Modelling a Complete Stirling Engine

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Abstract

The assumptions of second order Stirling engine models were reviewed. An ideal adiabatic plus simple heat exchanger model was developed. The model included the external components such as the fan, combustor, and preheater. The external heat transfer to the engine heater was modelled using a log-mean-temperature difference for a constant tube surface temperature. The performance of the model of the external components compared reasonably well to experimental data. The performance of the complete engine model was also compared to experimental data of the GPU-3. By adjusting the flow dissipation to better account for unsteady flow conditions and compressibility effects, the complete engine model was able to predict engine power and brake specific fuel consumption to within $\pm 14\%$ over a wide range of engine speeds and mean pressures. This analysis and others suggest that second order models of Stirling engines need to account for the gradient of the divergence of velocity term in the compressible momentum equation if the mean engine pressure is low enough (less than 3.0 MPa) and the engine speed is high enough (above 30 Hz).

Keywords: Stirling Engine; External Heat Transfer, 2nd Order, Compressibility

1. Introduction

The Stirling engine has, theoretically, some features that make it attractive: higher efficiency, lower emissions, quieter operation, and the ability to use multiple fuels. On the other hand, the Stirling engine has also had some difficulties to overcome such as seals, sustained high temperatures and difficult design to name a few. Overcoming the difficulties of designing and building a successful Stirling engine is facilitated by using a reasonably accurate model that is not time consuming. By including the external heat transfer and accounting for all of the terms of the momentum equation, a Stirling engine model can be developed from even the more basic analyses that is usable over a range of operating conditions.

2. Background

Modelling a Stirling engine has traditionally been divided into three levels of difficulty as described by Martini (1983). According to Martini, these are first order, second order and third order analysis, in increasing order of complexity. Second order analyses are of interest here since they offer greater accuracy than first order, but are fast enough to be used for design purposes, unlike third order ones.

Second order design methods start with a calculation of ideal power output then subtract decoupled losses due to fluid friction and thermal shorting, and then subtracting power losses and adding losses to either the heat input or heat rejection. The model for predicting the ideal power is usually more complicated than that used for the first order analysis.

There are many possible methods of determining the power output of a Stirling engine for a second order analysis. The two most common analyses used are the isothermal (Eid, 2009) and adiabatic (Rogdakis *et al*, 2012). Some others include the quasi-steady (Tlili, 2013), linear-harmonic analysis (LHA) (Choudhary, 2008), and finite-time-thermodynamics (FTT) (Cullen, 2011). More complicated models are 1-D nodal methods such as quasi 1-D compressible (Andersen, 2006) and method of characteristics (MOC) (Organ, 1982).

Loss mechanisms have to be taken into account since these are not included. These losses are determined either from decoupled loss models and then subtracted from the work indicated by the model used to find the ideal power and either added to the heater or cooler as additional heat loads. Martini (1983) discusses this method of dealing with losses and provides many models for estimating the losses. Some losses are conduction in the walls of the regenerator, cylinder and pistons, and heat loss in the regenerator due to non-ideal operation. Fluid losses can be due to friction, sudden contractions and expansions and bends. Since the loss mechanisms are accounted for assuming that they do not influence each other, the losses may very well be overestimated or underestimated depending on the

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nature of their interactions. The advantage of this method of accounting for losses is that it is easy to see which mechanisms have the most influence on the performance of the engine.

The isothermal, adiabatic, quasi-steady and linear-harmonic analyses are basically a finite volume solution of the mass and energy equations (Urieli and Berchowitz, 1984). The engine is divided into a minimum of 5 volumes: expansion space, heater, regenerator, cooler, and compression space. The isothermal model assumes that the gas is compressed and expanded isothermally, while the gas in the heater, the cooler and the regenerator is at the same temperature as the walls. The adiabatic and quasi-steady assume the gas is expanded and compressed adiabatically. LHA adds the assumption of representing variables as sinusoids. The quasi-steady model includes flow dissipation and heat transfer directly using empirically determined friction factors and heat transfer coefficients, while the isothermal and adiabatic models can add them after the cycle is complete and iterate to convergence.

The isothermal/adiabatic/quasi-steady types of models all assume the pressure is uniform, i.e. that the pressure drops in the heater, cooler and regenerator are negligible. This can be justified by examining the momentum equation. The normalized momentum equation is usually given, neglecting gravity and without assuming Stokes hypothesis, as (Potter and Foss, 1982):

$$\tilde{\rho} \frac{D\tilde{\mathbf{u}}}{Dt} = -\frac{1}{\gamma M_o^2} \nabla \tilde{P} + \frac{1}{Re} \left[\nabla^2 \tilde{\mathbf{u}} + \left(\frac{\beta}{\mu} + \frac{1}{3} \right) \nabla(\nabla \cdot \tilde{\mathbf{u}}) \right] \quad (1a)$$

where μ is the dynamic, or shear viscosity and β is the bulk viscosity. Stokes hypothesis is usually invoked, setting β equal to zero. Solving for the pressure gradient yields:

$$\nabla \tilde{P} = -\gamma M_o^2 \left(\tilde{\rho} \frac{D\tilde{\mathbf{u}}}{Dt} \right) + \frac{\gamma M_o^2}{Re} \left[\nabla^2 \tilde{\mathbf{u}} + \left(\frac{\beta}{\mu} + \frac{1}{3} \right) \nabla(\nabla \cdot \tilde{\mathbf{u}}) \right] \quad (1b)$$

or:

$$\nabla \tilde{P} \propto M_o^2 \quad (1c)$$

Since the Mach number is usually less than 0.1, the pressure gradient maybe assumed to be insignificant. Hence the pressure is assumed to depend only on time, and not location.

Finite-time-thermodynamics (FTT) is a development of classical thermodynamics. The ideal thermodynamic cycle is the starting point. Following Cullen (2011), for Stirling engines, the Schmidt analysis is used to calculate the ideal cycle work. A finite time constraint is imposed on heat transfer to and from the system. Within the system, reversibility is assumed. All irreversibility generation of the cycle is assumed at the system heat transfer boundaries. The power from the reversible model is corrected using the loss terms that generate irreversibility. Loss terms include fluid losses, thermal shorting and the effect of non-ideal heat exchangers. Again empirical friction factors and heat transfer coefficients are employed.

