Finite time blow up to critical semilinear wave equation outside the ball in 3-D

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Abstract

In this paper we study the initial boundary value problem of critical semilinear wave equation with small initial data in 3-D. Blow up result will be established no matter how small the data are. We find a radial and positive solution of the corresponding linear problem, then our blow up result follows from the idea of [23]. What is more, we get the upper bound of the lifespan estimate.

Keywords: Blow up, wave equation, exterior problem, critical power, lifespan estimate

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1. Introduction

We consider the initial boundary value problem of critical semilinear wave equation outside the unit ball in 3-D

$$\begin{cases} u_{tt} - \Delta u = |u|^p, & t > 0, \quad x \in B_1^c, \\ u(0, x) = \varepsilon u_0(x), \quad u_t(0, x) = \varepsilon u_1(x), & x \in B_1^c, \\ u|_{\partial B_1^c} = 0, \end{cases} \quad (1)$$

where

$$\Delta = \sum_{i=1}^3 \partial_{x_i}^2$$

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is the Laplace operator, $p = 1 + \sqrt{2}$, B_1^c denotes the exterior domain in $\mathbb{R}^3$ to the unit ball with the origin its center, and ε represents the smallness of the data. We assume that the data (u_0, u_1) satisfy a necessary compatibility condition arising from the Dirichlet boundary condition and

$$\begin{aligned} u_0(x) \geq 0, \quad u_1(x) \geq 0, \quad \forall x \in B_1^c, \\ \text{supp } u_0(x), u_1(x) \subset B_2 \triangleq \{x \in B_1^c, |x| < 2\}. \end{aligned} \quad (2)$$

Problem (1) can date back to the well-known Strauss' conjecture, which concerns the global and nonglobal existence of solutions to the Cauchy problem of semilinear wave equation

$$u_{tt} - \Delta u = |u|^p, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n. \quad (3)$$

The first work of the Cauchy problem (3) with small data is due to F. John [8], in which he considered the case $n = 3$ and regular data with compactly support and obtained two main results: (I) If $1 < p < 1 + \sqrt{2}$ and the data (u_0, u_1) are nonnegative, then the solution blows up in a finite time; (II) If $p > 1 + \sqrt{2}$, then the solution exists globally in time if the data are small enough. It implies that problem (3) with small data in $\mathbb{R}^3$ admits a critical power $p = 1 + \sqrt{2}$. After that, Strauss [17] conjectured that for each $n \geq 2$ there exists a critical power $p_c(n)$ for problem (3), which is the positive root of the quadratic equation

$$(n-1)p^2 - (n+1)p - 2 = 0.$$

And then many mathematicians have contributed to this conjecture, which has been completely solved recently. Glassey [3, 4] proved blow up for $1 < p < p_c(2)$ and global existence for $p > p_c(2)$. Sideris [14] established blow up result for
5 $1 < p < p_c(n)$ and $n \geq 4$, then the proof was simplified by Rammaha [12] and Jiao and Zhou [7]. Schaeffer [13] showed that the critical exponents $p = p_c(n)$ and $n = 2, 3$ also belong to the blow up case. In the opposite direction, Zhou [24] proved global existence for $p > p_c(4)$. Not much later, Georgiev, Lindblad and Sogge [2] showed the existence of global solutions for $p > p_c(n)$ and $n \geq 4$.
10 In 2001 Tataru [18] gave a simple proof of this result. Recently, the authors

[9] gave an elementary proof of Strauss' conjecture and furthermore obtained the sharp lower bound of lifespan estimate for $1 < p < p_c(n)$ and $n \geq 4$. The remaining part of Strauss' conjecture (blow up for $p = p_c(n)$ and $n \geq 4$) was solved by Yordanov and Zhang [20] and Zhou [25] independently.

