HIGHER REGULARITY OF THE FREE BOUNDARY IN THE ELLIPTIC SIGNORINI PROBLEM

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ABSTRACT. In this paper we study the higher regularity of the free boundary for the elliptic Signorini problem. By using a partial hodograph-Legendre transformation we show that the regular part of the free boundary is real analytic. The first complication in the study is the invertibility of the hodograph transform (which is only $C^{0,1/2}$) which can be overcome by studying the precise asymptotic behavior of the solutions near regular free boundary points. The second and main complication in the study is that the equation satisfied by the Legendre transform is degenerate. However, the equation has a subelliptic structure and can be viewed as a perturbation of the Baouendi-Grushin operator. By using the L^p theory available for that operator, we can bootstrap the regularity of the Legendre transform up to real analyticity, which implies the real analyticity of the free boundary.

1. INTRODUCTION

1.1. Problem set-up and known results. Let B_R be the Euclidean ball in $\mathbb{R}^n$ ($n \geq 3$) centered at the origin with radius $R > 0$. Let $B_R^+ = B_R \cap \{x_n > 0\}$, $B'_R = B_R \cap \{x_n = 0\}$ and $(\partial B_R)^+ = \partial B_R \cap \{x_n > 0\}$. Consider local minimizers of the Dirichlet functional

$$J(v) = \int_{B_R^+} |\nabla v|^2 dx$$

over the closed convex set

$$\mathcal{K} = \{v \in W^{1,2}(B_R^+) : v \geq 0 \text{ on } B'_R\},$$

i.e. functions $u \in \mathcal{K}$ which satisfy

$$J(u) \leq J(v), \text{ for any } v \in \mathcal{K} \text{ with } v - u = 0 \text{ on } (\partial B_R)^+.$$

This problem is known as the *(boundary) thin obstacle problem* or the *(elliptic) Signorini problem*. It was shown in [AC04] that the local minimizers u are of class $C_{\text{loc}}^{1,1/2}(B_R^+ \cup B'_R)$. Besides, u will satisfy

$$(1.1) \quad \Delta u = 0 \quad \text{in } B_R^+,$$

$$(1.2) \quad u \geq 0, \quad -\partial_{x_n} u \geq 0, \quad u \cdot \partial_{x_n} u = 0 \quad \text{on } B'_R.$$

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The boundary condition (1.2) is known as the *complementarity* or *Signorini boundary condition*. One of the main features of the problem is that the following sets are apriori unknown:

$$\begin{aligned}\Lambda_u &= \{x \in B'_R : u(x) = 0\} && \text{coincidence set} \\ \Omega_u &= \{x \in B'_R : u(x) > 0\} && \text{positivity set} \\ \Gamma_u &= \partial_{B'_R} \Omega_u, && \text{free boundary}\end{aligned}$$

where by $\partial_{B'_R}$ we understand the boundary in the relative topology of B'_R . The free boundary Γ_u sometimes is said to be *thin*, to indicate that it is (expected to be) of codimension two. One of the most interesting questions in this problem is the study of the structure and the regularity of the free boundary Γ_u .

To put our results in a proper perspective, below we give a brief overview of some of the known results in the literature. The proofs can be found in [ACS08, CSS08, GP09] and in Chapter 9 of [PSU12].

We start by noting that we can extend solutions u of the Signorini problem (1.1)–(1.2) to the entire ball B_R in two different ways: either by even symmetry in x_n variable or by odd symmetry. The even extension will be harmonic in $B_R \setminus \Lambda_u$, while the odd extension will be so in $B_R \setminus \Omega_u$. In a sense, those two extensions can be viewed as two different branches of a two-valued harmonic function. This gives a heuristic explanation for the monotonicity of Almgren's *frequency function*

$$N_{x_0}(u, r) = \frac{r \int_{B_r^+(x_0)} |\nabla u|^2}{\int_{\partial B_r(x_0)^+} u^2},$$

which goes back to Almgren's study of multi-valued harmonic functions [Alm00]. In particular, the limiting value

$$\kappa(x_0) = N_{x_0}(0_+, u)$$

for $x_0 \in \Gamma_u$ turns out to be a very effective tool in classifying free boundary points. By using the monotonicity of the frequency N_{x_0} , it can be shown that the rescalings

$$(1.3) \quad \tilde{u}_{r,x_0}(x) = \frac{u(x_0 + rx)}{\left(\int_{\partial B_r^+(x_0)} u^2\right)^{1/2}}.$$

converge, over subsequences $r = r_j \rightarrow 0_+$, to solutions $\tilde{u}_0$ of the Signorini problem (1.1)–(1.2) in $\mathbb{R}_+^n$. Such limits are known as *blowups* of u at x_0 . Moreover, it can be shown that such blowups will be homogeneous of degree $\kappa(x_0)$, regardless of the sequence $r_j \rightarrow 0_+$. It is readily seen from the definition that the mapping $x_0 \mapsto \kappa(x_0)$ is upper semicontinuous on Γ_u . Furthermore, it can be shown that $\kappa(x_0) \geq 3/2$ for every $x_0 \in \Gamma_u$ and, more precisely, that the following alternative holds:

$$\kappa(x_0) = 3/2 \quad \text{or} \quad \kappa(x_0) \geq 2.$$

This brings us to the notion of a regular point. A point $x_0 \in \Gamma_u$ is called *regular* if $\kappa(x_0) = 3/2$. By classifying all possible homogeneous solutions of homogeneity $3/2$, the above definition is equivalent to saying that the blowups of u at x_0 have the form

$$\tilde{u}_0(x) = c_n \operatorname{Re}(x_{n-1} + ix_n)^{3/2},$$

after a possible rotation of coordinate axes in $\mathbb{R}^{n-1}$.

In what follows, we will denote by $\mathcal{R}_u$ set of regular free boundary points, and call it the *regular set* of u :

$$\mathcal{R}_u = \{x_0 \in \Gamma_u : \kappa(x_0) = 3/2\}.$$

The upper semicontinuity of κ , and the gap of values between $3/2$ and 2 implies that $\mathcal{R}_u$ is a relatively open subset of Γ_u . Besides, it is known that $\mathcal{R}_u$ is locally a $C^{1,\alpha}$ regular $(n-2)$ -dimensional surface. In this paper, we are interested in the higher regularity of $\mathcal{R}_u$. Since the co-dimension of the free boundary Γ_u is 2 , this question is meaningful only when $n \geq 3$. In fact, in dimension $n = 2$ the complete characterization of the coincidence set and the free boundary was already found by Lewy [Lew72]: Λ_u is a locally finite union of closed intervals.

1.2. Notations. We will use fairly standard notations in this paper. By $\mathbb{R}^n$ we denote the n -dimensional Euclidean space of points $x = (x_1, \dots, x_n)$, $x_i \in \mathbb{R}$, $i = 1, \dots, n$. For any $x \in \mathbb{R}^n$ we denote $x' = (x_1, \dots, x_{n-1})$ and $x'' = (x_1, \dots, x_{n-2})$. We also identify x' with $(x', 0)$, thereby effectively embedding $\mathbb{R}^{n-1}$ into $\mathbb{R}^n$. Similarly, we identify x'' with $(x'', 0)$ and $(x'', 0, 0)$.

For $r > 0$,

$$\begin{aligned} B_r(x_0) &= \{x \in \mathbb{R}^n : |x - x_0| < r\}, \\ B_r^+(x_0) &= B_r(x_0) \cap \{x_n > 0\}, \\ B'_r(x_0) &= B_r(x_0) \cap \{x_n = 0\}, \\ B''_r(x_0) &= B'_r(x_0) \cap \{x_{n-1} = 0\}. \end{aligned}$$

If x_0 is the origin, we will simply write B_r , B_r^+ , B'_r and B''_r . Let $d(E, F) := \inf_{x \in E, y \in F} |x - y|$ be the Euclidean distance between two sets $E, F \subset \mathbb{R}^n$.

1.3. Assumptions. In this paper we are interested in local properties of the solutions and their free boundaries only near regular points and therefore, without loss of generality, we make the following assumptions.

We will assume that u solves the Signorini problem (1.1)–(1.2) in B_2^+ and that all free boundary points in B'_2 are regular, i.e.

$$(1.4) \quad \Gamma_u = \mathcal{R}_u.$$

Furthermore, we will assume that there exists $f \in C^{1,\alpha}(\overline{B''_2})$ with

$$(1.5) \quad f(0) = |\nabla_{x''} f(0)| = 0,$$

such that

$$(1.6) \quad \Gamma_u = \{(x'', x_{n-1}) \in B'_2 : x_{n-1} = f(x'')\},$$

$$(1.7) \quad \Lambda_u = \{(x'', x_{n-1}) \in B'_2 : x_{n-1} \leq f(x'')\}.$$

Next we assume $u \in C^{1,1/2}(B_2^+ \cup B'_2)$ and that

$$(1.8) \quad \partial_{x_{n-1}} u > 0 \quad \text{in } B_2^+ \cup \Omega_u;$$

$$(1.9) \quad \partial_{x_n} u < 0 \quad \text{in } B_2^+ \cup (\Lambda_u \setminus \Gamma_u), \quad \partial_{x_n} u = 0 \quad \text{on } \Omega_u \cup \Gamma_u.$$

Moreover, we will also assume the following nondegeneracy property for directional derivatives in a cone of tangential directions: for any $\eta > 0$, there exist $r_\eta > 0$ and $c_\eta > 0$ such that

$$(1.10) \quad \partial_\tau u(x) \geq c_\eta d(x, \Lambda_u) \quad \text{for } x \in \overline{B_{r_\eta}^+(x_0)} \text{ and } \tau \in C'_\eta(\nu'_{x_0}), |\tau| = 1,$$

for any $x_0 \in \Gamma_u \cap B'_1$, where ν'_{x_0} is the unit normal in $\mathbb{R}^{n-1}$ to Γ_u at x_0 outward to Λ_u and

$$C'_\eta(e_0) = \{x' \in \mathbb{R}^{n-1} : x' \cdot e_0 \geq \eta|x'|\},$$

for a unit vector $e_0 \in \mathbb{R}^{n-1}$.

We explicitly remark that if u is a solution to the Signorini problem, then the assumptions (1.5)-(1.10) hold at any regular free boundary point after a possible translation, rotation and rescaling of u (see e.g. [CSS08], [PSU12]).

1.4. Main results. Following the approach of Kinderlehrer and Nirenberg [KN77] in the classical obstacle problem, we will use the partial hodograph-Legendre transformation method to improve on the known regularity of the free boundary. The idea is to straighten the free boundary and then apply the boundary regularity of the solution to the transformed elliptic PDE. This works relatively simply for the classical obstacle problem, and allows to prove C^∞ regularity and even real analyticity of the free boundary.

In the Signorini problem, the free boundary Γ_u is of codimension two, and in order to straighten both Γ_u and Λ_u we need to make a partial hodograph transform in two variables. Namely, for u satisfying the assumptions in Section 1.3, consider the transformation

$$(1.11) \quad T : x \mapsto y = (x'', \partial_{x_{n-1}} u, \partial_{x_n} u), \quad x \in B_1^+ \cup B'_1.$$

Consider also the associated partial Legendre transform of u given by

$$(1.12) \quad v(y) = u(x) - x_{n-1}y_{n-1} - x_n y_n.$$

Formally, the inverse of T is given by

$$T^{-1} : y \mapsto (y'', -\partial_{y_{n-1}} v, -\partial_{y_n} v)$$

and we can recover the free boundary in the following way:

$$f(y'') = -\partial_{y_{n-1}} v(y'', 0, 0).$$

However, we note that the mapping T is only $C^{0,1/2}$ regular and even the local invertibility of such mapping is rather nontrivial. Besides, even if one has a local invertibility of T , the function v will satisfy a degenerate elliptic equation, and a priori it is not clear if the equation will have enough structure to be useful.

Concerning the first complication, we make a careful asymptotic analysis based on the precise knowledge of the blowups, and this does allow to establish the local invertibility of T .

Theorem 1.1. *Let u be a solution of the Signorini problem (1.1)–(1.2) in B_2^+ , satisfying the assumptions in Section 1.3. Then, there exists a small $\rho = \rho_u > 0$ such that the partial hodograph transformation T in (1.11) is injective in B_ρ^+ .*

Then via an asymptotic analysis of v at the straightened free boundary points, we observe that the fully nonlinear degenerate elliptic equation for v has a subelliptic structure, which can be viewed as a perturbation of the Baouendi-Grushin operator. Then using the L^p theory for the Baouendi-Grushin operator and a bootstrapping argument, we obtain the smoothness and even the real analyticity of v .

Theorem 1.2. *Let u be as in Theorem 1.1 and v be given by (1.12). Then exists $\delta = \delta_u > 0$ such that the mapping $y'' \mapsto \partial_{y_{n-2}} v(y'', 0, 0)$ is real analytic on B_δ'' . In particular, $\Gamma_u = \mathcal{R}_u$ is locally an analytic surface.*

1.5. Related problems. The Signorini problem is just one example of a problem with thin free boundaries. Many problems with thin free boundaries arise when studying problems for the fractional Laplacian and using the Caffarelli-Silvestre extension [CS07] to localize the problem at the expense of adding an extra dimension (which makes the free boundary “thin”). Thus, the Signorini problem can be viewed as an obstacle problem for half-Laplacian, see [CSS08]. We hope that the methods in this paper can be used to study the higher regularity of the free boundary in many such problems.

A different approach for the study of the higher regularity of thin free boundaries is being developed by De Silva and Savin [DSS12a, DSS12b, DSS14]. In particular, in [DSS14] they prove the C^∞ regularity of $C^{2,\alpha}$ free boundaries in the thin analogue of Alt-Caffarelli minimization problem [CRS10]. Their method is based on Schauder-type estimates rather than hodograph-Legendre transform used in this paper.

1.6. Outline of the paper. The paper is organized as follows:

- In Section 2 we study the so-called 3/2-homogeneous blowups of the solutions near regular points. This is achieved by a combination a boundary Hopf-type principle in domains with $C^{1,\alpha}$ slits, as well as Weiss- and Monneau-type monotonicity formulas.
- In Section 3 we introduce the partial hodograph transformation and show it is a homeomorphism in a neighborhood of the regular free boundary points. This is achieved by using the precise behavior of the solutions near regular free boundary points established in Section 2.
- In Section 4 we consider the corresponding Legendre transform v and show some basic regularity of v inherited from u .
- In Section 5 we study the fully nonlinear PDE satisfied by v , which is the transformed PDE of u . We show the linearization of the PDE is a perturbation of the Baouendi-Grushin operator. Using the L^p estimates available for this operator, and a bootstrapping argument, we obtain the smoothness of v , which in turn implies the smoothness of the free boundary.
- In Section 6, we give more careful estimates on the derivatives of v which imply that v , and consequently Γ_u , is real analytic.

2. 3/2-HOMOGENEOUS BLOWUPS OF SOLUTIONS

2.1. A stronger nondegeneracy property. We start our study by establishing a stronger nondegeneracy property for the tangential derivatives $\partial_\tau u$ than the one given in (1.10). Namely, we want to improve the lower bound in (1.10) to a multiple of $d(x, \Lambda_u)^{1/2}$. To achieve this, we construct a barrier function by using the $C^{1,\alpha}$ -regularity of Γ_u , to obtain a result that can be viewed as a version of the boundary Hopf lemma for the domains of the type $B_1 \setminus \Lambda_u$.

To proceed, we introduce some notations. For $\alpha \in (0, 1)$, let

$$\begin{aligned} \hat{f}_\alpha : [0, \infty) &\rightarrow \mathbb{R}, \quad \hat{f}_\alpha(r) = r^{1+\alpha} \\ \hat{\Lambda}_\alpha &= B'_1 \cap \{(x'', x_{n-1}) : x_{n-1} \leq \hat{f}_\alpha(|x''|)\}, \quad \hat{D}_\alpha = B_1 \setminus \hat{\Lambda}_\alpha, \end{aligned}$$

Lemma 2.1. *There exists a continuous function $\hat{h}_\alpha$ on B_1 and a small $\rho = \rho(n, \alpha) > 0$ such that*

$$(2.1) \quad \hat{h}_\alpha(0) = 0,$$

$$(2.2) \quad \hat{h}_\alpha \leq 0 \quad \text{on } B'_\rho \cap \hat{\Lambda}_\alpha,$$

$$(2.3) \quad \hat{h}_\alpha \leq 0 \quad \text{in } \mathcal{N}_{\rho^4}(\hat{\Lambda}_\alpha) \cap \partial B_\rho, \quad \mathcal{N}_r(E) = \{x : d(x, E) < r\}$$

$$(2.4) \quad \Delta \hat{h}_\alpha \geq 0 \quad \text{in } \hat{D}_\alpha,$$

and

$$(2.5) \quad \lim_{x_{n-1} \rightarrow 0^+} \frac{\hat{h}_\alpha(0, x_{n-1}, 0)}{(x_{n-1})^{1/2}} = 1.$$

Proof. Inspired by the construction in [LN07], we will show that the following function satisfies the conditions of the lemma:

$$\hat{h}_\alpha(x) = U(x) + \hat{g}_\alpha(U(x)) - 2\hat{f}_\alpha(|x|^{1/2}),$$

where

$$\begin{aligned} U(x) &= \operatorname{Re}(x_{n-1} + ix_n)^{1/2}, \\ \hat{g}_\alpha(r) &= 2\hat{f}_\alpha(r) + 2(2n-3)r \int_0^r \frac{\hat{f}_\alpha(s)}{s^2} ds \\ &= \left(2 + \frac{2(2n-3)}{\alpha}\right) r^{1+\alpha}, \quad r \in [0, 1]. \end{aligned}$$

We next verify each of the properties (2.1)–(2.5) in the lemma.

First of all, (2.1) is immediate.

Next, since $U(x', 0) = (x_{n-1}^+)^{1/2}$, we have

$$(2.6) \quad \hat{h}_\alpha(x', 0) = (x_{n-1}^+)^{1/2} + \hat{g}_\alpha((x_{n-1}^+)^{1/2}) - 2\hat{f}_\alpha(|x'|^{1/2}) \quad \text{on } B'_1.$$

Therefore, on $B'_1 \cap \hat{\Lambda}_\alpha$ one has

$$\begin{aligned} \hat{h}_\alpha(x', 0) &\leq |x''|^{(\alpha+1)/2} + \left(2 + \frac{2(2n-3)}{\alpha}\right) |x''|^{(\alpha+1)^2/2} - 2|x''|^{(\alpha+1)/2} \\ &= -|x''|^{(\alpha+1)/2} \left(1 - \left(2 + \frac{2(2n-3)}{\alpha}\right) |x''|^{\alpha+1}\right). \end{aligned}$$

Hence there exists $\rho = \rho(n, \alpha) > 0$ such that $\hat{h}_\alpha \leq 0$ on $B'_\rho \cap \hat{\Lambda}_\alpha$. This implies (2.2).

Further, to show (2.3), notice that

$$0 \leq U \leq (\rho^4 + \rho^{1+\alpha})^{1/2} \quad \text{in } \mathcal{N}_{\rho^4}(\hat{\Lambda}_\alpha) \cap \partial B_\rho,$$

which yields

$$\hat{h}_\alpha \leq (\rho^4 + \rho^{1+\alpha})^{1/2} + \left(2 + \frac{2(2n-3)}{\alpha}\right) (\rho^4 + \rho^{1+\alpha})^{(1+\alpha)/2} - 2\rho^{(1+\alpha)/2} \leq 0,$$

for small ρ , as claimed.

Next, we show (2.4). It is easy to check that $\hat{g}_\alpha$ and U satisfy

$$(2.7) \quad \hat{g}_\alpha''(r) = 2 \left(\hat{f}_\alpha''(r) + (2n-3) \frac{\hat{f}_\alpha'(r)}{r} \right);$$

$$(2.8) \quad \begin{aligned} \Delta U &= 0, \quad \text{in } B_1 \setminus \{(x'', x_{n-1}, 0) : x_{n-1} \leq 0\} \supseteq \hat{D}_\alpha; \\ |\nabla U|^2 &= \frac{1}{4}(x_{n-1}^2 + x_n^2)^{-1/2}, \quad \text{in } \hat{D}_\alpha \end{aligned}$$

Thus a direct computation shows that for $x \in \hat{D}_\alpha$

$$(2.9) \quad \begin{aligned} \Delta \hat{g}_\alpha(U) &= \hat{g}_\alpha''(U) |\nabla U|^2 + \hat{g}_\alpha'(U) \Delta U = \frac{1}{4} \hat{g}_\alpha''(U) (x_{n-1}^2 + x_n^2)^{-1/2}; \\ \Delta \hat{f}_\alpha(|x|^{1/2}) &= \frac{1}{4} \hat{f}_\alpha''(|x|^{1/2}) |x|^{-1} + \frac{2n-3}{4} \hat{f}_\alpha'(|x|^{1/2}) |x|^{-3/2}, \\ &= \frac{1}{4} \left(\hat{f}_\alpha''(|x|^{1/2}) + (2n-3) \frac{\hat{f}_\alpha'(|x|^{1/2})}{|x|^{1/2}} \right) |x|^{-1}, \\ (2.10) \quad &\leq \frac{1}{4} \left(\hat{f}_\alpha''(|x|^{1/2}) + (2n-3) \frac{\hat{f}_\alpha'(|x|^{1/2})}{|x|^{1/2}} \right) (x_{n-1}^2 + x_n^2)^{-1/2}. \end{aligned}$$

Combining (2.7)–(2.10), we obtain that for $x \in \hat{D}_\alpha$

$$(2.11) \quad \begin{aligned} (x_{n-1}^2 + x_n^2)^{1/2} \Delta \hat{h}_\alpha(x) &= (x_{n-1}^2 + x_n^2)^{1/2} [\Delta U + \Delta \hat{g}_\alpha(U) - 2\Delta \hat{f}_\alpha(|x|^{1/2})] \\ &\geq \frac{1}{2} \left(\hat{f}_\alpha''(U) + (2n-3) \frac{\hat{f}_\alpha'(U)}{U} \right) - \frac{1}{2} \left(\hat{f}_\alpha''(|x|^{1/2}) + (2n-3) \frac{\hat{f}_\alpha'(|x|^{1/2})}{|x|^{1/2}} \right). \end{aligned}$$

Since $U(x) \leq |x|^{1/2}$ and $\hat{f}_\alpha''(r) + (2n-3) \hat{f}_\alpha'(r)/r = (1+\alpha)(2n-3+\alpha)r^{\alpha-1}$ is decreasing on $(0, \infty)$, then by (2.11) we have $\Delta \hat{h}_\alpha \geq 0$ in $\hat{D}_\alpha$. This shows (2.4).