The quasi 1-D compressible CFD analysis is a one-dimensional simplification of a full three-dimensional CFD analysis. The quasi 1-D model uses either a finite-volume method (Andersen, 2006) or finite difference method (Gedeon, 2012) of CFD to solve the mass, momentum and energy equations in one dimension. Forces, e.g. friction, and heat transfer effects are modeled using empirical correlations since they cannot be explicitly included in the 1-D equations.

The method of characteristics is a classical method of solving hyperbolic differential equations. The most basic form of this used with fluid mechanics is the application of it to the mass and momentum equations for homentropic flow. Organ (1982) used this method in regards to Stirling engines. The conservation equations used are essentially the same as for the quasi 1-D model. They are recast into compatibility form for solving using velocity and pressure as state variables instead of the traditional MOC gas velocity and sonic velocity. If non-ideal heat exchangers are included, the addition of the energy equation results in a third compatibility equation, usually for entropy along a streamline (Poloni et al, 1987). As for the quasi 1-D CFD model, friction, and heat transfer effects are included by using empirical correlations.

All of the second order models use friction factors to determine the pressure losses in the heater, cooler and regenerator. All friction factor correlations employed were developed for steady, incompressible flow. For regenerators and tube bundles, the method of Kays (1964) for evaluating pressure loss is used. The method of evaluating pressure losses as used by Kays (1964) assumed that the flow was steady and incompressible; See Kays, 1950. This means that the $\nabla(\nabla \cdot \tilde{\mathbf{u}})$ term from Equation 1 and the unsteady part of the material derivative of velocity, $D\tilde{\mathbf{u}}/Dt$, are neglected. For 1-D flow, a factor of 4/3 can be applied to the $\nabla^2 \tilde{\mathbf{u}}$ term to better account for $\nabla(\nabla \cdot \tilde{\mathbf{u}})$. Models that employ only a small number of control volumes, e.g. adiabatic/quasi-steady, cannot account for this effect more accurately. The quasi 1-D models could evaluate the $\nabla(\nabla \cdot \tilde{\mathbf{u}})$ term and provide a more accurate estimate

of its magnitude if this term were included in the model equations. For engines running on air or hydrogen, the effect of this term may even be greater since the bulk viscosity for both of these gases is not zero, unlike helium for which it is zero. (Graves, 1999)

Equation 1b can provide further insight into the pressure loss mechanisms if Mach number (M_o) and Reynolds number (Re) are replaced by the following:

$$M_o^2 = \frac{(d_o \omega_o)^2}{\gamma P_o / \rho_o} \quad (2a)$$

$$Re = \frac{\rho_o (d_o \omega_o) L_o}{\mu} \quad (2b)$$

$$\rho_o = \frac{m_o}{V_o} \quad (2c)$$

where d_o = maximum displacement of drive

ω_o = engine speed (Hz)

P_o = reference pressure (cycle average)

L_o = length of gas circuit

V_o = mean volume of gas circuit

m_o = mass of gas in engine.

Then Equation 1b becomes:

$$\nabla \bar{P} = \left(\frac{d_o^2}{V_o} \right) \left(\omega_o^2 \frac{m_o}{P_o} \right) \left(-\bar{P} \frac{D\bar{u}}{Dt} \right) + \mu \left(\frac{d_o}{L_o} \right) \left(\frac{\omega_o}{P_o} \right) \left[\nabla^2 \bar{u} + \left(\frac{\beta}{\mu} + \frac{1}{3} \right) \nabla(\nabla \cdot \bar{u}) \right] \quad (3)$$

d_o , L_o , and V_o do not vary with engine speed or pressure. The first term only varies with engine speed if the heater and cooler tubes are maintained at the same temperatures using the Ideal gas law. The second term indicates that viscous losses increase proportionally with engine speed and the inverse of pressure.

Martini (1983) includes the performance of the GPU-3 calculated by GM as shown in his Figure 3-9. The model employed is the adiabatic and it uses empirical friction factors (Hargreaves, 1991). This model was developed by Philips and supplied to GM as part of the license agreement. Comparing Figures 3-8 and 3-9 of Martini, shows that this adiabatic model becomes increasingly inaccurate as the engine mean pressure is reduced for higher engine speeds. At the design point, the power is off by +5.4% and the efficiency is +14.6%, while at the lowest pressure, the power is off by 120% at 3000 rpm. In contrast, the adiabatic model as developed below is usable over all of the pressures tested.

Rogdakis *et al*, 2012 report on the analysis and experimental testing of the Solo V161 Stirling engine. The V161 is a kinematic alpha type. As tested, it ran on helium with mean pressures of 3.0 to 13.0 MPa at an engine speed of 1500 rpm. The model is a straightforward application of the Urieli and Berchowitz (1984) adiabatic model. While the performance of the external components was measured, they were not modelled. The measurements were used to find the heater and cooler loads for comparison to those predicted by the adiabatic model. The model prediction for power at the lowest pressure of 3.0 MPa was off by 3.4%. Due to the modest engine speed, the steady, incompressible friction factors seem to account for the flow dissipation.

Demko and Penswick (2005) modelled the NASA Technology Demonstration Converter (TDC) Stirling engine using Sage. The TDC is a small free-piston engine. The Sage software uses the quasi 1-D CFD finite difference model. The engine produces 50.7 W at a mean pressure of 2.5 MPa and engine speed of 81.2 Hz using helium. The completed, uncalibrated model overestimated power by 13.5% and efficiency by 5.9%. An artificial multiplier (1.4) was then applied to the pressure drop; the power was off by 3.6% and efficiency -1.5%. The low pressure and high frequency would indicate that the steady flow, incompressible friction factors need correcting and the $\nabla(\nabla \cdot \bar{u})$ term needs to be accounted for. The artificial multiplier used is close to the 4/3 as suggested. Note that this model does not include the external heat transfer.