15 For the Cauchy problem on asymptotically Euclidean non-trapping Riemannian manifolds, Sogge and Wang [16] obtained almost global existence for quadratic semilinear wave equation with small data in the case $n = 3$. Later, Wang and Yu [19] verified the Strauss' conjecture for semilinear wave equation on asymptotically Euclidean manifolds when $n = 3, 4$. Recently, Lindblad et al
20 [11] established an analog of the Strauss' conjecture on the Schwarzschild and Kerr (with small angular momentum) black hole backgrounds.

For the initial boundary value problem in exterior domain, Zhou and Han [26] established blow up result and lifespan estimate in the case $1 < p < p_c(n)$ and $n \geq 3$. The blow up result for $1 < p < p_c(2)$ was obtained by Li and Wang
25 [10] (see also [5]). When $p > p_c(n)$, it is known that we have global existence from the work of Du et al [1] for $n = 4$, Hidano et al [6] for $n = 3, 4$ and Smith, Sogge and Wang [15] for $n = 2$. What is more, for the critical case $p = p_c(n)$, we know that the lower bound of the lifespan estimate satisfies $T(\varepsilon) \geq \exp(C\varepsilon^{1-p})$ due to the work [1] for $n = 4$ and Yu [21] for $n = 3$. Recently, Zha and Zhou
30 [22] showed that $T(\varepsilon) \geq \exp(C\varepsilon^{-2})$ in the case $p = p_c(n)$ and $n = 4$.

In the present work, we are devoted to studying the initial boundary value problem for critical semilinear wave equation outside the unit ball in 3-D. Blow up result and the upper bound of lifespan estimate will be established, no matter how small the data are. In the case of the Cauchy problem in $\mathbb{R}^3$, one can use the positivity of the fundamental solution to the linear wave equation to get the blow up result. For the initial boundary value problem in 3-D, the key step is to seek a radial, positive and almost homogeneous solution of the linear wave equation

$$\begin{cases} v_{tt} - v_{rr} - \frac{2}{r}v_r = 0, & r \geq 1, \\ v(t, 1) = 0. \end{cases} \quad (4)$$

Then we obtain our blow up result and lifespan estimate by using the idea of [23].

Remark 1.1. *We consider the initial boundary value problem in the case that the supports of the initial data are not far away from the boundary. If they are sufficiently far away from the obstacle, the problem can be treated as in the free space case (Cauchy problem), according to the finite propagation of the support of the solution.*

Remark 1.2. *Unfortunately, we can't find a solution to the linear problem (4) in high dimensions ($n \geq 4$) which admits radial symmetry, positivity and homogeneity and hence we only consider the initial boundary value problem in 3-D. Blow up result for the initial boundary value problem of critical semilinear wave equation in exterior domain for $n \geq 4$ still remains open.*

We state our main result in the following theorem.

Theorem 1.3. *Assuming the data (u_0, u_1) satisfy (2), then the solution of problem (1) with $p = 1 + \sqrt{2}$ blows up in a finite time. What is more, there exists a positive constant C which is independent of ε such that the lifespan estimate satisfies*

$$T(\varepsilon) \leq \exp(C\varepsilon^{-\sqrt{2}(1+\sqrt{2})}). \quad (5)$$

Here and in the sequence, C denotes a positive constant which may change from line to line.

2. Preliminaries

In this section, we give some preliminary lemmas.

Lemma 2.1. *Let $w_0(x) = \frac{r-1}{r}$ with $r = |x|$, then it satisfies*

$$\begin{cases} \Delta w_0(x) = 0, & r \geq 1, \\ w_0(x)|_{\partial B_1^c} = 0, \end{cases} \quad (6)$$

and

$$\begin{aligned} 0 &\leq w_0(x) < 1, \\ w_0(x) &\rightarrow 1, \quad r \rightarrow \infty. \end{aligned} \tag{7}$$

It's easy to prove this lemma and we omit the details.