Finally, by (2.6), we have

$$\lim_{x_{n-1} \rightarrow 0^+} \frac{\hat{h}_\alpha(0, x_{n-1}, 0)}{(x_{n-1})^{1/2}} = 1.$$

This completes the proof of the lemma. $\square$

Using the function constructed in Lemma 2.1 as a barrier, we have the improvement of the nondegeneracy for nonnegative harmonic functions in $\hat{D}_\alpha$.

Lemma 2.2. *Let w be a nonnegative superharmonic function in $\hat{D}_\alpha$, $w \in C^{0,1/2}(B_1)$ and $w = 0$ on $\hat{\Lambda}_\alpha$. Moreover, suppose that w satisfies*

$$(2.12) \quad w(x) \geq c_0 d(x, \hat{\Lambda}_\alpha), \quad x \in B_1.$$

Then there exists $\rho = \rho(n, \alpha, c_0) > 0$ and $\varepsilon_0 = \varepsilon_0(n, \alpha, c_0) > 0$ such that

$$(2.13) \quad w(0, x_{n-1}, 0) \geq \varepsilon_0 (x_{n-1})^{1/2}, \quad 0 < x_{n-1} < \rho.$$

Proof. Let h and ρ be as in Lemma 2.1. Then by (2.12) and the continuity of h , there exists $\varepsilon_0 = \varepsilon_0(\rho, c_0) > 0$ such that

$$w(x) - \varepsilon_0 \hat{h}_\alpha(x) \geq 0, \quad x \in \partial B_\rho \setminus \mathcal{N}_{\rho^4}(\hat{\Lambda}_\alpha),$$

which combined with (2.2)–(2.3) gives that

$$w - \varepsilon_0 \hat{h}_\alpha \geq 0 \quad \text{on } \partial(\hat{D}_\alpha \cap B_\rho).$$

Then, by the maximum principle

$$w - \varepsilon_0 \hat{h}_\alpha \geq 0 \quad \text{on } \hat{D}_\alpha \cap B_\rho.$$

In particular,

$$\begin{aligned} w(0, x_{n-1}, 0) &\geq \varepsilon_0 (x_{n-1})^{1/2} + \left(2 + \frac{2(2n-3)}{\alpha}\right) (x_{n-1})^{(1+\alpha)/2} - 2(x_{n-1})^{(1+\alpha)/2} \\ &\geq \varepsilon_0 (x_{n-1})^{1/2}, \end{aligned}$$

for $0 < x_{n-1} < \rho$ with $\rho = \rho(n, \alpha) > 0$ small. $\square$

From Lemma 2.2, we obtain the following nondegeneracy property for the tangential derivatives of the solution to the elliptic Signorini problem.

Proposition 2.3. *Let u be a solution to the elliptic Signorini problem in B_2^+ satisfying the assumptions in Section 1.3. Then for each $x_0 \in \Gamma_u \cap B_1'$ and $\eta > 0$, there exist $\rho, \varepsilon_0 > 0$ depending on $n, \alpha, c_\eta, \|f\|_{C^{1,\alpha}(B_2')}$, such that*

$$(2.14) \quad \partial_\tau u(x_0 + t\nu'_{x_0}) \geq \varepsilon_0 t^{1/2}, \quad t \in (0, \rho), \quad \tau \in C'_\eta(\nu'_{x_0}).$$

Proof. For each $x_0 \in \Gamma_u \cap B_1'$, we can rotate a coordinate system in $\mathbb{R}^{n-1}$ so that $\nu'_{x_0} = e_{n-1}$. Then we have

$$B_1(x_0) \setminus \tilde{\Lambda}_{x_0, M} \subset B_1(x_0) \setminus \Lambda_u,$$

where

$$\begin{aligned} \tilde{\Lambda}_{x_0, M} &= \{(x'', x_{n-1}) \in B_1'(x_0) : x_{n-1} - (x_0)_{n-1} \leq M|x'' - (x_0)''|^{1+\alpha}\} \\ &\quad \text{with } M = \|f\|_{C^{1,\alpha}(B_2')}. \end{aligned}$$

Since u is harmonic in $B_1(x_0) \setminus \tilde{\Lambda}_{x_0, M}$ and $\partial_\tau u$ satisfies the nondegeneracy condition (1.10), then $\partial_\tau u$ satisfies the assumptions of Lemma 2.2, with a small difference that there is constant M in the definition of the set $\tilde{\Lambda}_{x_0, M}$ above. However, by a simple scaling, we can make $M = 1$. Thus, there exists $\rho = \rho(n, \alpha, c_\eta, M) > 0$ and $\varepsilon_0 = \varepsilon_0(n, \alpha, c_\eta, M) > 0$ such that (2.14) holds. $\square$

2.2. 3/2-homogeneous blowups. For our further study, we need to consider the following rescalings

$$(2.15) \quad u_{r, x_0}(x) = \frac{u(x_0 + rx)}{r^{3/2}}, \quad r > 0$$

with $x_0 \in \Gamma_u$. Note that these are different from rescalings (1.3) in the sense that the L^2 norm of u is not preserved under the rescaling, but it is better suited for the study of regular points. First, from the growth estimate (see e.g. [AC04])

$$(2.16) \quad \sup_{B_r^+(x_0)} |u| \leq C_u r^{3/2}$$

for $x_0 \in \Gamma_u \cap B_1'$, where $C_u = C(n, \|u\|_{L^2(B_2^+)})$ and $0 < r < 1$, we know the family $\{u_{r, x_0}\}_r$ is locally uniformly bounded. Moreover, by the interior $C^{1,1/2}$ estimate (see e.g. [AC04])

$$(2.17) \quad \|u\|_{C^{1,1/2}(\overline{B_r^+(x_0)})} \leq C_u, \quad 0 < r < 1$$

and the fact that $u = |\nabla u| = 0$ on Λ_u , we get $\{u_{r, x_0}\}_r$ is uniformly bounded in $C_{\text{loc}}^{1,1/2}(B_{1/r}^+ \cup B'_{1/r})$. Thus there exists $u_0 \in C^{1,1/2}(\mathbb{R}_+^n \cup \{x_n = 0\})$ such that

$u_{r,x_0} \rightarrow u_0$ in $C_{\text{loc}}^{1,\alpha}$, for any $\alpha \in (0, 1/2)$ over a certain subsequence $r = r_j \rightarrow 0$. It is also immediate to see that u_0 is a global solution of the Signorini problem, i.e., a solution of (1.1)–(1.2) in $\mathbb{R}_+^n$. Furthermore, it is important to note that u_0 is nonzero, because of the nondegeneracy provided by Proposition 2.3. Sometimes we will refer to the function u_0 as the *3/2-homogeneous blowup* to indicate the way it was obtained.

The following Weiss-type monotonicity formula, whose proof can be found in [GP09], implies that u_0 is a homogeneous global solution of the Signorini problem of degree $3/2$.

Lemma 2.4 (Weiss-type monotonicity formula). *Let u be a nonzero solution of the Signorini problem (1.1)–(1.2) in B_R^+ . For any $x_0 \in \Gamma_u$ and $0 < r < R - |x_0|$ define*

$$W_{x_0}(r, u) = \frac{1}{r^{n+1}} \int_{B_r^+(x_0)} |\nabla u|^2 - \frac{3/2}{r^{n+2}} \int_{\partial B_r(x_0)^+} u^2.$$

Then $r \mapsto W_{x_0}(r, u)$ is nondecreasing in $r \in (0, R - |x_0|)$. Moreover, for a.e. $r \in (0, R - |x_0|)$ we have

$$\frac{d}{dr} W_{x_0}(r, u) = \frac{2}{r^{n+3}} \int_{(\partial B_r(x_0))^+} ((x - x_0) \cdot \nabla u - (3/2)u)^2.$$

Furthermore, $W_{x_0}(r, u) \equiv \text{const}$ for $r_1 < r < r_2$ if and only if u is homogeneous of degree $3/2$ with respect to x_0 in $B_{r_2}(x_0) \setminus \overline{B_{r_1}(x_0)}$, i.e.

$$(x - x_0) \cdot \nabla u - (3/2)u = 0 \quad \text{in } B_{r_2}(x_0) \setminus \overline{B_{r_1}(x_0)}. \quad \square$$

Remark 2.5. The Weiss-type monotonicity formula above is specifically adjusted to work with rescalings (2.15). Namely, by a simple change of variables, one can show that

$$W_0(\rho, u_{r,x_0}) = W_{x_0}(r\rho, u).$$

Besides, by the definition of regular points, we also have that

$$W_{x_0}(0_+, u) = (N_{x_0}(0_+, u) - 3/2) \frac{1}{r^{n+2}} \int_{\partial B_r(x_0)^+} u^2 = 0, \quad \text{if } x_0 \in \mathcal{R}_u,$$

since $N_{x_0}(0_+, u) = \kappa(x_0) = 3/2$.

Proposition 2.6 (Unique type of 3/2-homogeneous blowup). *Let u be a solution of the Signorini problem (1.1)–(1.2) in B_2^+ , satisfying the assumptions in Section 1.3. Then there exist two positive constants c_u and C_u , depending only on u such that if $x_0 \in \Gamma_u \cap B_1'$ and that*

$$u_{r,x_0}(x) = \frac{u(x_0 + rx)}{r^{3/2}} \rightarrow u_0(x) \quad \text{in } C_{\text{loc}}^{1,\alpha}(\mathbb{R}_+^n \cup \mathbb{R}^{n-1})$$

over a sequence $r = r_j \rightarrow 0_+$, then

$$u_0(x) = C_0 \operatorname{Re}(x' \cdot \nu'_{x_0} + ix_n)^{3/2}$$

with a constant C_0 satisfying

$$c_u < C_0 < C_u.$$

Proof. We have already noticed at the beginning of Section 2.2 that u_0 is a nonzero global solution of the Signorini problem. Besides, by the Weiss-type monotonicity formula, we will have

$$W_0(\rho, u_0) = \lim_{r_j \rightarrow 0_+} W_0(\rho, u_{x_0, r_j}) = \lim_{r_j \rightarrow 0_+} W_{x_0}(\rho r_j, u) = W_{x_0}(0_+, u) = 0,$$

for any $\rho > 0$. Hence, by Lemma 2.4, u_0 is a homogeneous of degree $3/2$ in $\mathbb{R}_+^n \cup \mathbb{R}^{n-1}$. Then, by Proposition 9.9 in [PSU12], we must have the form

$$u_0 = C_0 \operatorname{Re}(x' \cdot e' + ix_n)^{3/2}$$

for some $C_0 > 0$ and a tangential unit vector $e' \in \partial B_1'$. We claim that $e' = \nu'_{x_0}$. Indeed, we have

$$\partial_\tau u_{r_j, x_0} \geq 0 \quad \text{in } \overline{B_R^+}, \text{ for any } \tau \in C'_\eta(\nu'_{x_0}),$$

provided $j \geq j_{R, h, x_0}$ so that $r_\eta(x_0)/r_j \geq R$. Passing to the limit $r_j \rightarrow 0$, we therefore have

$$\partial_\tau u_0 \geq 0 \quad \text{in } \mathbb{R}_+^n \cup \mathbb{R}^{n-1} \text{ for any } \tau \in C'_\eta(\nu'_{x_0}).$$

Hence, for any $\eta > 0$

$$(2.18) \quad \tau \cdot e' \geq 0, \quad \text{whenever } \tau \in C'_\eta(\nu'_{x_0}).$$

This is possible only if $e' = \nu'_{x_0}$. Thus, we have the claimed representation

$$u_0(x) = C_0 \operatorname{Re}(x' \cdot \nu'_{x_0} + ix_n)^{3/2}.$$

The estimates on constant C_0 now follow from the growth estimate (2.16) and the nondegeneracy Proposition 2.3, which are preserved under the $3/2$ -homogeneous blowup. $\square$

Remark 2.7. This proposition also holds for the rescaling family with varying centers:

$$u_{r_j, x_j}(x) = \frac{u(x_j + r_j x)}{r_j^{3/2}},$$

where $x_j \in \Gamma_u \cap B'_{1/2}$, $x_j \rightarrow x_0 \in \Gamma_u$ and $r_j \in (0, 1/2)$.

Indeed, by Lemma 2.4 we have

$$x \mapsto W_x(r, u) \searrow 0 \quad \text{pointwise on } \Gamma_u \cap \overline{B'_{1/2}} \text{ as } r \searrow 0_+.$$

Hence applying Dini's theorem from the classical analysis to the family of monotone continuous functions $\{x \mapsto W_x(r, u)\}_{0 < r < 1/2}$ on the compact set $\Gamma_u \cap \overline{B'_{1/2}}$, we have the above convergence is uniform on $\Gamma_u \cap \overline{B'_{1/2}}$. Hence passing to the limit $j \rightarrow \infty$, we obtain

$$W_0(\rho, u_0) = \lim_{r_j \rightarrow 0_+} W_0(\rho, u_{x_j, r_j}) = \lim_{r_j \rightarrow 0_+} W_{x_j}(\rho r_j, u) = 0,$$

for any $\rho > 0$. Arguing as in Proposition 2.6, we conclude that $u_0(x) = C_0 \operatorname{Re}(x' \cdot \nu'_{x_0} + ix_n)^{3/2}$.

In order to get the uniqueness of the blowup limit, we need to show that the constant C_0 in Proposition 2.6 does not depend on the subsequence r_j but only depends on x_0 . This is a consequence of the following Monneau-type monotonicity formula [GP09]. Without apriori knowledge on the free boundary, this formula is known to hold only at so-called singular points, i.e., $x_0 \in \Gamma_u$ with $\kappa(x_0) = 2m$, $m \in \mathbb{N}$. However, using the $C^{1, \alpha}$ regularity of the free boundary, we will be able to establish this result also at regular points.

Lemma 2.8 (Monneau-type monotonicity formula). *Let u be a solution of the Signorini problem (1.1)–(1.2) in B_2^+ , satisfying the assumptions in Section 1.3. For any $x_0 \in \Gamma_u \cap B_1'$, $0 < r < 1$, and a positive constant c_0 , we define*

$$M_{x_0}(r, u, c_0) = \frac{1}{r^{n+2}} \int_{(\partial B_r(x_0))^+} (u - c_0 u_{x_0})^2,$$

where

$$u_{x_0} = \operatorname{Re}(x' \cdot \nu'_{x_0} + i x_n)^{3/2}.$$

Then there exists a constant $\tilde{C}$ which depends on the $C^{1,\alpha}$ norm of f , C_u in (2.17), c_0 , and α , such that

$$r \mapsto M_{x_0}(r, u, c_0) + \tilde{C} r^\alpha$$

is monotone nondecreasing for $r \in (0, 1)$.

Proof. For simplicity we assume $x_0 = 0$ and write $u_0 = c_0 \operatorname{Re}(x_{n-1} + i x_n)^{3/2}$, $M_0(r, u) = M_0(r, u, c_0)$.

Letting $w = u - u_0$ and using the scaling properties of M_0 , we have

$$(2.19) \quad \frac{d}{dr} M_0(r, u) = \frac{2}{r^{n+3}} \int_{(\partial B_r)^+} w \left(x \cdot \nabla w - \frac{3}{2} w \right).$$

Next, we compute the Weiss energy functional

$$W_0(r, u) = \frac{1}{r^{n+1}} \int_{B_r^+} |\nabla(w + u_0)|^2 - \frac{3/2}{r^{n+2}} \int_{(\partial B_r)^+} (w + u_0)^2.$$

By Remark 2.5,

$$W_0(0_+, u) = W_0(r, u_0) = 0,$$

hence

$$(2.20) \quad W_0(r, u) = \frac{1}{r^{n+1}} \int_{B_r^+} (|\nabla w|^2 + 2 \nabla w \cdot \nabla u_0) - \frac{3/2}{r^{n+2}} \int_{(\partial B_r)^+} (w^2 + 2 w u_0).$$

An integration by parts gives

$$\int_{B_r^+} \nabla w \cdot \nabla u_0 = \int_{(\partial B_r)^+} w \partial_\nu u_0 + \int_{B_r'} w (-\partial_{x_n} u_0) - \int_{B_r^+} w \Delta u_0$$

Noticing that

$$\partial_\nu u_0 = \frac{x \cdot \nabla u_0}{r} \text{ on } (\partial B_r)^+ \quad \text{and} \quad \Delta u_0 = 0 \text{ in } B_r^+,$$

we have

$$(2.21) \quad \frac{2}{r^{n+1}} \int_{B_r^+} \nabla w \cdot \nabla u_0 = \frac{2}{r^{n+2}} \int_{(\partial B_r)^+} w (x \cdot \nabla u_0) + \frac{2}{r^{n+1}} \int_{B_r'} w (-\partial_{x_n} u_0).$$

Similarly, integrating by parts and using $\Delta w = \Delta u - \Delta u_0 = 0$ in B_r^+ , we obtain

$$(2.22) \quad \frac{1}{r^{n+1}} \int_{B_r^+} |\nabla w|^2 = \frac{1}{r^{n+2}} \int_{(\partial B_r)^+} w (x \cdot \nabla w) + \frac{1}{r^{n+1}} \int_{B_r'} w (-\partial_{x_n} w).$$

Combining (2.20)–(2.22), we have

$$\begin{aligned} W_0(r, u) &= \frac{1}{r^{n+2}} \int_{(\partial B_r)^+} w \left(x \cdot \nabla w - \frac{3}{2} w \right) + \frac{2}{r^{n+2}} \int_{(\partial B_r)^+} w \left(x \cdot u_0 - \frac{3}{2} u_0 \right) \\ &\quad + \frac{1}{r^{n+1}} \int_{B_r'} w (-\partial_{x_n} w) + 2w(-\partial_{x_n} u_0). \end{aligned}$$

Since u_0 is homogeneous of degree $3/2$, the second integral above is zero. Moreover, using $u(\partial_{x_n} u) = u_0(\partial_{x_n} u_0) = 0$ on B'_r , we have the third integral is equal to

$$\frac{1}{r^{n+1}} \int_{B'_r} u(-\partial_{x_n} u_0) + u_0(\partial_{x_n} u).$$

Recalling (2.19), we have

$$\frac{d}{dr} M_0(r, u) = \frac{2W_0(r, u)}{r} - \frac{2}{r^{n+2}} \int_{B'_r} (u(-\partial_{x_n} u_0) + u_0 \partial_{x_n} u).$$

Noticing that $W_0(r, u) > 0$ for $0 < r < 1$ by Lemma 2.4 and $\partial_{x_n} u \leq 0$, $u_0 \geq 0$ on B'_1 , we have

$$\frac{d}{dr} M_0(r, u) \geq \frac{2}{r^{n+2}} \int_{B'_r} u(\partial_{x_n} u_0).$$

Using the growth estimate (2.17) for u and explicit expression for u_0 , we have

$$\frac{d}{dr} M_0(r, u) \geq -\frac{2}{r^{n+2}} \cdot C_u c_0 r^2 H^{n-1}(B'_r \cap \{u > 0, x_{n-1} < 0\})$$

Since the free boundary $\Gamma_u = \partial_{B'_2} \{u > 0\}$ is a $C^{1,\alpha}$ graph and $0 \in \Gamma_u$, we have

$$H^{n-1}(B'_r \cap \{u > 0, x_{n-1} < 0\}) \leq c_0 r^{n-1+\alpha},$$

where c_1 is a constant depending on $\|f\|_{C^{1,\alpha}}$. Hence

$$\frac{d}{dr} M_0(r, u) \geq -2c_1 C_u c_0 r^{-1+\alpha},$$

which implies the claim of the lemma with $\tilde{C} = 2c_1 C_u c_0 / \alpha$. $\square$

We can now establish the main result of this section.

Theorem 2.9 (Uniqueness of $3/2$ -homogeneous blowup). *Let u be a solution of the Signorini problem in B_2^+ , satisfying the assumptions in Section 1.3. Then for $x_0 \in \Gamma_u \cap B'_1$, there exists a constant $C_{x_0} > 0$ such that*

$$\lim_{r \rightarrow 0_+} u_{r,x_0}(x) = C_{x_0} u_{x_0}(x) \text{ in } C_{\text{loc}}^{1,\alpha}(\mathbb{R}_+^n \cup \mathbb{R}^{n-1}),$$

where

$$u_{x_0}(x) = \text{Re}(x' \cdot \nu'_{x_0} + i x_n)^{3/2}.$$

Moreover, the function $x_0 \mapsto C_{x_0}$ is continuous on $\Gamma_u \cap B'_1$. Furthermore, for any $\alpha \in (0, 1/2)$ and $R > 0$,

$$(2.23) \quad \sup_{x_0 \in \Gamma_u \cap B'_{1/2}} \|u_{r,x_0} - C_{x_0} u_{x_0}\|_{C^{1,\alpha}(B_R^+ \cup B'_R)} \rightarrow 0 \quad \text{as } r \rightarrow 0_+.$$

Proof. We first show C_{x_0} does not depend on the converging sequences.

Given $x_0 \in \Gamma_u \cap B'_1$, let u_{r_j, x_0} be a converging sequence such that $u_{r_j, x_0} \rightarrow C_{x_0} u_{x_0}$ in $C_{\text{loc}}^{1,\alpha}$ for some C_{x_0} satisfying $c_u < C_{x_0} < C_u$. By Lemma 2.8, the mapping $r \mapsto M_{x_0}(r, u, C_{x_0}) + \tilde{C} r^\alpha$ is nonnegative, monotone nondecreasing on $(0, 1)$, hence

$$\begin{aligned} \lim_{r \rightarrow 0_+} M_{x_0}(r, u, C_{x_0}) + \tilde{C} r^\alpha &= \lim_{j \rightarrow \infty} M_{x_0}(r_j, u, C_{x_0}) + \tilde{C} r_j^\alpha \\ &= \lim_{j \rightarrow \infty} M_{x_0}(1, u_{r_j, x_0}, C_{x_0}) + \tilde{C} r_j^\alpha = 0. \end{aligned}$$

This implies C_{x_0} does not depend on the converging sequences u_{r_j, x_0} .