Dyson *et al* (2005) report on the results of performing a 2-D axisymmetric CFD analysis on a complete Stirling engine. The engine modelled was the same NASA TDC as for Demko and Penswick (2005). The complete compressible momentum equation including the $\nabla(\nabla \cdot \bar{u})$ term can be properly calculated in this type of model. Engine power and efficiency were off by at most 1%. This illustrates that when the $\nabla^2 \bar{u}$ and $\nabla(\nabla \cdot \bar{u})$ terms are correctly modeled, engine performance can be predicted accurately.

In addition to the use of empirical friction factors, all of the models mentioned use the assumption that the heater tube temperature is constant, spatially and temporally. There are few references in the literature that do include the external heat source in modelling the Stirling engine.

The FTT model (Cullen, 2011) already mentioned does include the external heat source to the Stirling engine. A thermal circuit is employed that contains the external convective heat transfer, the internal convective heat transfer and the conductive heat transfer through the heater tube walls. This model was validated by comparing it to the GPU-3 experimental data as given in Thieme, 1979 and Martini, 1983. Unfortunately, the geometry of the heater was incorrectly modelled. Comparing Figure 6.4 (Cullen, 2011) with Figure 8 (Johnson, 1981), shows that the free-flow area for the combustion gases is significantly smaller than modelled. Additionally, the exhaust gas temperature was assumed to be equal to ambient, (§6.5.2 of Cullen, 2011), but it was over 200°C hotter per Appendix F of Thieme (1979). These would both lead to an erroneous estimate of combustion mass flowrate.

There is one paper (Johnson *et al*, 1981) dealing directly with the external heat transfer to the Stirling engine heater. Johnson *et al* investigate the possibility of improving the heat transfer to the heater by using jet impingement. They create models for the unmodified and modified GPU-3 and compare it to experimental results. But they do not describe the model details.

In order to estimate the overall efficiency of a Stirling engine and the external heat source, the combustion process and the external heat transfer to the engine need to be included. This analysis provides a model of the combustion process and external heat transfer to the engine that can be easily interfaced with any of the 2nd order Stirling engine models that use the assumption of constant heater tube temperature while remaining fast enough to use for design purposes. By providing a simple correction to the momentum equation, the engine model can be made reasonably accurate over a range of operating conditions.

3. Methodology

The model to be employed is the adiabatic and simple heat exchanger model. (Urieli and Berchowitz, 1984) Adiabatic working spaces seem to be a little more realistic for a well-insulated engine than isothermal ones. The simple heat exchanger refinement is less complicated to implement than the quasi-steady. Since numerical integration for such a model is not time consuming, the LHA and FTT models main advantage, no need for numerical solution, is lost. Numerical integration can also deal with more complicated drive mechanisms like the offset-crank-slider or Ross yoke.

The specific engine to be modelled is the General Motors Ground Power Unit-3 (GPU-3). It is a single cylinder β -type, using a tube bundle heater and offset crank-slider drive, aka rhombic drive. Its dimensions and performance have been extensively documented in the literature, Thieme, 1979 and 1981, Johnson, 1981, and Martini, 1983. The dimensions including preheater, as used in the model, are given in Appendix A.

The complete engine plus auxiliaries is shown in Figure 1. The components include the Stirling engine, a fan, a heat exchanger used as a preheater, a combustor, and a fuel pump. Included within the Stirling engine are the heater, cooler, and water-to-air heat exchanger. The fan supplies the mass flow rate of atmospheric air to the system with a minimal pressure increase. The air is then heated in the preheater using the hot exhaust gases coming from the engine. The combustor burns the #1 diesel fuel and increases the gas temperature to its operating point. The resulting high temperature combustion gases pass over the heater of the Stirling engine, transferring the necessary energy to drive the engine.

The overall thermal efficiency of the system is the net usable shaft power divided by the total heat input to the system. The total heat input to the system is the amount energy released by the combustion of the fuel. The energy of the fuel is equal to the mass flow rate of fuel times the lower heating value (LHV) of the fuel.

Figure 1
Complete System Including Engine and Auxiliaries

The engine tests used Number 1 diesel fuel which has an LHV of 18590 BTU/lb (43236 kJ/kg). The power consumption of the fan and fuel pump is not included since they were not measured during testing.

The incoming air, fuel and products of combustion are modeled using the properties of air. The properties are assumed constant at an appropriate average temperature for each specific component. Due to the possible large temperature differences, the average specific heat at constant pressure (C_p) values are found once the inlet and exit temperatures are known. The new average C_p is then used to recalculate the inlet and exit temperatures. This is repeated until convergence, typically a few iterations.

3.1 Fan

The fan's power consumption is not included in this model. The air enters at atmospheric pressure with negligible velocity. The measured combustor pressure drop is 3 kPa which is used to find the pressure at the exit of the fan.

3.2 Preheater

The preheater is modeled as a plate-fin compact heat exchanger. Other than its inclusion in the drawings of the GPU-3 (Martini, 1983), no specific details of the preheater were found. The configuration used is the plain plate-fin surface 46.46T and its measured performance is given in Figure 10-37 of Kays (1964). Its size is determined by taking measurements from the known drawings. It is assumed to be stainless steel.

3.3 Combustor

The combustor model simply uses the LHV of the fuel to find the energy input into the complete engine. The products of combustion are modelled using the properties of air, such as specific heat. The mass of fuel burned is modelled as an equivalent mass of air since the mass of fuel is small compared to air. Starting with a simple control volume energy balance on the combustor:

$$\dot{Q}_{fuel} - \dot{Q}_{loss} = \dot{m}_{air} \Delta h_{air} + \dot{m}_{fuel} \Delta h_{fuel} \quad (4a)$$

where $\dot{Q}_{fuel} = \dot{m}_{fuel} \cdot LHV$ (rate of energy from fuel)

$\dot{Q}_{loss}$ = various heat losses to environment

$\dot{m}_{air}$ = mass flow rate of air

$\dot{m}_{fuel}$ = mass flow rate of fuel

Δh_{air} = enthalpy increase of air per mass

Δh_{fuel} = enthalpy increase of fuel per mass

the mass flow rate of fuel is solved for:

$$\dot{m}_{fuel} = \frac{\dot{m}_{air} \Delta h_{air} + \dot{Q}_{loss}}{LHV - \Delta h_{fuel}} \quad (4b)$$

Enthalpy changes are evaluated using variable specific heat data, i.e. $\Delta h = \int C_p(T) dT$. The specific heat of air is used to evaluate the enthalpy change of the fuel. The term $\dot{Q}_{loss}$ represents the heat loss to the ambient air in the form of radiation and convection as well as the heat absorbed by the fuel nozzle cooling system. Per Thieme (1981), these losses were constant over the range of engine speeds and pressures tested. The value used is 700 W.

