* while the finite supremum of the functional is called the *sharp constant*. For this HLS functional, we have the existence of extremizer pairs and can figure out all the extremizers for symmetric exponents $p = q = \frac{2Q}{2Q-\lambda}$. We will use the same notations for counterparts on the sphere.

3.1 Sharp HLS Inequalities

Theorem 3.1. [Sharp HLS on the Quaternionic Heisenberg Group]

Let $4 \leq \lambda < Q = 4n + 6, p = \frac{2Q}{2Q-\lambda}$, then $\forall f, g \in L^p(G)$,

$$\left| \iint_{G \times G} \frac{\overline{f(u)}g(v)}{|u^{-1}v|^\lambda} dudv \right| \leq C_\lambda \|f\|_p \|g\|_p \quad (3.1)$$

with sharp constant

$$\begin{aligned} C_\lambda &= \frac{2^{1-\frac{2n\lambda}{Q}} \pi^{2n+2} \Gamma(\frac{Q-\lambda}{2}) |S|^{1-\frac{2}{p}}}{\Gamma(\frac{2Q-\lambda}{4}) \Gamma(\frac{2Q-\lambda}{4} - 1)} \\ &= \left(\pi^{\frac{Q-2}{2}} \right)^{\frac{\lambda}{Q}} \frac{((\frac{Q-4}{2})!)^{1-\frac{\lambda}{Q}} \Gamma(\frac{Q-\lambda}{2})}{\Gamma(\frac{2Q-\lambda}{4}) \Gamma(\frac{2Q-\lambda}{4} - 1)}, \end{aligned} \quad (3.2)$$

where $|S| = \frac{2\pi^{2n+2}}{(2n+1)!}$ is the surface area of S . Moreover, all extremizers are given by

$$f \sim g \sim (|J_C \circ \sigma| |J_\sigma|)^{\frac{1}{p}} \sim ||q|^2 + w - 2q_0 \cdot \bar{q} + r_0|^{-\frac{2Q-\lambda}{2}}, \quad (3.3)$$

with $\sigma \in \text{Aut}(G), q_0 \in \mathbb{H}^n, r_0 \in \mathbb{H}$, satisfying $\text{Re } r_0 > |q_0|^2$, and we can choose the following correspondence: $\sigma = D_{\delta_0} \circ L_{u_0}$ or $D_{\delta} \circ C^{-1} \circ A_\xi \circ C$, where D_{δ_0}, D_δ are dilations, L_{u_0} is a left translation and A_ξ is a rotation in $Sp(n+1)$ s.t. $A_\xi^{-1}(0, \dots, 0, 1) = \frac{\xi}{|\xi|}$, with parameters $\delta_0 = (\text{Re } r_0 - |q_0|^2)^{-\frac{1}{2}}, u_0 = (q_0, -\text{Im } r_0)$ and $\xi = (\frac{2q_0}{r_0+1}, \frac{r_0-1}{r_0+1}), \delta = \sqrt{\frac{1 \mp |\xi|}{1 \pm |\xi|}}$.

Using the Cayley transform, we can give the sphere edition of last theorem: with the relation (2.5) between two distances on G and S , we have the following theorem, an equivalent sphere edition of sharp HLS inequality theorem, through the following correspondence between functions f on G and $\tilde{f}$ on S ,

$$\tilde{f}(\zeta) = f(C^{-1}\zeta)|J_{C^{-1}}|^{\frac{1}{p}}. \quad (3.4)$$

Theorem 3.2. [Sharp HLS on the Quaternionic Sphere]

Let $4 \leq \lambda < Q = 4n + 6, p = \frac{2Q}{2Q-\lambda}$, then $\forall f, g \in L^p(S)$,

$$\left| \iint_{S \times S} \frac{\overline{\tilde{f}(\zeta)} \tilde{g}(\eta)}{d_S^\lambda(\zeta, \eta)} d\zeta d\eta \right| \leq C'_\lambda \|f\|_p \|g\|_p \quad (3.5)$$

with sharp constant

$$\begin{aligned} C'_\lambda &= 2^{\frac{(4n+3)\lambda}{Q}} C_\lambda \\ &= \frac{2^{1+\frac{\lambda}{2}} \pi^{2n+2} \Gamma(\frac{Q-\lambda}{2}) |S|^{1-\frac{2}{p}}}{\Gamma(\frac{2Q-\lambda}{4}) \Gamma(\frac{2Q-\lambda}{4} - 1)} \\ &= \left(\frac{2\pi^{\frac{Q-2}{2}}}{(\frac{Q-4}{2})!} \right)^{\frac{\lambda}{Q}} \frac{2^{\frac{\lambda}{2}} (\frac{Q-4}{2})! \Gamma(\frac{Q-\lambda}{2})}{\Gamma(\frac{2Q-\lambda}{4}) \Gamma(\frac{2Q-\lambda}{4} - 1)}, \end{aligned} \quad (3.6)$$

where $|S| = \frac{2\pi^{2n+2}}{(2n+1)!}$ is the surface area of S . Moreover, all extremizers are given by

$$f \sim g \sim |J_\tau|^{\frac{1}{p}} \sim |1 - \xi \cdot \bar{\zeta}|^{-\frac{2Q-\lambda}{2}}, \quad (3.7)$$

with $\tau \in \text{Aut}(S), \xi \in \mathbb{H}^{n+1}, |\xi| < 1$, and we can choose the following correspondence: $\tau = C \circ D_{\delta_0} \circ L_{u_0} \circ C^{-1}$ or $C \circ D_\delta \circ C^{-1} \circ A_\xi$, where D_{δ_0}, D_δ are group dilations, L_{u_0} is a group left translation, with parameters $\delta_0 = \frac{|1+\xi_{n+1}|}{\sqrt{1-|\xi|^2}}, u_0 = (\frac{\xi'}{1+\xi_{n+1}}, -\text{Im} \frac{1-\xi_{n+1}}{1+\xi_{n+1}}), \delta = \sqrt{\frac{1+|\xi|}{1-|\xi|}}$, and A_ξ is a rotation in $Sp(n+1)$ s.t. $A_\xi^{-1}(0, \dots, 0, 1) = \frac{\xi}{|\xi|}$.

At first, we give several small *remarks* about the two theorems.

- Extremizer for (3.1) and (3.5) exists for all $0 < \lambda < Q$ and even nonsymmetric exponents. Several a bit standard methods can be used to prove the existence while compactness is the basic idea. We will still borrow the ingenious argument from [19] and give the proof for $\lambda \geq 4$.
- The large conformal symmetry group of the HLS inequality (3.1) consists of left-translations, dilations, rotations (on q variable) and inversion: $f(u) \mapsto f(\sigma_{inv} u) |u|^{-\frac{2Q}{p}}$ (a Kelvin-type transform), where σ_{inv} is the inversion transform (2.10). In other words, the inequality is invariant under the conformal action $f \mapsto f \circ \sigma |J_\sigma|^{\frac{1}{p}}, \forall \sigma \in \text{Aut}(G)$. Similarly, from the correspondence (3.4), the HLS inequality on the sphere (3.5) is invariant under the conformal action $f \mapsto f \circ \tau |J_\tau|^{\frac{1}{p}}, \forall \tau \in \text{Aut}(S)$. We can also see these from the following relations:

$$\begin{aligned} d_G(\sigma(u), \sigma(v)) &= (|J_\sigma(u)| |J_\sigma(v)|)^{\frac{1}{2Q}} d_G(u, v), \quad \forall \sigma \in \text{Aut}(G) \\ d_S(\tau(\zeta), \tau(\eta)) &= (|J_\tau(\zeta)| |J_\tau(\eta)|)^{\frac{1}{2Q}} d_S(\zeta, \eta), \quad \forall \tau \in \text{Aut}(S). \end{aligned} \quad (3.8)$$

- Moreover, in Theorem 3.1, modulo *only* constant multiples, left-translations and dilations, extremizer exists uniquely, i.e., sharp equality (3.1) holds if and only if

$$f = g = ((1 + |q|^2)^2 + |w|^2)^{-\frac{2Q-\lambda}{4}}.$$

Similarly, in Theorem 3.2, modulo *only* constant multiples, quaternionic rotations, and group dilations under the conjugate action of the Cayley transform, extremizer exists uniquely, i.e., sharp equality (3.5) holds if and only if

$$f = g = 1.$$

- For finding the explicit formula for extremizers, the core we want in this note, we select a special extremizer which is proved to be constant for $\lambda > 4$ and be in direct sum of subspaces $V_{0,0} \oplus_{j \geq k \geq 2} V_{j,k}$ (see (2.6)) for $\lambda = 4$. Fortunately, for $\lambda = 4$, we can recur to the Euler-Lagrange equation of first variation to restrict further the extremizer to be in $V_{0,0}$ which consists of constant functions. We can also see from the original functional of the inequality. For $\lambda < 4$, we say constant 1 is a “weak local” extremizer and guess it be global.

We leave the proofs of the two main theorems in later sections and want to give some related sharp inequalities first.

3.2 Related Sharp Sobolev-type Inequalities

We can also write above sharp HLS inequalities in dual CIS inequalities¹⁶ concerning intertwining operators, noticing that the fundamental solutions of intertwining operator $\mathcal{A}_d$ on S and $\mathcal{L}_d$ on G are a constant multiple of respectively $d_S(\zeta, \eta)^{d-Q}$ and $d_G(u, v)^{d-Q}$. See (2.13) and (2.19) and here we set $d \in (0, Q)$.

Corollary 3.3. [Sharp CIS Inequality]

(1) Let $0 < d \leq Q - 4, p' = \frac{2Q}{Q-d}$, then $\forall f \in W^{\frac{d}{2}, 2}(G)$,

$$\int_G \bar{f} \mathcal{L}_d f \geq \tilde{C}_d \|f\|_{p'}^2, \quad (3.9)$$

with sharp constant

$$\tilde{C}_d = (c_d C_{Q-d})^{-1} = \left(\frac{2^4 \pi^{\frac{Q}{2}-1}}{(\frac{Q}{2}-2)!} \right)^{-\frac{d}{Q}} \frac{\Gamma(\frac{Q-d}{4}) \Gamma(\frac{Q-d}{4} - 1)}{\Gamma(\frac{Q+d}{4}) \Gamma(\frac{Q+d}{4} - 1)}, \quad (3.10)$$

and all extremizers

$$f \sim | |q|^2 + w - 2q_0 \cdot \bar{q} + r_0 |^{-\frac{Q-d}{2}}, \quad (3.11)$$

with $q_0 \in \mathbb{H}^n, r_0 \in \mathbb{H}$, satisfying $\text{Re } r_0 > |q_0|^2$.

(2) Let $0 < d \leq Q - 4, p' = \frac{2Q}{Q-d}$, then $\forall f \in W^{\frac{d}{2}, 2}(S)$,

$$\int_S \bar{f} \mathcal{A}_d f \geq \tilde{C}'_d \|f\|_{p'}^2, \quad (3.12)$$

with sharp constant

$$\tilde{C}'_d = (c'_d C'_{Q-d})^{-1} = \left(\frac{2\pi^{\frac{Q}{2}-1}}{(\frac{Q}{2}-2)!} \right)^{-\frac{d}{Q}} \frac{\Gamma(\frac{Q-d}{4}) \Gamma(\frac{Q-d}{4} - 1)}{\Gamma(\frac{Q+d}{4}) \Gamma(\frac{Q+d}{4} - 1)}, \quad (3.13)$$

and all extremizers

$$f \sim |1 - \xi \cdot \bar{\zeta}|^{-\frac{Q-d}{2}}, \quad (3.14)$$

with $\xi \in \mathbb{H}^{n+1}, |\xi| < 1$.

Constant c_d, c'_d are defined by (2.21) and 2.14) in preliminary section, and C_{Q-d}, C'_{Q-d} are given by (3.2) and (3.6) in Theorem 3.1 and 3.2.

It's obvious that we only need to consider the inequality for real-valued, or even nonnegative functions. Besides, we can easily checked that there exists a similar conformal symmetry group, i.e. the inequalities are invariant under the action of the transformations $f \mapsto f \circ \sigma |J_\sigma|^{\frac{1}{p'}}$, $\forall \sigma \in \text{Aut}(G)$ (or $\text{Aut}(S)$).

Also, as in [19], we can give the endpoint limit case of sharp HLS inequality at $\lambda = Q$, using standard functional differentiation argument. The endpoint case corresponds to a LogS inequality. We list it in the sphere framework. For another endpoint $\lambda = 0$, we lack the sharp HLS inequality of small exponent to obtain corresponding sharp Log-HLS inequality.

¹⁶We call this fractional Sobolev inequality (FS) in [41].

Corollary 3.4. [Sharp LogS Inequality]

$\forall f \geq 0 \in L^2 \text{Log} L(S)$, normalized by $\int_S f^2 = |S|$,

$$\iint_{S \times S} \frac{|f(\zeta) - f(\eta)|^2}{d_S^Q(\zeta, \eta)} d\zeta d\eta \geq C \int_S f^2 \log f^2, \quad (3.15)$$

with sharp constant

$$C = \frac{2^{\frac{Q}{2}+3} \pi^{\frac{Q}{2}-1}}{Q \Gamma(\frac{Q}{4}-1) \Gamma(\frac{Q}{4})}, \quad (3.16)$$

and **some** extremizers

$$f \sim |1 - \xi \cdot \bar{\zeta}|^{-\frac{Q}{2}}, \quad (3.17)$$

satisfying normalized condition with $\xi \in \mathbb{H}^{n+1}$, $|\xi| < 1$.