Next we show that C_{x_0} depends continuously on $x_0 \in \Gamma_u \cap B'_1$. Fix $x_0 \in \Gamma_u \cap B'_1$. For any $\varepsilon > 0$, let $r_\varepsilon \in (0, 1/2)$ be such that

$$M_{x_0}(r_\varepsilon, u, C_{x_0}) + \tilde{C}r_\varepsilon^\alpha = \int_{(\partial B_1)^+} (u_{r_\varepsilon, x_0} - C_{x_0}u_{x_0})^2 + \tilde{C}r_\varepsilon^\alpha < \varepsilon.$$

For r_ε fixed, by the continuity of u we have $x \mapsto \int_{(\partial B_1)^+} u_{r_\varepsilon, x}$ is continuous on $\Gamma_u \cap B'_1$. Moreover, from the explicit formulation of u_x as well as the $C^{0, \alpha}$ continuity of $x \mapsto \nu'_x$, the function $x \mapsto \int_{(\partial B_1)^+} u_x$ is continuous. Therefore, there exists a positive δ_ε small enough, such that for all $x_1 \in \Gamma_u$ with $|x_1 - x_0| < \delta_\varepsilon$,

$$(2.24) \quad M_{x_1}(r_\varepsilon, u, C_{x_0}) = \int_{(\partial B_1)^+} (u_{r_\varepsilon, x_1} - C_{x_0}u_{x_1})^2 < 2\varepsilon.$$

For $0 < r < r_\varepsilon$, by Lemma 2.8

$$(2.25) \quad M_{x_1}(r, u, C_{x_0}) \leq M_{x_1}(r_\varepsilon, u, C_{x_0}) + \tilde{C}r_\varepsilon^\alpha < 3\varepsilon.$$

Let $r \rightarrow 0_+$ in (2.25), then

$$(2.26) \quad \begin{aligned} \lim_{r \rightarrow 0_+} M_{x_1}(r, u, C_{x_0}) &= \lim_{r \rightarrow 0_+} \int_{(\partial B_1)^+} (u_{r, x_1} - C_{x_0}u_{x_1})^2 \\ &= (C_{x_1} - C_{x_0})^2 \int_{(\partial B_1)^+} (u_{x_1})^2 < 3\varepsilon. \end{aligned}$$

By the explicit expression of u_x , there is a constant $c_n > 0$ such that $\int_{(\partial B_1)^+} (u_x)^2 \geq c_n > 0$ for any $x \in \Gamma_u \cap B'_1$. This together with (2.26) gives

$$|C_{x_1} - C_{x_0}| < \sqrt{\frac{3\varepsilon}{c_n}}, \quad \text{for } x_1 \in B'_{\delta_\varepsilon}(x_0) \cap \Gamma_u.$$

This shows the continuity of $x_0 \mapsto C_{x_0}$.

Finally, we show (2.27). In fact, $x_0 \mapsto M_{x_0}(r, u, C_{x_0}) + \tilde{C}r^\alpha$ is continuous on the compact set $\Gamma_u \cap \overline{B'_{1/2}}$ and it monotonically decreases to zero as r decreases to zero for each x_0 . Hence by Dini's theorem, $M_{x_0}(r, u, C_{x_0}) + \tilde{C}r^\alpha \rightarrow 0$ as $r \rightarrow 0_+$ uniformly on $\Gamma_u \cap \overline{B'_{1/2}}$. Thus one has

$$\sup_{x_0 \in \Gamma_u \cap \overline{B'_{1/2}}} \int_{(\partial B_1)^+} (u_{r, x_0} - C_{x_0}u_{x_0})^2 \rightarrow 0, \quad r \rightarrow 0_+.$$

It is not hard to see that $(u_{r, x_0} - C_{x_0}u_{x_0})^\pm$ are subharmonic in B_1 (after the even reflection of u about x_n). Hence by the $L^\infty - L^2$ estimates for the subharmonic functions we have

$$(2.27) \quad \sup_{x_0 \in \Gamma_u \cap \overline{B'_{1/2}}} \|u_{r, x_0} - C_{x_0}u_{x_0}\|_{L^\infty(B_{1/2}^+)} \rightarrow 0 \quad \text{as } r \rightarrow 0_+.$$

By an interpolation of Hölder spaces, i.e. for some absolute constant $C > 0$, $\|g\|_{C^{1, \alpha}} \leq C\|g\|_{L^\infty}^\lambda \|g\|_{C^{1, 1/2}}^{1-\lambda}$ for $\alpha \in (0, 1/2)$ and $\lambda = \lambda(\alpha) \in (0, 1)$, as well as a rescaling argument we obtain (2.23). $\square$

The continuous dependence on x_0 of C_{x_0} gives the uniqueness of the blowups with varying centers.

Corollary 2.10. *Let u be a solution of the Signorini problem (1.1)–(1.2) in B_2^+ , satisfying the assumptions in Section 1.3. Let $x_j, x_0 \in \Gamma_u \cap B'_{1/2}$, such that $x_j \rightarrow x_0$ as $j \rightarrow \infty$. Let $r_j \rightarrow 0$ as $j \rightarrow \infty$. Then for any $\alpha \in (0, 1/2)$ and u_{x_0} defined in Theorem 2.9,*

$$u_{r_j, x_j} \rightarrow C_{x_0} u_{x_0} \text{ in } C_{\text{loc}}^{1, \alpha}(\mathbb{R}_+^n \cup \mathbb{R}^{n-1}).$$

Proof. This follows from the uniform continuity of $x_0 \mapsto \int_{(\partial B_1)^+} C_{x_0} u_{x_0}$ and Theorem 2.9. $\square$

3. PARTIAL HODOGRAPH TRANSFORM

Let u be a solution of the Signorini problem in B_2^+ satisfying the assumptions in Section 1.3. Following the idea in the classical obstacle problem [KN77], we would like to use the method of partial hodograph-Legendre transforms to study the higher regularity of the free boundary in the Signorini problem. Since Γ_u has codimension 2, the most natural hodograph transformation to consider is the one with respect to variable x_{n-1} and x_n :

$$(3.1) \quad T = T^u : B_1^+ \cup B'_1 \rightarrow \mathbb{R}^n,$$

$$(3.2) \quad y = T(x) = (x'', \partial_{x_{n-1}} u, \partial_{x_n} u).$$

(The reader can easily check that if we do the partial hodograph transform in x_{n-1} variable only, it will still straighten Γ_u , however, the image of B^+ will not have a flat boundary and this will render this transformation rather useless.)

By doing so, we hope that there exists a small neighborhood B_ρ of origin, such that T is one-to-one on $B_\rho^+ \cup B'_\rho$. However, due to the $C^{1,1/2}$ regularity of u , the mapping T is only $C^{0,1/2}$ near the origin. Hence the simple inverse function theorem (which typically requires the transformation T to be from class C^1) cannot be applied here. Instead, we will make use of the blowup profiles, which contain enough information to catch the behavior of the solution near the free boundary points.

Before stating the main results, we make several observations.

- (i) By the assumptions in Section 1.3, $\partial_{x_{n-1}} u > 0$ in $B_1^+ \cup \Omega_u$ and $\partial_{x_n} u \leq 0$ on B'_1 . This together with the complementary boundary condition gives us

$$T(\Lambda_u \cap B'_1) \subset \{y \in \mathbb{R}^n : y_{n-1} = 0, y_n \leq 0\};$$

$$T(\Omega_u \cap B'_1) \subset \{y \in \mathbb{R}^n : y_{n-1} > 0, y_n = 0\};$$

$$T(\Gamma_u \cap B'_1) \subset \{y \in \mathbb{R}^n : y_{n-1} = y_n = 0\};$$

$$T(B_1^+) \subset \{y \in \mathbb{R}^n : y_{n-1} > 0, y_n < 0\}.$$

- (ii) If we extend u across $\{x_n = 0\}$ to B_1 by the even symmetry or odd symmetry in x_n , then the resulting function will satisfy

$$\Delta u = 0 \text{ in } B_1 \setminus \Lambda_u, \quad \text{for even extension in } x_n;$$

$$\Delta u = 0 \text{ in } B_1 \setminus \overline{\Omega_u}, \quad \text{for odd extension in } x_n.$$

Hence u is analytic in $(B_1^+ \cup B'_1) \setminus \Gamma_u$. From (3.2), T is also analytic in $(B_1^+ \cup B'_1) \setminus \Gamma_u$.

- (iii) To better understand the nature of T , consider the solution $u_0(x) = \text{Re}(x_1 + ix_2)^{3/2}$ of the Signorini problem and find an explicit formula for T^{u_0} . A simple

FIGURE 1. The partial hodograph transform T . Shown for even extension of u in x_n variable.

computation shows that using complex notations the mapping T^{u_0} is given by

$$x_1 + ix_2 \mapsto \frac{3}{2}(x_1 - ix_2)^{1/2},$$

where by the latter we understand the appropriate branch. Loosely speaking, this tells that T behaves like $\sqrt{z}$ function in the last two variables.

The observation above also suggests us to compose T with the mapping

$$\psi : \mathbb{R}^n \rightarrow \mathbb{R}^n;$$

$$(y'', y_{n-1}, y_n) \mapsto (y'', y_{n-1}^2 - y_n^2, -2y_{n-1}y_n),$$

which can be expressed, by using complex notations (denote $\psi(y)$ by w), as

$$w'' = y'', \quad w_{n-1} + iw_n = (y_{n-1} - iy_n)^2.$$

More explicitly, alongside T , we consider the transformation $T_1 = T_1^u = \psi \circ T$

$$(3.3) \quad T_1 : B_1^+ \cup B_1' \rightarrow \mathbb{R}^n,$$

$$w = T_1(x) = (x_1, \dots, x_{n-2}, (\partial_{x_{n-1}} u)^2 - (\partial_{x_n} u)^2, -2(\partial_{x_{n-1}} u)(\partial_{x_n} u)).$$

Now T_1 maps B_1^+ to the upper half space $\{w_n > 0\}$ and B_1' to the hyperplane $\{w_n = 0\}$. Moreover, T_1 straightens the free boundary Γ_u as T and satisfies

$$T_1(\Lambda_u) \subset \{w \in \mathbb{R}^n : w_{n-1} \leq 0, w_n = 0\} = \Lambda_0$$

$$T_1(\Omega_u) \subset \{w \in \mathbb{R}^n : w_{n-1} > 0, w_n = 0\} = \Omega_0,$$

$$T_1(\Gamma_u) \subset \{w \in \mathbb{R}^n : w_{n-1} = 0, w_n = 0\} = \Gamma_0.$$

- (iv) The advantage of T_1 is that by arguing as in (ii) above and making odd and even extensions of u in x_n , we can extend it to a mapping on B_1 . Moreover, this extension will be the same in both cases (unlike for the mapping T). In particular, this makes T_1 a single-valued mapping on B_1 , which is real analytic in $B_1 \setminus \Gamma_u$.

In what follows, we will prove the injectivity of T_1 in a neighborhood of the origin. For that we will need to make the following direct computations.

FIGURE 2. The modified partial hodograph transform $T_1 = \psi \circ T$

Proposition 3.1. *Let $\hat{u}_0 : \mathbb{R}^2 \rightarrow \mathbb{R}$ be given by*

$$\hat{u}_0(x_1, x_2) = \operatorname{Re}(x_1 + ix_2)^{3/2} = r^{3/2} \cos(3\theta/2),$$

for $x_1 + ix_2 = re^{i\theta}$, $\theta \in (-\pi, \pi]$. Then we have the following identities

$$\begin{aligned} (\partial_{x_1} \hat{u}_0, \partial_{x_2} \hat{u}_0) &= (3r^{1/2}/2)(\cos(\theta/2), -\sin(\theta/2)) \\ D^2 \hat{u}_0(x_1, x_2) &= (3r^{-1/2}/4) \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ \sin(\theta/2) & -\cos(\theta/2) \end{pmatrix} \\ T_1^{\hat{u}_0}(x_1, x_2) &= (9/4)(x_1, x_2). \end{aligned}$$

Proof. Direct computation. □

From now on, we will use a slightly different notation from Theorem 2.9 to denote the blowup limit, i.e. for $x_0 \in \Gamma_u$, we let

$$u_{x_0}(x) = C_{x_0} \operatorname{Re}(x' \cdot \nu'_{x_0} + ix_n)^{3/2}.$$

The following proposition is a consequence of Theorem 2.9 and the C^1 regularity of f .

Proposition 3.2. *For any $\varepsilon > 0$, there exists $\delta > 0$ depending on ε and u , such that for all $0 < r < \delta$ and $x_0 \in \Gamma_u \cap B'_\delta$, we have:*

- (i) $\|u_{r,x_0} - u_0\|_{C^1(B_2^+ \cup B_2')} < \varepsilon$;
- (ii) u_{r,x_0} has a harmonic extension at any $x \in B'_2$ with $|x_{n-1}| > 1/4$ (by making even or odd reflection about x_n). Hence if we let

$$A_+^{1/2,1} := \{(x'', x_{n-1}, x_n) : |x''| \leq 1, 1/2 \leq \sqrt{x_{n-1}^2 + x_n^2} \leq 1, x_n \geq 0\},$$

then for any multi-index $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| = k \geq 2$ we have

$$\|\partial^\alpha u_{r,x_0} - \partial^\alpha u_0\|_{L^\infty(A_+^{1/2,1})} < C(n, k)\varepsilon.$$

Proof. (i) Given any $\varepsilon > 0$, by (2.23) there is a positive constant δ_1 depending on u such that for any $0 < r < \delta_1$,

$$\sup_{x_0 \in \Gamma_u \cap B'_{1/2}} \|u_{r,x_0} - u_{x_0}\|_{C^1(B_2^+ \cup B_2')} < \varepsilon/2.$$

On the other hand, there exists $\delta_2 > 0$ depending on ε , modulus of continuity of C_{x_0} and $C^{1,\alpha}$ norm of f such that

$$\sup_{x_0 \in \Gamma_u \cap B'_{\delta_2}} \|u_{x_0} - u_0\|_{C^1(B_2^+ \cup B_2')} < \varepsilon/2.$$

Taking $\delta = \min\{\delta_1, \delta_2, 1/2\}$ we proved (i).

(ii) We observe that for small enough r and $|x_0|$, the rescaled free boundary

$$\Gamma_{u_{r,x_0}} = \{(x'', x_{n-1}, 0) : x_{n-1} = f_{r,x_0}(x'')\}, \quad f_{r,x_0}(x'') := \frac{f(x_0'' + rx'') - f(x_0'')}{r}$$

satisfies $|f_{r,x_0}(x'')| < 1/4$ when $|x''| < 2$. This follows immediately from the assumption $f(0) = |\nabla'' f(0)| = 0$. The rest of (ii) then follows from the estimates for the higher order derivatives of harmonic functions. $\square$

The next proposition is a consequence of Proposition 3.1 and Proposition 3.2, which is useful to understand the transformation T . Since T fixes the first $n-2$ coordinates, it is more convenient to work on a “tubular” neighborhood of Γ_u defined as follows: First we consider the projection map

$$p : B_1 \rightarrow \Gamma_u, \quad x \mapsto (x'', f(x''), 0)$$

Since $f \in C^{1,\alpha}(B_2'')$, p is continuous in B_1 . Moreover, it is easy to verify that for some constant $c_1 = c_1(n, \|f\|_{C^1}) \geq 1$,

$$(3.4) \quad d(x, \Gamma_u) \leq |x - p(x)| \leq c_1 d(x, \Gamma_u).$$

Next for $\delta \in (0, 1/2)$, we let

$$\mathcal{O}_\delta := \{x : p(x) \in B'_\delta \cap \Gamma_u, |x - p(x)| < \delta\};$$

$$\mathcal{O}_\delta^+ = \mathcal{O}_\delta \cap \{x_n > 0\}; \quad \mathcal{O}'_\delta = \mathcal{O}_\delta \cap \{x_n = 0\}.$$

By the continuity of p and (3.4), $\mathcal{O}_\delta$ is a tubular neighborhood of the part of the free boundary Γ_u lying in B_δ .

Proposition 3.3. *Given any $\varepsilon > 0$, let $\delta = \delta_\varepsilon$ be the same constant as in Proposition 3.2. Then for any $x \in (\mathcal{O}_\delta^+ \cup \mathcal{O}'_\delta) \setminus \Gamma_u$, we have*

$$\begin{aligned} \text{(i)} \quad & \left| \frac{|T(x) - T(p(x))|^2}{|x - p(x)|} - \frac{9C_0^2}{4} \right| < 2C_u \varepsilon; \\ \text{(ii)} \quad & \left| |x - p(x)| \det(DT)(x) - \left(-\frac{9C_0^2}{16} \right) \right| < 2C_0 n \varepsilon. \end{aligned}$$

Proof. (i) First we observe that

$$(3.5) \quad |T(x) - T(p(x))|^2 = (\partial_{x_{n-1}} u)^2(x) + (\partial_{x_n} u)^2(x).$$

For $x \in (\mathcal{O}_\delta^+ \cup \mathcal{O}'_\delta) \setminus \Gamma_u$, let

$$d = |x - p(x)| \in (0, \delta), \quad \xi = \frac{x - p(x)}{d} \in \partial B_1.$$

Applying Proposition 3.2(i) to $u_{d,p(x)}$ we have,

$$(3.6) \quad |\nabla u_{d,p(x)}(\xi) - \nabla u_0(\xi)| < \varepsilon.$$

On the other hand, by the computation in Proposition 3.1,

$$(3.7) \quad (\partial_{x_{n-1}} u_0)^2(\zeta) + (\partial_{x_n} u_0)^2(\zeta) = \frac{9C_0^2}{4}, \quad \text{for any } \zeta \in \partial B_1.$$

(3.6) together with (3.7) implies

$$\left| (\partial_{x_{n-1}} u_{d,p(x)})^2(\xi) + (\partial_{x_n} u_{d,p(x)})^2(\xi) - \frac{9C_0^2}{4} \right| \leq 2C_u \varepsilon.$$

Rescaling back to u and using (3.5), we obtain (i).

(ii) The proof of (ii) is similar to that of (i). In $(\mathcal{O}_\delta^+ \cup \mathcal{O}_\delta') \setminus \Gamma_u$, T is a smooth mapping, and

$$\det(DT)(x) = \partial_{x_{n-1}x_{n-1}} u(x) \partial_{x_n x_n} u(x) - (\partial_{x_{n-1}x_n} u)^2(x).$$

Given $x \in (\mathcal{O}_\delta^+ \cup \mathcal{O}_\delta') \setminus \Gamma_u$, consider $u_{d,p(x)}$ and rescaled point $\xi \in \partial B_1$ as above. By Proposition 3.2(ii),

$$|D^2 u_{d,p(x)}(\xi) - D^2 u_0(\xi)| \leq n\varepsilon.$$

This together with the explicit expression for $D^2 u_0$ in Proposition 3.1 gives

$$(3.8) \quad \left| \det DT^{u_{d,p(x)}}(\xi) - \left(-\frac{9C_0^2}{16} \right) \right| < 2C_0 n\varepsilon.$$

It is easy to check the following rescaling property

$$\det(DT^{u_{r,x_0}})(\xi) = r \det(DT)(x_0 + r\xi).$$

This combined with (3.8) gives (ii). $\square$

Now we are ready to prove the main theorem of the section. Let

$$\mathcal{Q} = \{y \in \mathbb{R}^n : y_{n-1} \geq 0, y_n \leq 0\}.$$

Theorem 3.4. *There exists a small constant $\delta = \delta_u > 0$, such that T is a homeomorphism from $\mathcal{O}_\delta^+ \cup \mathcal{O}_\delta'$ to $T(\mathcal{O}_\delta^+ \cup \mathcal{O}_\delta')$, which is relatively open in $\mathcal{Q}$. Moreover, if we extend T to B_1^- via the even (odd) reflection of u about x_n , then it is a diffeomorphism from $\mathcal{O}_\delta \setminus \Lambda_u$ ($\mathcal{O}_\delta \setminus \bar{\Omega}_u$) onto an open subset in $\mathbb{R}^n$.*

Proof. By observation (iii) and (iv), instead of T , we consider the map

$$T_1 = \psi \circ T : B_1 \rightarrow \mathbb{R}^n,$$

which is first defined in $B_1^+ \cup B_1'$ as (3.3) and then extended to B_1 via even or odd reflection of u in x_n variable. We will first show T_1 is a homeomorphism from $\mathcal{O}_\delta$ to $T_1(\mathcal{O}_\delta)$ for sufficiently small δ . This is divided into three steps.

Step 1. There exists $\delta = \delta_u > 0$, such that for any $0 < r < \delta$ and $x_0 \in \Gamma_u \cap B_\delta'$, $T_1^{u_{r,x_0}}$ is injective in $A^{1/2,1}$. Here

$$A^{1/2,1} := \{(x'', x_{n-1}, x_n) : |x''| \leq 1, 1/2 \leq \sqrt{x_{n-1}^2 + x_n^2} \leq 1\}.$$

In fact, by Proposition 3.2 and the definition of T_1 , for any $\varepsilon > 0$, there exists $\delta = \delta_{\varepsilon,u} > 0$ such that

$$\|T_1^{u_{r,x_0}} - T_1^{u_0}\|_{C^1(A^{1/2,1})} < \varepsilon, \quad 0 < r < \delta, \quad x_0 \in \Gamma_u \cap B_\delta'.$$

From Proposition 3.1,

$$T_1^{u_0}(x) = \left(x'', \frac{9C_0}{4}x_{n-1}, \frac{9C_0}{4}x_n \right)$$

is a nondegenerate linear map. Hence if we take $\varepsilon = \varepsilon(C_0)$ sufficiently small, $T_1^{u_r, x_0}$ will be injective on the compact set $A^{1/2, 1}$.

Step 2. T_1 is injective in $\mathcal{O}_\delta$ for δ chosen as above.

It is clear that T_1 is injective on Γ_u , since it maps $(x'', f(x''), 0)$ to $(x'', 0, 0)$. Thus, it will be sufficient if we show the injectivity of T_1 in $\mathcal{O}_\delta \setminus \Gamma_u$.