3.4 Stirling Engine

The engine model is the ideal adiabatic plus simple heat exchanger model developed by Urieli and Berchowitz (1984). Some changes to the code include adding the rhombic drive equations, increasing the number of engine dimensions, an iterative procedure to adjust the mean cycle pressure by changing the mass of gas, adding three volumes, adding loss mechanisms and revising the energy balance.

The original code determined the mass of the working gas by applying the Schmidt analysis. The mass of gas from the Schmidt analysis is used as the initial value for an iterative scheme. The complete adiabatic model is run and the resulting cycle average pressure is found. The new mass of gas is adjusted based off of:

$$m_{\text{new}} = \left(1 + \frac{P_{\text{mean}} - \bar{P}}{P_{\text{mean}}}\right) \cdot m_{\text{old}} \quad (5)$$

where m_{new} = new mass value

P_{mean} = desired mean pressure

$\bar{P}$ = cycle averaged pressure

m_{old} = old mass value.

The adjusted mass of gas is found to be about 10% less than that found using the Schmidt analysis. This is primarily due to the offset-crank-slider mechanism. Reducing the mass of working gas reduces the power output of the adiabatic model accordingly.

The three volumes added to the model are shown in Figure 2. The volumes added are the cooler-regenerator manifold, regenerator-heater manifold, and the appendix gap. The two manifolds do not really impact the model. They were added just to better represent the geometry. The appendix gap was added to be able to include its losses. The temperature of the appendix gap (T_a) is the log-mean temperature difference using the expansion space temperature (T_e) and the compression space temperature (T_c). The revised set of equations is given in Appendix B.

Figure 2
Adiabatic Model Volumes and Temperatures

The loss mechanisms are flow dissipation, non-ideal regenerator loss, appendix gap loss, regenerator wall thermal short, cylinder wall thermal short, displacer wall thermal short, displacer gas internal convection, displacer internal radiation thermal short, and gas circuit hysteresis.

The flow dissipation was modelled starting with the loss model of Urieli (2012). The original loss model accounts for shear stress by using friction factors. The Reynolds friction factor (f_r) for the heater and cooler tubes used is the Blasius relation assuming steady turbulent flow:

$$f_r = 0.0791 Re^{0.75} \quad (6)$$

$$f_r = 24 \text{ if } Re < 2000$$

The steady flow Reynolds friction factor for the regenerator mesh taken from Kays and London (1964) is:

$$f_r = 54 + 1.43 Re^{0.78} \quad (7)$$

Pressure losses due to sudden contraction, sudden expansion and acceleration were added following Kays and London (1964) for heat exchangers. Equation 8 is derived from Equation 2.26a (Kays, 1964), using the Fanning friction factor (f_f). It includes an unsteady term and losses due to bends:

$$\Delta P = \frac{(\rho u)^2}{2\rho_{\text{in}}} (K_{\text{in}} - 1 - \sigma_{\text{in}}^2) + \frac{(\rho u)^2}{2\rho_{\text{out}}} (K_{\text{out}} + 1 + \sigma_{\text{out}}^2) + 4f_f \frac{L\rho u^2}{D_{\text{hyd}}^2} + L\omega \frac{\partial(\rho u)}{\partial \theta} + \sum K_b \frac{\rho u^2}{2} \quad (8)$$

where K_{in} = inlet contraction coefficient

K_{out} = outlet expansion coefficient

σ_{in} = inlet ratio of free-flow area to frontal area

σ_{out} = outlet ratio of free-flow area to frontal area
 K_b = bend loss coefficient
 L = length of component
 ω = angular velocity of drive
 θ = crank angle.

The partial derivative ($\partial/\partial\theta$) is approximated by a backward finite difference.

The loss due to the non-ideal operation of the regenerator is modelled exactly as in Urieli (1984). The effectiveness (ϵ) of the regenerator is derived as:

$$\epsilon = \frac{NTU}{NTU + 1} \quad (9)$$

where NTU = number of transfer units. The NTU is found using the Stanton number (St). The same empirical correlation for a mesh used by Urieli (1984) from Kays and London (1964) is used:

$$St = 0.46 \frac{Re^{-0.4}}{Pr} \quad (10)$$

The appendix gap is modelled as a non-ideal regenerator as well. The effectiveness of the appendix gap functioning as an annular gap regenerator without a mesh is evaluated using the same ϵ -NTU relationship, Eq. 9, (Urieli, 2012). The Reynolds friction factor (f_r) is found from Equation 6. The Stanton number for the appendix gap is found using the friction factor of the annular gap and then applying Reynolds simple analogy.

$$St = \frac{f_r}{2Re \cdot Pr} \quad (11)$$

The appendix gap loss is then the loss of heat due to the less-than-ideal effectiveness of the appendix gap.

The thermal shorting losses in the regenerator wall, cylinder wall, and displacer wall are estimated by assuming 1-D conduction through a constant thickness wall. The heater and cooler wall temperatures are used for the regenerator thermal short. The cooler and heater gas temperature are adjusted by $\pm 10\%$ for use with the displacer and cylinder walls to account for finite rates of heat transfer. The thermal conductivity of the walls is taken at the log-mean-temperature difference of the appropriate end temperatures.