Now we give a simple proof of Corollary 3.3 and 3.4. First, we assume the validity of Theorem 3.1 and 3.2.

Proof of Corollary 3.3. Because of relationship between $\mathcal{L}_d$ and $\mathcal{A}_d$ and similar arguments, we only prove, for example, on the group. We may restrict functions to be positive, however, our proof goes for all complex ones. Note the exponent p' in this dual theorem is the conjugate of that p in the main theorems given in last subsection under the correspondence $\lambda + d = Q$. We use $\langle \cdot, \cdot \rangle$ for L^2 -inner product with associated measure in the context, then by Plancherel formula, Cauchy-Schwartz inequality and properties (2.19) and (2.20) of intertwining operators, we have

$$\begin{aligned} \langle f, g \rangle^2 &= \langle \hat{f}, \hat{g} \rangle^2 = \left\langle \left(|2\lambda|^{\frac{d}{2}} \frac{\Gamma(k + \frac{Q+d}{4} - 1)}{\Gamma(k + \frac{Q-d}{4} - 1)} \right)^{\frac{1}{2}} \hat{f}, \left(|2\lambda|^{\frac{d}{2}} \frac{\Gamma(k + \frac{Q+d}{4} - 1)}{\Gamma(k + \frac{Q-d}{4} - 1)} \right)^{-\frac{1}{2}} \hat{g} \right\rangle^2 \\ &\leq \langle \hat{f}, \widehat{\mathcal{L}_d f} \rangle \langle \hat{g}, \widehat{\mathcal{L}_d^{-1} g} \rangle \\ &= \langle \hat{f}, \widehat{\mathcal{L}_d f} \rangle \langle \hat{g}, c_d |u|^{\widehat{d-Q}} * g \rangle. \end{aligned}$$

So, we have the following equivalent form

$$|\langle f, g \rangle|^2 \leq c_d \langle f, \mathcal{L}_d f \rangle \langle g, |u|^{d-Q} * g \rangle,$$

and from the L^p norm characterization and sharp HLS inequality (3.1), we have

$$\begin{aligned} \|f\|_{p'}^2 &\leq c_d \int_G \bar{f} \mathcal{L}_d f \sup_{\|g\|_p=1} \iint_{G \times G} \frac{\overline{g(u)} g(v)}{d_G^{Q-d}(u, v)} du dv \\ &= c_d C_{Q-d} \int_G \bar{f} \mathcal{L}_d f, \end{aligned}$$

and “=” holds if and only if g is an extremizer for the sharp HLS inequality (3.1) and $g \sim f^{p'-1}$, i.e. together with Theorem 3.1, this inequality is sharp with sharp constants (3.10) and all extremizers are given by (3.11) in the corollary. Corollary 3.3 is then proved. $\square$

Actually, from the obvious opposite direction, we can see the equivalence of the sharp CIS-type inequality and HLS inequality. In applications, it depends on cases to choose for convenience which formula to use.

Proof of Corollary 3.4. Step 1. *Inequality (3.15).* From the normalized assumption,

$$\iint_{S \times S} \frac{f^2(\zeta) + f^2(\eta)}{d_S^\lambda(\zeta, \eta)} d\zeta d\eta = 2C'_\lambda |S|^{\frac{2}{p}}.$$

Subtracting two mutiple of the sharp HLS inequality (3.5) and taking limitation $\lambda \rightarrow Q$, we get

$$\begin{aligned} \iint_{S \times S} \frac{|f(\zeta) - f(\eta)|^2}{d_S^Q(\zeta, \eta)} d\zeta d\eta &\geq \lim_{\lambda \rightarrow Q} 2C'_\lambda (|S|^{\frac{2}{p}} - \|f\|_p^2) \\ &= \frac{2^{\frac{Q}{2}+3} \pi^{2n+2}}{\Gamma(\frac{Q}{4}-1) \Gamma(\frac{Q}{4})} \lim_{\lambda \rightarrow Q} \frac{|S|^{\frac{2}{p}} - \|f\|_p^2}{Q - \lambda}. \end{aligned}$$

It's easy to check¹⁷

$$\frac{|S|^{\frac{2}{p}} - \|f\|_p^2}{Q - \lambda} = \frac{p - 2}{Q - \lambda} \frac{|S|^{\frac{2}{p}} - \|f\|_p^2}{p - 2} \xrightarrow{\lambda \rightarrow Q} \left(-\frac{2}{Q}\right) \left(-\frac{\int_S f^2 \log f^2}{2}\right) = \frac{\int_S f^2 \log f^2}{Q}.$$

Constant (3.16) is got, then the theorem is proved after checking equality can be achieved by any function of (3.17), which is the limit of extremizers (3.7) for the sharp HLS inequality (3.5) as $\lambda \rightarrow Q$.

Step 2. *Sharpness*. Actually, if we denote

$$h \sim |1 - \xi \cdot \bar{\eta}|^{-\frac{Q}{2}}, \quad h_\lambda \sim |1 - \xi \cdot \bar{\eta}|^{-\frac{2Q-\lambda}{2}},$$

both satisfying the normalized condition, then

$$h_\lambda = \left(\frac{|S|}{\int_S h^{\frac{4}{p}}} \right)^{\frac{1}{2}} h^{\frac{2}{p}}, \quad h_\lambda \rightrightarrows h(\lambda \rightarrow Q)$$

and we have

$$\iint_{S \times S} \frac{|h_\lambda(\zeta) - h_\lambda(\eta)|^2}{d_S^\lambda(\zeta, \eta)} d\zeta d\eta = 2C'_\lambda (|S|^{\frac{2}{p}} - \|h_\lambda\|_p^2).$$

As $\lambda \rightarrow Q$, by care checking of the commutation of integral and limitation, this integral inequality converges to (3.15), replacing f by h , because of the uniformly convergence $h_\lambda \rightrightarrows h$. Another way to prove the sharpness of (3.15) is using the extremizing function¹⁸ $f(\zeta) = \sqrt{1 - \epsilon} + \epsilon \operatorname{Re} \zeta_{n+1}$ formerly used by [19] for the Heisenberg group problem. $\square$

Remark. What about *uniqueness*? This is open. Take any extremizer h for (3.15), then if we can prove some approximation of h to the extremizing submanifold E_λ of the sharp HLS (3.5), consisting of all extremizers (3.3), e.g. the L^p convergence $d_p(h, E_\lambda) \rightarrow 0$, then h can only be the form of (3.17). Actually, in [41], we have a stability like

$$C'_\lambda \|f\|_p^2 - \iint_{S \times S} \frac{\overline{f(\zeta)} f(\eta)}{d_S(\zeta, \eta)} d\zeta d\eta \gtrsim_\lambda d_p^2(f, E_\lambda), \quad (3.18)$$

and if the constant in the right side have low bound or is infinitesimal with low order < 1 as $\lambda \rightarrow Q$, then we have the uniqueness conclusion for our LogS inequality, after replacing f with h , but unfortunately the best constant for (3.18) is still unknown.

The rest of this paper is given to the proof of our main results of sharp HLS inequalities on the quaternionic group and sphere (Theorem 3.1 and 3.2). In order to express clearly the idea thread, we split it into steps.

4 Existence of Extremizers

For completeness, we here prove a partial result for our purpose using an ingenious Frank-Lieb argument in [19], which can be easily transformed to our case, considering compactness argument, Fatou lemma and a refined HLS. Positive definiteness is very important here to simplify the problem to a L^2 estimate. Concentration compactness from [40] is an alternative method, which works perfectly for general non-symmetric exponents, see [24] for the Heisenberg group case.

Existence of extremizers for $\lambda \geq 4$. We now demonstrate Frank-Lieb method on group G , considered to be a relatively simpler way. First, we note that the kernel — negative power of distance $|u|^{-\lambda}$ is positive definite for $\lambda > 4$ and semi-positive definite for $\lambda = 4$; Actually, it's the fundamental solution of intertwining operators $\mathcal{L}_d^{-1} \sim |u|^{-\lambda}$, with $d = Q - \lambda$, see (2.20) and we will also compute the eigenvalues of the distance kernel in Lemma 5.5 in subsection 5.4. So, the maximum problem can be considered in the quadratic form (for $f \sim g$). Now we assume $\lambda > 4$ as $\lambda = 4$ case comes from continuity argument¹⁹.

¹⁷For any $1 < p < 2$, function $|f|^p \log |f|$, the derivative of $|f|^p$ relative to p , is controlled by $|f|^2 \log |f|$ when $|f| \gtrsim 1$ and by $|f| \log^{1/2} |f|$ when $|f| \lesssim 1$.

¹⁸Here we compute the difference functional using Lemma 5.5 as both the two sides in (3.15) vanish as $\epsilon \rightarrow 0$ to order ϵ^2 .

¹⁹We find the formula of extremizers in our main theorem is continuous with regard to λ , but pay attention that this *only* tells from the result for $\lambda > 4$ that for $\lambda = 4$ the corresponding functions are (but *not necessarily*) extremizers. Then are they right all the extremizers? The answer is yes, which completes the proof of our range and we will see this later.

Lemma 4.1. *Let $4 < \lambda < Q$, then there exists a positive definite, real-valued, even and homogeneous of order $-\frac{Q+\lambda}{2}$ kernel $k \in L_w^{\frac{2Q}{Q+\lambda}}$ satisfying*

$$|u|^{-\lambda} = k * k.$$

Proof. Fix $d = Q - \lambda$, then from the positive definiteness and relation (2.19), we have for $\lambda > 4$ that as operators $\sqrt{|u|^{-\lambda}} \sim \mathcal{L}_d^{-\frac{1}{2}}$, which is again positive definite and self-adjoint, and the corresponding convolution kernel

$$k \sim \mathcal{L}_d^{-\frac{1}{2}} \delta_0 \sim \mathcal{L}_d^{-\frac{1}{2}} \mathcal{L}_{\frac{d}{2}} |u|^{-\frac{Q+\lambda}{2}} \sim m(|2\mathcal{T}|^{-1} \mathcal{L}) |u|^{-\frac{Q+\lambda}{2}}, \quad \text{where} \quad m(\cdot) = \sqrt{\frac{\Gamma(\cdot + \frac{2-d}{4}) \Gamma(\cdot + \frac{4+d}{8})}{\Gamma(\cdot + \frac{2+d}{4}) \Gamma(\cdot + \frac{4-d}{8})}},$$

where δ_0 is the standard Dirac function at zero. From the commutation of $m(|2\mathcal{T}|^{-1} \mathcal{L})$ with inversion and dilation, it transmits the even and homogeneous nature of $|u|^{-\frac{Q+\lambda}{2}}$ to k , which is also real-valued from the self-adjointness. The weak Lebesgue property comes from the Marcinkiewicz interpolation and the following multiplier lemma due to Müller, Ricci and Stein [42]. We only need to check the Marcinkiewicz condition for the spectral multiplier $m(\cdot)$ of the essentially self-adjoint operator $|2\mathcal{T}|^{-1} \mathcal{L}$:

$$|(x\partial_x)^l \log m(x)| = \left| (x\partial_x)^l \int_{\mathbb{R}} \phi(t - (x + \frac{1}{2})) (\log \Gamma)'(t) dt \right| \lesssim_l 1,$$

where

$$\phi(t) = \begin{cases} 1/2 & |t| \leq d/8 \\ -1/2 & d/8 < |t| \leq d/4 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad (\log \Gamma)'(t) = \log t - \frac{1}{2t} - 2 \int_{\mathbb{R}^+} \frac{s ds}{(s^2 + t^2)(e^{2\pi s} - 1)}.$$

Then this lemma is proved after above easy computation. $\square$

Lemma 4.2. *The multiplier $m(\cdot)$ is strong-type (p, p) for any $1 < p < \infty$, i.e.,*

$$\|m(|2\mathcal{T}|^{-1} \mathcal{L})f\|_p \lesssim \|f\|_p \quad \forall f \in \mathcal{S}(G),$$

if m satisfies the following Marcinkiewicz condition

$$|(x\partial_x)^l \log m(x)| \lesssim_l 1 \quad \forall x \in [n, \infty), \quad l \in \mathbb{N}.$$

Then the sharp problem changes to that of

$$\|f * k\|_q \lesssim \|f\|_2, \quad q = p' = \frac{2Q}{\lambda} \text{ is the conjugate exponent.} \quad (4.1)$$

Here L^2 is a special Hilbert space with good compactness. When using the compactness argument for above inequality, we should be alert as the weak-limit of extremizing sequence is easy to vanish. So, the following “quaternionic edition” of an enforced estimate called “refined HLS” inequality kills this bug perfectly. See [3] for the analogue on the Heisenberg group, which was first used by Gérard [23] to prove the existence of extremizers of the Sobolev inequality on $\mathbb{R}^n$.

Lemma 4.3. *Let $4 \leq \lambda < Q$, $q = \frac{2Q}{\lambda}$, $k \sim \mathcal{L}_d^{-\frac{1}{2}} \delta$, then*

$$\|f * k\|_q \lesssim \|f\|_2^{\frac{\lambda}{Q}} \left(\sup_{\beta > 0} \beta^{\frac{\lambda}{4}} \|e^{-\beta \mathcal{L}} \beta \mathcal{L}(f * k)\|_{\infty} \right)^{\frac{Q-\lambda}{Q}},$$

with $e^{-\beta \mathcal{L}}$ being the heat semigroup of the sublaplacian.