Indeed, suppose there exist $x_1, x_2 \in \mathcal{O}_\delta \setminus \Gamma_u$ such that $T_1(x_1) = T_1(x_2)$. Necessarily $p(x_1) = p(x_2)$. Let $x_0 = p(x_i)$, $d_i = |x_i - p(x_i)|$, $i = 1, 2$ and w.l.o.g assume $d_1 \geq d_2$. A simple rescaling gives

$$T_1^{u_{d_1, x_0}}(\xi_1) = T_1^{u_{d_1, x_0}}(\xi_2), \quad \xi_i = \frac{x_i - x_0}{d_1} \in \overline{B_1}, \quad i = 1, 2.$$

On the other hand, $T_1(x_1) = T_1(x_2)$ implies

$$(\partial_{x_{n-1}} u)^2(x_1) + (\partial_{x_n} u)^2(x_1) = (\partial_{x_{n-1}} u)^2(x_2) + (\partial_{x_n} u)^2(x_2).$$

Then recalling (3.5), by Proposition 3.3(i) for small enough ε (by choosing δ small in step 1) one has

$$(3.9) \quad \frac{1}{2} \leq \frac{d_2}{d_1} \leq 1.$$

Note that $d_2/d_1 = |x_2 - x_0|/d_1 = |\xi_2|$ and ξ_1, ξ_2 have first $n-2$ coordinates equal to zero, thus $\xi_1, \xi_2 \in A^{1/2, 1}$. But this contradicts the injectivity of $T_1^{u_{d_1, x_0}}$ on $A^{1/2, 1}$ obtained in step 1.

Step 3. T_1 is a homeomorphism from $\mathcal{O}_\delta$ to $T_1(\mathcal{O}_\delta)$.

Now $T_1 : \mathcal{O}_\delta \rightarrow \mathbb{R}^n$ is an injective map. Moreover, it is continuous because of the continuity of Du . Thus by the Brouwer invariance of domain theorem $T_1(\mathcal{O}_\delta)$ is open and T_1 is a homeomorphism between $\mathcal{O}_\delta$ and $T_1(\mathcal{O}_\delta)$.

Now we proceed to show $T : \mathcal{O}_\delta^+ \cup \mathcal{O}_\delta' \rightarrow T(\mathcal{O}_\delta^+ \cup \mathcal{O}_\delta')$ is a homeomorphism for δ chosen as above. Indeed, recalling $T_1 = \psi \circ T$ we obtain the injectivity of T from the injectivity of T_1 . Next we note that ψ is injective in $\mathcal{Q}$, which contains $T(B_1^+ \cup B_1')$ by observation (i), thus $T(U) = \psi^{-1}(T_1(U)) \cap \mathcal{Q}$ for any subset $U \subset B_1^+ \cup B_1'$. Thus, T is an open map from $\mathcal{O}_\delta^+ \cup \mathcal{O}_\delta'$ to $\mathcal{Q}$, since T_1 is open and ψ is continuous. Combining with the continuity of T , we obtain that T is an homeomorphism from $\mathcal{O}_\delta^+ \cup \mathcal{O}_\delta'$ to $T(\mathcal{O}_\delta^+ \cup \mathcal{O}_\delta') \subset \mathcal{Q}$.

Finally, we notice that T is smooth on $\mathcal{O}_\delta \setminus \Lambda_u$ (or $\mathcal{O}_\delta \setminus \overline{\Omega}_u$) after even (or odd) extension of u about x_n . Moreover, by Proposition 3.3(ii), $\det(DT)$ is nonvanishing there for sufficiently small δ . Hence T is a diffeomorphism by the implicit function theorem. $\square$

Next we construct a space $\mathcal{M}$ by gluing two copies of $\mathcal{O}_\delta^+ \cup \mathcal{O}_\delta'$, $\mathcal{O}_\delta^- \cup \mathcal{O}_\delta'$ together properly and extend u as well as T to $\mathcal{M}$. The advantage of doing this is, the straightened free boundary $T(\Gamma_u)$ will be contained in $T(\mathcal{M})$, which is an open neighborhood of the origin in $\mathbb{R}^n$. This transfers the boundary singularity into the interior singularity, which will be easier for us to deal with.

More precisely, we fix the δ chosen in Theorem 3.4 and consider $\{\mathcal{O}^i; i = 1, 2, 3, 4\}$, a family of subsets in $\mathbb{R}^n$, where

$$\mathcal{O}^1 = \mathcal{O}^3 := \mathcal{O}_\delta^+ \cup \mathcal{O}_\delta', \quad \mathcal{O}^2 = \mathcal{O}^4 := \mathcal{O}_\delta^- \cup \mathcal{O}_\delta'.$$

FIGURE 3. Manifold $\mathcal{M}$ and extension of T .

Define $u^i : \mathcal{O}^i \rightarrow \mathbb{R}$ as follows: $u^1 = u$, where u is the solution to the Signorini problem satisfying the assumptions in the introduction; u^2 is the even reflection of u about x_n ; $u^3 = -u^1$ and $u^4 = -u^2$. Let $T^i = T^{u^i} : \mathcal{O}^i \rightarrow \mathbb{R}^n$ be the corresponding partial hodograph-Legendre transform defined in (3.2).

Consider the disjoint union of $\mathcal{O}^i$: $\sqcup_{i=1}^4 \mathcal{O}^i$, and denote the elements of it by (x, i) for $x \in \mathcal{O}^i$, $i = 1, \dots, 4$. Define

$$\begin{aligned} \sqcup_{i=1}^4 u^i : \sqcup_{i=1}^4 \mathcal{O}^i &\rightarrow \mathbb{R}, & (\sqcup_{i=1}^4 u^i)((y, j)) &= u^j(y); \\ \sqcup_{i=1}^4 T^i : \sqcup_{i=1}^4 \mathcal{O}^i &\rightarrow \mathbb{R}^n, & (\sqcup_{i=1}^4 T^i)((y, j)) &= T^j(y). \end{aligned}$$

Now we define an equivalence relation on $\sqcup_{i=1}^4 \mathcal{O}^i$ as follows:

$$(x, i) \sim (y, j) \quad \text{iff} \quad (\sqcup_{i=1}^4 T^i)((x, i)) = (\sqcup_{i=1}^4 T^i)((y, j)).$$

It is easy to check from the definition of T^i and Theorem 3.4 that this equivalence relation identifies the points (x, i) and (x, j) if (i) $x \in \Lambda_u$, $i = 2, j = 3$ or $i = 4, j = 1$; (ii) $x \in \overline{\Omega}_u$, $i = 1, j = 2$ or $i = 3, j = 4$. In particular, for $x \in \Gamma_u$, (x, i) and (x, j) are identified for all $i, j = 1, 2, 3, 4$.

Let $\mathcal{M} := \sqcup_{i=1}^4 \mathcal{O}^i / \sim$ denote the quotient space. Consider on $\mathcal{M}$:

$$\begin{aligned} \overline{u} &:= \overline{\sqcup_{i=1}^4 u^i} : \mathcal{M} \rightarrow \mathbb{R}, & \overline{u}(\overline{(y, j)}) &= u^j(y); \\ \overline{T} &:= \overline{\sqcup_{i=1}^4 T^i} : \mathcal{M} \rightarrow \mathbb{R}^n, & \overline{T}(\overline{(x, i)}) &= T(x, i). \end{aligned}$$

It is immediate that $\overline{T}$ is continuous and injective. Moreover, it is open from $\mathcal{M}$ to $\mathbb{R}^n$ by Theorem 3.4 and the special way we glue $\mathcal{O}^i$. Hence we obtain that $\overline{T}$ is a homeomorphism from $\mathcal{M}$ to $\overline{T}(\mathcal{M})$. In particular, $\overline{T}(\mathcal{M})$ is an open neighborhood of the origin in $\mathbb{R}^n$, which contains the straightened free boundary $T(\Gamma_u)$.

We still denote the set $\{\overline{(x, i)} \in \mathcal{M} : x \in \Gamma_u\}$ by Γ_u . It is not hard to observe that $\mathcal{M} \setminus \Gamma_u$ is a double cover of $\mathcal{O}_\delta \setminus \Gamma_u \subset \mathbb{R}^n$ with the covering map $\varphi : \overline{(x, i)} \mapsto x$. Hence $\mathcal{M} \setminus \Gamma_u$ can be given a smooth structure which makes φ into a local diffeomorphism. In the local coordinate charts $\varphi_\alpha : U_\alpha \rightarrow \mathcal{O}_\delta \setminus \Gamma_u$, we have $(\overline{u} \circ \varphi_\alpha^{-1})(x) = u(x)$,

where u is the extended function via the even or odd reflection about Λ_u or Ω_u , hence $\bar{u}$ is continuous on $\mathcal{M}$, smooth in $\mathcal{M} \setminus \Gamma_u$ and $\Delta \bar{u} = 0$ there. Similarly, one can compute $(\bar{T} \circ \varphi_\alpha^{-1})(x) = (x'', \partial_{n-1}u(x), \partial_nu(x))$, which is a diffeomorphism on $\varphi_\alpha(U_\alpha)$ by Theorem 3.4 (apply Theorem 3.4 for the extended u). This shows that $\bar{T} : \mathcal{M} \setminus \Gamma_u \rightarrow \bar{T}(\mathcal{M} \setminus \Gamma_u)$ is a diffeomorphism.

From now on, with slight abuse of the notation we will still denote $\bar{u}$ by u and $\bar{T}$ by T . In the following, we will simply write $\partial_k u$, DT , etc. while having in mind that we are taking the derivatives in the local coordinates.

4. PARTIAL LEGENDRE TRANSFORM AND A NONLINEAR PDE

In this section we study the partial Legendre transform of u and the fully nonlinear PDE it satisfies. We let

$$\mathcal{U} := T(\mathcal{M}),$$

which is an open neighborhood of the origin and

$$\mathcal{P} := T(\Gamma_u)$$

be the straightened free boundary.

4.1. Partial Legendre transform. For $y \in \mathcal{U}$, we define the partial Legendre transform of u by the identity

$$(4.1) \quad v(y) = u(x) - x_{n-1}y_{n-1} - x_n y_n,$$

where $x = T^{-1}(y) \in \mathcal{M}$. It is immediate to check the following properties of v :

- (i) v is odd about y_{n-1} and even about y_n .
- (ii) v is continuous in $\mathcal{U}$, smooth in $\mathcal{U} \setminus \mathcal{P}$ and $v = 0$ on $\mathcal{P}$.
- (iii) A direct computation shows that in $\mathcal{U} \setminus \mathcal{P}$

$$(4.2) \quad \begin{aligned} \partial_{y_i} v &= \partial_{x_i} u, \quad i = 1, \dots, n-2 \\ \partial_{y_{n-1}} v &= -x_{n-1}, \quad \partial_{y_n} v = -x_n. \end{aligned}$$

Hence T^{-1} can be written as

$$x = T^{-1}(y) = (y_1, \dots, y_{n-2}, -\partial_{y_{n-1}} v, -\partial_{y_n} v).$$

The Jacobian matrix of v is then

$$(4.3) \quad DT^{-1}(y) = \begin{pmatrix} I_{n-2} & 0 \\ A & B \end{pmatrix}, \quad DT = \begin{pmatrix} I_{n-2} & 0 \\ -B^{-1}A & B^{-1} \end{pmatrix},$$

where

$$\begin{aligned} A &= \begin{pmatrix} -\partial_{y_{n-1}y_1} v & -\partial_{y_{n-1}y_2} v & \dots & -\partial_{y_{n-1}y_{n-2}} v \\ -\partial_{y_n y_1} v & -\partial_{y_n y_2} v & \dots & -\partial_{y_n y_{n-2}} v \end{pmatrix}, \\ B &= \begin{pmatrix} -\partial_{y_{n-1}y_{n-1}} v & -\partial_{y_{n-1}y_n} v \\ -\partial_{y_n y_{n-1}} v & -\partial_{y_n y_n} v \end{pmatrix}. \end{aligned}$$

Since $u \in C^1(\mathcal{M} \setminus \Gamma_u)$ and its differential has an continuous extension to Γ_u , this together with the continuity of T^{-1} and (4.2) imply that $v \in C^1(\mathcal{U})$.

- (iv) The restriction of T^{-1} to $\mathcal{P}$ is given by

$$T^{-1}(y'', 0, 0) = (y'', -\partial_{y_{n-1}} v(y'', 0, 0), 0) \in \Gamma_u,$$

which gives a local parametrization of the free boundary Γ_u . Thus, the regularity of the free boundary is directly related to the regularity of $\partial_{y_{n-1}} v$, restricted to $\mathcal{P}$.

4.2. Fully nonlinear equation for v . A direct computation using (4.2) and (4.3) shows that

$$\begin{aligned} \partial_{x_i x_i} u &= \partial_{y_i y_i} v - (\partial_{y_{n-1} y_i} v, \partial_{y_n y_i} v) \begin{pmatrix} \partial_{y_{n-1} y_{n-1}} v & \partial_{y_{n-1} y_n} v \\ \partial_{y_{n-1} y_n} v & \partial_{y_n y_n} v \end{pmatrix}^{-1} \begin{pmatrix} \partial_{y_i y_{n-1}} v \\ \partial_{y_i y_n} v \end{pmatrix} \\ &\quad i = 1, \dots, n-2 \\ \begin{pmatrix} \partial_{x_{n-1} x_{n-1}} u & \partial_{x_{n-1} x_n} u \\ \partial_{x_n x_{n-1}} u & \partial_{x_n x_n} u \end{pmatrix} &= - \begin{pmatrix} \partial_{y_{n-1} y_{n-1}} v & \partial_{y_{n-1} y_n} v \\ \partial_{y_n y_{n-1}} v & \partial_{y_n y_n} v \end{pmatrix}^{-1}. \end{aligned}$$

Since $\Delta u = 0$ in $\mathcal{M} \setminus \Gamma_u$, the Legendre function v satisfies the following fully nonlinear equation in $\mathcal{U} \setminus \mathcal{P}$

$$\begin{aligned} (4.4) \quad \tilde{F}(D^2 v) &= \sum_{i=1}^{n-2} \partial_{y_i y_i} v - \text{tr} \begin{pmatrix} \partial_{y_{n-1} y_{n-1}} v & \partial_{y_{n-1} y_n} v \\ \partial_{y_{n-1} y_n} v & \partial_{y_n y_n} v \end{pmatrix}^{-1} \\ &\quad - \sum_{i=1}^{n-2} (\partial_{y_i y_{n-1}} v, \partial_{y_i y_n} v) \begin{pmatrix} \partial_{y_{n-1} y_{n-1}} v & \partial_{y_{n-1} y_n} v \\ \partial_{y_{n-1} y_n} v & \partial_{y_n y_n} v \end{pmatrix}^{-1} \begin{pmatrix} \partial_{y_i y_{n-1}} v \\ \partial_{y_i y_n} v \end{pmatrix} = 0. \end{aligned}$$

Multiplying both sides of (4.4) by

$$-J(v) = -\det \begin{pmatrix} \partial_{y_{n-1} y_{n-1}} v & \partial_{y_{n-1} y_n} v \\ \partial_{y_{n-1} y_n} v & \partial_{y_n y_n} v \end{pmatrix},$$

we can write it in the form

$$\begin{aligned} F(D^2 v) &= -J(v) \tilde{F}(D^2 v) = \partial_{y_{n-1} y_{n-1}} v + \partial_{y_n y_n} v - J(v) \sum_{i=1}^{n-2} \partial_{y_i y_i} v \\ &\quad + \sum_{i=1}^{n-2} ((\partial_{y_i y_{n-1}} v)^2 \partial_{y_n y_n} v - 2 \partial_{y_i y_{n-1}} v \partial_{y_i y_n} v \partial_{y_{n-1} y_n} v + (\partial_{y_i y_n} v)^2 \partial_{y_{n-1} y_{n-1}} v) = 0, \end{aligned}$$

which can be further rewritten as

$$(4.5) \quad F(D^2 v) = \partial_{y_{n-1} y_{n-1}} v + \partial_{y_n y_n} v - \sum_{i=1}^{n-2} \det(V^i) = 0,$$

where V^i , $i = 1, \dots, n-2$, is the 3×3 matrix

$$V^i = \begin{pmatrix} \partial_{y_i y_i} v & \partial_{y_i y_{n-1}} v & \partial_{y_i y_n} v \\ \partial_{y_{n-1} y_i} v & \partial_{y_{n-1} y_{n-1}} v & \partial_{y_{n-1} y_n} v \\ \partial_{y_n y_i} v & \partial_{y_n y_{n-1}} v & \partial_{y_n y_n} v \end{pmatrix}_{3 \times 3}.$$

4.3. Blowup of v at $\mathcal{P}$. In order to study the asymptotic of higher derivatives of v at the straightened free boundary $\mathcal{P}$, we study the blowup of v at points on $\mathcal{P}$.

Let v_{y_0} be the Legendre function of u_{x_0} as in (4.1), where $y_0 = T(x_0) = (x_0'', 0, 0) \in \mathcal{P}$. It is not hard to compute that

$$(4.6) \quad v_{y_0}(y) = -\frac{4}{27C_{x_0}^2} \left(\left(\frac{y_{n-1}}{(\nu'_{x_0})_{n-1}} \right)^3 - 3 \left(\frac{y_{n-1}}{(\nu'_{x_0})_{n-1}} \right) y_n^2 \right) + (\nu''_{x_0} \cdot y'') \left(\frac{y_{n-1}}{(\nu'_{x_0})_{n-1}} \right)$$

where

$$(4.7) \quad \nu'_{x_0} = (\nu''_{x_0}, (\nu'_{x_0})_{n-1}) = \frac{(-\nabla f(x_0''), 1)}{\sqrt{1 + |\nabla f(x_0'')|^2}}$$

is the unit outer normal of Λ_u at x_0 . In particular, at the origin,

$$v_0(y) = \left(-\frac{4}{27C_0^2} \right) (y_{n-1}^3 - 3y_{n-1}y_n^2).$$

For $y = (y'', y_{n-1}, y_n) \in \mathbb{R}^n$ and $r > 0$, we consider the non-isotropic dilation

$$(4.8) \quad \delta_r y = (r^2 y'', r y_{n-1}, r y_n)$$

and the rescaling at y_0 (note $v(y_0) = 0$)

$$(4.9) \quad v_{r,y_0}(y) = \frac{v(y_0 + \delta_r y)}{r^3}.$$

From (4.1), (4.9) and (2.15), one can easily check that

$$(4.10) \quad v_{r,y_0}(y) = u_{r^2,x_0}(x) - x_{n-1}y_{n-1} - x_n y_n, \quad x = (T^{u_{r^2,x_0}})^{-1}(y).$$

Here rescaling family u_{r^2,x_0} and $T^{u_{r^2,x_0}}$ are defined on $\mathcal{M}_{r^{-2},x_0}$, where $\mathcal{M}_{r^{-2},x_0}$ are the topological spaces obtained by gluing four rescaled copies

$$\mathcal{O}_{r^{-2},x_0}^i := (\mathcal{O}^i - x_0)/r^2$$

together as in the construction of $\mathcal{M}$.

To study the convergence of v_{r,y_0} to v_{y_0} , we first show a lemma which concerns about in the local coordinate charts the uniform convergence of $(T^{u_{r^2,x_0}})^{-1}$ to $(T^{u_{x_0}})^{-1}$. The following two facts are easy to verify:

- (i) $T^{u_{r^2,x_0}}$ is bijective and $T^{u_{r^2,x_0}}(\mathcal{M}_{r^{-2},x_0}) = \delta_{r^{-1}}(\mathcal{U})$, where $\delta_{r^{-1}}$ is the non-isotropic dilation in (4.8). In particular, we have $T^{u_{x_0}}(\mathcal{M}_{\infty,x_0}) = \mathbb{R}^n$.
- (ii) Let

$$\mathcal{Q}^1 := \{y \in \mathbb{R}^n : y_{n-1} \geq 0, y_n \leq 0\}, \quad \mathcal{Q}^2 := e^{i\pi/2}\mathcal{Q}^1, \quad \mathcal{Q}^3 := e^{i\pi/2}\mathcal{Q}^2, \quad \mathcal{Q}^4 := e^{i\pi/2}\mathcal{Q}^3$$

and let

$$\mathcal{O}_{r^{-1},x_0}^{i,*} := \{\overline{(x,i)} \in \mathcal{M}_{r,x_0} : x \in \mathcal{O}_{r^{-1},x_0}^i\}, \quad i = 1, 2, 3, 4.$$

Then for any sufficiently small r and $x \in \Gamma_u \cap \mathcal{O}_\delta$, we have $T^{u_{r^2,x_0}}(\mathcal{O}_{r^{-2},x_0}^{i,*}) \subset \mathcal{Q}^i$, i.e. $T^{u_{r^2,x_0}}$ always maps the copy $\mathcal{O}_{r^{-2},x_0}^{i,*}$ into the corresponding “quarter” domain $\mathcal{Q}^i$.

Let $\varphi^i : \mathcal{O}_{r^{-1},x_0}^{i,*} \rightarrow \mathcal{O}_{r^{-1},x_0}^i$ with $\varphi^i(\overline{(x,i)}) = x$ denote the restriction of the covering map to $\mathcal{O}_{r^{-1},x_0}^{i,*}$.

Lemma 4.1. *Let $y_0 \in \mathcal{P}$ and $x_0 = T^{-1}(y_0)$. Then*

$$\varphi^i \circ (T^{u_{r^2,x_0}})^{-1} \rightarrow \varphi^i \circ (T^{u_{x_0}})^{-1}, \quad r \rightarrow 0_+$$

uniformly in any compact subset $K \subset \mathcal{Q}^i$, $i = 1, 2, 3, 4$. Moreover, this convergence is also uniform for y_0 varying in a compact subset of $\mathcal{P}$.

Proof. Given $y_0 \in \mathcal{P}$ and a compact set $K \subset \mathcal{Q}^i$, by (i) and (ii) above, there is a positive $r_1 = r_1(y_0, K)$, such that $K \subset T^{u_{r^2,x_0}}(\mathcal{O}_{r^{-2},x_0}^{i,*})$ for any $r < r_1$.

By the rescaled version of Proposition 3.2(i), there exists $r_0 \in (0, r_1)$ small such that $\varphi^i \circ (T^{u_{r^2,x_0}})^{-1}(K) \subset \mathcal{O}_{r_0^{-2},x_0}^i$, which is a bounded subset in $\mathbb{R}^n$, for any $0 < r < r_0$.