The displacer gas internal convection loss is modeled as a thermal circuit from the expansion space gas, in through the displacer wall, to the internal gas, out through the displacer wall to the compression space gas. The Nu to/from the displacer ends is assumed to be 4.0, i.e. laminar. The gas in the core of the displacer is assumed to be stationary; therefore it only conducts the heat loss.

The displacer internal radiation thermal short models the radiation heat transfer that takes place in the displacer core, using the standard method for radiation heat transfer between surfaces (Incropera, 1990). View factors for the circular ends (coaxial parallel disks) and cylindrical side are calculated. The heat loss from the hot end to the cold end is then found assuming that the cylindrical side wall is reradiating. As with the cylinder and displacer wall thermal shorts, the displacer end temperatures are assumed to be $\pm 10\%$ of the compression and expansion space temperatures to account for finite heat transfer.

The gas circuit hysteresis loss is modelled following Bailey et al (2007). The hysteresis losses (Q_{hys}) are found using the following equations:

$$Q_{hys} = Q_o \frac{\pi}{2} \frac{1}{y} \left(\frac{\cosh y \sinh y - \sin y \cos y}{\cosh^2 y - \sin^2 y} \right) \quad (12a)$$

$$Q_o = P_{avg} V_{avg} \left(\frac{P_{amp}}{P_{avg}} \right)^2 \frac{(y - 1)}{y} \quad (12b)$$

$$y = 0.49 Pe_{\omega}^{0.43} \quad (12c)$$

$$Pe_{\omega} = \frac{\omega D_{hyd}^2}{4\alpha_p} \quad (12d)$$

$$D_{hyd} = \frac{4V_o}{A_o} \quad (12e)$$

where P_{max} = maximum pressure of cycle

P_{min} = minimum pressure of cycle

$P_{avg} = (P_{max} + P_{min})/2$

$P_{amp} = (P_{max} - P_{min})/2$

V_{max} = maximum volume of cycle

V_{min} = minimum volume of cycle

$V_{avg} = (V_{max} + V_{min})/2$

α_{mid} = thermal diffusivity at mid-stroke

V_{mid} = volume at mid-stroke

$A_{w,mid}$ = wetted area at mid-stroke.

Pe_o is the Peclet number based on the angular velocity. D_{hyd} is the hydraulic diameter. Q_o is a normalizing term. Equation 12c is an adjustment of the original, theoretical result to better agree with experimental results. For this model, the compression space, compression space clearance volume and cooler are considered as one volume. The expansion space, expansion space clearance volume and heater are considered as a second volume.

The net work, heater heat absorption and cooler heat rejection as calculated by the adiabatic model are adjusted using the losses mentioned. The flow dissipation and hysteresis losses reduce the amount of work produced. The cylinder, displacer, and regenerator wall thermal shorts, the displacer internal radiation and gas convection losses, and appendix gap and regenerator losses are all added to both the heater and cooler as additional heat loads. The flow dissipation in the heater is subtracted from the heater load as a reheat affect. The flow dissipation in the cooler and regenerator is added to the cooler load.

3.5 Stirling Engine Heater

The high temperature heat exchanger on the Stirling engine is configured as a bank of tubes in cross-flow. The tubes are arranged vertically in a radial pattern, similar to the heater of the GPU-3. See either Martini (1983) or Urieli (1984) for the layout and dimensions of the GPU-3 heater. The combustion gases flow into the heater axially and exit radially. The flow is analogous to a shell-and-tube heat exchanger with one shell (combustion gases) and two tube passes (Stirling engine gas).

The gas velocity over the heater tubes is initially found by determining the heat transfer coefficient (U) from Newton's law of heating since the surface area, gas and surface temperatures and the rate of heat transfer are known. The conduction resistance of the heater tube walls is also included. Since a large range of gas velocities is to be examined, the extra effect of heating due to high velocity viscous dissipation is accounted for by using the free-stream stagnation temperature instead of the free-stream static temperature (Kays, 2005). The velocity is extracted from the heat transfer coefficient using a Nu correlation for a single cylinder in cross-flow, which is recommended for flow over a bank of cylinders if the Reynolds number is no more than 1000. The correlation used is Equation 7.45 from Incropera (1990). It is of the form:

$$\overline{Nu}_D = c Re_D^n Pr^{\frac{1}{4}} \quad (13)$$

where c and n are based on the Reynolds number (Re). The velocity used for the Re is the mass averaged velocity in the gap between the heater tubes.

The temperature of the combustion products after flowing over the Stirling engine heater tubes is initially estimated by applying an energy balance to the fluid, given that the amount of heat transfer to the engine is known. The net rate of heat transfer ($\dot{Q}$) is then calculated using Newton's law of heating and an appropriate form of the log-mean-temperature difference (LMTD):

$$\dot{Q} = U \cdot A_w \cdot LMTD \quad (14)$$

For the assumption of a constant heater tube surface temperature ($T_{h,s}$), the LMTD can be shown to be:

$$LMTD = \frac{(T_{h,out} - T_{h,in})}{\ln \left(\frac{T_{h,out} - T_{h,s}}{T_{h,in} - T_{h,s}} \right)} \quad (15)$$

where $T_{h,out}$ and $T_{h,in}$ are the temperatures of the combustion gases exiting and entering the heater, respectively. The gas velocity and exit temperature are iterated until the rate of heat transfer using both the energy balance and LMTD methods is the same.

Radiation from the combustion products to the heater tubes is excluded. The products of combustion would contain N_2 and O_2 which can be ignored. CO_2 and H_2O do need to be accounted for, but at the partial pressures encountered in this system, their emissivities would be very low, resulting in very low amounts of radiative heat transfer (Incropera, 1990).

3.6 Stirling Engine Cooling System

The Stirling engine cooler is modeled as a compact heat exchanger arranged in a staggered tube bank. A constant tube surface temperature is assumed. The Nu correlation used is Equation 7.57 (Incropera, 1990):

$$\overline{Nu} = c Re_D^n Pr^{0.36} \left(\frac{Pr}{Pr_s} \right)^{1/4} \quad (16)$$

with c and n depending on Re and the tube bank configuration. The thermal resistance of the cooler tubes is assumed negligible. A source of constant temperature water was used during testing, so the cooling system water-air radiator is not included.