Proof. We borrow the idea from the proof of the Euclidean and Heisenberg group analogue. First we prove the following inequality

$$\|g\|_q \lesssim \|\mathcal{L}^{\frac{d}{2}} g\|_2 (B_g^{\lambda})^{\frac{d}{Q}}$$

where $B_g^{\lambda} := \sup_{\beta > 0} \beta^{\frac{\lambda}{4}} \|e^{-\beta \mathcal{L}} \beta \mathcal{L} g\|_{\infty}$, then the lemma is proved from the L^2 bound $\|\mathcal{L}^{\frac{d}{2}}(f * k)\|_2 \lesssim \|f\|_2$, which is again got by the spectral multiplier theorem — Lemma 4.2. For the inequality of g , we write $g \in \mathcal{S}(G)$ into an integral of the heat flow of the sublaplacian operator,

$$g = \int_0^{\infty} e^{-t\mathcal{L}} \mathcal{L} g dt = \left(\int_0^A + \int_A^{\infty} \right) e^{-t\mathcal{L}} \mathcal{L} g dt = g_1^A + g_2^A,$$

leaving A a constant to be fixed later. From the following pointwise estimate

$$\begin{aligned} |g_1^A| &= \left| \int_0^A (-t)^{\frac{d}{2}-1} \left(e^{-t\mathcal{L}} (-t\mathcal{L})^{1-\frac{d}{2}} \right) \mathcal{L}^{\frac{d}{2}} f dt \right| \lesssim A^{\frac{d}{2}} |M(\mathcal{L}^{\frac{d}{2}} g)| \\ |g_2^A| &\lesssim B_g^\lambda \int_A^\infty t^{-(1+\frac{\lambda}{4})} dt \lesssim A^{-\frac{\lambda}{4}} B_g^\lambda, \end{aligned}$$

where M is the maximal function operator and the first estimate follows from the rapidly decreasing of the heat kernel of the sublaplacian, we have after taking $A = (B_g^\lambda M(\mathcal{L}^{\frac{d}{2}} g))^{\frac{d}{\lambda}}$ that $|g| \lesssim (B_g^\lambda)^{\frac{d}{\lambda}} (M(\mathcal{L}^{\frac{d}{2}} g))^{\frac{\lambda}{d}}$. Then the lemma is proved from the (p, p) -boundness of M . For the property of the maximal function operator on groups, see e.g. [44]. $\square$

We *remark* here that the g inequality is actually a special case of an embedding relation between Lebesgue, homogeneous Sobolev and Besov norms; More precisely, for any $1 \leq r \leq \infty, 0 < s < \frac{Q}{r}, q = \frac{rQ}{Q-rs}$, we have

$$\|f\|_q \lesssim \|f\|_{\dot{W}^{s,r}(G)}^{1-\frac{sr}{Q}} \|f\|_{\dot{B}_{\infty,\infty}^{s-\frac{Q}{r}}}^{\frac{sr}{Q}}.$$

Then our lemma follows from the following characterization of the Besov norm

$$\|f\|_{\dot{B}_{\infty,\infty}^{-s}} \sim \sup_{t>0} t^{\frac{s}{2}} \|(t\mathcal{L})^k e^{-t\mathcal{L}} f\|_\infty \sim B_f^{2s} \quad \forall k \in \mathbb{N}^+, s > 0,$$

and see [20] for the definition and properties of the Besov norms and spaces on general stratified Lie groups, where both continuous and discrete wavelet characterization was specially given. We now give a compactness lemma due to Brezis and Lieb, and see e.g. [39] for its application to classical HLS.

Lemma 4.4. *For a norm-bounded sequence $\{f_j\} \subset L^p$ on a measure space, i.e., $\|f_j\|_p \lesssim 1$, if pointwise convergence happens, i.e., $f_j \rightarrow f$, a.e., then we have a refined Fatou-type formula for L^p norm*

$$\int |f_j|^p - |f|^p - |f_j - f|^p \xrightarrow{j \rightarrow \infty} 0.$$

Proposition 4.5. *Let $\{f_j\}$ be any normalized extremizing sequence, i.e., $\|f_j\|_2 = 1$ and $I(f_j) := \frac{\|f_j * k\|_q}{\|f_j\|_2}$ converges to the supremum of this functional, then after moving onto a subsequence, there exist two sequences $\{u_j\} \subset G$ and $\{\delta_j\} \subset \mathbb{R}^+$, s.t. after conformal transformation $f_j \mapsto \delta_j^{Q/2} f_j(\delta_j u_j^{-1} \cdot)$, we get a new strong L^2 convergent sequence, with the limit being an extremizer.*

Proof. Step 1. Under the conformal actions of dilations and left translations, we can get a new sequence f_j , which weakly converges to a non-vanishing function f . This follows from refined HLS — Lemma 4.3. Actually, $B_{g_j}^\lambda \gtrsim 1$ with $g_j = f_j * k$, then there exist $\{u_j\}, \{\delta_j\}$, s.t., $|\delta_j^{\frac{\lambda}{4}} e^{-\delta_j \mathcal{L}} \delta_j \mathcal{L} g_j(u_j)| \gtrsim 1$. If we denote by ϕ the Schwartz kernel associated to $e^{-\mathcal{L}} \mathcal{L}$, then we have

$$|\delta_j^{\frac{\lambda}{4}} f_j * k * \phi_{\delta_j^{\frac{1}{2}}}(u_j)| \gtrsim 1,$$

where ϕ_δ is the dilation given by $\delta^{-Q} \phi(\delta^{-1} \cdot)$, and denote $D_\delta f = f(\delta \cdot)$, then

$$\begin{aligned} \delta_j^{\frac{\lambda}{4}} f_j * k * \phi_{\delta_j^{\frac{1}{2}}}(u_j) &= \delta_j^{\frac{\lambda}{4}} f_j * (D_{\delta_j^{-\frac{1}{2}}}(D_{\delta_j^{\frac{1}{2}}} k * \phi))(u_j) = \delta_j^{-\frac{Q}{4}} f_j * D_{\delta_j^{-\frac{1}{2}}}(k * \phi)(u_j) \\ &= \int_G \delta_j^{\frac{Q}{4}} f_j(u_j \delta_j^{\frac{1}{2}} v) (k * \phi)(v^{-1}) dv =: \int_G f_{u_j}^{\delta_j}(k * \phi) \tilde{\cdot}. \end{aligned}$$

Here $\tilde{\cdot}$ denotes inversion $\tilde{f}(v) := f(v^{-1})$ and we will still use f_j for the new sequence $f_{u_j}^{\delta_j}$ in last equality, which is still normalized. From Alaoglu's compactness principle, we know there exists a function $f \in L^2$, s.t., $f_j \xrightarrow[j]{weak L^2} f$ and it follows from the weak Young inequality and the fact $k \in L_w^{\frac{2Q}{Q+\lambda}}$ (Lemma 4.1) that $k * \phi \in L^2$, which tells then

$$\left| \int_G f_j(k * \phi) \right| \rightarrow \left| \int_G f(k * \phi) \right| \gtrsim 1$$

and finally $f \neq 0$.

Step 2. For above new subsequence, we have $f_j * k \rightarrow f * k$, a.e.. For this, we recall from the proof of Lemma 4.1 that $k \sim m(|2\mathcal{T}|^{-1}\mathcal{L})|u|^{-\frac{Q+\lambda}{2}}$ and $T := m(|2\mathcal{T}|^{-1}\mathcal{L})$ is an operator of strong-type (p, p) , especially $L^2 \rightarrow L^2$, so $\{Tf_j\}$ is again weak compact and the weak limit is nothing but Tf . Then the problem is equivalent to

$$Tf_j * |u|^{-\frac{Q+\lambda}{2}} \rightarrow Tf * |u|^{-\frac{Q+\lambda}{2}}, \text{ a.e..}$$

Now, we prove a stronger²⁰ L^r_{loc} convergence for any $r < \frac{2Q}{\lambda}$. Denote

$$k_1 = \chi_{|u| \geq \epsilon} |u|^{-\frac{Q+\lambda}{2}}, \quad k_2 = \chi_{|u| \leq \epsilon} |u|^{-\frac{Q+\lambda}{2}},$$

then

$$\|T(f_j - f) * k\|_{L^r_{loc}} \leq \|T(f_j - f) * k_1\|_{L^r_{loc}} + \|T(f_j - f) * k_2\|_{L^r_{loc}}.$$

For the first term, the fact that $k_1 \in L^2$ and the weak L^2 convergence give $Tf_j * k_1 \rightarrow Tf * k_1$ and then L^r_{loc} convergence comes from the (bounded) dominated convergence theorem; For the second term, it follows from Young inequality that $\|T(f_j - f) * k_2\|_r \leq \|T(f_j - f)\|_2 \|k_2\|_{r^*}$, where $r^* = \frac{2r}{r+2} < \frac{2Q}{Q+\lambda}$ and $\|k_2\|_{r^*} \lesssim \epsilon^{\frac{2Q-\lambda r}{r+2}} \xrightarrow{\epsilon \rightarrow 0} 0$.

Step 3. Furthermore, $f_j \xrightarrow{L^2} f$. Now we borrow Lemma 4.4 and get

$$\begin{aligned} C := \supremum &\leftarrow \frac{\|f_j * k\|_q^2}{\|f_j\|_2^2} = \frac{(\|f * k\|_q^q + \|(f_j - f) * k\|_q^q + o(1))^{2/q}}{\|f\|_2^2 + \|f_j - f\|_2^2 + o(1)} \\ &\leq \frac{\|f * k\|_q^2 + \|(f_j - f) * k\|_q^2 + o(1)}{\|f\|_2^2 + \|f_j - f\|_2^2 + o(1)} \leq C + o(1), \end{aligned}$$

from which we easily see f is an extremizer as all inequalities above reach equality. However, we have a stronger conclusion that $\|f\|_2 = 1$ and $\|f_j - f\|_2 \rightarrow 0$, which follows from the equality condition of the first inequality above. Here for clarity, we use contradiction argument. Assume $\|f\|_2^2 = 1 - \epsilon$ with $\epsilon > 0$, then put the fact that f is an extremizer and $f_j - f$ is also an extremizing sequence into

$$\|f_j * k\|_q^q = \|(f_j - f) * k\|_q^q + \|f * k\|_q^q + o(1),$$

we get a strict inequality, i.e., the formula

$$C^{q/2} = ((1 - \epsilon)C)^{q/2} + (\epsilon C)^{q/2} + o(1),$$

is impossible for any positive ϵ .

Now the proposition is proved and therefore from the conjugate relation of the sharp problem (4.1) and sharp HLS (3.1), there also exists for the sharp HLS an extremizer $(f * k)^{q-1}$, where f is the extremizer we get in this proposition. The question for (3.5) is also answered. $\square$

Now, for Theorem 3.1 and 3.2, it suffices to figure out all extremizers. We proved an *uniqueness*, modulo the conformal symmetry group and also compute the *explicit formula* of all extremizers.

5 Uniqueness and Formulas of Extremizers

Fix $\lambda \geq 4$. For simplicity of computation, we deal with the sharp inequality on the framework of quaternionic sphere. From (1) in Lemma 5.5 or the fundamental solution of intertwining operators (2.13), we know that the integral kernel $d_S(\zeta, \eta)^{-\lambda}$ is positive definite for $\lambda > 4$ and semi-positive for $\lambda = 4$. Therefore we can restrict the sharp problem to $f = g$ case, especially for $\lambda > 4$, for which, by standard argument, we can further restrict the extremizer to be a complex multiple of a positive real-valued function. We conclude these in the following lemma.

Lemma 5.1. *For $\lambda > 4$, any extremizer pair (f, g) of the sharp HLS is given by $f \sim g \sim h$, where h is a positive quadratic extremizer. For $\lambda = 4$, any quadratic extremizer $f \sim h$, where h is a positive one.*

²⁰ L^r convergence implies pointwise a.e. convergence for a subsequence. So does local Lebesgue convergence using a classical diagonally choosing method.

Proof. We set $\langle \cdot, \cdot \rangle_\lambda := \langle \cdot * d_S^{-\lambda}, \cdot \rangle$, which is positive definite for $\lambda > 4$, then for extremizer pair (f, g) , we have $I(f, g) \leq (I(f)I(g))^{1/2}$ with equality if and only if $f \sim g$. Besides, for $\lambda \geq 4$, any quadratic extremizer must be a complex multiple of a positive one. For any $f = a + ib$, with a, b being respectively the real and image part, $\langle f, f \rangle_\lambda = \langle a, a \rangle_\lambda + \langle b, b \rangle_\lambda \leq \langle |f|, |f| \rangle_\lambda$ by the symmetry of the integral kernel and Cauchy-Schwartz inequality, which gives $I(f) \leq I(|f| = \sqrt{a^2 + b^2})$ with equality if and only if for a.e. $(\zeta, \eta) \in S \times S$, $f(\eta) = 0$ or $(a(\zeta), b(\zeta)) = c(a(\eta), b(\eta))$ with $c > 0$, which implies finally after fixing a point $b(\eta) \neq 0$ (otherwise $b \equiv 0$, then $f = a$ with fixed sign) that $f = \left(\frac{a(\eta)}{b(\eta)} + i\right) \text{sign } b(\eta) |b|$. So, modulo a complex constant, the quadratic extremizer must be real-valued and non-negative. Now, we compute the first variation around any non-negative extremizer h ,

$$\left. \frac{\partial}{\partial t} \right|_{t=0} I(h + t\varphi) = 2\|h\|_p^{-2} \langle h * d_S^{-\lambda} - \langle h, h \rangle_\lambda \|h\|_p^{-p} h^{p-1}, \varphi \rangle, \quad \forall \varphi \in L^p,$$

which gives the Euler-Lagrange equation from the critical property of extremizers

$$h^{p-1}(\zeta) \sim \int_S \frac{h(\eta)}{|1 - \zeta \cdot \bar{\eta}|^{\frac{\lambda}{2}}} d\eta, \quad (5.1)$$

which tells h is positive a.e.. Then the lemma is proved. $\square$

Now, we will first care the quadratic extremizer for $\lambda \geq 4$.