We know from the C_{loc}^1 convergence of u_{r^2,x_0} to u_{x_0} that

$$T^{u_{r^2,x_0}} \circ (\varphi^i)^{-1} \rightarrow T^{u_{x_0}} \circ (\varphi^i)^{-1} \text{ uniformly on } \mathcal{O}_{r_0^{-2},x_0}^i.$$

Moreover, the limit $\varphi^i \circ (T^{u_{x_0}})^{-1}$ is continuous on $\mathcal{Q}^i$ by a direct computation. Therefore, $\varphi^i \circ (T^{u_{r^2, x_0}})^{-1} \rightarrow \varphi^i \circ (T^{u_{x_0}})^{-1}$ uniformly in K .

By Theorem 2.9 and the continuous dependence of $(T^{u_{x_0}})^{-1}$ on x_0 , we have the above convergence is uniform for y_0 in any compact subset of $\mathcal{P}$. $\square$

Next we show the following compactness results.

Proposition 4.2. *Let K be a compact subset in $\mathbb{R}^n \setminus \{y_{n-1} = y_n = 0\}$. Then for any multi-index α ,*

$$\partial_y^\alpha v_{r, y_0} \rightarrow \partial_y^\alpha v_{y_0}, \quad r \rightarrow 0_+$$

uniformly in K . Moreover, the above convergence is also uniform for y_0 varying in a compact subset of $\mathcal{P}$.

Proof. Given $y_0 \in \mathcal{P}$, let $x_0 = T^{-1}(y_0) \in \Gamma_u$. For $|\alpha| = 0$ and 1, using (4.10) and (4.2), we can easily conclude from Lemma 4.1 together with the uniform convergence of u_{r^2, x_0} to u_{x_0} that, $\partial^\alpha v_{r, y_0}$ converges to $\partial^\alpha v_{y_0}$ uniformly in K .

For $|\alpha| \geq 2$, using (4.2) and (4.3) one can express $\partial^\alpha v_{r, y_0}$ in terms of $\partial^{\alpha'} u_{r^2, x_0}$ with $|\alpha'| \leq |\alpha|$, i.e. for fixed α ,

$$(4.11) \quad \partial_y^\alpha v_{r, y_0}(y) = f_\alpha(\partial_x^{\alpha'} u_{r^2, x_0}(x))|_{x=(T^{u_{r^2, x_0}})^{-1}(y)},$$

where f_α is some polynomial. For $K \subset \mathbb{R}^n \setminus \{y_{n-1} = y_n = 0\}$ compact, $(T^{u_{x_0}})^{-1}(K)$ is also compact and $(T^{u_{x_0}})^{-1}(K) \cap \Gamma_{u_{x_0}} = \emptyset$. By the local uniform convergence of $(T^{u_{r^2, x_0}})^{-1}$ to $(T^{u_{x_0}})^{-1}$ (Lemma 4.1) as well as the flatness of the free boundary $\Gamma_{u_{x_0}}$ (i.e. the Hausdorff distance between $\Gamma_{u_{r^2, x_0}}$ and $\Gamma_{u_{x_0}}$ goes to zero as r goes to zero), there exists $K' \subset \mathbb{R}^n$ compact and $r_0 = r_0(K, K')$ small, such that for all $r < r_0$, we have

$$(4.12) \quad (\varphi^i \circ (T^{u_{r^2, x_0}})^{-1})(K \cap \mathcal{Q}^i) \subset K', \quad i = 1, 2, 3, 4$$

and

$$(4.13) \quad K' \cap \Gamma_{u_{r^2, x_0}} = \emptyset.$$

Note that (4.13) implies that for any $r < r_0$ and $i = 1, 2, 3, 4$, $u_{r^2, x_0} \circ (\varphi^i)^{-1}$ are harmonic in K' . Thus for any multi-index α , we have $\partial^\alpha(u_{r^2, x_0} \circ (\varphi^i)^{-1}) \rightarrow \partial^\alpha(u_{x_0} \circ (\varphi^i)^{-1})$ uniformly. This combined with (4.11), (4.12) and Lemma 4.1 gives the conclusion.

Due to Theorem 2.9 and Lemma 4.1, the above convergence is uniform in y_0 varying the compact subset of $\mathcal{P}$. $\square$

From Proposition 4.2 one can get continuous extension of higher order (properly weighted) derivatives of v at $\mathcal{P}$.

Corollary 4.3. *For each $y_0 \in \mathcal{P}$, $x_0 = T^{-1}(y_0)$, we extend the following functions to y_0 by setting*

- (i) $\sqrt{y_{n-1}^2 + y_n^2} \partial_{ij} v(y_0) = 0, \quad i, j = 1, \dots, n-2;$
- (ii) $\partial_{i,n-1} v(y_0) = \frac{(\nu_{x_0})_i}{(\nu_{x_0})_{n-1}} = \partial_i f(x_0''), \quad \partial_{i,n} v(y_0) = 0, \quad i = 1, \dots, n-2;$
- (iii) $\partial_{\alpha,\beta} v(y_0) = 0, \quad \alpha, \beta = n-1, n;$
- (iv) $\partial_{n-1,n-1,n-1} v(y_0) = -\frac{8}{9(\nu_{x_0})_{n-1}^3 C_{x_0}^2}, \quad \partial_{n-1,n,n} v(y_0) = \frac{8}{9(\nu_{x_0})_{n-1} C_{x_0}^2};$
 $\partial_{n-1,n-1,n} v(y_0) = 0, \quad \partial_{n,n,n} v(y_0) = 0.$

Then after such extension the above functions are continuous on $\mathcal{U}$.

Proof. The proof is based on the following two facts (a) (b) and a blowup argument.

(a) For $i, j \in \{1, \dots, n-2\}$, $\alpha, \beta, \gamma \in \{n-1, n\}$,

$$\begin{aligned} \partial_{i,j} v_{r,y_0}(y) &= r(\partial_{i,j} v)(y_0 + ry), \quad \partial_{i,\alpha} v_{r,y_0}(y) = (\partial_{i,\alpha} v)(y_0 + ry), \\ \partial_{\alpha,\beta} v_{r,y_0}(y) &= r^{-1}(\partial_{\alpha,\beta} v)(y_0 + ry), \quad \partial_{\alpha,\beta,\gamma} v_{r,y_0}(y) = (\partial_{\alpha,\beta,\gamma} v)(y_0 + ry). \end{aligned}$$

(b) From the $C^{1,\alpha}$ regularity of Γ_u and the explicit expression of v_{y_0} , we have the map $y_0 \mapsto \partial^\alpha v_{y_0}$ is continuous from $\mathcal{P}$ to $C_0(K)$, where $K \subset \mathbb{R}^n \setminus \{y_{n-1} = y_n = 0\}$ compact. This together with Proposition 4.2 gives that for any multi-index α

$$\lim_{\substack{y_0 \in \mathcal{P}, y_0 \rightarrow y_0 \\ r \rightarrow 0_+}} \|\partial^\alpha v_{r,y_0} - \partial^\alpha v_{y_0}\|_{C_0(K)} = 0$$

We proceed to show the extended functions are continuous at $y_0 \in \mathcal{P}$. First they are continuous on $\mathcal{P}$ from the $C^{0,\alpha}$ dependence of ν'_{x_0} on x_0 . Next for $y \notin \mathcal{P}$, we use y'' to denote $(y'', 0, 0)$ and let

$$r = |y - y''| = \sqrt{y_{n-1}^2 + y_n^2}, \quad \eta = \frac{y - y''}{r} = \frac{(0, y_{n-1}, y_n)}{r}.$$

Then

$$v(y) = v_{r,y''}(\eta), \quad \eta \in \{y'' = 0, y_{n-1}^2 + y_n^2 = 1\}.$$

As $y \rightarrow y_0$, we have $r \rightarrow 0_+$ and $y'' \rightarrow y_0$. Thus by (b) above, for a fixed $K \subset \mathbb{R}^n \setminus \{y_{n-1} = y_n = 0\}$, which is compact and contains the set $\{y'' = 0, y_{n-1}^2 + y_n^2 = 1\}$, we have

$$(4.14) \quad \partial^\alpha v_{r,y''} \rightarrow \partial^\alpha v_{y_0} \text{ in } C_0(K) \text{ as } y \rightarrow y_0.$$

From the explicit expression of v_{y_0} (see (4.6)) we have

$$\begin{aligned} D^2 v_{y_0}(y) &= \begin{pmatrix} 0 & (\nu'_{x_0})^t / (\nu'_{x_0})_{n-1} & 0 \\ \nu''_{x_0} / (\nu'_{x_0})_{n-1} & -8y_{n-1}/9(\nu'_{x_0})_{n-1}^3 C_{x_0}^2 & 8y_n/9(\nu'_{x_0})_{n-1} C_{x_0}^2 \\ 0 & 8y_n/9(\nu'_{x_0})_{n-1} C_{x_0}^2 & 8y_{n-1}/9(\nu'_{x_0})_{n-1} C_{x_0}^2 \end{pmatrix} \\ \partial_{n-1,n-1,n-1} v_{y_0} &\equiv -\frac{8}{9(\nu'_{x_0})_{n-1}^3 C_{x_0}^2}, \quad \partial_{n-1,n,n} v_{y_0} \equiv \frac{8}{9(\nu'_{x_0})_{n-1} C_{x_0}^2} \\ \partial_{n-1,n-1,n} v_{y_0} &= \partial_{n,n,n} v_{y_0} \equiv 0 \end{aligned}$$

This together with (a) and (4.14) gives (i)-(iv). $\square$

5. SMOOTHNESS

5.1. Subelliptic structure. In this section we show the fully nonlinear equation $F(D^2v) = 0$ in (4.5) has a subelliptic structure in $\mathcal{U}$.

Let $\mathcal{S}^{n \times n}$ be the space of $n \times n$ symmetric matrices and we may consider F as a smooth function on $\mathcal{S}^{n \times n}$. Let $F_{ij}(P) = \frac{\partial F}{\partial p_{ij}}(P)$ for $P = (p_{ij}) \in \mathcal{S}^{n \times n}$ be the linearization of F at P . A direct computation shows that for $i, j = 1, \dots, n-2$ and $y \in \mathcal{U} \setminus \mathcal{P}$, the linearization of F at D^2v has the form

$$\begin{aligned}
F_{ij}(D^2v) &= 0, \quad i \neq j \\
F_{ii}(D^2v) &= -(\partial_{n-1,n-1}v \partial_{nn}v - (\partial_{n-1,n}v)^2) \\
&= -\left(\frac{\partial_{n-1,n-1}v}{y_{n-1}} \frac{\partial_{nn}v}{y_{n-1}}\right) y_{n-1}^2 + \frac{(\partial_{n-1,n}v)^2}{y_n^2} y_n^2 \\
F_{i,n-1}(D^2v) &= \partial_{n-1,i}v \partial_{nn}v - \partial_{n-1,n}v \partial_{n,i}v \\
&= \left(\partial_{n-1,i}v \frac{\partial_{nn}v}{y_{n-1}}\right) y_{n-1} - \left(\partial_{n,i}v \frac{\partial_{n-1,n}v}{y_n}\right) y_n \\
F_{i,n}(D^2v) &= -(\partial_{n-1,i}v \partial_{n,n-1}v - \partial_{n-1,n-1}v \partial_{n,i}v) \\
&= \left(\partial_{n-1,i}v \frac{\partial_{n,n-1}v}{y_n}\right) y_n - \left(\frac{\partial_{n-1,n-1}v}{y_{n-1}} \partial_{n,i}v\right) y_{n-1} \\
F_{n-1,n-1}(D^2v) &= 1 - \sum_{i=1}^{n-2} (\partial_{ii}v \partial_{nn}v - \partial_{i,n}v \partial_{n,i}v) \\
F_{n-1,n}(D^2v) &= \sum_{i=1}^{n-2} (\partial_{ii}v \partial_{n,n-1}v - \partial_{i,n-1}v \partial_{n,i}v) \\
F_{n,n}(D^2v) &= 1 - \sum_{i=1}^{n-2} (\partial_{ii}v \partial_{n-1,n-1}v - \partial_{i,n-1}v \partial_{n-1,i}v).
\end{aligned}$$

Observe that one can write $\{F_{ij}(D^2v)\}_{i,j} = ABA^t$, with A and B symmetric matrices of the following forms:

$$A = \begin{pmatrix} Y & 0 & 0 & \cdots & 0 \\ 0 & Y & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & Y & 0 \\ 0 & 0 & 0 & \cdots & I_2 \end{pmatrix}_{n \times 2(n-1)}$$

$$B = (b_{ij}) = \begin{pmatrix} B_0 & 0 & 0 & \cdots & B_1 \\ 0 & B_0 & 0 & \cdots & B_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & B_0 & B_{n-2} \\ (B_1)^t & (B_2)^t & \cdots & \cdots & B_{n-1} \end{pmatrix}_{2(n-1) \times 2(n-1)}$$

where

$$Y = (y_{n-1}, y_n),$$

$$B_0 = \begin{pmatrix} b_0(y) & 0 \\ 0 & \widetilde{b}_0(y) \end{pmatrix}, \quad B_i = \begin{pmatrix} b_{i,1}(y) & b_{i,2}(y) \\ \widetilde{b}_{i,1}(y) & \widetilde{b}_{i,2}(y) \end{pmatrix}, \quad i = 1, \dots, n-1$$

with

$$\begin{aligned} b_0(y) &= -\frac{\partial_{n-1,n-1}v}{y_{n-1}} \frac{\partial_{nn}v}{y_{n-1}}, & \widetilde{b}_0(y) &= \frac{(\partial_{n-1,n}v)^2}{y_n^2}; \\ b_{i,1}(y) &= \partial_{n-1,i}v \frac{\partial_{nn}v}{y_{n-1}}, & \widetilde{b}_{i,1}(y) &= -\partial_{n,i}v \frac{\partial_{n-1,n}v}{y_n}, \\ b_{i,2}(y) &= -\partial_{n,i}v \frac{\partial_{n-1,n-1}v}{y_{n-1}}, & \widetilde{b}_{i,2}(y) &= \partial_{n-1,i}v \frac{\partial_{n,n-1}v}{y_n}, \end{aligned}$$

for $i = 1, \dots, n-2$, and for $i = n-1$,

$$B_{n-1} = \begin{pmatrix} F_{n-1,n-1} & F_{n-1,n} \\ F_{n,n-1} & F_{n,n} \end{pmatrix}.$$

Note that $B(y)$ is smooth in $\mathcal{U} \setminus \mathcal{P}$ due to the smoothness of v there. Moreover, by Corollary 4.3(iii)(iv) and the intermediate value theorem,

$$\begin{aligned} \frac{\partial_{n-1,n-1}v(y)}{y_{n-1}} &\rightarrow -\frac{8}{9(\nu'_{x_0})_{n-1}^3 C_{x_0}^2}; & \frac{\partial_{n,n}v(y)}{y_{n-1}} &\rightarrow \frac{8}{9(\nu'_{x_0})_{n-1} C_{x_0}^2}; \\ \frac{\partial_{n-1,n}v(y)}{y_n} &\rightarrow \frac{8}{9(\nu'_{x_0})_{n-1} C_{x_0}^2}, \end{aligned}$$

thus $B(y)$ has a continuous extension on $\mathcal{P}$. In particular,

$$(5.1) \quad B(0) = a^2 I_{2(n-1)}, \quad a = \frac{8}{9C_0^2}.$$

Hence, $B(y)$ is positive definite in a small neighborhood of the origin, which implies that the linearized operator $F_{ij}(D^2v)\partial_{ij}$ has a subelliptic structure near the origin. Moreover, using (5.1) we have

$$AB(0)A^t = \begin{pmatrix} a^2(y_{n-1}^2 + y_n^2)I_{n-2} & 0 \\ 0 & I_2 \end{pmatrix}$$

which is the coefficient matrix for the *Baouendi-Grushin type operator*:

$$\mathcal{L}_a = a^2(y_{n-1}^2 + y_n^2) \sum_{i=1}^{n-3} \partial_{i,i}^2 + \partial_{n-1,n-1}^2 + \partial_{n,n}^2.$$

This indicates us to view the linearization of F in a neighborhood of the origin as a perturbation of Baouendi-Grushin type operator.

5.2. L^p estimates. We first recall the classical L^p estimate for the Baouendi-Grushin operator. For $(x, t) \in \mathbb{R}^{m+n}$, $x \in \mathbb{R}^m$, $t \in \mathbb{R}^n$, the Baouendi-Grushin operator is

$$\mathcal{L}_0 = \Delta_x + |x|^2 \Delta_t, \quad |x| = \sqrt{|x_1|^2 + \dots + |x_m|^2}.$$

In order to study the weak solution associated with $\mathcal{L}_0$, it is natural to consider the following function space associated with the Hörmander vector fields

$$\begin{aligned} X_i &= \partial_{x_i} \quad i = 1, \dots, m; \\ X_{m+ij} &= x_i \partial_{t_j} \quad i = 1, \dots, m; \quad j = 1, \dots, n. \end{aligned}$$

For $k, p \geq 1$, $\Omega \subset \mathbb{R}^{m+n}$ a bounded open subset, we define

$$(5.2) \quad M^{k,p}(\Omega) = \{u \in L^p(\Omega) : X_{j_1} \cdots X_{j_s} u \in L^p(\Omega), \\ 1 \leq s \leq k, \quad j_1, \dots, j_s \in \{1, \dots, m+mn\}\}.$$

By Theorem 1 in [Xu90], $M^{k,p}(\Omega)$ is a separable Banach space for $1 \leq p < \infty$ with the norm

$$\|u\|_{M^{k,p}(\Omega)} = \|u\|_{p,\Omega} + \sum_{1 \leq s \leq k} \|X_{j_1} \cdots X_{j_s} u\|_{p,\Omega}.$$

Denote by $M_0^{k,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $M^{k,p}(\Omega)$. It is not hard to prove by using mollifier that if $u \in M^{k,p}(\Omega)$ and has compact support in Ω , then $u \in M_0^{k,p}(\Omega)$, $1 \leq p < \infty$.

In this paper only the spaces with $k = 1, 2$ are involved. We list them below separately:

$$\begin{aligned} M^{1,p}(\Omega) &= \{u \in L^p(\Omega) : \partial_{x_i} u, |x| \partial_{t_j} u \in L^p(\Omega)\}; \\ M^{2,p}(\Omega) &= \{u \in L^p(\Omega) : \partial_{x_i} u, \partial_{t_j} u \in L^p(\Omega); \\ &\quad \partial_{x_i x_j}^2 u, |x|^2 \partial_{t_i t_j}^2 u, |x| \partial_{t_i x_j}^2 u \in L^p(\Omega)\}. \end{aligned}$$

To simplify the notation, we will denote

$$\begin{aligned} \|\partial_w^1 u\|_{p,\Omega} &:= \|\nabla_x u\|_{p,\Omega} + \| |x| \nabla_t u \|_{p,\Omega} \\ \|\partial_w^2 u\|_{p,\Omega} &:= \sum_{i,j=1,\dots,m} \|\partial_{x_i x_j}^2 u\|_{p,\Omega} + \sum_{\substack{i=1,\dots,m \\ j=1,\dots,n}} \| |x| \partial_{x_i t_j}^2 u \|_{p,\Omega} \\ &\quad + \sum_{i,j=1,\dots,n} \| |x|^2 \partial_{t_i t_j}^2 u \|_{p,\Omega} + \sum_{j=1,\dots,n} \|\partial_{t_j} u\|_{p,\Omega}. \end{aligned}$$

We will need the Sobolev embedding theorem for $M_0^{1,p}(\Omega)$ and the L^p estimate for $\mathcal{L}_0$. Similar results for more general subelliptic operators can be found in lots of literature like [Xu90] and [SC88]. Since our case is much simpler, we provide a relatively shorter and self-contained proof in the Appendix.