3.7 Solution

The system analysis is done numerically by assuming values for some variables and iterating to convergence. Due to the transcendental nature of Equation 15, it is not possible to get an analytical solution for the tube surface temperature directly. Appendix C shows the steps in finding the performance of the external components. The analysis of an operating point is begun using an assumed heater tube inner surface temperature, an assumed cooler tube temperature, and the engine speed. The temperature of the products of combustion and the air and fuel mass flow rates are not known a priori. The adiabatic plus simple heat exchanger model is run, and the average heater and cooler gas temperatures, heater and cooler heat loads, and indicated power and efficiency of the engine are found. Next the temperature of the products of combustion and the static fan pressure ratio are assumed. The fuel and air mass flow rates, temperature of the gas after passing over the Stirling engine heater, and the performance of the preheater and cooler are found. The combustion gas temperature and gas velocity through the heater tube gap are adjusted until the desired air/fuel mass ratio, heater tube surface temperatures, and cooler water flow rate are attained. The assumed heater tube inner surface temperature and cooler tube temperature used in the adiabatic model may have to be fine-tuned to achieve the known tube surface temperature and the heater gas temperature.

4. Results and Discussion

4.1 External System Evaluation

The performance of the model of the external components of the engine is compared to the experimentally determined performance as given in Johnson (1981). The data used is from Figure 14. The engine was run using hydrogen at a mean pressure of 6.9 MPa, at an engine speed of 3000 rpm with an air/fuel ratio of 26:1. The measured heat into the engine (22454 W) was used instead of running the engine model to eliminate any error introduced by the engine model. Table 1 shows the results. The preheater exhaust temperature is a little high, but given the unknown configuration of the original, 17% difference is acceptable. The heat transfer coefficient is low. Still it is within 20% of the measured value which is reasonable. As a result of the lower heat transfer coefficient, the combustion temperature is higher. The rest of the quantities agree within 10% of the measured values.

The variable with the largest amount of uncertainty in the modelling of the external system is the heat transfer coefficient. Hargreaves (1991) reports that Philips while developing the 1-98 engine created heat transfer correlations similar to Equation 13 but without the Pr correction. The values in Table 1 in brackets are found using the Philips correlation. While the temperatures are significantly in error, the flow rates (which are the most important) are still within 10% of the measured values.

Quantity	Model	Experimental	Error (%)
Air Preheat Temperature T_2 (°C)	1047 [741]	1037	0.96 [-28.5]
Combustion Temperature T_3 (°C)	2297 [1997]	2108	8.97 [-5.3]
Engine Exhaust Temperature T_4 (°C)	1285 [910]	1191	7.89 [-23.6]
Preheater Exhaust Temperature T_5 (°C)	343 [243]	293	17.1 [-17.1]
Heat transfer coefficient (W/m ² -°C)	253 [462]	301	-15.9 [53.2]
Fuel Flow Rate (g/s)	0.68 [0.63]	0.66	3.03 [-4.5]
Air Flow Rate (g/s)	17.83 [16.39]	17.51	1.83 [-6.4]
Preheater effectiveness	0.81	0.80	1.25

Table 1
External Model Comparison to Experimental
(Numbered temperatures refer to Figure 1)

4.2 Complete Engine Evaluation

The experimentally measured performance is from Figure 3-8 (Martini, 1983). Figure 3-8 is from the tests that General Motors performed on the GPU-3 in 1969. The engine power % error ranges from -1.9% to +40.9%. The brake specific fuel consumption (bsfc) % error ranges from -35.1% to -1.8%. The drive efficiency is calculated using the brake and indicated power measurements from Figures 6 and 15 of Thieme, 1981. The drive efficiency varied from 69% (at 1.72 MPa) to 84% (at 6.9 MPa). Comparing the power and bsfc for each pressure, the model performance at the lowest pressure has significant error. On the other hand, for the three higher pressures, the engine power % error ranges from -1.9% to +6.1% and the bsfc % error ranges from -9.7% to -1.8%. If the measured power values at 1.72 MPa are used instead of the model power values, the model bsfc becomes significantly better, with an error range of -11.0% to -1.5%. This indicates that the engine model is not calculating the power output correctly at the lowest pressure, but the heat input to the engine is reasonable, albeit low. Some heat losses maybe underestimated or not included.

One possible cause of the model over predicting the engine power, especially at low mean pressures, could be the use of steady flow friction factors instead of ones for oscillating flow. According to Ju *et al*, 1998, the regenerator friction factor for oscillating flow is significantly higher than that of steady flow. Ju *et al* measured the pressure drop in a regenerator of a pulse tube cryocooler. The working gas was helium, the operating frequency was 50 Hz and the mean pressure varied from 0.6 to 0.9 MPa. The resulting cycle-averaged oscillating flow friction factor was compared with a steady flow friction factor. The pressure drop was found to be at least 2 times larger using the cycle averaged oscillating friction factor than the pressure drop from the steady flow friction factor.

The complete engine simulation for all engine speeds and pressures was repeated using a regenerator pressure drop multiplied by a correction factor of two. For the three higher pressures, the engine power % error ranges from -7.8% to +3.3% and the bsfc % error ranges from -7.6% to +6.2%. For the lowest pressure, the engine power % error

improves to +2.2% to +15.8%. The bsfc % error also improves to a range of -13.4% to -21.5%. This is an improvement, but there is still something missing from the engine model.

The complete engine simulation for all engine speeds and pressures was run again with the shear stress pressure drop multiplied by a factor of 4/3 to account for the $\frac{4}{3}(\frac{V}{V - \frac{4}{3}V})$ term of Equation 1. The engine power % error ranges from -14.1% to +12.7% and the bsfc % error ranges from -12.7% to +13.4%. for all pressures. Figures 3a-3d show the engine output and the bsfc as a function of engine speed for the four mean pressures.