5.1 Zero Center-Mass Condition

The sharp HLS inequality on group is invariant under translation, dilation and constant multiple, while the sphere-edition is preserved by the action of quaternionic unitary group $Sp(n+1)$. So, through the Cayley transform, we can intuitionally try breaking the huge conformal symmetry by assuming that the extremizer satisfies a certain zero center-mass condition. Indeed, the conformal invariant group plays a critical role here, while is absent for most of other inequalities.

Lemma 5.2. *For any extremizer of the sharp HLS inequality (3.5), there exists a conformal transformation γ s.t. by the conformal action $\gamma : h \mapsto \tilde{h} = |J_{\gamma^{-1}}|^{\frac{1}{p}} h \circ \gamma^{-1}$, we get another extremizer $\tilde{h}$ satisfying zero center-mass condition, i.e.,*

$$\int_S \zeta |\tilde{h}|^p(\zeta) d\zeta = 0. \quad (5.2)$$

Proof. Step 1. For any integrable function f with nonzero mean, there is a conformal transformation γ on S , s.t., $\int_S \gamma f = 0$. For this, we define a conformal map on $S \setminus \{A^{-1}o\}$:

$$\gamma_A^\delta = A^* \circ \mathcal{C} \circ D_\delta \circ \mathcal{C}^{-1} \circ A \quad (5.3)$$

with any $A \in Sp(n+1)$ and dilation D_δ for $\delta > 0$ on G . Actually, take $\xi = A^{-1}(0, \dots, 0, 1)$, then the map γ_A^δ is (δ, ξ) -determinate, but independent of the choice of A , so we can use notation $\gamma_\xi^\delta = \gamma_A^\delta$: from diagram

$$\zeta \xrightarrow{A} \left(\frac{2q}{1 + |q|^2 - w}, \frac{1 - |q|^2 + w}{1 + |q|^2 - w} \right) \xrightarrow{D_\delta \circ \mathcal{C}^{-1}} (\delta q, \delta^2 w) \xrightarrow{\mathcal{C}} \left(\frac{2\delta q}{1 + \delta^2(|q|^2 - w)}, \frac{1 - \delta^2(|q|^2 - w)}{1 + \delta^2(|q|^2 - w)} \right) \xrightarrow{A^*} \gamma_\xi^\delta(\zeta) \quad (5.4)$$

and unitary property of A

$$\zeta \cdot \bar{\xi} = A\zeta \cdot (0, \dots, 0, 1) = \frac{1 - |q|^2 + w}{1 + |q|^2 - w},$$

we give the explicit form

$$\gamma_\xi^\delta(\zeta) = \frac{2\delta}{1 + \zeta \cdot \bar{\xi} + \delta^2(1 - \zeta \cdot \bar{\xi})} (\zeta - (\zeta \cdot \bar{\xi})\xi) + \frac{1 + \zeta \cdot \bar{\xi} - \delta^2(1 - \zeta \cdot \bar{\xi})}{1 + \zeta \cdot \bar{\xi} + \delta^2(1 - \zeta \cdot \bar{\xi})} \xi \quad (5.5)$$

with boundary limits

$$\gamma_\xi^\delta(\zeta) \xrightarrow{\delta \rightarrow 0} \xi, \quad \gamma_\xi^\delta(\zeta) \xrightarrow{\delta \rightarrow 1} \zeta. \quad (5.6)$$

Note that as $\delta \rightarrow 1$, the convergence is uniform in (ξ, ζ) on total $S \times S$, and as $\delta \rightarrow 0$, the convergence is uniform on any slightly small subset

$$E = \{(\zeta, \xi) \in S \times S : |1 + \zeta \cdot \bar{\xi}| \geq \epsilon > 0\}.$$

For any f with $\int_S f = 1$, take

$$F(r\xi) = \int_S \gamma_\xi^{1-r}(\zeta) f(\zeta) d\zeta,$$

defined originally on unit ball minus origin point, then we have limits both at origin and on boundary, uniformly in ξ :

$$F(r\xi) \xrightarrow[r \rightarrow 0]{\xi} \int_S \zeta f(\zeta) d\zeta, \quad F(r\xi) \xrightarrow[r \rightarrow 1]{\xi} \xi. \quad (5.7)$$

The notation $\xrightarrow[r \rightarrow r_0]{\xi}$ means the convergence is uniformly in ξ when $r \rightarrow r_0$. Actually, the $r \rightarrow 0$ case is a direct consequence of the $\delta \rightarrow 1$ case in (5.6), while the $r \rightarrow 1$ case comes from the $\delta \rightarrow 0$ case in (5.6) and

$$\begin{aligned} |F(r\xi) - \xi| &\leq \left(\int_{E|_S} + \int_{[(S \times S) \setminus E]|_S} \right) |\gamma_\xi^{1-r}(\zeta) - \xi| f(\zeta) d\zeta \\ &= \left(\int_{d_S(\zeta, -\xi) \geq \epsilon} + \int_{d_S(\zeta, -\xi) \leq \epsilon} \right) |\gamma_\xi^{1-r}(\zeta) - \xi| f(\zeta) d\zeta \\ &= I_1 + I_2, \end{aligned}$$

with $I_2 \xrightarrow[r \rightarrow 0]{\xi} 0$ and $I_1 \xrightarrow[r \rightarrow 1]{\xi} 0$. So, F can be extended to be a continuous function on closed unit ball in $\mathbb{H}^{n+1} \simeq \mathbb{R}^{4n+4}$ and $F(\xi) = \xi$ on the sphere. By Brouwer's fixed point theorem, F has at least one zero point, i.e., there exists a pair (δ_0, ξ_0) s.t. $\int_S \gamma_{\xi_0}^{\delta_0}(\zeta) f(\zeta) d\zeta = 0$.

Step 2. Take $f = |h|^p$ with h the (positive) extremizer, then from above arguments, there exists a conformal transformation $\gamma := \gamma_{\xi_0}^{\delta_0}$, s.t., $\int_S \gamma(\zeta) |h|^p(\zeta) d\zeta = 0$. Change the variables, then we get $\int_S \zeta |\tilde{h}|^p(\zeta) d\zeta = 0$. Besides, from the conformal symmetry, see the second item in the remark following Theorem 3.2, $\tilde{h}$ is still an (positive) extremizer for (3.5). $\square$

The zero center-mass condition is critical here in restricting our extremizers into a small, particular function class.

5.2 Second Variation and Inverse Inequality

We now return to prove the main theorem. Now, from Lemma 5.2, we can assume the positive quadratic extremizer h satisfies (5.2). Then we will try to prove h can only be constant function.

We first compute the second variation around h of the functional $I(f)$ associated to the sharp HLS inequality (3.5)

$$\frac{\partial^2}{\partial t^2} \Big|_{t=0} I(h + t\varphi) = 2\|h\|_p^{-2-p} \left(\langle \varphi, \varphi \rangle_\lambda \|h\|_p^p - (p-1) \langle h, h \rangle_\lambda \langle h^{p-2}, |\varphi|^2 \rangle \right),$$

for any $\varphi \in L^p$, $\langle h^{p-1}, \varphi \rangle = 0$, where we use the same notation $\langle \cdot, \cdot \rangle_\lambda$ for the inner product as that in Lemma 5.1 and the extremizer gives non-positive second variation, so

$$\iint_{S \times S} \frac{\overline{\varphi(\zeta)} \varphi(\eta)}{|1 - \zeta \cdot \bar{\eta}|^{\frac{p}{2}}} d\zeta d\eta \int_S h^p - (p-1) \iint_{S \times S} \frac{h(\zeta) h(\eta)}{|1 - \zeta \cdot \bar{\eta}|^{\frac{p}{2}}} d\zeta d\eta \int_S h^{p-2} |\varphi|^2 \leq 0 \quad (5.8)$$

for all φ satisfying $\int_S h^{p-1} \varphi = 0$. Substitute $\varphi(\zeta) = h\zeta_j^1, h\zeta_j^2 (1 \leq j \leq n+1)$, all satisfying $\int_S h^{p-1} \varphi = 0$, into the second variation inequality (5.8) and summing the results, then we get

$$\iint_{S \times S} \frac{h(\zeta) \sum_{j=1}^{n+1} (\bar{\zeta}_j^1 \eta_j^1 + \bar{\zeta}_j^2 \eta_j^2) h(\eta)}{|1 - \zeta \cdot \bar{\eta}|^{\frac{p}{2}}} d\zeta d\eta \leq (p-1) \iint_{S \times S} \frac{h(\zeta) h(\eta)}{|1 - \zeta \cdot \bar{\eta}|^{\frac{p}{2}}} d\zeta d\eta.$$

From the symmetry of left side integrand on (ζ, η) and $\bar{\zeta} \cdot \mathbb{C} \eta + \bar{\eta} \cdot \mathbb{C} \zeta = \bar{\zeta} \cdot \eta + \bar{\eta} \cdot \zeta$, we have

$$\iint_{S \times S} \frac{h(\zeta) (\bar{\zeta} \cdot \eta + \bar{\eta} \cdot \zeta) h(\eta)}{|1 - \zeta \cdot \bar{\eta}|^{\frac{p}{2}}} d\zeta d\eta \leq 2(p-1) \iint_{S \times S} \frac{h(\zeta) h(\eta)}{|1 - \zeta \cdot \bar{\eta}|^{\frac{p}{2}}} d\zeta d\eta. \quad (5.9)$$

In following subsections 5.4 and 5.5, we use quaternionic analogue of Funk-Hecke formula to prove that extremizer satisfying (5.9) can only be constant (Proposition 5.6). Concerning this involves very complicated computation, we leave it in independent subsections.

Actually, we **claim** here: For any $4 \leq \lambda < Q = 4n + 6$ and for h , which is the extremizer discussed above, we have the following inverse second variation inequality (ISV)

$$\iint_{S \times S} \frac{\overline{h(\zeta)}(\bar{\zeta} \cdot \eta + \bar{\eta} \cdot \zeta)h(\eta)}{|1 - \zeta \cdot \bar{\eta}|^{\frac{\lambda}{2}}} d\zeta d\eta \geq 2(p-1) \iint_{S \times S} \frac{\overline{h(\zeta)}h(\eta)}{|1 - \zeta \cdot \bar{\eta}|^{\frac{\lambda}{2}}} d\zeta d\eta. \quad (5.10)$$

Moreover, “=” holds only when $h \equiv \text{constant}$.

5.3 Sharp Constants and Formulas of Extremizers

Assume the **claim** holds, then we’ve proved an uniqueness of the extremizer. We need to take care for $\lambda = 4$ case. The **claim** tells that any quadratic extremizer can only be almost constant. Then we said quadratic extremizer is still the only case for pair.

Lemma 5.3. For $\lambda = 4$, any extremizer pair (f, g) for $I(f, g)$ is quadratic, i.e. $f \sim g$.

Proof. We consider on the sphere. From Lemma 5.2 and the **claim**, we know any quadratic extremizer $f \sim |J_\tau|^{1/p}, \tau \in \text{Aut}(S)$. Given an extremizer pair (f, g) , the fact $\langle f, g \rangle_\lambda \leq \langle f, f \rangle_\lambda^{1/2} \langle g, g \rangle_\lambda^{1/2}$ gives $I(f, g) \leq (I(f)I(g))^{1/2}$, so (f, f) and (g, g) are again extremizer pairs and can only be the form of quadratic extremizer, i.e., we may assume $f = 1$ and $g = |J_\tau|^{1/p}$ for some $\tau \in \text{Aut}(S)$. But from the equality condition of Cauchy-Schwartz inequality, when restricted onto the positive definite subspace $V := \bigoplus_{j \geq k=0} V_{j,k}$, we have $f|_V \sim g|_V$. However, by simple computation, $g|_{V_{1,0}} = 0$ if and only if $g \sim 1$. Besides, we can also see from the Euler-Lagrange equation for non-quadratic functional $I(f, g)$,

$$f * d_S^{-\lambda} \sim g^{p-1}, \quad g * d_S^{-\lambda} \sim f^{p-1}.$$

So, any extremizer pair for $\lambda \geq 4$ should be also quadratic. $\square$

We can give the sharp constants and all extremizers now with the *a priori* **claim**.