Lemma 2.2. *Let*

$$w_1(t, x) = \frac{(t+2+r)^{\sqrt{2}-1} - (t+4-r)^{\sqrt{2}-1}}{r}, \tag{8}$$

then $w_1(t, x)$ solves

$$\begin{cases} v_{tt} - v_{rr} - \frac{2}{r}v_r = 0, & r \geq 1, \\ v(t, 1) = 0, \end{cases} \tag{9}$$

and satisfies for $1 < r \leq t+2$

$$w_1(t, x) \sim w_0(x)(1+t)^{\sqrt{2}-2}. \tag{10}$$

Here the notation $f(x) \sim g(x)$ means that there exists a positive constant C such that

$$C^{-1}g(x) \leq f(x) \leq Cg(x). \tag{11}$$

Proof. Since

$$rw_1(t, x) = (t+2+r)^{\sqrt{2}-1} - (t+4-r)^{\sqrt{2}-1}, \tag{12}$$

then $rw_1(t, x)$ solves

$$(rw_1)_{tt} - (rw_1)_{rr} = 0, \tag{13}$$

and this implies

$$w_{1tt} - w_{1rr} - \frac{2}{r}w_{1r} = 0, \tag{14}$$

which verifies that $w_1(t, x)$ solves equation (9).

Next we demonstrate the proof of (10). First for $1 < r \leq \frac{t+4}{2}$, we have

$$\begin{aligned} w_1(t, x) &= \frac{(t+2+r)^{\sqrt{2}-1} - (t+4-r)^{\sqrt{2}-1}}{r} \\ &= \frac{\int_{-1}^1 \frac{d}{d\theta} [t+3+\theta(r-1)]^{\sqrt{2}-1} d\theta}{r} \\ &= \frac{(\sqrt{2}-1)(r-1) \int_{-1}^1 [t+3+\theta(r-1)]^{\sqrt{2}-2} d\theta}{r} \\ &\sim w_0(x)(1+t)^{\sqrt{2}-2}. \end{aligned} \tag{15}$$

For $\frac{t+4}{2} < r \leq t+2$, on one hand it is easy to get

$$\begin{aligned} w_1(t, x) &= \frac{(t+2+r)^{\sqrt{2}-1} - (t+4-r)^{\sqrt{2}-1}}{r} \\ &\leq \frac{(t+2+r)^{\sqrt{2}-1}}{r} \\ &\leq C(1+t)^{\sqrt{2}-2}. \end{aligned} \quad (16)$$

On the other hand we have

$$\begin{aligned} (t+2+r)^{\sqrt{2}-1} &\geq \left(t+2+\frac{t+4}{2}\right)^{\sqrt{2}-1} \\ &= \left(\frac{3t+8}{2}\right)^{\sqrt{2}-1} \end{aligned} \quad (17)$$

and

$$\begin{aligned} (t+4-r)^{\sqrt{2}-1} &\leq \left(t+4-\frac{t+4}{2}\right)^{\sqrt{2}-1} \\ &= \left(\frac{t+4}{2}\right)^{\sqrt{2}-1}. \end{aligned} \quad (18)$$

By combining (17) and (18) we come to

$$\begin{aligned} w_1(t, x) &= \frac{(t+2+r)^{\sqrt{2}-1} - (t+4-r)^{\sqrt{2}-1}}{r} \\ &\geq C(1+t)^{\sqrt{2}-2}. \end{aligned} \quad (19)$$

It is obvious that

$$w_0(x) = \frac{r-1}{r} \sim C, \quad \text{for } \frac{t+4}{2} < r \leq t+2 \quad (20)$$

for some positive constant C , we then conclude from (16), (19) and (20) that

$$\begin{aligned} w_1(t, x) &= \frac{(t+2+r)^{\sqrt{2}-1} - (t+4-r)^{\sqrt{2}-1}}{r} \\ &\sim w_0(x)(1+t)^{\sqrt{2}-2}, \quad \frac{t+4}{2} < r \leq t+2. \end{aligned} \quad (21)$$

⁵⁰ Hence estimate (10) comes from (15) and (21).