Figure 3a
GPU-3 Power and bsfc as Engine Speed Varies
(6.9 MPa Mean Engine Pressure)

Figure 3b
GPU-3 Power and bsfc as Engine Speed Varies
(5.17 MPa Mean Engine Pressure)

Figure 3c
GPU-3 Power and bsfc as Engine Speed Varies
(3.45 MPa Mean Engine Pressure)

Figure 3d
GPU-3 Power and bsfc as Engine Speed Varies
(1.72 MPa Mean Engine Pressure)

Most of the power predictions are lower than the measured values. This would indicate that the decoupled loss mechanisms are actually less than predicted. The most significant losses are flow dissipation and hysteresis. The notable exception to the less than expected power is at an engine speed of 10 Hz and a pressure of 1.72 MPa. It is +12.7% higher. Considering the small amount of power produced, smaller loss mechanisms that have been ignored may be the difference for this operating condition. Two of these might be the flow dissipation in the appendix gap and the leakage past the displacer seal.

Figure 4a shows that the coefficient ($\omega_o^2 \cdot m_o / P_o$) of Equation 3 increases as engine speed varies for 6.9 MPa and 1.72 MPa. For the same engine speed, this term does not vary with pressure. P_o is a function of the drive geometry, gas circuit geometry, heater and cooler temperatures and mass of the gas, m_o , but the ratio (m_o / P_o) does not change. Figure 4a indicates that the contribution of the unsteady and acceleration ($\partial \vec{D} \vec{u} / \partial t$) effects does not change since its coefficient does not change as pressure is lowered.

Figure 4b shows the coefficient (ω_o / P_o) of Equation 3 as the engine speed varies for all four pressures.

Figure 4a
Coefficient of Unsteady and Acceleration Term in Momentum Equation

Figure 4b
Coefficient of Dissipation and Compressibility Term in Momentum Equation

As pressure decreases, the magnitude of this ratio increases. This reveals that the contribution of dissipation $(\nabla^2 \bar{u})$ and compressibility $(\nabla(\nabla \cdot \bar{u}))$ to the pressure gradient increases as pressure is reduced. Comparing the values for this coefficient at 50 Hz shows the increasing magnitude of the dissipation and compressibility term as pressure is decreased.

The modeling of the GPU-3 indicates that flow dissipation losses increase significantly below a mean engine pressure of 3.45 MPa. Figure 5 shows the engine output and bsfc for 3.45 MPa and 1.72 MPa mean pressures with and without the corrections for compressibility and oscillatory flow. For all engine speeds, the uncorrected model is sufficiently accurate at 3.45 MPa. At 1.72 MPa, the uncorrected model is reasonably accurate below 1500 rpm, but it is off by as much as 35% for power and 41% for bsfc at higher speeds. When corrected, the model is off by at most 13% at 1.72 MPa. The analysis of the SOLO 161 (Rogdakis *et al*, 2012) also shows that the steady, incompressible friction factors adequately account for flow dissipation for pressures of at least 3.0 MPa at an engine speed of 1500 rpm. No correction was necessary to account for compressibility or oscillating flow in modeling the SOLO 161. On the other hand, the modeling of the TDC (Demko and Penswick, 2005) implies that a correction for compressibility is needed at 2.5 MPa and 81 Hz engine speed. This suggests a rule of thumb that for engines operating below a mean pressure of 3.0 MPa and at engine speeds above 1500 rpm, the effects of unsteady, compressible flow need accounting for. For models of the isothermal/adiabatic type multiplying the steady, incompressible friction factor by

4/3 seems to be an adequate correction. For models that can evaluate the $(\nabla(\nabla \cdot \mathbf{u}))$ term from Equation 3, this term needs to be included.

Figure 5
Comparison of power and bsfc for models with and without pressure corrections
Left column: 3.45 MPa mean engine pressure
Right column: 1.72 MPa mean engine pressure
Top row: With compressibility and oscillatory flow corrections
Bottom row: No corrections

Figures 6a and 6b show the decoupled losses as a percentage of the total loss for 6.9 MPa and 1.72 MPa. At 6.9 MPa, regenerator ineffectiveness, flow dissipation, and regenerator wall losses are the largest. At 1.72 MPa, the regenerator wall conduction and flow dissipation are the two biggest losses. The wall conduction losses are constant over all pressures and engine speeds since they are mainly dependent on temperature and geometry, but become a larger percentage of the total loss as engine speed and pressure are reduced. For example, at 1.72 MPa and 10 Hz, the regenerator wall conduction is still the same magnitude (800 W), but is 69% of the losses. The loss due to non-ideal regenerator operation is more significant at higher pressures due to the resulting larger mass flow rates that reduce the amount of time for heat transfer. Not shown in Figures 6a and 6b are the displacer internal heat losses which were much less than 1% of the total loss.

Figure 6a
GPU-3 Decoupled Losses
(6.9 MPa Mean Pressure)

Figure 6b
GPU-3 Decoupled Losses
(1.72 MPa Mean Pressure)

Figures 7a and 7b show the external energy balance as a function of engine speed for 6.9 MPa and 1.72 MPa. At 6.9 MPa, the engine output decreases as a fraction of the total energy input, while the cycle heat rejection remains fairly steady as engine speed increases. The exhaust losses increase from 15% to over 23%. For 1.72 MPa, the engine output peaks at an engine speed of 30 Hz (1800 rpm). The cycle heat rejection increases as engine speed increases, while exhaust losses are constant. The losses to the environment become significant at low pressures and low engine speeds, being more than 20% at 10 Hz. These trends are similar to those from Figure 11 of Thieme, 1979.

Figure 7a
GPU-3 Energy Balance
(6.9 MPa Mean Pressure)

Figure 7b
GPU-3 Energy Balance
(1.72 MPa Mean Pressure)

5. Conclusions

The assumptions involving the momentum equation for second order Stirling engine models were reviewed. An ideal adiabatic plus simple heat exchanger model was developed. The model included the external components such as the fan, combustor, and preheater. The external heat transfer to the engine heater was modelled using a log-mean-temperature difference for a constant tube surface temperature. The performance of the model of the external components compared reasonably well to experimental data. The performance of the complete engine model was also compared to experimental data of the GPU-3. By adjusting the flow dissipation to better account for unsteady flow conditions and compressibility effects, the complete engine model was able to predict engine power and brake specific fuel consumption to within $\pm 14\%$ over a range of engine speeds and mean pressures.