1. *Extremizers* for (3.1) and (3.5). For $\lambda \geq 4$, from Lemma 5.1, 5.2, 5.3 and the **claim**, we know that all extremizer pairs on the sphere are given by $f \sim g \sim |J_\tau|^{1/p}$, where τ is any conformal transformation in $\text{Aut}(S)$, and correspondingly all extremizers on the group are given by $f \sim g \sim |J_C \circ \sigma J_\sigma|^{1/p}$, $\sigma \in \text{Aut}(G)$ from the relation (3.4) and definitions of the conformal groups. The explicit formulas now follows from our calculation in Appendix B. Note that some transformations are assimilated in extremizer formulas.
2. *Sharp constants* for (3.1) and (3.5). Using the fact that constant 1 is an extremizer, we have from the formula of $\lambda_{0,0}(K_1^{\frac{\lambda}{4}})$ in (5.16) Lemma 5.5 that the sharp constants

$$\begin{aligned} C'_\lambda &= 2^{\frac{\lambda}{2}} |S|^{1-\frac{2}{p}} \int_S \frac{1}{|1 - \eta_{n+1}|^{\frac{\lambda}{2}}} d\eta = |S|^{1-\frac{2}{p}} \lambda_{0,0}(K_1^{\frac{\lambda}{4}}) \\ &= 2^{\frac{\lambda}{2}} |S|^{1-\frac{2}{p}} \frac{2\pi^{2n+2} \Gamma(2n - \frac{\lambda}{2} + 3)}{\Gamma(2n - \frac{\lambda}{4} + 2) \Gamma(2n - \frac{\lambda}{4} + 3)} \\ &= \left(\frac{2\pi^{\frac{Q-2}{2}}}{(\frac{Q-4}{2})!} \right)^{\frac{\lambda}{Q}} \frac{2^{\frac{\lambda}{2}} (\frac{Q-4}{2})! \Gamma(\frac{Q-\lambda}{2})}{\Gamma(\frac{2Q-\lambda}{4}) \Gamma(\frac{2Q-\lambda}{4} - 1)}, \end{aligned}$$

and then

$$\begin{aligned} C_\lambda &= 2^{-\frac{(4n+3)\lambda}{Q}} C'_\lambda = 2^{-\frac{2n\lambda}{Q}} |S|^{1-\frac{2}{p}} \frac{2\pi^{2n+2} \Gamma(2n - \frac{\lambda}{2} + 3)}{\Gamma(2n - \frac{\lambda}{4} + 2) \Gamma(2n - \frac{\lambda}{4} + 3)} \\ &= \left(\frac{\pi^{\frac{Q-2}{2}}}{2^{\frac{Q-8}{2}}} \right)^{\frac{\lambda}{Q}} \frac{((\frac{Q-4}{2})!)^{1-\frac{\lambda}{Q}} \Gamma(\frac{Q-\lambda}{2})}{\Gamma(\frac{2Q-\lambda}{4}) \Gamma(\frac{2Q-\lambda}{4} - 1)}. \end{aligned}$$

The two main theorems are then proved after we check the **claim** at the end of subsection 5.2, which tells that any extremizer h that satisfying zero center-mass condition can only be constant. In the remaining part of this paper, we focus on proving this and before that we need to introduce the Funk-Hecke formula.

5.4 Quaternionic Funk-Hecke Formula and Eigenvalues

In this subsection, we are going to derive an quaternionic analogue of classical real or complex Funk-Hecke theorems, which concerns the integral operators associated with kernel of the form $K(\zeta \cdot \bar{\eta})$. We then give eigenvalues of integral operators with two useful kernels. We left the proof of the **claim** to next subsection.

Lemma 5.4. [Quaternionic Funk-Hecke Formula]

Let K be a function on the unit ball in $\mathbb{H}$, s.t. the following integral exists, e.g. by assuming $K \in L^1(B(0,1))$. Then any integral operator on S with kernel of the form $K(\zeta \cdot \bar{\eta})$ is diagonal w.r.t the decomposition (2.6), and the eigenvalue on $V_{j,k}$ is given by

$$\lambda_{j,k}(K) = \frac{2\pi^{2n}k!}{(j-k+1)(k+2n-1)!} \int_0^{\frac{\pi}{2}} \sin^{4n-1} \theta \cos^{j-k+3} \theta P_k^{(2n-1,j-k+1)}(\cos 2\theta) d\theta \\ \times \int_{|u|=1} K(\cos \theta u) \frac{\sin(j-k+1)\phi}{\sin \phi} du, \quad (5.11)$$

with $\operatorname{Re} u = \cos \phi$ ($\phi \in [0, \pi]$) and du is the Lebesgue measure on $\mathbb{S}^3$.

Proof. From Schur's lemma and the irreducibility of bispherical subspace $V_{j,k}$, we see the integral operator K with kernel $K(\zeta \cdot \bar{\eta})$ is diagonal with regards to the decomposition (2.6) with eigenvalues denoted by $\lambda_{j,k}$. Now, we compute the eigenvalues. Assume $\{Y_{j,k}^\mu\}_{\mu=1}^{m_{j,k}}$ is a normalized orthogonal basis of $V_{j,k}$, where $m_{j,k} = \dim V_{j,k}$, then as footnoted in the definition of zonal harmonics in the preliminary section and in abuse of notation, the reproducing kernel of projection operator onto $V_{j,k}$ is given by

$$\sum_{\mu=1}^{m_{j,k}} Y_{j,k}^\mu(\zeta) \overline{Y_{j,k}^\mu(\eta)} = Z_{j,k}(\zeta \cdot \bar{\eta}) = \overline{Z_{j,k}(\eta \cdot \bar{\zeta})} = Z_{j,k}^\eta(\zeta).$$

By the definition of spectrum $\lambda_{j,k}$,

$$\int_S K(\zeta \cdot \bar{\eta}) Z_{j,k}(\eta \cdot \bar{\zeta}) d\eta = \lambda_{j,k} Z_{j,k}(1),$$

which implies

$$\lambda_{j,k} = Z_{j,k}^{-1}(1) \int_S K(\zeta \cdot \bar{\eta}) Z_{j,k}(\eta \cdot \bar{\zeta}) d\eta \\ = Z_{j,k}^{-1}(1) \int_S K(\overline{\eta_{n+1}}) Z_{j,k}(\eta_{n+1}) d\eta. \quad (5.12)$$

In polar coordinates for $\eta = (\eta_1, \eta_2, \dots, \eta_{n+1})$,

$$\begin{aligned} \eta_1 &= u_1 \sin \theta_n \sin \theta_{n-1} \dots \sin \theta_1, & |u_i| &= 1, \quad u_i = (u_i^1, u_i^2, u_i^3, u_i^4), \\ \eta_2 &= u_2 \sin \theta_n \sin \theta_{n-1} \dots \cos \theta_1, & u_i^4 &= \sin \phi_i^3 \sin \phi_i^2 \sin \phi_i^1, \\ \eta_i &= \dots & u_i^3 &= \sin \phi_i^3 \sin \phi_i^2 \cos \phi_i^1, \\ \eta_n &= u_n \sin \theta_n \cos \theta_{n-1}, & u_i^2 &= \sin \phi_i^3 \cos \phi_i^2, \\ \eta_{n+1} &= u_{n+1} \cos \theta_n, & u_i^1 &= \cos \phi_i^3, \end{aligned}$$

with $\theta_i \in [0, \frac{\pi}{2}]$, $\phi_i^3, \phi_i^2 \in [0, \pi]$, $\phi_i^1 \in [0, 2\pi]$, $\forall 1 \leq i \leq n+1$, we have the invariance measure on the sphere (11.7.3 (2) in [45])

$$d\eta = \prod_{i=1}^n (\sin^{4i-1} \theta_i \cos^3 \theta_i d\theta_i) \prod_{j=1}^{n+1} du_j, \quad du_j = \sin^2 \phi_j^3 \sin \phi_j^2 d\phi_j^3 d\phi_j^2 d\phi_j^1. \quad (5.13)$$

Putting the formula (2.7) for zonal harmonics²¹ and invariance measure (5.13) together into (5.12), and from the formulas

$$|\mathbb{S}^{4n-1}| = \frac{2\pi^{2n}}{(2n)!}, \quad P_k^{(2n-1,j-k+1)}(1) = \frac{(k+2n-1)!}{k!(2n-1)!}$$

²¹ $Z_{j,k}(\zeta)$ only depends on ζ_{n+1} as η is the north pole.

(22.2.1 in [1]), finally we get

$$\begin{aligned} \lambda_{j,k} &= \frac{2\pi^{2n}k!}{(j-k+1)(k+2n-1)!} \int_0^{\frac{\pi}{2}} \sin^{4n-1} \theta_n \cos^{j-k+3} \theta_n P_k^{(2n-1, j-k+1)}(\cos 2\theta_n) d\theta_n \\ &\quad \times \int_{\mathbb{S}^3} K(\cos \theta_n u_{n+1}) \frac{\sin(j-k+1)\phi_{n+1}^3}{\sin \phi_{n+1}^3} du_{n+1}. \end{aligned}$$

□

In order to prove the **claim** (Proposition 5.6), it suffices to compute eigenvalues of kernels for functions of two forms $K_1^\alpha(q) = |1-q|^{-2\alpha}$, $K_2^\alpha(q) = |q|^2|1-q|^{-2\alpha}$, noting that $\zeta \cdot \bar{\eta} + \eta \cdot \bar{\zeta} = 2\operatorname{Re} \zeta \cdot \bar{\eta} = 1 + |\zeta \cdot \bar{\eta}|^2 - |1 - \zeta \cdot \bar{\eta}|^2$. Define²² the following Gamma functions series for any real numbers pair (a, b, c) satisfying $c > a + b$ by

$$A(a, b, c) := \sum_{\mu \geq 0} \frac{\Gamma(\mu+a)\Gamma(\mu+b)}{\mu!\Gamma(\mu+c)} = \frac{\Gamma(a)\Gamma(b)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad (5.14)$$

Lemma 5.5. [Eigenvalues]

Given $-1 < \alpha < \frac{Q}{4}$ and denote after fixing an integer pair $j \geq k \geq 0$

$$(a, b, c) := (j + \alpha, k + \alpha - 1, j + k + \frac{Q}{2} - 1). \quad (5.15)$$

(1) The eigenvalues of integral operators with kernels $K_1^\alpha(\zeta \cdot \bar{\eta}) = |1 - \zeta \cdot \bar{\eta}|^{-2\alpha}$ are given by

$$\begin{aligned} \lambda_{j,k}(K_1^\alpha) &= \frac{2\pi^{2n+2}}{\Gamma(\alpha)\Gamma(\alpha-1)} A(a, b, c) \\ &= 2\pi^{2n+2} \frac{\Gamma(\frac{Q}{2} - 2\alpha)}{\Gamma(\alpha)\Gamma(\alpha-1)} \frac{\Gamma(j+\alpha)}{\Gamma(j+\frac{Q}{2}-\alpha)} \frac{\Gamma(k+\alpha-1)}{\Gamma(k+\frac{Q}{2}-\alpha-1)}. \end{aligned} \quad (5.16)$$

(2) The eigenvalues of integral operators with kernels $K_2^\alpha(\zeta \cdot \bar{\eta}) = |\zeta \cdot \bar{\eta}|^2|1 - \zeta \cdot \bar{\eta}|^{-2\alpha}$ are given by

$$\lambda_{j,k}(K_2^\alpha) = C_{j,k}^\alpha \lambda_{j,k}(K_1^\alpha), \quad (5.17)$$

with

$$C_{j,k}^\alpha = 1 - (\alpha-2)(c-a-b) \left(\frac{1}{(a-1)(c-a)} + \frac{1}{(b-1)(c-b)} - (\alpha-2) \frac{c-a-b+1}{(a-1)(b-1)(c-a)(c-b)} \right). \quad (5.18)$$

In the singular points $\alpha = 0, 1, 2$, above formulas can be viewed as limits, fixing j and k .

Proof. (1) Putting $K = K_1^\alpha$ into Lemma 5.4, we get

$$\begin{aligned} \lambda_{j,k}(K_1^\alpha) &= \frac{4\pi^{2n+1}k!}{(j-k+1)(k+2n-1)!} \int_0^{\frac{\pi}{2}} \sin^{4n-1} \theta \cos^{j-k+3} \theta P_k^{(2n-1, j-k+1)}(\cos 2\theta) d\theta \\ &\quad \times \int_0^\pi (1 + \cos^2 \theta - 2 \cos \phi \cos \theta)^{-\alpha} (\cos(j-k)\phi - \cos(j-k+2)\phi) d\phi. \end{aligned} \quad (5.19)$$

Using Gegenbauer polynomials, we have the following fact ((5.11) in [19] from [1])

$$\begin{aligned} &\int_0^\pi (1 + \cos^2 \theta - 2 \cos \phi \cos \theta)^{-\alpha} \cos(j-k)\phi d\phi \\ &= \frac{\pi}{\Gamma^2(\alpha)} \sum_{\mu \geq 0} \cos^{|j-k|+2\mu} \theta \frac{\Gamma(\mu+\alpha)\Gamma(\mu+|j-k|+\alpha)}{\mu!(\mu+|j-k|)!}, \end{aligned} \quad (5.20)$$

²²The second equality property for $A(a, b, c)$ comes from 15.1.1 and 15.1.20 in [1].

which, from (5.19) and $j \geq k \geq 0$, gives

$$\lambda_{j,k}(K_1^\alpha) = \frac{4\pi^{2n+2}k!}{(j-k+1)(k+2n-1)!\Gamma^2(\alpha)} \sum_{\mu \geq 0} \frac{\Gamma(\mu+\alpha)}{\mu!} \quad (5.21)$$

$$\begin{aligned} & \times \left(\frac{\Gamma(\mu+|j-k|+\alpha)}{(\mu+|j-k|)!} \int_0^{\frac{\pi}{2}} \sin^{4n-1} \theta \cos^{2(j-k)+3+2\mu} \theta P_k^{(2n-1,j-k+1)}(\cos 2\theta) d\theta \right. \\ & \quad \left. - \frac{\Gamma(\mu+|j-k|+2+\alpha)}{(\mu+|j-k|+2)!} \int_0^{\frac{\pi}{2}} \sin^{4n-1} \theta \cos^{2(j-k)+5+2\mu} \theta P_k^{(2n-1,j-k+1)}(\cos 2\theta) d\theta \right) \\ & =: \frac{4\pi^{2n+2}k!}{(j-k+1)(k+2n-1)!\Gamma^2(\alpha)} (I_1 - I_2). \end{aligned} \quad (5.22)$$