Lemma 2.3. *Let*

$$w_2(t, x) = \frac{(t+4-r)^{\sqrt{2}-2} - (t+2+r)^{\sqrt{2}-2}}{r}, \quad (22)$$

then we have

$$\partial_t w_1(t, x) = (1 - \sqrt{2})w_2(t, x) \quad (23)$$

and

$$w_2(t, x) \leq Cw_0(x)(2+t)^{-1}(t+3-r)^{\sqrt{2}-2}, \quad 1 < r \leq t+2. \quad (24)$$

Proof. We first consider the case $1 < r \leq \frac{t+4}{2}$. As in (15) we have

$$\begin{aligned} w_2(t, x) &= \frac{(t+4-r)^{\sqrt{2}-2} - (t+2+r)^{\sqrt{2}-2}}{r} \\ &= \frac{\int_{-1}^1 \frac{d}{d\theta} [t+3-\theta(r-1)]^{\sqrt{2}-2} d\theta}{r} \\ &= \frac{(2-\sqrt{2})(r-1) \int_{-1}^1 [t+3-\theta(r-1)]^{\sqrt{2}-3} d\theta}{r} \\ &\sim w_0(x)(2+t)^{\sqrt{2}-3} \\ &\leq Cw_0(x)(2+t)^{-1}(3+t-r)^{\sqrt{2}-2}. \end{aligned} \quad (25)$$

For $\frac{t+4}{2} < r \leq t+2$, since $w_0(x) = \frac{r-1}{r} \sim C$, then it holds

$$\begin{aligned} w_2(t, x) &= \frac{(t+4-r)^{\sqrt{2}-2} - (t+2+r)^{\sqrt{2}-2}}{r} \\ &= \frac{(t+4-r)^{\sqrt{2}-2}}{r} \left[1 - \left(\frac{t+2+r}{t+4-r} \right)^{\sqrt{2}-2} \right] \\ &\leq Cw_0(x)(2+t)^{-1}(t+3-r)^{\sqrt{2}-2}. \end{aligned} \quad (26)$$

The desired inequality (24) follows from (25) and (26).

Lemma 2.4. *Let u solve problem (1), then we have*

$$\int_{B_1^c} w_0(x) |u|^{1+\sqrt{2}} dx \geq C\varepsilon^{1+\sqrt{2}} (t+1)^{1-\sqrt{2}}, \quad (27)$$

where $w_0(x) = \frac{r-1}{r}$.

Proof. For the proof, we refer the reader to [26], in which they employed a special test function to obtain the desired estimate for $n \geq 3$.

55 3. Proof of the Main Result

Now we are in a position to demonstrate the proof of the main result. Following the idea of [23], the key step is to use the test function $w_1(t, x)$ defined above to construct two auxiliary functions: $G(t)$ and $H(t)$. What is more, we obtain some estimates for $G(t)$ and $H(t)$ by using the properties of $w_1(t, x)$ and

60 $w_2(t, x)$.

3.1. The auxiliary function $G(t)$

Lemma 3.1. *Let $u(t, x)$ solve problem (1) with assumption (2) and $w_1(t, x)$ be defined as in (8). Set*

$$G(t) = \int_0^t (t - \tau)(1 + \tau) \int_{B_1^c} w_1(\tau, x) |u(\tau, x)|^{1+\sqrt{2}} dx d\tau, \quad (28)$$

then it holds

$$G'(t) > C(\ln(2+t))^{-\sqrt{2}}(2+t) \left(\int_0^t (2+\tau)^{-3} G(\tau) d\tau \right)^{1+\sqrt{2}}, \quad t \geq 1, \quad (29)$$

where $C > 0$ is a constant independent of ε .