Lemma 5.1 (Sobolev embedding for $M_0^{1,p}(\Omega)$). *Let Ω be a bounded domain in $\mathbb{R}^{m+n}$.*

- (i) *If $1 \leq p < m + 2n$, then $M_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for p, q satisfying $\frac{1}{q} + \frac{1}{m+2n} = \frac{1}{p}$. i.e. there exists $C = C(p) > 0$ such that for all $u \in M_0^{1,p}(\Omega)$*

$$\|u\|_{q,\Omega} \leq C (\|\nabla_x u\|_{p,\Omega} + \| |x| \nabla_t u \|_{p,\Omega}).$$

- (ii) *If $p > m + 2n$, then $M_0^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$, i.e. there exists $C = C(p, n, m) > 0$ such that for all $u \in M_0^{1,p}(\Omega)$,*

$$\|u\|_{\infty,\Omega} \leq C (\|\nabla_x u\|_{p,\Omega} + \| |x| \nabla_t u \|_{p,\Omega}).$$

Proof. See Appendix. □

Theorem 5.2 (L^p estimates for $\mathcal{L}_0$). *Let Ω be a domain in $\mathbb{R}^{m+n}$, $m \geq 2$ and m is even. Let $u \in M_0^{2,p}(\Omega)$ with $1 < p < \infty$. Then there is a positive constant C_p which only depends on p such that*

$$(5.3) \quad \|\partial_w^2 u\|_{p,\Omega} \leq C_p \|\mathcal{L}_0 u\|_{p,\Omega}.$$

Proof. See Appendix. $\square$

Next we state the local L^p estimates for the perturbed operator

$$(5.4) \quad \mathcal{L} := \sum_{i,j=1,\dots,m+n} \ell_{ij}(x,t) \partial_{ij}.$$

where $\{\ell_{ij}(x,t)\}_{(m+n) \times (m+n)}$ can be decomposed into the form ABA^t with

$$A = \begin{pmatrix} I_m & 0 & \cdots & 0 \\ 0 & X & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & X \end{pmatrix}_{(m+n) \times (m+mn)}, \quad X = (x_1, \dots, x_m)_{1 \times m}$$

and $B = (b_{ij})_{(m+mn) \times (m+mn)}$ a positive definite matrix with continuous entries and for some small positive δ_0 it satisfies

$$(5.5) \quad |b_{ij}(x,t) - \delta_{ij}| \leq \delta_0$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

From now on, we will work on the following scale-invariant “cylinder” (w.r.t $\mathcal{L}_0$) centered at the origin: for $r > 0$

$$\mathcal{C}_r = \{(x,t) \in \mathbb{R}^{m+n} : |x| < r, |t| < r^2\}.$$

Proposition 5.3. *Let $u \in M^{2,p}(\mathcal{C}_r)$ satisfy*

$$(5.6) \quad \mathcal{L}u = f \text{ a.e. in } \mathcal{C}_r, \quad f \in L^p(\mathcal{C}_r), \quad 1 < p < \infty,$$

where $\mathcal{L}$ is a perturbed Baouendi-Grushin operator given by (5.4). Then if (5.5) is satisfied for sufficiently small $\delta_0 = \delta_0(p, m, n) > 0$, there exists $C = C(p) > 0$ such that for any $\sigma \in (0, 1)$,

$$(5.7) \quad \|\partial_w^2 u\|_{p,\mathcal{C}_{\sigma r}} \leq \frac{C}{(1-\sigma)^2 r^2} (r^2 \|f\|_{p,\mathcal{C}_r} + \|u\|_{p,\mathcal{C}_r}).$$

Proof. We write

$$\mathcal{L}_0 u = (\mathcal{L}_0 - \mathcal{L})u + \mathcal{L}u.$$

If $\text{supp } u \subseteq \mathcal{C}_r$, then by (5.3)

$$\begin{aligned} \|\partial_w^2 u\|_{p,\mathcal{C}_r} &\leq C_p (\|\mathcal{L}u\|_{p,\mathcal{C}_r} + \|(\mathcal{L}_0 - \mathcal{L})u\|_{p,\mathcal{C}_r}) \\ &\leq C_p \left(\|\mathcal{L}u\|_{p,\mathcal{C}_r} + \sup_{\mathcal{C}_r} |b_{ij}(x,t) - b_{ij}(0,0)| \|\partial_w^2 u\|_{p,\mathcal{C}_r} \right) \\ &\leq C_p (\|\mathcal{L}u\|_{p,\mathcal{C}_r} + \delta_0 \|\partial_w^2 u\|_{p,\mathcal{C}_r}). \end{aligned}$$

Hence if we choose $\delta_0 = 1/(2C_p)$ in (5.5), for any $u \in M_0^{2,p}(\mathcal{C}_r)$ we have

$$(5.8) \quad \|\partial_w^2 u\|_{p,\mathcal{C}_r} \leq \frac{C_p}{1 - C_p \delta_0} \|\mathcal{L}u\|_{p,\mathcal{C}_r} \leq 2C_p \|\mathcal{L}u\|_{p,\mathcal{C}_r}.$$

Now we remove the compact support condition. Let $\sigma \in (0, 1)$ be fixed and let $\sigma' = (1+\sigma)/2$. Let $\eta(x,t) = \eta_1(|x|)\eta_2(|t|)$ be a smooth cut-off function in $\mathcal{C}_r$, where

$0 \leq \eta_i \leq 1$ satisfy $\eta_1 = 1$ when $|x| \leq \sigma r$, $\eta_1 = 0$ when $|x| \geq \sigma' r$; $\eta_2 = 1$ when $|t| < \sigma r^2$, $\eta_2 = 0$ when $|t| > \sigma' r^2$. Moreover, $|\eta'_1| \leq 2/(1-\sigma')r$, $|\eta'_2| \leq 2/(1-\sigma')r^2$, $|\eta''_1| \leq 4/(1-\sigma')^2 r^2$, $|\eta''_2| \leq 4/(1-\sigma')^2 r^4$. Let $v = u\eta^2$, then

$$\mathcal{L}v = \eta \mathcal{L}u + [\mathcal{L}, \eta^2]u.$$

By (5.8) we have

$$(5.9) \quad \|\partial_w^2 u\|_{p, \mathcal{C}_{\sigma r}} \leq 2C_p (\|\mathcal{L}u\|_{p, \mathcal{C}_{\sigma' r}} + \|[\mathcal{L}, \eta^2]u\|_{p, \mathcal{C}_{\sigma' r}}).$$

Compute $[\mathcal{L}, \eta^2]u$ directly. Using the estimates for the coefficient matrix b_{ij} with δ_0 chosen less than 1, as well as the cut-off functions, we obtain the following (for simplicity we write $\|\cdot\|_p = \|\cdot\|_{p, \mathcal{C}_{\sigma' r}}$)

$$(5.10) \quad \begin{aligned} \|[\mathcal{L}, \eta^2]u\|_p &\leq C \left(\frac{1}{(1-\sigma')r} \|\nabla_x u\|_p + \frac{1}{(1-\sigma')r^2} \| |x|^2 \nabla_t u \|_p \right) + \\ &\quad + \frac{C}{(1-\sigma')^2 r^2} \|u\|_p + \frac{C}{(1-\sigma')r} \| |x| \nabla_t u \|_p, \end{aligned}$$

where C is some absolute constant.

Now using the interpolation between the classical Sobolev spaces (for $r = 1$) and Young's inequality we have for any $\varepsilon > 0$,

$$\begin{aligned} (1-\sigma) \|\nabla_x u\|_{p, \mathcal{C}_\sigma} &\leq \varepsilon (1-\sigma)^2 \|D_{x,x}^2 u\|_{p, \mathcal{C}_\sigma} + \frac{C}{\varepsilon} \|u\|_{p, \mathcal{C}_\sigma}, \\ (1-\sigma) \| |x|^2 \nabla_t u \|_{p, \mathcal{C}_\sigma} &\leq \varepsilon (1-\sigma)^2 \| |x|^2 D_{t,t}^2 u \|_{p, \mathcal{C}_\sigma} + \frac{C}{\varepsilon} \| |x|^2 u \|_{p, \mathcal{C}_\sigma}, \\ (1-\sigma) \| |x| \nabla_t u \|_{p, \mathcal{C}_\sigma} &\leq \varepsilon (1-\sigma)^2 \| |x|^2 D_{t,t}^2 u \|_{p, \mathcal{C}_\sigma} + \frac{C}{\varepsilon} \|u\|_{p, \mathcal{C}_\sigma}. \end{aligned}$$

Hence by rescaling and then taking the supreme in σ , we have

$$(5.11) \quad \begin{aligned} \sup_{0 < \sigma < 1} (1-\sigma)r \|\nabla_x u\|_{p, \mathcal{C}_{\sigma r}} \\ \leq \varepsilon \sup_{0 < \sigma < 1} (1-\sigma)^2 r^2 \|D_{x,x}^2 u\|_{p, \mathcal{C}_{\sigma r}} + \frac{C}{\varepsilon} \sup_{0 < \sigma < 1} \|u\|_{p, \mathcal{C}_{\sigma r}}; \end{aligned}$$

$$(5.12) \quad \begin{aligned} \sup_{0 < \sigma < 1} (1-\sigma) \| |x|^2 \nabla_t u \|_{p, \mathcal{C}_{\sigma r}} \\ \leq \varepsilon \sup_{0 < \sigma < 1} (1-\sigma)^2 r^2 \| |x|^2 \nabla_t^2 u \|_{p, \mathcal{C}_{\sigma r}} + \frac{C}{\varepsilon r^2} \sup_{0 < \sigma < 1} \| |x|^2 u \|_{p, \mathcal{C}_{\sigma r}}. \end{aligned}$$

$$(5.13) \quad \begin{aligned} \sup_{0 < \sigma < 1} (1-\sigma)r \| |x| \nabla_t u \|_{p, \mathcal{C}_{\sigma r}} \\ \leq \varepsilon \sup_{0 < \sigma < 1} (1-\sigma)^2 r^2 \| |x|^2 \nabla_t^2 u \|_{p, \mathcal{C}_{\sigma r}} + \frac{C}{\varepsilon} \sup_{0 < \sigma < 1} \|u\|_{p, \mathcal{C}_{\sigma r}}. \end{aligned}$$

Combining (5.9)-(5.13) and choosing ε , $\delta_0 < 1/(2C_p)$ small enough, depending on p , we obtain the inequality (5.7). $\square$

5.3. Smoothness at the free boundary points. In this section we show that the Legendre function v which satisfies the fully nonlinear PDE (4.5) is smooth in a neighborhood of the origin.

We will work on the non-isotropic cylinder at the origin:

$$\mathcal{C}_r = \{(y'', y_{n-1}, y_n) : |y''| < r^2, \sqrt{y_{n-1}^2 + y_n^2} < r\}, \quad n \geq 3, r > 0.$$

Before proving the main theorem, we make the following two remarks:

- (i) By Corollary 4.3 and the discussion in Section 5.1, there is $r_0 > 0$ small enough such that $v \in M^{2,p}(\mathcal{C}_{r_0})$ for any $1 \leq p < \infty$ and the linearized operator $F_{ij}(D^2v)\partial_{ij}$ can be viewed as a perturbation of the Baouendi-Grushin type operator in $\mathcal{C}_{r_0}$.
- (ii) We note the following rescaling property: if v solves (4.5) in $\mathcal{C}_{r_0}$, then $\tilde{v}(y) = c_0 v(\delta_r y)/r^3$ with $c_0 > 0$ and δ_r the non-isotropic dilation in (4.8) will solve

$$\partial_{y_{n-1}y_{n-1}}\tilde{v} + \partial_{y_n y_n}\tilde{v} - (c_0)^{-2} \sum_{i=1}^{n-2} \det(\tilde{V}^i) = 0 \quad \text{in } \mathcal{C}_{r_0/r}.$$

Hence by multiplying a nonzero constant, we may assume that the coefficient matrix $F_{ij}(D^2v)$ is of the form ABA^t in $\mathcal{C}_{r_0}$ with B continuous and satisfying (5.5) for sufficiently small δ_0 , where δ_0 is chosen such that the L^p estimate (Proposition 5.3) applies.

The idea to show the smoothness is then to apply iteratively (5.7) to the first order difference quotient of $\partial^\alpha v$, but each step we need to be careful that the non-homogeneous RHS coming from differentiation is bounded in L^p .

For notation simplicity, in what follows we will discuss the case when $n = 3$. Then equation (4.5) is simply

$$(5.14) \quad F(D^2v) = \partial_{22}v + \partial_{33}v - \det(D^2v) = 0.$$

The arguments for $n > 3$ are the same.

Theorem 5.4. *Let v be the Legendre function of u defined in (4.1). Let $r_0 > 0$ such that in $\mathcal{C}_{r_0}$, Corollary 4.3 holds for v and $F_{ij}(D^2v)$ can be written in the form of (5.4) with b_{ij} satisfying (5.5) for sufficiently small δ_0 . Then v is smooth at the origin.*

Proof. We let

$$\Delta_i^h v := \frac{v(\cdot + he_i) - v}{h}, \quad h \neq 0, \quad i = 1, 2, 3$$

denote the first difference quotient of v in e_i direction.

Step 1: Show $\partial_i v \in M^{2,p}(\mathcal{C}_r)$ for any $1 < p < \infty$, $0 < r < r_0$. By Corollary 4.3, it is enough to show it for $\partial_1 v$.

In fact, $v \in M^{2,p}(\mathcal{C}_{r_0})$ implies that $\Delta_1^h v \in M^{2,p}(\mathcal{C}_{r_0-h})$ and $\Delta_1^h v = 0$ on $\mathcal{C}'_{r_0-h}$ for $0 < h < r \ll r_0$. Moreover, by taking Δ_1^h on both sides of (5.14) we get $\Delta_1^h v$ satisfies

$$\tilde{F}_{ij}(D^2v, h)\partial_{ij}\Delta_1^h v = 0$$

with

$$\tilde{F}_{ij}(D^2v, h) = \int_0^1 F_{ij}(D^2v + t(D^2v(\cdot + he_1) - D^2v)) dt.$$

Since a translation in e_1 direction does not change the subelliptic structure of the operator (by Corollary 4.3 and the $C^{0,\alpha}$ dependence of ν_{x_0} on x_0), i.e. $\tilde{F}_{ij}(D^2v, h)\partial_{ij}$

is still a perturbed Baouendi-Grushin operator in the form of (5.4), with b_{ij} satisfying (5.5) in $\mathcal{C}_{r_0-h}$. Then by Proposition 5.3 there exists $C = C(p) > 0$ such that for any $\sigma \in (0, 1)$

$$(5.15) \quad \|\partial_w^2 \Delta_1^h v\|_{p, \mathcal{C}_{\sigma(r_0-h)}} \leq \frac{C}{(1-\sigma)^2(r_0-h)^2} \|\Delta_1^h v\|_{p, \mathcal{C}_{r_0-h}}.$$

Note that the RHS of (5.15) is uniformly bounded in h . Moreover, $\|\partial_w^2 \Delta_1^h v\|_p = \|\Delta_1^h(\partial_w^2 v)\|_p$ and $\|\partial_w^2 v\|_{p, \mathcal{C}_{r_0}} < \infty$ (here we slightly abuse of the notation to let $\partial_w^2 v$ denote the weighted second order derivatives and first order derivatives w.r.t. y_{n-1} and y_n). Thus $\partial_1 v \in M^{2,p}(\mathcal{C}_r)$ for any $0 < r < r_0$ with $\|\partial_1 v\|_{M^{2,p}(\mathcal{C}_r)}$ depending on $r_0 - r$, r_0 and p .

Step 2: Show $\partial_{11} v \in M^{2,p}(\mathcal{C}_r)$, $0 < r < r_0$.

Take ∂_1 to both sides of (5.14). From step 1, $\partial_1 v \in M^{2,p}(\mathcal{C}_r)$ and it satisfies

$$(5.16) \quad F_{ij}(D^2 v) \partial_{ij} \partial_1 v = 0.$$

Applying Δ_1^h to (5.16) with $0 < h << r_0/8$, then $\Delta_1^h \partial_1 v \in M^{2,p}(\mathcal{C}_{r-h})$ and it satisfies

$$(5.17) \quad F_{ij}(D^2 v) \partial_{ij} (\Delta_1^h \partial_1 v) = f \quad \text{in } \mathcal{C}_{r-h}$$

with

$$f = -\Delta_1^h(F_{ij}) \partial_{ij} \partial_1 v(\cdot + h e_1).$$

To estimate the $\|f\|_p$, we first notice that f (up to a translation $\tau_{e_1 h}$) is a summation of the following terms:

$$\begin{aligned} I_1 &= \partial_{111} v \partial_{\beta' \gamma'} v \Delta_1^h \partial_{\beta \gamma} v, \\ I_2 &= \partial_{11\beta} v (\Delta_1^h \partial_{1\gamma} v \partial_{\beta' \gamma'} v + \partial_{1\gamma} v \Delta_1^h \partial_{\beta' \gamma'} v), \\ I_3 &= \partial_{1\beta\gamma} v (\Delta_1^h \partial_{11} v \partial_{\beta' \gamma'} v + \partial_{11} v \Delta_1^h \partial_{\beta' \gamma'} v), \quad \beta, \gamma, \beta', \gamma' = 2, 3. \end{aligned}$$

Next, since $(y_2^2 + y_3^2)^{-1/2} \in L^s(\mathcal{C}_r \cap \{y_1 = \text{const}\})$ for any $s \in (1, 2)$, then by Hölder's inequality, for $q' \in (1, 2)$ satisfying $\frac{1}{q'} = \frac{1}{p} + \frac{1}{s}$ we have

$$(5.18) \quad \|(y_2^2 + y_3^2)^{1/2} \partial_{111} v\|_{q', \mathcal{C}_r} \leq \|(y_2^2 + y_3^2)^{-1/2}\|_{s, \mathcal{C}_r} \|(y_2^2 + y_3^2) \partial_{111} v\|_{p, \mathcal{C}_r};$$

$$(5.19) \quad \|\partial_{11\beta} v\|_{q', \mathcal{C}_r} \leq \|(y_2^2 + y_3^2)^{-1/2}\|_{s, \mathcal{C}_r} \|(y_2^2 + y_3^2)^{1/2} \partial_{11\beta} v\|_{p, \mathcal{C}_r}, \quad \beta = 2, 3.$$

Apply Hölder to estimate I_1 . For some q satisfying $1/q = 1/q' + 1/p$ we have

$$(5.20) \quad \|I_1\|_{q, \mathcal{C}_{r-h}} \leq \left\| (y_2^2 + y_3^2)^{1/2} \partial_{111} v \right\|_{q', \mathcal{C}_{r-h}} \left\| (y_2^2 + y_3^2)^{-1/2} \partial_{\beta' \gamma'} v \right\|_{\infty, \mathcal{C}_{r-h}} \left\| \Delta_1^h \partial_{\beta \gamma} v \right\|_{p, \mathcal{C}_{r-h}}.$$

By Corollary 4.3(iv), the second term on the RHS of (5.20) is bounded. From the boundedness of $\|\partial_w^2 \partial_1 v\|_{p, \mathcal{C}_r}$ shown in step 1, the third term is uniformly bounded in h . Hence combining (5.18) we have $\|I_1\|_{q, \mathcal{C}_{r-h}}$ is uniformly bounded in h . Similarly by using Corollary 4.3, (5.19) and step 1, we have $\|I_2\|_{q, \mathcal{C}_{r-h}}, \|I_3\|_{q, \mathcal{C}_{r-h}}$ are uniformly bounded in h . Therefore, applying the L^p estimate (Proposition 5.3) to (5.17), one can find a constant C_q independent of h , such that for any $\sigma \in (0, 1)$,

$$\|\partial_w^2 (\Delta_1^h \partial_1 v)\|_{q, \mathcal{C}_{\sigma(r-h)}} \leq \frac{C_q}{(1-\sigma)^2} \|f\|_{q, \mathcal{C}_{r-h}} + \frac{C_q}{(r-h)^2} \|\Delta_1^h \partial_1 v\|_{q, \mathcal{C}_{r-h}}.$$

Since the RHS is uniformly bounded in h , this implies

$$(5.21) \quad \|\partial_w^2 \partial_{11} v\|_{q, \mathcal{C}_{\sigma r}} \leq C_{q, \sigma, r}.$$

From (5.21) we know $(y_2^2 + y_3^2)^{1/2} \partial_{111} v, \partial_{11\beta} v \in M^{1,q}(\mathcal{C}_{\sigma r})$, $\beta = 2, 3$. Multiplying a cut-off function to extend the functions to $\mathbb{R}^n$ and applying the Sobolev embedding lemma 5.1(i) we have, for $q_1 = 4q/(4 - q)$ (with 4 the homogeneous dimension associated with $F_{ij}(D^2 v) \partial_{ij}$),

$$\|(y_2^2 + y_3^2)^{1/2} \partial_{111} v\|_{q_1, \mathcal{C}_{\sigma r}}, \|\partial_{11\beta} v\|_{q_1, \mathcal{C}_{\sigma r}} \leq C_{q, \sigma, r}.$$

Repeat the above arguments starting from (5.20) with q' replaced by q_1 (note $q_1 > q'$ if $p > 4$). After finite steps m (which only depends on the dimension) we will get $(y_2^2 + y_3^2)^{1/2} \partial_{111} v, \partial_{11\beta} v \in M^{1,q}(\mathcal{C}_{\sigma^m r})$ with q larger than the homogeneous dimension 4, and hence by embedding lemma 5.1(ii) are in $L^\infty(\mathcal{C}_{\sigma^m r})$. Applying Proposition 5.3 again we obtain $\|\partial_w^2 \partial_{11} v\|_{p, \mathcal{C}_{\sigma^m r}} < C_{p, \sigma, r, m}$ for $1 < p < \infty$. Noting that $r \in (0, r_0)$ is chosen arbitrary, we complete the proof for step 2.

Step 3. Show $\partial^\alpha v \in M^{2,p}(\mathcal{C}_r)$, $|\alpha| = 2$ with $\alpha_2 + \alpha_3 \geq 1$.

First from step 1, $\|\partial^\alpha v\|_{p, \mathcal{C}_r} < C_{p, r, r_0}$ for $|\alpha| = 3$ with $\alpha_2 + \alpha_3 \geq 2$. This together with the boundedness of $\|\partial_w^2 \partial_{11} v\|_{p, \mathcal{C}_r}$ obtained in step 2 gives

$$(5.22) \quad \|\partial^\alpha v\|_{p, \mathcal{C}_r} < C_{p, r, r_0} \quad \text{for all } \alpha \text{ with } |\alpha| = 3.$$

Next we estimate $\Delta_k^h \partial_\beta v$ with $k = 1, 2, 3$ and $\beta = 2, 3$. Similar as (5.17), $\Delta_k^h \partial_\beta v$ satisfies

$$F_{ij} \partial_{ij} (\Delta_k^h \partial_\beta v) = f$$

with

$$f = -\Delta_k^h (F_{ij} \partial_{ij} \partial_\beta v(\cdot + h e_k)).$$

By (5.22) we immediately have $\|f\|_{p, \mathcal{C}_{r-h}}$ is uniformly bounded in h for any $1 < p < \infty$. Applying the L^p estimate (Proposition 5.3) we have $\|\partial_w^2 (\Delta_k^h \partial_\beta v)\|_{p, \mathcal{C}_{r-h}}$ is uniformly bounded in h , and therefore completes the proof for step 3.

Step 4. Show $\partial^\alpha v \in M^{2,p}(\mathcal{C}_r)$ for $|\alpha| = k > 2$, with $\|\partial^\alpha v\|_{M^{2,p}(\mathcal{C}_r)}$ depending on $p, r, r_0, |\alpha|$ and dimension.