Using the following Rodrigues' formula ((22.11.1) in [1]),

$$P_k^{(2n-1,j-k+1)}(t) = \frac{(-1)^k}{2^k k!} (1-t)^{-(2n-1)} (1+t)^{-(j-k+1)} \frac{d^k}{dt^k} \{(1-t)^{2n+k-1} (1+t)^{j+1}\},$$

and by changing variable $\cos 2\theta = t$ and integrating by part, we get

$$\begin{aligned} & \int_0^{\frac{\pi}{2}} \sin^{4n-1} \theta \cos^{2(j-k)+3+2\mu} \theta P_k^{(2n-1,j-k+1)}(\cos 2\theta) d\theta \\ & = \frac{(-1)^k 2^{-(\mu+j+2n+2)}}{k!} \int_{-1}^1 (1+t)^\mu \frac{d^k}{dt^k} \{(1-t)^{2n+k-1} (1+t)^{j+1}\} dt \\ & = \chi_{\mu \geq k} \frac{2^{-(\mu+j+2n+2)} \mu!}{k! (\mu-k)!} \int_{-1}^1 (1+t)^{\mu+j-k+1} (1-t)^{2n+k-1} dt \\ & = \chi_{\mu \geq k} \frac{\mu! B(\mu+j-k+2, 2n+k)}{2k! (\mu-k)!} \\ & = \chi_{\mu \geq k} \frac{\mu! \Gamma(\mu+j-k+2) \Gamma(2n+k)}{2k! (\mu-k)! \Gamma(\mu+j+2n+2)}, \end{aligned} \quad (5.23)$$

where B is the Beta function. Inserting (5.23) into (5.21), we get

$$\begin{aligned} I_1 &= \frac{1}{2} \sum_{\mu \geq k} \frac{\Gamma(\mu+\alpha) \Gamma(\mu+|j-k|+\alpha)}{\mu! (\mu+|j-k|)!} \frac{\mu! \Gamma(\mu+j-k+2) \Gamma(2n+k)}{k! (\mu-k)! \Gamma(\mu+j+2n+2)}, \\ I_2 &= \frac{1}{2} \sum_{\mu+1 \geq \max\{k,1\}} \frac{\Gamma(\mu+\alpha) \Gamma(\mu+|j-k|+2+\alpha)}{\mu! (\mu+|j-k|+2)!} \frac{(\mu+1)! \Gamma(\mu+1+j-k+2) \Gamma(2n+k)}{k! (\mu+1-k)! \Gamma(\mu+1+j+2n+2)}. \end{aligned} \quad (5.24)$$

Now, we get the difference of the two terms.

For $k \geq 1$,

$$\begin{aligned} I_1 - I_2 &= \frac{1}{2} \sum_{\mu \geq k} \frac{\mu! \Gamma(\mu+j-k+2) \Gamma(2n+k)}{k! (\mu-k)! \Gamma(\mu+j+2n+2)} \\ & \quad \times \left(\frac{\Gamma(\mu+\alpha) \Gamma(\mu+|j-k|+\alpha)}{\mu! (\mu+|j-k|)!} - \frac{\Gamma(\mu-1+\alpha) \Gamma(\mu+|j-k|+1+\alpha)}{(\mu-1)! (\mu+|j-k|+1)!} \right) \\ &= \frac{1}{2} \sum_{\mu \geq k} \frac{\mu! \Gamma(\mu+j-k+2) \Gamma(2n+k)}{k! (\mu-k)! \Gamma(\mu+j+2n+2)} \frac{(\alpha-1)(j-k+1) \Gamma(\mu+\alpha-1) \Gamma(\mu+j-k+\alpha)}{\mu! (\mu+j-k+1)!} \\ &= \frac{(\alpha-1)(j-k+1) \Gamma(2n+k)}{2k!} \sum_{\mu \geq k} \frac{\Gamma(\mu+\alpha-1) \Gamma(\mu+j-k+\alpha)}{(\mu-k)! \Gamma(\mu+j+2n+2)} \\ &= \frac{(\alpha-1)(j-k+1) \Gamma(2n+k)}{2k!} \sum_{\mu \geq 0} \frac{\Gamma(\mu+k+\alpha-1) \Gamma(\mu+j+\alpha)}{\mu! \Gamma(\mu+j+k+2n+2)}; \end{aligned}$$

For $k = 0$, as computation above

$$\begin{aligned} I_1 - I_2 &= \frac{(\alpha-1)(j+1) \Gamma(2n)}{2} \sum_{\mu \geq 1} \frac{\Gamma(\mu+\alpha-1) \Gamma(\mu+j+\alpha)}{\mu! \Gamma(\mu+j+2n+2)} + \frac{\Gamma(2n) \Gamma(\alpha) \Gamma(j+\alpha)(j+1)}{2\Gamma(j+2n+2)} \\ &= \frac{(\alpha-1)(j+1) \Gamma(2n)}{2} \sum_{\mu \geq 0} \frac{\Gamma(\mu+\alpha-1) \Gamma(\mu+j+\alpha)}{\mu! \Gamma(\mu+j+2n+2)} \end{aligned}$$

So, together with (5.14), (5.15) and (5.21), we get

$$\begin{aligned}\lambda_{j,k}(K_1^\alpha) &= \frac{2\pi^{2n+2}}{\Gamma(\alpha)\Gamma(\alpha-1)} A(a, b, c) \\ &= 2\pi^{2n+2} \frac{\Gamma(\frac{Q}{2} - 2\alpha)}{\Gamma(\alpha)\Gamma(\alpha-1)} \frac{\Gamma(j+\alpha)}{\Gamma(j+\frac{Q}{2}-\alpha)} \frac{\Gamma(k+\alpha-1)}{\Gamma(k+\frac{Q}{2}-\alpha-1)}.\end{aligned}$$

The first part of the theorem is then proved.

(2) Putting $K = K_2^\alpha$ into Lemma 5.4, we get

$$\begin{aligned}\lambda_{j,k}(K_2^\alpha) &= \frac{4\pi^{2n+1}k!}{(j-k+1)(k+2n-1)!} \int_0^{\frac{\pi}{2}} \sin^{4n-1} \theta \cos^{j-k+5} \theta P_k^{(2n-1, j-k+1)}(\cos(2\theta)) d\theta \\ &\quad \times \int_0^\pi (1 + \cos^2 \theta - 2 \cos \phi \cos \theta)^{-\alpha} (\cos(j-k)\phi - \cos(j-k+2)\phi) d\phi.\end{aligned}\quad (5.25)$$

Compare this with (5.19), (5.21) and (5.23), we see μ there is now substituted by $\mu+1$, so repeating the same computation in (1), we obtain an analogue of (5.21), with

$$\begin{aligned}I_1 &= \frac{1}{2} \sum_{\mu+1 \geq \max\{k,1\}} \frac{\Gamma(\mu+\alpha)\Gamma(\mu+|j-k|+\alpha)}{\mu!(\mu+|j-k|)!} \frac{(\mu+1)!\Gamma(\mu+1+j-k+2)\Gamma(2n+k)}{k!(\mu+1-k)!\Gamma(\mu+1+j+2n+2)} \\ I_2 &= \frac{1}{2} \sum_{\mu+2 \geq \max\{k,2\}} \frac{\Gamma(\mu+\alpha)\Gamma(\mu+|j-k|+2+\alpha)}{\mu!(\mu+|j-k|+2)!} \frac{(\mu+2)!\Gamma(\mu+2+j-k+2)\Gamma(2n+k)}{k!(\mu+2-k)!\Gamma(\mu+2+j+2n+2)}.\end{aligned}\quad (5.26)$$

Then for $k \geq 2$,

$$\begin{aligned}\lambda_{j,k}(K_2^\alpha) &= \frac{2\pi^{2n+2}}{\Gamma(\alpha)\Gamma(\alpha-1)} \sum_{\mu \geq 0} \frac{(\mu+k)(\mu+j+1)\Gamma(\mu+k+\alpha-2)\Gamma(\mu+j+\alpha-1)}{\mu!\Gamma(\mu+j+k+2n+2)} \\ &= \frac{2\pi^{2n+2}}{\Gamma(\alpha)\Gamma(\alpha-1)} \sum_{\mu \geq 0} \frac{\Gamma(\mu+k+\alpha-1)\Gamma(\mu+j+\alpha)}{\mu!\Gamma(\mu+j+k+2n+2)} \frac{(\mu+k)(\mu+j+1)}{(\mu+k+\alpha-2)(\mu+j+\alpha-1)}.\end{aligned}\quad (5.27)$$

Noting that using the notation (5.15)

$$\begin{aligned}\frac{(\mu+k)(\mu+j+1)}{(\mu+k+\alpha-2)(\mu+j+\alpha-1)} &= \frac{(\mu+a-1-(\alpha-2))(\mu+b-1-(\alpha-2))}{(\mu+a-1)(\mu+b-1)} \\ &= 1 - (\alpha-2) \left(\frac{1}{\mu+a-1} + \frac{1}{\mu+b-1} - (\alpha-2) \frac{1}{(\mu+a-1)(\mu+b-1)} \right),\end{aligned}\quad (5.28)$$

so, from (5.14), we have

$$\lambda_{j,k}(K_2^\alpha) = C_{j,k}^\alpha \lambda_{j,k}(K_1^\alpha), \quad (5.29)$$

with

$$C_{j,k}^\alpha = 1 - (\alpha-2)(c-a-b) \left(\frac{1}{(a-1)(c-a)} + \frac{1}{(b-1)(c-b)} - (\alpha-2) \frac{c-a-b+1}{(a-1)(b-1)(c-a)(c-b)} \right). \quad (5.30)$$

For $k < 2$ ($k = 0, 1$), we only prove for $k = 1$ as we can check similarly that the special cases can be formally integrated into the general formula. We denote by $(I_1 - I_2)_i$ the obvious terms for K_i^α ($i=1,2$), see (5.24) and (5.26), then

$$\begin{aligned}(I_1 - I_2)_2 &= C_{j,k}^\alpha (I_1 - I_2)_1 + (\alpha-1)(j+1)(\alpha-2) \\ &\quad \times \left(\frac{\Gamma(\alpha-1)\Gamma(j+\alpha)}{\Gamma(j+2n+3)} + \frac{\Gamma(\alpha)\Gamma(j+\alpha-1)}{\Gamma(j+2n+3)} - (\alpha-2) \frac{\Gamma(\alpha-1)\Gamma(j+\alpha-1)}{\Gamma(j+2n+3)} \right) \\ &\quad - \frac{\Gamma(j+2)}{\Gamma(j+2n+3)} \left(\frac{(\alpha-1)j\Gamma(\alpha)\Gamma(j+\alpha)}{(j+1)!} - \frac{\Gamma(\alpha)\Gamma(j+\alpha-1)}{(j-1)!} \right) \\ &= C_{j,k}^\alpha (I_1 - I_2)_1.\end{aligned}$$

The second part of the theorem is proved. $\square$

5.5 Proof of the Claim and Theorem 3.1 and 3.2

For simplicity, we first prove a proposition about a general quadratic ISV-type inequality, which is almost equivalent to the **claim**, with $\alpha = \frac{\lambda}{4}$.

Proposition 5.6. [Quadratic ISV Inequality]

Let $1 \leq \alpha < \frac{Q}{4}$, then for any f on S , s.t. the following integrals exist, we have

$$\iint_{S \times S} \frac{\overline{f(\zeta)}(\bar{\zeta} \cdot \eta + \bar{\eta} \cdot \zeta)f(\eta)}{|1 - \zeta \cdot \bar{\eta}|^{2\alpha}} d\zeta d\eta \geq \frac{2\alpha}{\frac{Q}{2} - \alpha} \iint_{S \times S} \frac{\overline{f(\zeta)}f(\eta)}{|1 - \zeta \cdot \bar{\eta}|^{2\alpha}} d\zeta d\eta, \quad (5.31)$$

and here $\alpha \geq 1$ is sharp²³; Moreover, when $\alpha > 1$, equality holds if and only if f is a constant function, and when $\alpha = 1$, equality holds if and only if $f \in V_{0,0} \oplus_{j \geq k \geq 2} V_{j,k}$.

Proof. We always first consider the problem for a test function, and then extends to bigger function spaces and all singular points of Gamma functions are interpreted as limits here. From $\zeta \cdot \bar{\eta} + \eta \cdot \bar{\zeta} = 1 + |\zeta \cdot \bar{\eta}|^2 - |1 - \zeta \cdot \bar{\eta}|^2$, to prove the quadratic inequality in the proposition, it suffices to prove (not necessary) that

$$\sum_{j \geq k \geq 0} (\lambda_{j,k}(K_1^\alpha) + \lambda_{j,k}(K_2^\alpha) - \lambda_{j,k}(K_1^{\alpha-1})) \|f_{j,k}\|_2^2 \geq \sum_{j \geq k \geq 0} \frac{2\alpha}{\frac{Q}{2} - \alpha} \lambda_{j,k}(K_1^\alpha) \|f_{j,k}\|_2^2, \quad (5.32)$$

where $f = \sum_{j \geq k \geq 0} f_{j,k}$ ($f_{j,k} \in V_{j,k}$) is the bispherical harmonic decomposition (2.6) of f . From the formulas of eigenvalues in Lemma 5.5, we can check (5.32) by meticulous computation and for completion, we do it here.