Proof. By direct computation, we have

$$\begin{aligned} G'(t) &= \int_0^t (1 + \tau) \int_{B_1^c} w_1(\tau, x) |u(\tau, x)|^{1+\sqrt{2}} dx d\tau, \\ G''(t) &= (1 + t) \int_{B_1^c} w_1(t, x) |u(t, x)|^{1+\sqrt{2}} dx. \end{aligned} \quad (30)$$

Multiplying both sides of the equation in (1) with $w_1(t, x)$, we come to

$$\int_{B_1^c} w_1(u_{tt} - \Delta u) dx = \int_{B_1^c} w_1 |u|^{1+\sqrt{2}} dx. \quad (31)$$

By using integration by parts and the boundary conditions we arrive at

$$\begin{aligned} \int_{B_1^c} w_1 \Delta u dx &= \int_{B_1^c} \Delta w_1 u dx \\ &= \int_1^\infty \int_{\mathbb{S}^2} \left(\partial_r^2 w_1 + \frac{2}{r} \partial_r w_1 \right) u r^2 d\omega dr \\ &= \int_1^\infty \int_{\mathbb{S}^2} w_{1tt} u r^2 d\omega dr \\ &= \int_{B_1^c} \partial_t^2 w_1 u dx, \end{aligned} \quad (32)$$

and this yields

$$\begin{aligned} \int_{B_1^c} w_1(u_{tt} - \Delta u) dx &= \int_{B_1^c} w_1 u_{tt} - w_{1tt} u dx \\ &= \frac{d}{dt} \int_{B_1^c} (w_1 u_t - w_{1t} u) dx. \end{aligned} \quad (33)$$

On the other hand from (23) it follows

$$\begin{aligned}\int_{B_1^c} w_1 u_t - w_{1t} u dx &= \frac{d}{dt} \int_{B_1^c} w_1 u dx - 2 \int_{B_1^c} w_{1t} u dx \\ &= \frac{d}{dt} \int_{B_1^c} w_1 u dx + 2(\sqrt{2} - 1) \int_{B_1^c} w_2 u dx.\end{aligned}\quad (34)$$

Hence by combining (31), (33) and (34) we obtain

$$\frac{d^2}{dt^2} \int_{B_1^c} w_1 u dx + 2(\sqrt{2} - 1) \frac{d}{dt} \int_{B_1^c} w_2 u dx = \int_{B_1^c} w_1 |u|^{1+\sqrt{2}} dx. \quad (35)$$

Integrating (35) over $[0, t]$ three times yields

$$\begin{aligned}& \int_0^t \int_{B_1^c} w_1 u dx d\tau + 2(\sqrt{2} - 1) \int_0^t (t - \tau) \int_{B_1^c} w_2 u dx d\tau \\ &= \frac{1}{2} \int_0^t (t - \tau)^2 \int_{B_1^c} w_1 |u|^{1+\sqrt{2}} dx d\tau + \varepsilon t \int_{B_1^c} w_1(0, x) u_0(x) dx \\ &+ \frac{\varepsilon t^2}{2} \int_{B_1^c} (w_1(0, x) u_1(x) + (\sqrt{2} - 1) w_2(0, x) u_0(x)) dx.\end{aligned}\quad (36)$$

By combining (36) and (2) we come to

$$\begin{aligned}& \int_0^t \int_{B_1^c} w_1 u dx d\tau + 2(\sqrt{2} - 1) \int_0^t (t - \tau) \int_{B_1^c} w_2 u dx d\tau \\ &\geq \frac{1}{2} \int_0^t (t - \tau)^2 \int_{B_1^c} w_1 |u|^{1+\sqrt{2}} dx d\tau.\end{aligned}\quad (37)$$

By Hölder inequality and finite propagation speed of wave equations we obtain

$$\begin{aligned}& \int_0^t \int_{B_1^c} w_1 u dx d\tau \\ &= \int_0^t \int_{B_1^c \cap |x| \leq \tau+2} w_1 u dx d\tau \\ &\leq (G'(t))^{\frac{1}{1+\sqrt{2}}} \left(\int_0^t \int_{B_1^c \cap |x| \leq \tau+2} (1 + \tau)^{-\frac{1}{\sqrt{2}}} w_1 dx d\tau \right)^{\frac{\sqrt{2}}{1+\sqrt{2}}}.\end{aligned}\quad (38)$$