This is done in the same way as for $|\alpha| = 2$. More precisely, we first consider the equation of $\Delta_1^h \partial_1^{k-1} v$ in $\mathcal{C}_r$

$$F_{ij} (D^2 v) \Delta_1^h (\partial_1^{k-1} v) + l.o.t. = 0$$

with $\|l.o.t.\|_{q, \mathcal{C}_r}$ bounded uniformly in h for some $q \in (1, 2)$. Then we apply the Sobolev embedding lemma and the L^p estimate iteratively to obtain the boundedness of $\|l.o.t.\|_{\infty, \mathcal{C}_{r'}}$ with some $r' \in (0, r)$, as well as the boundedness of $\|\partial_w^2 \partial_1^k v\|_{p, \mathcal{C}_{r'}}$ for any $1 < p < \infty$. In particular, this combined with the fact that $\partial^\alpha v \in M^{2,p}(\mathcal{C}_r)$ for all $|\alpha| \leq k - 1$ gives that $\|\partial^\alpha v\|_{p, \mathcal{C}_{r'}}$ is bounded with $|\alpha| = k + 1$ for any $1 < p < \infty$. Next, we consider the equation for $\Delta_j^h \partial^\alpha v$, $j = 1, 2, 3$, $|\alpha| = k - 1$ and $\alpha_2 + \alpha_3 \geq 1$. Similar as in step 3, one easily get the uniform boundedness of $\|l.o.t.\|_{p, \mathcal{C}_{r'}}$ for any $1 < p < \infty$ due to the boundedness of $\|\partial^\alpha v\|_{p, \mathcal{C}_{r'}}$, $|\alpha| = k + 1$. Applying L^p estimate again we obtain the conclusion. $\square$

Corollary 5.5. *Let u be a solution to the Signorini problem in B_2^+ , and let Γ_u be the regular set of the free boundary. Then Γ_u is smooth.*

Proof. Take $x_0 \in \Gamma_u$ a regular free boundary point, in a coordinate chart centered at x_0 , by [ACS08] Γ_u can be locally expressed as the graph of a $C^{1,\alpha}$ function $y_2 = f(y_1)$ with $f(0) = f'(0) = 0$. Consider in a neighborhood of x_0 the partial hodograph-Legendre transform and the corresponding Legendre function v defined in section 4.1. By Theorem 5.4, v is smooth at the origin. Since

$$f(y_1) = -\partial_2 v(y_1, 0, 0), \quad f(0) = -\partial_2 v(0).$$

Hence the smoothness of v at the origin implies the smoothness of f at 0. $\square$

6. REAL ANALYTICITY

In this section, we show that the Legendre transform v is analytic in a neighborhood of the origin (Theorem 1.2).

Theorem 6.1. *Let v be the Legendre transform defined in Section 4. Then v is real analytic in a neighborhood of the origin.*

We first make some more assumptions and observations.

1. For simplicity we work on $\mathbb{R}^3$. By the scaling invariant property mentioned at the beginning of Section 5.3 we may assume v is smooth in $\mathcal{C}_2$ and solves the fully nonlinear equation (5.14) there. Denote

$$M_0 := \sup_{|\alpha| \leq 4} \|\partial_w^2 \partial^\alpha v\|_{\infty, \mathcal{C}_2}.$$

From Corollary 4.3(iii)(iv) and the intermediate value theorem, we have

$$(6.1) \quad \left\| \frac{\partial_{\beta\beta} v}{y_2} \right\|_{\infty, \mathcal{C}_2}, \left\| \frac{\partial_{1\beta\beta} v}{y_2} \right\|_{\infty, \mathcal{C}_2}, \left\| \frac{\partial_{23} v}{y_3} \right\|_{\infty, \mathcal{C}_2}, \left\| \frac{\partial_{123} v}{y_3} \right\|_{\infty, \mathcal{C}_2} \leq M_0, \quad \beta = 2, 3.$$

2. The following multi-index notation will be used:

For a multi-index $\alpha = (\alpha_j)_{j=1, \dots, n} \in \mathbb{Z}_+^n$,

$$\alpha! = \prod_{j=1}^n (\alpha_j)! \quad (0! = 1)$$

Let $\alpha = (\alpha_j)$, $\beta = (\beta_j)$ be two multi-index in $\mathbb{Z}_+^n$, we say $\alpha \leq \beta$ iff $\alpha_j \leq \beta_j$, $j = 1, \dots, n$; and $\alpha < \beta$ iff $\alpha \leq \beta$, $|\alpha| < |\beta|$, where $|\alpha| = \sum_1^n \alpha_j$.

3. The strategy to prove the analyticity is as follows: Given $\alpha \in \mathbb{Z}_+^3$, taking ∂^α on both sides of (5.14) and using the summation convention on i, j s, we obtain that $\partial^\alpha v$ satisfies

$$(6.2) \quad F_{ij}(D^2 v) \partial_{ij}(\partial^\alpha v) + \sum_{\substack{\alpha^1 + \alpha^2 + \alpha^3 = \alpha \\ \alpha^i < \alpha, i=1,2,3}} \frac{\alpha!}{\alpha^1! \alpha^2! \alpha^3!} \sum_{\sigma \in S_3} \text{sgn}(\sigma) (\partial^{\alpha^1} \partial_{1\sigma_1} v) (\partial^{\alpha^2} \partial_{2\sigma_2} v) (\partial^{\alpha^3} \partial_{3\sigma_3} v) = 0,$$

where S_3 is the set of all permutations of $\{1, 2, 3\}$. We will apply Proposition 5.3 for (6.2) to get a fine estimate of the L^p norm of $\partial^\alpha v$. In order to do so, usually one needs a sequence of domains with properly shrinking radius as well as the corresponding sequence of cut-off functions. In this paper, we use the trick introduced

in [Kat96] to avoid this technical trouble. In the following we take and fix a cut-off function $\eta \in C^\infty(\mathbb{C}_2)$ which satisfies

$$\begin{aligned} \eta(y) &= \eta^1(y_1)\eta^2(y_2, y_3), \quad 0 \leq \eta^i \leq 1, \\ \eta^1 &= \eta^2 = 1 \text{ if } |y_1| \leq 1, \sqrt{y_2^2 + y_3^2} \leq 1, \\ \eta^1 &= \eta^2 = 0 \text{ if } |y_1| > \frac{3}{2}, \sqrt{y_2^2 + y_3^2} > \frac{3}{2}, \\ |\partial \eta^i|, |\partial^2 \eta^i| &\leq C_\eta, \quad i = 1, 2. \end{aligned}$$

We will estimate $\|\eta^{|\alpha|-2} \partial_w^2 \partial^\alpha v\|_{p, \mathbb{C}_{3/2}}$ with $|\alpha| = k$ by $\|\eta^{|\alpha|-2} \partial_w^2 \partial^\alpha v\|_{p, \mathbb{C}_{3/2}}$ with $|\alpha| < k$.

4. From now on we will simply write $\|\cdot\|_p$ if the integral domain is $\mathbb{C}_{3/2}$. We will fix a p larger than the homogeneous dimension 4. By a universal constant we mean an absolute constant which only depends on C_η , M_0 , dimension (which is 3 in our setting) or p chosen, in particular independent of k .

5. The following observation will be useful in the proof:

$$\text{supp}(\partial_j \eta) \subset \{y : y_2^2 + y_3^2 > 1\} \quad \text{for } j = 2, 3,$$

which implies

$$|\partial_j \eta(y)| \leq (y_2^2 + y_3^2)^{1/2} C_\eta \quad \text{for } j = 2, 3.$$

Hence for multi-index α with $|\alpha| = k \geq 4$,

$$(6.3) \quad \left| \|\partial_w^1 (\eta^{|\alpha|-2} \partial^\alpha u)\|_p - \|\eta^{|\alpha|-2} \partial_w^1 \partial^\alpha u\|_p \right| \leq |\alpha| C_\eta \|\eta^{|\alpha|-3} (y_2^2 + y_3^2)^{1/2} \partial^\alpha v\|_p$$

$$(6.4) \quad \left| \|\partial_w^2 (\eta^{|\alpha|-2} \partial^\alpha u)\|_p - \|\eta^{|\alpha|-2} \partial_w^2 \partial^\alpha u\|_p \right| \leq 2|\alpha| C_\eta \|\eta^{|\alpha'|-2} \partial_w^2 \partial^{\alpha'} u\|_p + |\alpha|^2 C_\eta \|\eta^{|\alpha''|-2} \partial_w^2 \partial^{\alpha''} u\|_p$$

for some α' , α'' with $|\alpha'| = k-1$ and $|\alpha''| = k-2$.

The following proposition is the main proposition of this section.

Proposition 6.2. *There exist universal constants R , $0 < R < 1$ and $\bar{C} > 2$ such that for any $k \geq 4$*

- (i) $\|\eta^{k-2} \partial_w^2 \partial_1^k v\|_p \leq R^{-(k-4)} k^{k-4},$
- (ii) $\|\eta^{k-2} \partial_w^2 \partial^\alpha v\|_p \leq R^{-(k-3)} k^{k-3}, \quad \text{for all } \alpha \text{ with } |\alpha| = k \text{ and } \alpha_2 + \alpha_3 \geq 1.$

Theorem 6.1 will follow from Proposition 6.2. Indeed, by Proposition 6.2 there exists a universal $R > 0$ such that

$$\|\partial^\alpha v\|_{p, \mathbb{C}_1} \leq R^{-|\alpha|} |\alpha|^{|\alpha|}, \quad |\alpha| \geq 4.$$

Hence by the classical Sobolev embedding $W^{1,p} \hookrightarrow L^\infty$ for $p > 4$ chosen above, one has

$$(6.5) \quad \sup_{y \in \mathbb{C}_1} |\partial^\alpha v(y)| \leq C R^{-|\alpha|} |\alpha|^{|\alpha|}.$$

Hence v is in Gevrey class G^1 , which is the same as the class of real analytic functions.

Before proving Proposition 6.2, we first show a lemma on the L^∞ -norm of $\partial^\alpha v$, which roughly speaking is a consequence of the Sobolev embedding lemma (Lemma 5.1(ii)).

Lemma 6.3. *For $k \geq 5$, assume Proposition 6.2 holds for $k-1$ and k . Then there is a universal constant $C_2 > M_0 > 0$ such that*

- (i) $\|\eta^{k-2}\partial_1^k v\|_\infty, \|\eta^{k-2}\partial_1^k \partial_\beta v\|_\infty \leq C_2 R^{-(k-4)} k^{k-4}, \quad \beta = 2, 3;$
- (ii) $\|\eta^{k-2}\partial^\alpha v\|_\infty \leq C_2 R^{-(k-3)} k^{k-3}, \quad |\alpha| = k+1, \alpha_2 + \alpha_3 \geq 2;$
- (iii) $\|\eta^{k-2}\partial_1^{k-2}\partial_{\beta\beta} v/y_2\|_\infty, \|\eta^{k-2}\partial_1^{k-2}\partial_{\beta\gamma} v/y_3\|_\infty \leq C_2 R^{-(k-3)} k^{k-3},$
for $\beta, \gamma = 2, 3$ and $\beta \neq \gamma$.

Proof. (i) First by Lemma 6.3(ii), there is a universal constant C such that

$$\|\eta^{k-2}\partial_1^k \partial_\beta v\|_\infty \leq C \|\partial_w^1 (\eta^{k-2}\partial_1^k \partial_\beta v)\|_p, \quad \beta = 2, 3.$$

Applying (6.3) to the RHS of above inequality, we have

$$\begin{aligned} \|\eta^{k-2}\partial_1^k \partial_\beta v\|_\infty &\leq C (\|\eta^{k-2}\partial_w^1 \partial_1^k \partial_\beta v\|_p + C_\eta k \|\eta^{k-3}\partial_w^2 \partial_1^{k-1} v\|_p) \\ &\leq C (\|\eta^{k-2}\partial_w^2 \partial_1^k v\|_p + C_\eta k \|\eta^{k-3}\partial_w^2 \partial_1^{k-1} v\|_p) \end{aligned}$$

By Proposition 6.2(i) for $k-1$ and k , there exists a universal constant C_2 , which can be chosen larger than M_0 such that

$$(6.6) \quad \|\eta^{k-2}\partial_1^k \partial_\beta v\|_\infty \leq C_2 R^{-(k-4)} k^{k-4}.$$

The estimate for $\|\eta^{k-2}\partial_1^k v\|_\infty$ follows from the classical Sobolev embedding, (6.3), (6.6) and the assumption.

(ii) Similar to (i).

(iii) Use Corollary 4.3 and (ii) above. $\square$

Proof of Proposition 6.2. This is done by induction. Assume (i) and (ii) hold for $4, \dots, k-1$. We want to show they hold for k .

Let α be a multi-index with $|\alpha| = k$. From (6.2), $\eta^{k-2}\partial^\alpha v$ satisfies the following equation

$$F_{ij}\partial_{ij}(\eta^{k-2}\partial^\alpha v) = I + II,$$

where

$$I = - \sum_{\substack{\alpha^1 + \alpha^2 + \alpha^3 = \alpha \\ 0 \leq \alpha^i < \alpha, i=1,2,3}} \frac{\alpha!}{\alpha^1! \alpha^2! \alpha^3!} \sum_{\sigma \in S_3} \text{sgn}(\sigma) (\partial^{\alpha^1} \partial_{1\sigma_1} v) (\partial^{\alpha^2} \partial_{2\sigma_2} v) (\partial^{\alpha^3} \partial_{3\sigma_3} v) \eta^{k-2}$$

$$II = 2F_{ij}\partial_i(\eta^{k-2})\partial_j\partial^\alpha v + F_{ij}\partial_{ij}(\eta^{k-2})\partial^\alpha v.$$

By the L^p estimate for the perturbed Baouendi-Grushin operator (Proposition 5.3), there is a universal $C_3 > 0$ such that

$$(6.7) \quad \|\partial_w^2(\eta^{k-2}\partial^\alpha v)\|_p \leq C_3 (\|I\|_p + \|II\|_p + \|\eta^{k-2}\partial^\alpha v\|_p).$$

The estimate of $\|II\|_p$ is standard. In fact,

$$\left\| \sum_{i,j} F_{ij}\partial_i(\eta^{k-2})\partial_j\partial^\alpha v \right\|_p \leq 2M_0^2(k-2)C_\eta \|\eta^{k-3}\partial_w^2(\partial^{\tilde{\alpha}} v)\|_p,$$

for some $\tilde{\alpha} < \alpha$ with $|\tilde{\alpha}| = k-1$. By the induction assumption (i)(ii) for $k-1$, the above RHS is bounded by

$$2M_0^2(k-2)C_\eta \begin{cases} R^{-(k-5)}(k-1)^{k-5} & \text{if } \alpha' = (k-1, 0, 0) \\ R^{-(k-4)}(k-1)^{k-4} & \text{otherwise.} \end{cases}$$

Similarly there is some $\tilde{\alpha} < \alpha$ with $|\tilde{\alpha}| = k - 2$ such that

$$\begin{aligned} \left\| \sum_{i,j} F_{ij} \partial_{ij} (\eta^{k-2}) \partial^\alpha v \right\|_p &\leq 2M^2(k-2)(k-3)C_\eta \|\eta^{k-4} \partial_w^2 (\partial^{\tilde{\alpha}} v)\|_p \\ &\leq 2M_0^2(k-2)(k-3)C_\eta \begin{cases} R^{-(k-6)}(k-2)^{k-6} & \text{if } \alpha'' = (k-2, 0, 0) \\ R^{-(k-5)}(k-2)^{k-5} & \text{otherwise.} \end{cases} \end{aligned}$$

Hence

$$\begin{aligned} \|II\|_p &\leq \|2 \sum_{i,j} F_{ij} \partial_i (\eta^{k-2}) \partial_j \partial^\alpha v\|_p + \left\| \sum_{i,j} F_{ij} \partial_{ij} (\eta^{k-2}) \partial^\alpha v \right\|_p \\ (6.8) \quad &\leq 6M_0^2 C_\eta \begin{cases} R^{-(k-5)} k^{k-4} & \text{if } \alpha = (k, 0, 0) \\ R^{-(k-4)} k^{k-3} & \text{otherwise.} \end{cases} \end{aligned}$$

Next we estimate $\|I\|_p$.

Proof of (i). Let $\ell_1 = |\alpha_1|$, $\ell_2 = |\alpha_2|$, $\ell_3 = |\alpha_3|$. Then

$$\|I\|_p \leq \sum_{\substack{\ell_1 + \ell_2 + \ell_3 = k \\ 0 \leq \ell_i < k}} \frac{k!}{\ell_1! \ell_2! \ell_3!} \sum_{\sigma \in S_3} \left\| (\partial_1^{\ell_1} \partial_{1\sigma_1} v) (\partial_1^{\ell_2} \partial_{2\sigma_2} v) (\partial_1^{\ell_3} \partial_{3\sigma_3} v) \eta^{k-2} \right\|_p.$$

We discuss the following two cases:

Case 1. $\ell_1 \geq \max\{\ell_2, \ell_3\}$.

If $\sigma_1 = 1$, then

$$\begin{aligned} &\left\| (\partial_1^{\ell_1} \partial_{1\sigma_1} v) (\partial_1^{\ell_2} \partial_{2\sigma_2} v) (\partial_1^{\ell_3} \partial_{3\sigma_3} v) \eta^{k-2} \right\|_p \\ &\leq \left\| \eta^{\ell_1-2} (y_2^2 + y_3^2) \partial_1^{\ell_1} \partial_{11} v \right\|_p \left\| \eta^{\ell_2} \frac{\partial_1^{\ell_2} \partial_{2\sigma_2} v}{(y_2^2 + y_3^2)^{1/2}} \right\|_\infty \left\| \eta^{\ell_3} \frac{\partial_1^{\ell_3} \partial_{3\sigma_3} v}{(y_2^2 + y_3^2)^{1/2}} \right\|_\infty \\ &\leq \left\| \eta^{\ell_1-2} \partial_w^2 \partial_1^{\ell_1} v \right\|_p \left\| \eta^{\ell_2} \frac{\partial_1^{\ell_2} \partial_{2\sigma_2} v}{(y_2^2 + y_3^2)^{1/2}} \right\|_\infty \left\| \eta^{\ell_3} \frac{\partial_1^{\ell_3} \partial_{3\sigma_3} v}{(y_2^2 + y_3^2)^{1/2}} \right\|_\infty, \end{aligned}$$

which is by Lemma 6.3(iii) and the induction assumption (i) for $k-1$ bounded by

$$R^{-(\ell_1-4)} \ell_1^{\ell_1-4} C_2 R^{-(\ell_2-1)} \ell_2^{\ell_2-1} C_2 R^{-(\ell_3-1)} \ell_3^{\ell_3-1} = (C_2)^2 R^{-(k-6)} \ell_1^{\ell_1-4} \ell_2^{\ell_2-1} \ell_3^{\ell_3-1}.$$

If $\sigma_1 \neq 1$, then

$$\begin{aligned} &\left\| (\partial_1^{\ell_1} \partial_{1\sigma_1} v) (\partial_1^{\ell_2} \partial_{2\sigma_2} v) (\partial_1^{\ell_3} \partial_{3\sigma_3} v) \eta^{k-2} \right\|_p \\ &\leq \left\| \eta^{\ell_1-2} (y_2^2 + y_3^2)^{1/2} \partial_1^{\ell_1} \partial_{1\sigma_1} v \right\|_p \left\| \eta^{\ell_2+\ell_3} \frac{(\partial_1^{\ell_2} \partial_{2\sigma_2} v) (\partial_1^{\ell_3} \partial_{3\sigma_3} v)}{(y_2^2 + y_3^2)^{1/2}} \right\|_\infty, \end{aligned}$$

which is by Lemma 6.3(iii)(i) and the induction assumption (i) for $k-1$ bounded by

$$(C_2)^2 R^{-(k-7)} \ell_1^{\ell_1-4} \ell_2^{\ell_2-1} \ell_3^{\ell_3-1}.$$

Hence

$$\begin{aligned} & \sum_{\substack{\ell_1+\ell_2+\ell_3=k \\ 0 \leq \ell_i < k \\ \ell_1 \geq \max\{\ell_2, \ell_3\}}} \frac{k!}{\ell_1! \ell_2! \ell_3!} \sum_{\sigma \in S_3} \left\| (\partial_1^{\ell_1} \partial_{1\sigma_1} v) (\partial_1^{\ell_2} \partial_{2\sigma_2} v) (\partial_1^{\ell_3} \partial_{3\sigma_3} v) \eta^{k-2} \right\|_p \\ & \leq \sum_{\substack{\ell_1+\ell_2+\ell_3=k \\ 0 \leq \ell_i < k \\ \ell_1 \geq \max\{\ell_2, \ell_3\}}} \frac{k!}{\ell_1! \ell_2! \ell_3!} 6(C_2)^2 R^{-(k-6)} \ell_1^{\ell_1-4} \ell_2^{\ell_2-1} \ell_3^{\ell_3-1}. \end{aligned}$$

By Stirling's formula and the fact that $\ell_3 \geq \max\{\ell_1, \ell_2\} \implies \ell_3 \geq k/3$, there is a universal constant $C_4 > 0$ such that

$$\sum_{\substack{\ell_1+\ell_2+\ell_3=k \\ 0 \leq \ell_i < k \\ \ell_1 \geq \max\{\ell_2, \ell_3\}}} \frac{k!}{\ell_1! \ell_2! \ell_3!} \ell_1^{\ell_1-4} \ell_2^{\ell_2-1} \ell_3^{\ell_3-1} \leq C_4 k^{k-4}.$$

Hence

$$\begin{aligned} & \sum_{\substack{\ell_1+\ell_2+\ell_3=k \\ 0 \leq \ell_i < k \\ \ell_1 \geq \max\{\ell_2, \ell_3\}}} \frac{k!}{\ell_1! \ell_2! \ell_3!} \sum_{\sigma \in S_3} \left\| (\partial_1^{\ell_1} \partial_{1\sigma_1} v) (\partial_1^{\ell_2} \partial_{2\sigma_2} v) (\partial_1^{\ell_3} \partial_{3\sigma_3} v) \eta^{k-2} \right\|_p \\ & \leq 6(C_2)^2 C_4 R^{-(k-6)} k^{k-4}. \end{aligned}$$

Case 2. $\ell_2 \geq \max\{\ell_1, \ell_3\}$ or $\ell_3 \geq \max\{\ell_1, \ell_2\}$.