When $\alpha > 1$, from Lemma 5.5 and recalling that $\lambda_{j,k}(K_1^\alpha) > 0$ ($\forall j \geq k \geq 0$), (5.32) holds if (not only if) we can prove that, $\forall j \geq k \geq 0$,

$$2 - (\alpha - 2)(c - a - b) \left(\frac{1}{(a-1)(c-a)} + \frac{1}{(b-1)(c-b)} + \frac{c-a-b+1}{(a-1)(b-1)(c-a)(c-b)} \right) \geq \frac{2\alpha}{\frac{Q}{2} - \alpha},$$

which is

$$(\alpha - 2) \left(\frac{1}{(a-1)(c-a)} + \frac{1}{(b-1)(c-b)} + \frac{c-a-b+1}{(a-1)(b-1)(c-a)(c-b)} \right) \leq \frac{2}{\frac{Q}{2} - \alpha}, \quad (5.33)$$

as $c - a - b = \frac{Q}{2} - 2\alpha > 0$.

For $\alpha > 2$, it suffices to check

$$\frac{2}{(\alpha - 2)(\frac{Q}{2} - \alpha)} - \left(\frac{1}{(a-1)(c-a)} + \frac{1}{(b-1)(c-b)} \right) \geq \frac{c-a-b+1}{(a-1)(b-1)(c-a)(c-b)}.$$

which becomes, after substituting (a, b, c) (see (5.15)),

$$\begin{aligned} & \frac{j(\alpha - 2) + k(\frac{Q}{2} - \alpha) + kj}{(\alpha - 2)(\frac{Q}{2} - \alpha)(\alpha - 2 + k)(\frac{Q}{2} - \alpha + j)} + \frac{(k-1)(\alpha - 2) + (j+1)(\frac{Q}{2} - \alpha) + (j+1)(k-1)}{(\alpha - 2)(\frac{Q}{2} - \alpha)(\alpha - 2 + j + 1)(\frac{Q}{2} - \alpha + k - 1)} \\ & \geq \frac{\frac{Q}{2} - 2\alpha + 1}{(\alpha - 2 + k)(\frac{Q}{2} - \alpha + j)(\alpha - 2 + j + 1)(\frac{Q}{2} - \alpha + k - 1)} \end{aligned} \quad (5.34)$$

To prove (5.34), first note that the denominator of left side is smaller than that in the right side, i.e.,

$$(\alpha - 2)(\frac{Q}{2} - \alpha) \leq \min\{(\alpha - 2 + k)(\frac{Q}{2} - \alpha + j), (\alpha - 2 + j + 1)(\frac{Q}{2} - \alpha + k - 1)\},$$

which is equivalent to

$$(k-1)(\alpha - 2) + (j+1)(\frac{Q}{2} - \alpha) + (j+1)(k-1) \geq 0,$$

²³For any $\alpha < 1$, there exists a function, s.t., the inequality fails.

and we easily see this as the left side is bigger than $\frac{Q}{2} - \alpha - 1 - (\alpha - 2) = \frac{Q}{2} - 2\alpha + 1 \geq 1$. Then compare the numerators, and we find that the sum of numerators in the left side is larger than that in the right side, i.e.,

$$(j+k-1)(\alpha-2) + (k+j+1)(\frac{Q}{2}-\alpha) + kj + (j+1)(k-1) \geq \frac{Q}{2} - 2\alpha + 1, \quad (5.35)$$

which is just

$$(\frac{Q}{2}-3)j + (\frac{Q}{2}-1)k + 2jk \geq 0.$$

This inequality is obvious and become equality if and only if $j = k = 0$, and in this special case, inequalities above (5.33), (5.34) and (5.35) all become equalities. So, (5.32) holds and reaches equality only when $f_{j,k} = 0$, $\forall (j,k) \neq (0,0)$, and therefore, for $\alpha > 2$, recalling $V_{0,0} = \{\text{constant functions}\}$, we have proved that (5.31) holds and becomes equality if and only if f is a constant function.

For $1 < \alpha < 2$, for (5.33), it suffices to prove the inverse of (5.34),

$$\begin{aligned} & \frac{j(\alpha-2) + k(\frac{Q}{2}-\alpha) + kj}{(\alpha-2)(\frac{Q}{2}-\alpha)(\alpha-2+k)(\frac{Q}{2}-\alpha+j)} + \frac{(k-1)(\alpha-2) + (j+1)(\frac{Q}{2}-\alpha) + (j+1)(k-1)}{(\alpha-2)(\frac{Q}{2}-\alpha)(\alpha-2+j+1)(\frac{Q}{2}-\alpha+k-1)} \\ & \leq \frac{\frac{Q}{2} - 2\alpha + 1}{(\alpha-2+k)(\frac{Q}{2}-\alpha+j)(\alpha-2+j+1)(\frac{Q}{2}-\alpha+k-1)}. \end{aligned}$$

For $k \geq 1$, it's obvious by checking sign that a strict inequality holds. For $k = 0$, it's also obvious and “=” holds if and only if $j = 0$, checking sign of the first term and using (5.35) for $j \geq 1$. The $\alpha = 2$ case is obvious from similar or limitation argument. So, Proposition 5.6 is first proved for $\alpha > 1$.

When $\alpha \leq 1$, from Lemma 5.5, (5.32) holds if we can prove, $\forall j \geq k \geq 0$,

$$\begin{aligned} & \left(\frac{j(\alpha-2) + k(\frac{Q}{2}-\alpha) + kj}{(\alpha-2)(\frac{Q}{2}-\alpha)(\alpha-2+k)(\frac{Q}{2}-\alpha+j)} + \frac{(k-1)(\alpha-2) + (j+1)(\frac{Q}{2}-\alpha) + (j+1)(k-1)}{(\alpha-2)(\frac{Q}{2}-\alpha)(\alpha-2+j+1)(\frac{Q}{2}-\alpha+k-1)} - \right. \\ & \left. \frac{\frac{Q}{2} - 2\alpha + 1}{(\alpha-2+k)(\frac{Q}{2}-\alpha+j)(\alpha-2+j+1)(\frac{Q}{2}-\alpha+k-1)} \right) \frac{\Gamma(k+\alpha-1)}{\Gamma(\alpha-1)} \leq 0. \end{aligned} \quad (5.36)$$

For $\alpha = 1$, (5.36) holds and equality holds if and only if $k = j = 0$ or $k \geq 2$. When $\alpha < 1$, we find it still holds for $k = 0$ and equality holds if and only if $j = 0$. For $k = 1$, it still holds for $j \leq j_\alpha = \frac{3}{2} - \frac{Q}{4} + \frac{\frac{Q}{4}-\frac{1}{2}}{1-\alpha}$, reaches equality if $j = j_\alpha$ (if $j_\alpha \in \mathbb{N}$), and especially holds strictly for $j = 1$, while the opposite strict inequality holds for $j > j_\alpha$. Note that $j_\alpha \in (1, \infty)$ and is strict increasing for $\alpha \in (0, 1)$. For $k \geq 2$, the opposite inequality holds strictly. So, we have proved that (5.32) and whence (5.31) holds for $\alpha = 1$, but *fails* for some functions in the case $0 < \alpha < 1$.

Then Proposition 5.6 is proved for $\alpha \geq 1$, which is a sharp range. $\square$

Proof of the claim. The $\lambda > 4$ case is already contained in Proposition 5.6. For $\lambda = 4$, from Lemma 5.5, we know that the eigenvalue $\lambda_{j,k}(|1-\zeta \cdot \bar{\eta}|^{-\frac{\lambda}{2}})$ vanishes for $k \geq 1$ and is positive for $k = 0$, so from Euler-Lagrange equation (5.1) and Proposition 5.6, h can only be constant. Another way to see this is considering directly the original inequality as for any positive extremizer h satisfying zero center-mass condition, we have

$$\iint_{S \times S} \frac{h(\zeta)h(\eta)}{d_S(\zeta, \eta)^\lambda} d\zeta d\eta = \iint_{S \times S} \frac{h_0(\zeta)h_0(\eta)}{d_S(\zeta, \eta)^\lambda} d\zeta d\eta, \quad \text{but} \quad \|h\|_p^2 \geq \|h_0\|_p^2,$$

where $h_0 = \frac{1}{|S|} \int_S h$ (mean of h on S) is the projection of h onto $V_{0,0}$. $\square$

Proof of Theorem 3.1 and 3.2. The **claim** is now proved and together with Lemma 5.1, 5.2, 5.3 tell that all extremizer pairs on the sphere is given by $f \sim g \sim |J_\tau|^{1/p}$, $\tau \in \text{Aut}(S)$, and the sharp constants and explicit formulas are proved in subsection 5.3 and Appendix B. $\square$

6 A Remark for Small λ

Weak local extremizer for $\lambda < 4$. For $\lambda < 4$, we primarily believe constant function should still be a global extremizer as we failed to find any counterexample and maybe strengthened analysis is needed. But what the extremizer could be is still unknown by us so far. However, we remark substitutively that constant

function is indeed a *weak local* extremizer. Actually, we proved that (5.8) holds strictly for $h \equiv 1$ and all φ satisfying $\int_S \varphi = 0$, $\int_S \varphi(\zeta) \zeta d\zeta = 0$. Note the second assumption of φ comes from the conformal symmetry group. If we decompose it w.r.t (2.6), $\varphi = \sum_{j,k} Y_{j,k}$ with $Y_{j,k} \in V_{j,k}$, then $Y_{0,0} = Y_{1,0} = 0$, and it suffices to prove

$$\sum_{j,k} (\lambda_{j,k}^\alpha - (p-1)\lambda_{0,0}^\alpha) |S| \|Y_{j,k}\|_2^2 < 0,$$

where $\lambda_{j,k}^\alpha$ is the eigenvalues of operator $|1 - \zeta \cdot \bar{\eta}|^{-2\alpha}$ with $\alpha = \frac{\lambda}{4}$. From (5.16), we can check that

$$\lambda_{j,k}^\alpha - (p-1)\lambda_{0,0}^\alpha$$

is positive when $j = k = 0$, zero when $j = 1, k = 0$, and strictly negative otherwise. So, we have proved a strictly negative second variation and therefore “weak local” maximum at constant function 1. A stronger concept of *local* extremizer means, any function h is called a local extremizer of functional $I(f)$, if there exists a small $\delta > 0$, s.t, in the δ -neighbourhood of h , $I(h)$ reaches a (local) maximum. Unfortunately, we have no uniformly norm expansion near 1.

Appendix

A Intertwining Operators

We give in this appendix a proof of the spectrum characterization of the intertwining operators defined on the sphere at the end of section 2. We use the idea of [5] from [33].

Proposition A.1. *Any operator $\mathcal{A}_d$ on the sphere S satisfies the intertwining property, i.e.,*

$$|J_\tau|^{\frac{Q+d}{2Q}} (\mathcal{A}_d F) \circ \tau = \mathcal{A}_d \left(|J_\tau|^{\frac{Q-d}{2Q}} (F \circ \tau) \right), \quad (\text{A.1})$$

for any $F \in C^\infty(S)$ and $\tau \in \text{Aut}(S)$, if and only if it's diagonal with respect to the bispherical harmonic decomposition with its spectrum, modulo a constant, almost uniquely given by

$$\lambda_{j,k}(\mathcal{A}_d|_{V_{j,k}}) = \frac{\Gamma(j + \frac{Q+d}{4}) \Gamma(k + \frac{Q+d}{4} - 1)}{\Gamma(j + \frac{Q-d}{4}) \Gamma(k + \frac{Q-d}{4} - 1)}.$$

Proof. Sufficiency. From the eigenvalues of the kernel $K = d_S(\zeta, \eta)^{d-Q}$, it suffices to prove this kernel satisfies an inverse intertwining relation, i.e.,

$$K(|J_\tau|^{\frac{Q+d}{2Q}} F \circ \tau) = |J_\tau|^{\frac{Q-d}{2Q}} (KF) \circ \tau,$$

for any conformal transformation and test function, which is obvious from changing variables and the relations (3.8): the left side equals

$$\begin{aligned} & \int_S d_S^{d-Q}(\zeta, \eta) |J_\tau(\eta)|^{\frac{Q+d}{2Q}} (F \circ \tau)(\eta) d\eta \\ &= \int_S |J_{\tau^{-1}}(\tau(\zeta)) J_{\tau^{-1}}(\eta)|^{\frac{d-Q}{2Q}} d_S^{d-Q}(\tau(\zeta), \eta) |J_\tau(\tau^{-1}(\eta))|^{\frac{Q+d}{2Q}} F(\eta) |J_{\tau^{-1}}(\eta)| d\eta \\ &= |J_\tau(\zeta)|^{\frac{Q-d}{2Q}} \int_S d_S^{d-Q}(\tau(\zeta), \eta) F(\eta) d\eta, \end{aligned}$$

which is the right side.