From (10) we know that

$$w_1(t, x) \sim w_0(x)(1+t)^{\sqrt{2}-2} \leq C(1+t)^{\sqrt{2}-2}, \quad (39)$$

which in turn gives by combining (38)

$$\begin{aligned}
& \int_0^t \int_{B_1^c} w_1 u dx d\tau \\
& \leq (G'(t))^{\frac{1}{1+\sqrt{2}}} \left(\int_0^t \int_{B_1^c \cap |x| \leq \tau+2} (1+\tau)^{-\frac{1}{\sqrt{2}}} w_1 dx d\tau \right)^{\frac{\sqrt{2}}{1+\sqrt{2}}} \\
& \leq C (G'(t))^{\frac{1}{1+\sqrt{2}}} \left(\int_0^t (1+\tau)^{-\frac{1}{\sqrt{2}} + \sqrt{2} - 2 + 3} d\tau \right)^{\frac{\sqrt{2}}{1+\sqrt{2}}} \\
& \leq C (G'(t))^{\frac{1}{1+\sqrt{2}}} (1+t)^{3-\sqrt{2}}.
\end{aligned} \tag{40}$$

In the same way, by Hölder inequality one has

$$\begin{aligned}
& \int_0^t (t-\tau) \int_{B_1^c} w_2 u dx d\tau \\
& \leq (G'(t))^{\frac{1}{1+\sqrt{2}}} \\
& \quad \times \left(\int_0^t (t-\tau)^{\frac{1+\sqrt{2}}{\sqrt{2}}} \int_{B_1^c \cap |x| \leq \tau+2} w_1 \left(\frac{w_2}{w_1} \right)^{\frac{1+\sqrt{2}}{\sqrt{2}}} (1+\tau)^{-\frac{1}{\sqrt{2}}} dx d\tau \right)^{\frac{\sqrt{2}}{1+\sqrt{2}}}.
\end{aligned} \tag{41}$$

It's easy to get from (10) in Lemma 2.2 and (24) in Lemma 2.3

$$\begin{aligned}
& \int_{B_1^c \cap |x| \leq \tau+2} w_1 \left(\frac{w_2}{w_1} \right)^{\frac{1+\sqrt{2}}{\sqrt{2}}} (1+\tau)^{-\frac{1}{\sqrt{2}}} dx \\
& \leq C \int_0^{\tau+2} (1+\tau)^{\sqrt{2}-2 - \frac{(\sqrt{2}-2)(1+\sqrt{2})}{\sqrt{2}} - \frac{1+\sqrt{2}}{\sqrt{2}} - \frac{1}{\sqrt{2}} + 2} (3+\tau-r)^{\frac{(\sqrt{2}-2)(1+\sqrt{2})}{\sqrt{2}}} dr \\
& = C \int_0^{\tau+2} (3+\tau-r)^{-1} dr \\
& = C \ln(3+\tau).
\end{aligned} \tag{42}$$

Then by combining (41) and (42) we come to

$$\begin{aligned}
& \int_0^t (t-\tau) \int_{B_1^c} w_2 u dx d\tau \\
& \leq C (G'(t))^{\frac{1}{1+\sqrt{2}}} \left(\int_0^t (t-\tau)^{\frac{1+\sqrt{2}}{\sqrt{2}}} \ln(3+\tau) d\tau \right)^{\frac{\sqrt{2}}{1+\sqrt{2}}} \\
& \leq C (G'(t))^{\frac{1}{1+\sqrt{2}}} (\ln(3+t))^{\frac{\sqrt{2}}{1+\sqrt{2}}} (1+t)^{3-\sqrt{2}}.
\end{aligned} \tag{43}$$

Hence it follows from the second equality in (30), inequalities (37), (40) and (43) that

$$\begin{aligned}
& (G'(t))^{\frac{1}{1+\sqrt{2}}} (\ln(2+t))^{\frac{\sqrt{2}}{1+\sqrt{2}}} (1+t)^{3-\sqrt{2}} \\
& \geq C \int_0^t (t-\tau)^2 (1+\tau)^{-1} G''(\tau) d\tau,
\end{aligned} \tag{44}$$

where the fact $\ln(2+t) > 1$ when $t \geq 1$ has been used.