We only discuss when $\ell_2 \geq \max\{\ell_1, \ell_3\}$. Similarly we consider $\sigma_2 = 1$ and $\sigma_2 \neq 1$. If $\sigma_2 = 1$, the estimate is indeed included in Case 1. If $\sigma_2 \neq 1$, then we simply have

$$\begin{aligned} & \left\| (\partial_1^{\ell_1} \partial_{1\sigma_1} v) (\partial_1^{\ell_2} \partial_{2\sigma_2} v) (\partial_1^{\ell_3} \partial_{3\sigma_3} v) \eta^{k-2} \right\|_p \\ & \leq \left\| \eta^{\ell_1} \partial_1^{\ell_1} \partial_{1\sigma_1} v \right\|_{\infty} \left\| \eta^{\ell_2} \partial_1^{\ell_2} \partial_{2\sigma_2} v \right\|_p \left\| \eta^{\ell_3-1} \partial_1^{\ell_3} \partial_{3\sigma_3} v \right\|_{\infty} \\ & \leq (C_2)^2 R^{-(k-8)} \ell_1^{\ell_1-2} \ell_2^{\ell_2-4} \ell_3^{\ell_3-2}. \end{aligned}$$

Arguing similarly as in Case 1 we have

$$\begin{aligned} & \sum_{\substack{\ell_1+\ell_2+\ell_3=k \\ 0 \leq \ell_i < k \\ \ell_1 \leq \max\{\ell_2, \ell_3\}}} \frac{k!}{\ell_1! \ell_2! \ell_3!} \sum_{\sigma \in S_3} \left\| (\partial_1^{\ell_1} \partial_{1\sigma_1} v) (\partial_1^{\ell_2} \partial_{2\sigma_2} v) (\partial_1^{\ell_3} \partial_{3\sigma_3} v) \eta^{k-2} \right\|_p \\ & \leq 12(C_2)^2 C_4 R^{-(k-6)} k^{k-4}. \end{aligned}$$

Combining the above two cases we have,

$$(6.9) \quad \|I\|_p \leq 18(C_2)^2 C_4 R^{-(k-6)} k^{k-4}.$$

We combine (6.7), (6.8) for $\alpha = (k, 0, 0)$ and (6.9), and use the induction assumption (i) for $\alpha = (k-1, 0, 0)$ to estimate $\|\eta^{k-2}\partial^\alpha v\|_p$. Then

$$\begin{aligned}
 (6.10) \quad & \|\partial_w^2(\eta^{k-2}\partial^\alpha v)\|_p \\
 & \leq C_3 \left(18(C_2)^2 C_4 R^{-(k-6)} k^{k-4} + 6M_0^2 C_\eta R^{-(k-5)} k^{k-4} + (k-1)^{k-5} R^{-(k-5)} \right) \\
 & \leq C_5 R^{-(k-5)} k^{k-4},
 \end{aligned}$$

where $C_5 = C_3(18(C_2)^2 C_4 + 6M_0^2 C_\eta + 1)$ is a universal constant. By (6.4) for $\alpha = (k, 0, 0)$,

$$\begin{aligned}
 & \|\eta^{k-2}\partial_w^2\partial_1^k v\|_p \\
 & \leq \|\partial_w^2(\eta^{k-2}\partial_1^k v)\|_p + 2C_\eta k \|\eta^{k-3}\partial_w^2\partial_1^{k-1} v\|_p + C_\eta k^2 \|\eta^{k-4}\partial_w^2\partial_1^{k-2} v\|_p.
 \end{aligned}$$

Thus combining (6.10) with induction assumption (i) for $k-1$ and $k-2$, we have

$$\begin{aligned}
 (6.11) \quad & \|\eta^{k-2}\partial_w^2\partial_1^k v\|_p \\
 & \leq C_5 R^{-(k-6)} k^{k-4} + 2C_\eta R^{-(k-5)} k(k-1)^{k-5} + C_\eta R^{-(k-6)} k^2(k-2)^{k-6} \\
 & \leq C_6 R^{-(k-5)} k^{k-4},
 \end{aligned}$$

where $C_6 = C_5 + 3C_\eta$. Choosing $R = C_6^{-1}$ we proved (i).

Proof of (ii). The key step is to estimate $\|I\|_p$, which is done similarly as for (i).

More precisely, Lemma 6.3(i) and (6.11) (with $R = C_6^{-1}$) imply

$$\|\eta^{k-2}\partial_1^k \partial_\ell v\|_p \leq C_2 R^{-(k-4)} k^{k-4}, \quad \ell = 1, 2, 3.$$

This together with the induction assumption (ii) up to $k-1$ yields, for any multi-index α , $3 \leq |\alpha| \leq k+1$,

$$(6.12) \quad \|\eta^{|\alpha|-3}\partial^\alpha v\|_p \leq C_2 R^{-(|\alpha|-5)} (|\alpha|-1)^{|\alpha|-5}.$$

In the estimate of $\|I\|_p$, we first consider when $|\alpha^1| \geq \max\{|\alpha^2|, |\alpha^3|\}$: by Lemma 6.3(ii) and (6.12),

$$\begin{aligned}
 & \left\| (\partial^{\alpha^1} \partial_{1\sigma_1} v) (\partial^{\alpha^2} \partial_{2\sigma_2} v) (\partial^{\alpha^3} \partial_{3\sigma_3} v) \eta^{k-2} \right\|_p \\
 & \leq \left\| \eta^{|\alpha^1|-1} \partial^{\alpha^1} \partial_{1\sigma_1} v \right\|_p \left\| \eta^{|\alpha^2|-1} \partial^{\alpha^2} \partial_{2\sigma_2} v \right\|_\infty \left\| \eta^{|\alpha^3|-1} \partial^{\alpha^3} \partial_{3\sigma_3} v \right\|_\infty \\
 & \leq (C_2)^3 R^{-(k-7)} (|\alpha^1|+1)^{|\alpha^1|-3} (|\alpha^2|)^{|\alpha^2|-2} (|\alpha^3|)^{|\alpha^3|-2}.
 \end{aligned}$$

Hence

$$\begin{aligned}
 (6.13) \quad & \sum_{\substack{\alpha^1+\alpha^2+\alpha^3=\alpha \\ 0 \leq |\alpha^i| < k \\ |\alpha^1| \geq \max\{|\alpha^2|, |\alpha^3|\}}} \frac{\alpha!}{\alpha^1! \alpha^2! \alpha^3!} \sum_{\sigma \in S_3} \left\| (\partial^{\alpha^1} \partial_{1\sigma_1} v) (\partial^{\alpha^2} \partial_{2\sigma_2} v) (\partial^{\alpha^3} \partial_{3\sigma_3} v) \eta^{k-2} \right\|_p \\
 & \leq 6(C_2)^3 R^{-(k-7)} \sum_{\substack{\alpha^1+\alpha^2+\alpha^3=\alpha \\ 0 \leq |\alpha^i| < k \\ |\alpha^1| \geq \max\{|\alpha^2|, |\alpha^3|\}}} \frac{\alpha!}{\alpha^1! \alpha^2! \alpha^3!} (|\alpha^1|+1)^{|\alpha^1|-3} (|\alpha^2|)^{|\alpha^2|-2} (|\alpha^3|)^{|\alpha^3|-2}
 \end{aligned}$$

Since for α with $|\alpha| = k$ fixed, we have the following identity (e.g. Proposition 2.1 in [Kat96]):

$$\sum_{\substack{\alpha^1 + \alpha^2 + \alpha^3 = \alpha \\ 0 \leq \alpha^i \leq \alpha}} \frac{\alpha!}{\alpha^1! \alpha^2! \alpha^3!} = \sum_{\substack{\ell_1 + \ell_2 + \ell_3 = k \\ 0 \leq \ell_i \leq k}} \frac{k!}{\ell_1! \ell_2! \ell_3!},$$

then if we let $|\alpha^1| = \ell_1$, $|\alpha^2| = \ell_2$, $|\alpha^3| = \ell_3$, the RHS of (6.13) is bounded by

$$6(C_2)^3 R^{-(k-7)} \sum_{\substack{\ell_1 + \ell_2 + \ell_3 = k \\ 0 \leq \ell_i \leq k \\ \ell_1 \geq \max\{\ell_2, \ell_3\}}} \frac{k!}{\ell_1! \ell_2! \ell_3!} (\ell_1 + 1)^{\ell_1 - 3} \ell_2^{\ell_2 - 2} \ell_3^{\ell_3 - 2}.$$

By Stirling's formula and if we still use C_4 to denote the universal constant from it, then the above quantity is bounded by

$$6(C_2)^3 C_4 R^{-(k-7)} k^{k-3}.$$

The arguments for the case $|\alpha^1| \leq \max\{|\alpha^2|, |\alpha^3|\}$ are exactly the same. Hence

$$\|I\|_p \leq 18(C_2)^3 C_4 R^{-(k-7)} k^{k-3}.$$

The rest of the proof for (ii) is the same as for (i) and we do not repeat here. $\square$

7. APPENDIX

In the Appendix, we give a short proof for Lemma 5.1 and Theorem 5.2.

Proof of Lemma 5.1. We first prove (i). Suppose that $u \in C_0^1(\Omega)$ and extend u to be zero outside Ω . For every $\sigma_1 \in \mathbb{R}^m$, $\sigma_2 \in \mathbb{R}^n$ with $|\sigma_1|, |\sigma_2| \leq 1$,

$$(7.1) \quad u(x, t) = - \int_0^\infty \partial_s u(x + \sigma_1 s, t + \sigma_2 \gamma(s)) ds,$$

where $\gamma : [0, \infty) \rightarrow \mathbb{R}$ is a C^1 function satisfying $\gamma(0) = 0$, $\dot{\gamma}(s) = |x + \sigma_1 s|$. Let

$$f(x, t) := |\nabla_x u(x, t)| + |x| |\nabla_t u(x, t)|,$$

then by a direct computation

$$|\partial_s u(x + \sigma_1 s, t + \sigma_2 \gamma(s))| \leq f(x + \sigma_1 s, t + \sigma_2 \gamma(s)).$$

Hence

$$|u(x, t)| \leq c_n \int_0^\infty \int_{|\sigma_2| \leq 1} f(x + \sigma_1 s, t + \sigma_2 \gamma(s)) d\sigma_2 ds,$$

which by a change of variable $\eta_2 = \sigma_2 \gamma(s)$ gives

$$(7.2) \quad |u(x, t)| \leq c_n \int_0^\infty \frac{1}{|\gamma(s)|^n} \int_{|\eta_2| \leq \gamma(s)} f(x + \sigma_1 s, t + \eta_2) d\eta_2 ds.$$

By Young's inequality for convolution and Fubini, for $1 \leq p < q < \infty$,

$$\|u(x, \cdot)\|_q \leq c_n \int_0^\infty \frac{1}{|\gamma(s)|^{n(1/p-1/q)}} \|f(x + \sigma_1 s, \cdot)\|_p ds$$

Observe that for $x \neq 0$,

$$\begin{aligned}\gamma(s) &= \int_0^s \dot{\gamma}(\tau) d\tau = \int_0^s |x + \sigma_1 \tau| d\tau \\ &\geq \int_0^s \left| x + \frac{-x}{|x|} \tau \right| d\tau \\ &= \int_0^s ||x| - \tau| d\tau \geq \frac{s^2}{4},\end{aligned}$$

for any $\sigma_1 \in \mathbb{R}^m$ with $|\sigma_1| = 1$, and for $x = 0$, $\gamma(s) = s^2/2$. Hence if we let

$$v(x) := \|u(x, \cdot)\|_q, \quad g(x) := \|f(x, \cdot)\|_p$$

after integrating over $\sigma_1 \in S^{m-1}$, by Fubini and a change of variable we have

$$\begin{aligned}(7.3) \quad v(x) &\leq c_n \int_0^\infty \frac{1}{s^{2n(1/p-1/q)}} g(x + \sigma_1 s) ds \\ &= c_n c_m \int_0^\infty \int_{|\sigma_1|=1} \frac{1}{s^{2n(1/p-1/q)}} g(x + \sigma_1 s) d\sigma ds \\ &= c_{n,m} \int_{\mathbb{R}^m} \frac{1}{|\eta_1|^{2n(1/p-1/q)+(m-1)}} g(x + \eta_1) d\eta_1\end{aligned}$$

By the Hardy-Littlewood-Sobolev inequality, for the chosen q satisfying

$$\frac{1}{q} + \frac{1}{m+2n} = \frac{1}{p}$$

we have

$$\|v\|_q \leq C_{n,m,p} \|g\|_p.$$

Next we prove (ii). Since this property is local in nature, given $(x, t) \in \Omega$ we may assume by multiplying a characteristic function that $u = 0$ in $\Omega \setminus B_1(x, t)$. Hence the integration in (7.1) can be written from 0 to M for some $M > 0$ large enough.

Applying Hölder's inequality to (7.2) we have

$$|u(x, t)| \leq c_n \int_0^M \frac{1}{|\gamma(s)|^{n/p}} \|f(x + \sigma_1 s, \cdot)\|_p ds.$$

Then arguing similarly as in (7.3) we have

$$|u(x, t)| \leq c_{n,m} \int_{\{\eta_1 \in \mathbb{R}^m : |\eta_1| \leq M\}} \frac{1}{|\eta_1|^{2n/p+(m-1)}} g(x + \eta_1) d\eta_1.$$

Then (ii) follows from Hölder's inequality because $p > m + 2n$. $\square$

Next we give a short proof for the L^p estimate for Baouendi-Grushin operator $\mathcal{L}_0$. The idea is to treat $\mathcal{L}_0$ as the projected operator of sub-Laplacian on the Heisenberg-Reiter type group onto certain quotient space. The transference method in [CW76] links the L^p estimate on the group to the L^p estimate for $\mathcal{L}_0$ in the quotient space.

Proof of Theorem 5.2. We give a proof for $m = 2$. The proof for general m is the same. We extend u to all $\mathbb{R}^{m+n}$ by setting $u = 0$ in $\mathbb{R}^{m+n} \setminus \Omega$.

Consider $\mathbb{R}^2 \times \mathbb{R}^{n \times 2} \times \mathbb{R}^n$ equipped with the group law

$$(x, y, t) \circ (\xi, \eta, \tau) = \left(x + \xi, y + \eta, t + \tau + \frac{1}{2}(y\xi - \eta x) \right)$$

where $\mathbb{R}^{n \times 2}$ is the space of $n \times 2$ real matrices, and $y\xi, \eta x$ are understood as the matrix multiplication. Let $G := (\mathbb{R}^2 \times \mathbb{R}^{n \times 2} \times \mathbb{R}^n, \circ)$. G is an example of Heisenberg-Reiter group, which is by [Rei74] a nilpotent Lie group of step 2, with dilation $\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t)$ and homogeneous dimension $Q = 2 + 2n + 2n = 2 + 4n$. It is also immediate that the Lebesgue measure $dx dy dt$ is a left and right Haar measure on G .

Now we recall several facts about the nilpotent Lie groups with dilations. A direct computation shows that the horizontal vector fields in the Lie algebra $\mathfrak{g}$ that agree at the origin with ∂_{x_i} and $\partial_{y_{i,j}}$ are

$$\begin{aligned} X_j &= \partial_{x_j} + \frac{1}{2} \sum_{s=1}^n y_{sj} \partial_{t_s}, \quad j = 1, 2 \\ Y_{i,j} &= \partial_{y_{i,j}} - \frac{1}{2} x_j \partial_{t_i}, \quad i = 1, \dots, n; j = 1, 2. \end{aligned}$$

Consider the sub-Laplacian in G

$$\Delta_{\mathcal{H}} := \sum_{j=1,2} X_j^2 + \sum_{\substack{i=1,\dots,n; \\ j=1,2}} Y_{i,j}^2.$$

It is easy to check that $X_i, Y_{i,j}$ are Hörmander vector fields, then $\Delta_{\mathcal{H}}$ is hypoelliptic. By Theorem 2.1 in [Fol75], there exists a unique fundamental solution of type 2 (i.e. smooth away from 0 and homogeneous of degree $2 - Q$) for $\Delta_{\mathcal{H}}$, which we denote by Ψ . For $u \in C_0^\infty(G)$ we have $u = (\Delta_{\mathcal{H}} u) * \Psi$. Since X_j and $Y_{i,j}$ are left-invariant, then

$$(7.4) \quad P(X, Y)u = (\Delta_{\mathcal{H}} u) * P(X, Y)\Psi,$$

where P is a quadratic polynomial in X_j and $Y_{i,j}$. By L_p estimates for the singular integral in homogeneous groups (see for example XIII 5.3 in [Ste93]), we have

$$(7.5) \quad \|w * P(X, Y)\Psi\|_p \leq C_p \|w\|_p, \quad \forall w \in L^p(G),$$

and in particular, using (7.4) we have

$$(7.6) \quad \|P(X, Y)u\|_p \leq C_p \|\Delta_{\mathcal{H}} u\|_p.$$

Now let $H := \{(0, y, 0) : y \in \mathbb{R}^{n \times 2}\}$. One can easily verify that H is a closed subgroup of G with a bi-invariant measure the Lebesgue measure $dh = dy = dy_{1,1} \dots dy_{n,2}$. Let $H \backslash G = \{Hg : g \in G\}$ be the quotient space and $\pi : g \mapsto Hg$ the natural quotient mapping. Given $f \in C_0^\infty(G)$, define

$$\bar{f}(Hg) = \int_H f(h \circ g) dh$$

By Theorem 15.21 in [HR79], the above correspondence $f \mapsto \bar{f}$ defines a linear mapping of $C_0^\infty(G)$ onto $C_0^\infty(H \backslash G)$.

Identify $H \backslash G$ with $\mathbb{R}^2 \times \{0\} \times \mathbb{R}^n$. A direct computation gives

$$(7.7) \quad \pi \left((x, y, t + \frac{1}{2} yx) \right) = \pi((0, y, 0) \circ (x, 0, t)) = (x, 0, t).$$

Consider the following vector fields, which are the ‘push-down’ vector fields of $X_j, Y_{i,j}$ on $H \backslash G \cong \mathbb{R}^2 \times \{0\} \times \mathbb{R}^n$

$$\begin{aligned} X_1^b &= \partial_{x_1}, \quad X_2^b = \partial_{x_2}, \\ Y_{i,j}^b &= -x_j \partial_{t_i}, \quad i = 1, \dots, n; j = 1, 2. \end{aligned}$$

Note that the Baouendi-Grushin operator can be written as

$$\mathcal{L}_0 = \sum_{j=1,2} (X_j^b)^2 + \sum_{\substack{i=1,\dots,n; \\ j=1,2}} (Y_{i,j}^b)^2.$$

To see that they are indeed the push-down vector fields on $H \setminus G$, we notice that for $u \in C_0^\infty(G)$,

$$\begin{aligned} \int_{\mathbb{R}^{2n}} X_1 u((0, y, 0) \circ (x, 0, t)) dy &= \int_{\mathbb{R}^{2n}} \frac{d}{d\tau} u((0, y, 0) \circ (x, 0, t) \circ (\tau, 0, 0))|_{\tau=0} dy \\ &= \frac{d}{d\tau} \int_{\mathbb{R}^{2n}} u((0, y, 0) \circ (x, 0, t) \circ (\tau, 0, 0)) dy |_{\tau=0} \\ &= \frac{d}{d\tau} \bar{u}(x + \tau, 0, t)|_{\tau=0} \\ &= X_1^b \bar{u}((x, 0, t)). \end{aligned}$$

Similarly, using (7.7) one can check that

$$\begin{aligned} X_j^b \bar{u}(Hg) &= \overline{X_j u}(Hg), \\ Y_{i,j}^b \bar{u}(Hg) &= \overline{Y_{i,j} u}(Hg), \quad i = 1 \dots, n; j = 1, 2, \end{aligned}$$

hence

$$(7.8) \quad \mathcal{L}_0 \bar{u}(Hg) = \overline{\Delta_{\mathcal{H}} u}(Hg),$$

$$(7.9) \quad P(X^b, Y^b) \bar{u}(Hg) = \overline{P(X, Y) u}(Hg).$$

Since the Lebesgue measure $dg = dx dy dt$ and $dh = dy$ are bi-invariant on G and H correspondingly, then by Theorem 15.24 in [HR79], there exists a unique right-invariant measure (up to a constant) on $H \setminus G \cong \mathbb{R}^2 \times \{0\} \times \mathbb{R}^n$, which is necessarily the Lebesgue measure $dx dt$.

Let R be the representation of G acting on $L^p(H \setminus G)$ given by the right translation, i.e. given $\bar{f} \in L^p(H \setminus G)$

$$(7.10) \quad R_g(\bar{f})(Hg') = \bar{f}(Hg'g).$$

It is easy to check that $\{R_g\}$ are bounded from $L^p(H \setminus G)$ to $L^p(H \setminus G)$ by 1. Given $\bar{u} \in C_0^\infty(H \setminus G)$, $1 < p < \infty$, we consider the following linear map T on $L^p(H \setminus G)$:

$$(7.11) \quad T\bar{w} := \int_G (R_{g^{-1}} \bar{w})(P(X, Y)\Psi(g)) dg, \quad \bar{w} \in L^p(H \setminus G).$$

Notice that G is locally compact and by (7.5) the convolution operator $w \mapsto w * P(X, Y)\Psi$ is bounded in $L^p(G)$. Thus the transference method (Theorem 2.4 in [CW76]) applies and we have for $\bar{w} \in C_0^\infty(H \setminus G)$,

$$(7.12) \quad \|T\bar{w}\|_p \leq C_p \|\bar{w}\|_p, \quad 1 < p < \infty,$$

in particular if we take $\bar{w} = \mathcal{L}_0 \bar{u}$, then

$$(7.13) \quad \|T\mathcal{L}_0 \bar{u}\|_p \leq C_p \|\mathcal{L}_0 \bar{u}\|_p.$$

In the end we show that $T\mathcal{L}_0 \bar{u} = P(X^b, Y^b) \bar{u}$. Indeed, by (7.10) and (7.11)

$$T\mathcal{L}_0 \bar{u}(Hg_0) = \int_G (\mathcal{L}_0 \bar{u})(Hg_0 g^{-1})(P(X, Y)\Psi(g)) dg$$

By (7.8) and Fubini, then using (7.4) and (7.9) we have

$$\begin{aligned}
T\bar{u}(Hg_0) &= \int_G \left(\int_H \Delta_{\mathcal{H}} u(h \circ g_0 \circ g^{-1}) dh \right) P(X, Y) \Psi(g) dg \\
&= \int_H \left(\int_G \Delta_{\mathcal{H}} u(h \circ g_0 \circ g^{-1}) P(X, Y) \Psi(g) dg \right) dh \\
&= \int_H (\Delta_{\mathcal{H}} u * P(X, Y) \Psi)(h \circ g_0) dh \\
&= \int_H P(X, Y) u(h \circ g_0) dh \\
&= P(X^b, Y^b) \bar{u}(Hg_0).
\end{aligned}$$

Observing that $\partial_{x_i x_j}^2$, $|x|^2 \partial_{t_i t_j}^2$, $x_k \partial_{x_i t_j}^2$, and ∂_{t_j} can be written as $P(X^b, Y^b)$ for some quadratic polynomial P , we complete the proof of the theorem. $\square$

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