Necessity. Again the diagonal comes from Schur's lemma and we now compute the spectrums. Choose $\delta > 0$, and from the diagram (5.4), we have

$$\tau_\delta = \mathcal{C} \circ D_\delta \circ \mathcal{C}^{-1} : \zeta = (\zeta', \zeta_{n+1}) \mapsto \tau_\delta(\zeta) = \left(\frac{2\delta\zeta'}{1 + \zeta_{n+1} + \delta^2(1 - \zeta_{n+1})}, \frac{1 + \zeta_{n+1} - \delta^2(1 - \zeta_{n+1})}{1 + \zeta_{n+1} + \delta^2(1 - \zeta_{n+1})} \right).$$

We choose the special zonal spherical harmonics for the intertwining relation (A.1)

$$F = Z_{j,k}, \quad \tau = \tau_\delta, \quad F_\delta^a(\zeta) := |J_{\tau_\delta}(\zeta)|^{\frac{a}{Q}} (Z_{j,k} \circ \tau_\delta)(\zeta),$$

where $Z_{j,k}$ is given in (2.7) and the Jacobian of τ_δ is given by

$$|J_{\tau_\delta}| = |(2\delta)^{-1}(1 + \zeta_{n+1} + \delta^2(1 - \zeta_{n+1}))|^{-Q}.$$

Then for operator $\mathcal{A}_d$, we always have

$$\lambda_{j,k} F_\delta^{\frac{Q+d}{2}} = \mathcal{A}_d F_\delta^{\frac{Q-d}{2}}. \quad (\text{A.2})$$

Note that by comparing the highest order of $\text{Re } \zeta_{n+1}$, $|\zeta_{n+1}|$, we have

$$\left. \frac{\partial}{\partial \delta} \right|_{\delta=1} F_\delta^a = A^a Z_{j+1,k} + B^a Z_{j,k+1} + \text{Remainder}, \quad (\text{A.3})$$

where we omit the subscripts j, k of A and B just for simplicity. Then from this property and differentiation of (A.2) with regards to δ at point $\delta = 1$, we have

$$\lambda_{j,k} A^{\frac{Q+d}{2}} Z_{j+1,k} + \lambda_{j,k} B^{\frac{Q+d}{2}} Z_{j,k+1} = \lambda_{j+1,k} A^{\frac{Q-d}{2}} Z_{j+1,k} + \lambda_{j,k+1} B^{\frac{Q-d}{2}} Z_{j,k+1},$$

which gives

$$\frac{\lambda_{j+1,k}}{\lambda_{j,k}} = \frac{A^{\frac{Q+d}{2}}}{A^{\frac{Q-d}{2}}}, \quad \frac{\lambda_{j,k+1}}{\lambda_{j,k}} = \frac{B^{\frac{Q+d}{2}}}{B^{\frac{Q-d}{2}}}. \quad (\text{A.4})$$

Now we compute A, B . We first give several formulas

$$\begin{aligned} \frac{\sin(j-k+1)\phi}{\sin \phi} &= \sum_{i=0}^{\lfloor \frac{j-k}{2} \rfloor} a_i^{j-k} \cos^{j-k-2i} \phi \quad \text{with} \quad a_i^{j-k} = (-1)^i \binom{j-k-i}{i} 2^{j-k-2i}, \\ P_n^{\alpha,\beta}(x) &= \sum_{i=0}^n \binom{n+\alpha}{i} \binom{n+\beta}{n-i} \left(\frac{x+1}{2}\right)^i \left(\frac{x-1}{2}\right)^{n-i} \quad \text{for } x \in \mathbb{R}, \\ P_n^{\alpha,\beta}(2x-1) &= \binom{2n+\alpha+\beta}{n} x^n + \dots, \\ (P_n^{\alpha,\beta})'(2x-1) &= \frac{n}{2} \binom{2n+\alpha+\beta}{n} x^{n-1} + \dots \end{aligned}$$

As only ζ_{n+1} is related, in abuse of notation, we denote by just ζ this last quaternion coordinate, then

$$\begin{aligned} \left. \frac{\partial}{\partial \delta} \right|_{\delta=1} F_\delta^a &= a \text{Re } \zeta \left(\sum_{i=0}^{\lfloor \frac{j-k}{2} \rfloor} a_i^{j-k} (\text{Re } \zeta)^{j-k-2i} |\zeta|^{2i} \right) P_k^{2n-1,j-k+1}(2|\zeta|^2-1) \\ &+ \sum_{i=0}^{\lfloor \frac{j-k}{2} \rfloor} a_i^{j-k} \left(\chi_{i < \frac{j-k}{2}} 2(j-k-2i)(2(\text{Re } \zeta)^2 - |\zeta|^2 - 1)(\text{Re } \zeta)^{j-k-2i-1} |\zeta|^{2i} \right. \\ &\quad \left. + \chi_{i > 0} 2i(\text{Re } \zeta)^{j-k-2i+1} |\zeta|^{2i-2} (|\zeta|^2 - 1) \right) P_k^{2n-1,j-k+1}(2|\zeta|^2-1) \\ &+ \left(\sum_{i=0}^{\lfloor \frac{j-k}{2} \rfloor} a_i^{j-k} (\text{Re } \zeta)^{j-k-2i} |\zeta|^{2i} \right) (P_k^{2n-1,j-k+1})'(2|\zeta|^2-1) \end{aligned}$$

and after assuming that $j-k$ is positive even and comparing the coefficients of the highest order terms $(\text{Re } \zeta)^{j-k} |\zeta|^{2k}$ and $(\text{Re } \zeta) |\zeta|^{j+k}$ in (A.3), we have

$$\begin{aligned} \binom{2n+j+k}{k} a_0^{j-k} (a+2j) &= \binom{2n+j+1+k}{k} a_0^{j+1-k} A^a, \\ \binom{2n+j+k}{k} ((a+j+k) a_{\frac{j-k}{2}}^{j-k} - 2a_{\frac{j-k-2}{2}}^{j-k}) &= \binom{2n+j+1+k}{k} A^a a_{\frac{j-k}{2}}^{j+1-k} + \binom{2n+j+k+1}{k+1} B^a a_{\frac{j-k-2}{2}}^{j-k-1}, \end{aligned}$$

which gives

$$A^a = \frac{2n+j+1}{2n+j+k+1} (j + \frac{a}{2}), \quad B^a = \frac{k+1}{2n+j+k+1} (k + \frac{a}{2} - 1)$$

and from the recursion relation (A.4), we proved the proposition. $\square$

B The Jacobian Determinants of Conformal Transformations and Formulas of Extremizers

Proposition B.1. *The Jacobian of conformal transformations on the sphere and group are given by*

$$|J_{\tau_\xi}| \sim |1 - \xi \cdot \bar{\zeta}|^{-Q}, \quad |J_{\mathcal{C} \circ (\sigma_{q_0}^{r_0})}| \sim |q|^2 + w - 2q_0 \cdot \bar{q} + r_0|^{-Q}, \quad (\text{B.1})$$

for any $\xi \in \mathbb{H}^{n+1}$ satisfying $|\xi| < 1$, and $q_0 \in \mathbb{H}^n, r_0 \in \mathbb{H}$ satisfying $\text{Re } r_0 > |q_0|^2$. The correspondence between the conformal transformation and its parameters is not unique, but we can choose $\tau_\xi = \mathcal{C} \circ D_\delta \circ \mathcal{C}^{-1} \circ A_\xi$, where $\delta = \sqrt{\frac{1+|\xi|}{1-|\xi|}}$ and $A_\xi \in Sp(n+1)$ s.t. $A_\xi^{-1}(0, \dots, 0, 1) = \frac{\xi}{|\xi|}$, and $\sigma_{q_0}^{r_0} = D_{\delta_0} \circ L_{u_0}$, where $\delta_0 = (\text{Re } r_0 - |q_0|^2)^{-\frac{1}{2}}$ and $u_0 = (q_0, -\text{Im } r_0)$. Moreover, all extremizers for the sharp HLS are $1/p$ power of the Jacobians above.

Proof. We can prove directly for the Jacobians, but we want more attention on extremizers. A similar computation was done for the Heisenberg group and complex sphere in [32, 5].

Extremizers on the sphere. On the one hand, for any extremizer h of (3.5), after the conformal action in Lemma 5.2, $\tilde{h}$ is a constant function, which means $h = |J_\gamma|^{\frac{1}{p}}$ is the *only* possible²⁴ form of any extremizer, with γ given by (5.3). To compute, first note $|J_\gamma| \sim |J_{\mathcal{C}^{-1}}(A\zeta)| |J_{\mathcal{C}}(A\gamma(\zeta))|$, then from the formula (2.3) for Jacobian of the Cayley transform

$$\begin{aligned} h &\sim \left(\frac{(1 + |\delta q|^2)^2 + |\delta^2 w|^2}{(1 + |q|^2)^2 + |w|^2} \right)^{-\frac{2Q-\lambda}{4}} \\ &= \left| \frac{1 + |\delta q|^2 \pm \delta^2 w}{1 + |q|^2 \pm w} \right|^{-\frac{2Q-\lambda}{2}} \\ &\sim \left| 1 - \frac{\delta^2 - 1}{\delta^2 + 1} \frac{1 - |q|^2 \pm w}{1 + |q|^2 \mp w} \right|^{-\frac{2Q-\lambda}{2}} \\ &= |1 - \xi \cdot \bar{\zeta}|^{-\frac{2Q-\lambda}{2}}, \end{aligned}$$

with $(q, w) = \mathcal{C}^{-1}(A\zeta)$, $\xi = A^{-1}(0, \dots, \frac{\delta^2-1}{\delta^2+1})$, satisfying $|\xi| < 1$.

On the other hand, given any $|\xi| < 1$, we can inverse the above process: there exists one $A \in Sp(n+1)$ s.t. $A\xi = |\xi|(0, \dots, 0, 1)$, so, through the Cayley transform, we have

$$\begin{aligned} |J_{\mathcal{C}}|^{\frac{1}{p}} |1 - \xi \cdot \overline{(A^{-1}\zeta)}|^{-\frac{2Q-\lambda}{2}} &= |J_{\mathcal{C}}|^{\frac{1}{p}} \left| 1 - |\xi| \frac{1 - |q|^2 \pm w}{1 + |q|^2 \mp w} \right|^{-\frac{2Q-\lambda}{2}} \\ &\sim ((1 + |q|^2)^2 + |w|^2)^{-\frac{2Q-\lambda}{4}} \left(\frac{(1 + |\delta q|^2)^2 + |\delta^2 w|^2}{(1 + |q|^2)^2 + |w|^2} \right)^{-\frac{2Q-\lambda}{4}} \\ &= ((1 + |\delta q|^2)^2 + |\delta^2 w|^2)^{-\frac{2Q-\lambda}{4}}, \end{aligned}$$

with $(q, w) = \mathcal{C}^{-1}\zeta$, $\delta^2 = \frac{1+|\xi|}{1-|\xi|}$, which tells from the relation formula (3.4) that the counterpart on the group of function

$$|1 - \xi \cdot \overline{(A^{-1}\zeta)}|^{-\frac{2Q-\lambda}{2}}$$

is just the δ -dilation of a particular extremizer which corresponds to a constant function on the sphere. Because of the rotation-invariant of (3.5), any function with formula (3.7) is indeed an extremizer.

Then, combine the two direction arguments, we have proved that functions with formulas (3.7) are *right all* the extremizers for (3.5) and the first formula in (B.1) is then proved. Moreover, we note that the set of Jacobians of all conformal transformations γ_ξ^δ in Lemma 5.2 ergod the set of Jacobians $|J_\gamma|$, which tells the correspondence part of the proposition on the sphere.

²⁴ Actually they are, as constants are proved to be extremizers.

Extremizers on the group. For any $|\xi| < 1$,

$$\begin{aligned} |J_C|^{\frac{1}{p}} |1 - \xi \cdot \bar{\xi}|^{-\frac{2Q-\lambda}{2}} &\sim ((1 + |q|^2)^2 + |w|^2)^{-\frac{2Q-\lambda}{4}} \left| 1 - \xi \cdot \left(\frac{2q}{1 + |q|^2 - w}, \frac{1 - |q|^2 + w}{1 + |q|^2 - w} \right) \right|^{-\frac{2Q-\lambda}{2}} \\ &\sim \left| 1 + |q|^2 + w - (\xi_1, \xi_2, \dots, \xi_n) \cdot \overline{(2q)} - \xi_{n+1} \cdot (1 - |q|^2 - w) \right|^{-\frac{2Q-\lambda}{2}} \\ &\sim |q|^2 + w - 2q_0 \cdot \bar{q} + r_0|^{-\frac{2Q-\lambda}{2}}, \end{aligned}$$

with $q_0 = \frac{(\xi_1, \dots, \xi_n)}{1 + \xi_{n+1}}, r_0 = \frac{1 - \xi_{n+1}}{1 + \xi_{n+1}}$, satisfying $\text{Rer}_0 > |q_0|^2$. Then, because of the bijection between $\{\xi : |\xi| < 1\}$ and $\{(q_0, p_0) : \text{Rep}_0 > |q_0|^2\}$, we have proved that all extremizers of (3.1) are explicitly given by (3.3). Moreover, we remark that all Jacobians $|J_C|$ are ergoded by dilations and left translations from $|J_C|$, with the correspondence $\sigma_{q_0}^{r_0} = D_{\delta_0} \circ L_{u_0}$, where $\delta_0 = (\text{Rer}_0 - |q_0|^2)^{-\frac{1}{2}}, u_0 = (q_0, -\text{Im}r_0)$. Actually, for any $\delta_0 > 0, u_0 = (q_0, w_0) \in G$,

$$\begin{aligned} |J_C(\delta_0(u_0^{-1}u))| &\sim ((1 + |\delta_0(q - q_0)|^2)^2 + |\delta_0^2(w - w_0 - 2\text{Im}q_0 \cdot \bar{q})|^2)^{-\frac{Q}{2}} \\ &= |1 + |\delta_0(q - q_0)|^2 + \delta_0^2(w - w_0 - 2\text{Im}q_0 \cdot \bar{q})|^{-Q} \\ &\sim |q|^2 + w - 2q_0 \cdot \bar{q} + \delta_0^{-2} + |q_0|^2 - w_0|^{-Q}. \end{aligned}$$

Then we get another part of this proposition and Theorem 3.1 and 3.2. $\square$

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