By integration by parts it is easy to get

$$\begin{aligned} \int_0^t (t-\tau)^2 (1+\tau)^{-1} G''(\tau) d\tau &= \int_0^t \partial_\tau^2 [(t-\tau)^2 (1+\tau)^{-1}] G(\tau) d\tau \\ &= 2(1+t)^2 \int_0^t (1+\tau)^{-3} G(\tau) d\tau. \end{aligned} \quad (45)$$

Hence (44) and (45) yield

$$\begin{aligned} &(G'(t))^{\frac{1}{1+\sqrt{2}}} (\ln(2+t))^{\frac{\sqrt{2}}{1+\sqrt{2}}} (1+t)^{3-\sqrt{2}} \\ &\geq C(1+t)^2 \int_0^t (1+\tau)^{-3} G(\tau) d\tau, \quad t \geq 1, \end{aligned} \quad (46)$$

which in turn gives

$$G'(t) \geq C(\ln(2+t))^{-\sqrt{2}} (1+t) \left(\int_0^t (1+\tau)^{-3} G(\tau) d\tau \right)^{1+\sqrt{2}}, \quad t \geq 1, \quad (47)$$

which is the desired inequality (29) and we finish the proof of Lemma 3.1.

65 3.2. The auxiliary function $H(t)$

Set

$$H(t) = \int_0^t (2+\tau)^{-3} G(\tau) d\tau, \quad (48)$$

then we have

$$H'(t) = (2+t)^{-3} G(t), \quad (49)$$

which implies

$$G(t) = (2+t)^3 H'(t) \quad (50)$$

and

$$((2+t)^3 H'(t))' > C(\ln(2+t))^{-\sqrt{2}} (2+t) H^p(t). \quad (51)$$

For $G(t)$, we can establish the following estimate by combining (10) and Lemma 2.4

$$\begin{aligned} G(t) &= \int_0^t (t-\tau)(1+\tau) \int_{B_1^c} w_1(\tau, x) |u(\tau, x)|^{1+\sqrt{2}} dx d\tau \\ &\geq C \int_0^t (t-\tau)(1+\tau)^{\sqrt{2}-1} \int_{B_1^c} w_0(x) |u|^{1+\sqrt{2}} dx d\tau \\ &\geq C\varepsilon^{1+\sqrt{2}} \int_0^t (t-\tau) d\tau \\ &\geq C\varepsilon^{1+\sqrt{2}} t^2. \end{aligned} \quad (52)$$

Then from the definition of $H(t)$ we have

$$\begin{aligned} H(t) &\geq C\varepsilon^{1+\sqrt{2}} \int_0^t (2+\tau)^{-3} \tau^2 d\tau \\ &\geq C\varepsilon^{1+\sqrt{2}} \ln(2+t), \quad t \geq 2. \end{aligned} \tag{53}$$

Also from (49) one gets

$$\begin{aligned} H'(t) &= (2+t)^{-3} G(t) \\ &\geq C\varepsilon^{1+\sqrt{2}} (2+t)^{-3} t^2 \\ &\geq C\varepsilon^{1+\sqrt{2}} (2+t)^{-1}, \quad t \geq 2. \end{aligned} \tag{54}$$

The remaining part of the proof is similar to that given in [23] so we shall be brief.

3.3. Obtain $H_0(t)$ and $H_1(t)$ from $H(t)$ by changing of variables

Set

$$\tau = \ln(t+2)$$

and

$$H_0(\tau) = H(t) = H(e^\tau - 2), \tag{55}$$

then (51), (53) and (54) become

$$\begin{aligned} H_0''(\tau) + 2H_0'(\tau) &\geq C_1 \tau^{-\sqrt{2}} H_0^{1+\sqrt{2}}(\tau), \\ H_0(\tau) &\geq C_2 \varepsilon^{1+\sqrt{2}} \tau, \\ H_0'(\tau) &\geq C_2 \varepsilon^{1+\sqrt{2}}, \end{aligned} \tag{56}$$

where C_1, C_2 are positive constants independent of ε .

Set

$$H_1(s) = \varepsilon H_0(\varepsilon^{-(2+\sqrt{2})} s), \tag{57}$$

then (56) yields

$$\begin{aligned} \varepsilon^{2+\sqrt{2}} H_1''(s) + 2H_1'(s) &\geq C_1 s^{-\sqrt{2}} H_1^{1+\sqrt{2}}(s), \\ H_1(s) &\geq C_2 s, \\ H_1'(s) &\geq C_2. \end{aligned} \tag{58}$$

70 3.4. Introduce auxiliary functions $H_2(s)$ and $H_3(s)$

Choosing s_0, δ independent of ε such that

$$C_1, C_2 \ll s_0 \ll \frac{1}{\delta},$$

and assuming that $H_3(s)$ solves the ordinary differential equation

$$\begin{aligned} H_3'(s) &= \delta H_3^{\frac{2+\sqrt{2}}{2}}(s), \quad s \geq s_0, \\ H_3(s_0) &= \frac{C_2}{4}. \end{aligned} \tag{59}$$

Then it is easy to see that there exists $s_1 > 0$ which is independent of ε such that $H_3(s)$ blows up.

Let $H_2(s) = sH_3(s)$. If δ is small enough, then we have

$$\begin{aligned} \varepsilon^{2+\sqrt{2}} H_2''(s) + 2H_2'(s) &\leq C_1 s^{-\sqrt{2}} H_2^{1+\sqrt{2}}(s), \\ H_2(s_0) &\leq C_2 s_0, \\ H_2'(s_0) &\leq C_2. \end{aligned} \tag{60}$$

Next we introduce a comparison lemma which is due to the work of Zhou and Han [23].

Lemma 3.2. (Zhou and Han [23], Lemma 2.1) Suppose that $K(t)$ and $h(t)$ are all positive C^2 functions and satisfy

$$\begin{aligned} a(t)K''(t) + K'(t) &\geq b(t)K^{1+\alpha}(t), \quad \forall t \geq 0, \\ a(t)h''(t) + h'(t) &\leq b(t)h^{1+\alpha}(t), \quad \forall t \geq 0 \end{aligned} \tag{61}$$

with $\alpha \geq 0$, $a(t), b(t) > 0, \forall t \geq 0$. If $K(0) > h(0)$ and $K'(0) > h'(0)$, then we have

$$\begin{aligned} K'(t) &> h'(t), \quad \forall t \geq 0, \\ K(t) &> h(t), \quad \forall t \geq 0. \end{aligned} \tag{62}$$

By combining (58), (60) and Lemma 3.2 we come to

$$H_1(s) \geq H_2(s) = sH_3(s), \quad s \geq s_0, \tag{63}$$

75 which means that the lifespan of $H_1(s)$ has an upper bound s_1 , and hence the lifespan of $H_0(s)$ has an upper bound $\varepsilon^{-\sqrt{2}(1+\sqrt{2})}s_1$ by (57). Finally we get the desired lifespan estimate (5) from (55).

4. Conclusion

In this work we establish blow up result and lifespan estimate for the exterior
80 problem of critical semilinear wave equation in 3-D. For this purpose, two special
test functions are introduced. In high dimensions($n \geq 4$), we can't find the
corresponding test functions which admit "good behavior" hence our method
doesn't work. Our next goal is to study the initial boundary value problem of
critical semilinear wave equation in exterior domain in high dimensions($n \geq 4$).

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