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8. Nomenclature

Symbol	Definition
A	Area
bsfc	Brake specific fuel consumption
c	Empirical constant
C _p	Constant pressure specific heat
d	Displacement
D	Diameter
f _f	Fanning friction factor
f _r	Reynolds friction factor
h	Specific enthalpy
K	Loss coefficient
L	Length
LHV	Lower heating value
LMTD	Log-mean-temperature-difference
m	Mass
M	Mach number
n	Empirical exponent
NTU	Number of transfer units
Nu	Nusselt number
P	Static pressure
Pe	Peclet number
Pr	Prandtl number
Q	Heat transfer
Re	Reynolds number
St	Stanton number
t	Time
T	Temperature
u	Velocity
U	Heat transfer convection coefficient
V	Volume

Symbol	Definition
α	Thermal diffusivity
β	Bulk viscosity
γ	Ratio of specific heats
ε	Effectiveness
θ	Crank angle
μ	Shear or dynamic viscosity
ρ	Density
σ	Ratio of free-flow area to frontal area
ω	Angular velocity

Subscript	Definition
298	Reference temperature
ω	Angular velocity
a	Appendix gap
air	Air
amp	Amplitude
avg	Average
c	Compression space
b	Bend
D	Diameter
e	Expansion space
fuel	Fuel
h	Heater
hyd	Hydraulic
hys	Hysteresis
in	Inlet
loss	Loss
max	Maximum
mean	Mean
mid	Mid-stroke
min	Minimum
new	New
o	Reference
old	Old
out	Outlet
s	Surface
w	Wall

Superscript	
.	Rate
~	Normalized
—	Average

Appendix A

GPU-3 Engine Dimensions

Parameter	Value
connecting rod length (m)	4.600e-002
eccentricity (m)	2.080e-002
crank radius (m)	1.380e-002
piston diameter (m)	6.990e-002
piston length (m)	5.370e-002
compression clearance volume (m ³)	21.47e-006
expansion clearance volume (m ³)	21.04e-006
buffer space volume (m ³)	400.0e-006
cylinder wall thickness (m)	3.800e-003
displacer length (m)	4.350e-002
displacer rod diameter (m)	0.952e-002
displacer wall thickness (m)	1.590e-003
cylinder bore clearance (m)	1.000e-004
appendix gap width (m)	2.500e-004
cooler inner pipe diameter (m)	1.080e-003
cooler heat transfer length (m)	3.550e-002
cooler total pipe length (m)	4.610e-002
number of cooler tubes	312
regenerator housing outer diameter (m)	2.646e-002
regenerator housing inner diameter (m)	2.260e-002
regenerator length (m)	2.260e-002
number of regenerators	8
cooler-to-regenerator manifold length (m)	0.0807e-002
regenerator-to-heater manifold length (m)	0.2293e-002
regenerator void volume (m ³)	5.030e-006
regenerator matrix void factor	0.697
wire diameter (m)	4.000e-005
heater tube inner diameter (m)	3.020e-003
heater heat transfer length (m)	15.54e-002
heater total pipe length (m)	24.53e-002
number of heater tubes	40
preheater height (m)	0.094
preheater thickness (m)	0.035
preheater center diameter (m)	0.232

Appendix B

Equation Set

Ideal Adiabatic plus Simple Heat Exchanger including cooler-regenerator manifold, regenerator-heater manifold and appendix gap (Based off of Urieli, 1984)

Pressure, P

$$dP = \frac{-\gamma P \left(\frac{dV_c}{T_{ck}} + \frac{dV_s}{T_{hs}} \right)}{\left[\frac{V_c}{T_{ck}} + \frac{V_s}{T_{hs}} + \gamma \left(\frac{V_k}{T_k} + \frac{V_{ri}}{T_{ri}} + \frac{V_r}{T_r} + \frac{V_{ro}}{T_{ro}} + \frac{V_a}{T_a} \right) \right]}$$

$$P = \frac{MR}{\left(\frac{V_c}{T_c} + \frac{V_k}{T_k} + \frac{V_{ri}}{T_{ri}} + \frac{V_r}{T_r} + \frac{V_{ro}}{T_{ro}} + \frac{V_h}{T_h} + \frac{V_s}{T_s} + \frac{V_a}{T_a} \right)}$$

Mass, m

$$dm_c = \frac{1}{RT_{ck}} \left(P dV_c + \frac{dV_c dP}{\gamma} \right) \quad m_c = \frac{PV_c}{RT_c}$$

$$dm_s = \frac{1}{RT_{hs}} \left(P dV_s + \frac{dV_s dP}{\gamma} \right) \quad m_s = \frac{PV_s}{RT_s}$$

$$dm_k = \frac{m_k}{P} dP \quad m_k = \frac{PV_k}{RT_k}$$

$$dm_r = \frac{m_r}{P} dP \quad m_r = \frac{PV_r}{RT_r}$$

$$dm_h = \frac{m_h}{P} dP \quad m_h = \frac{PV_h}{RT_h}$$

$$dm_{ri} = \frac{m_{ri}}{P} dP \quad m_{ri} = \frac{PV_{ri}}{RT_{ri}}$$

$$dm_{ro} = \frac{m_{ro}}{P} dP \quad m_{ro} = \frac{PV_{ro}}{RT_{ro}}$$

$$dm_a = \frac{m_a}{P} dP \quad m_a = \frac{PV_a}{RT_a}$$

Mass Flows, $\dot{m}$

$$\dot{m}_{ck} = -dm_c$$

$$\dot{m}_{kr} = \dot{m}_{ck} - dm_k$$

$$\dot{m}_{rir} = \dot{m}_{kr} - dm_{ri}$$

$$\dot{m}_{rro} = \dot{m}_{rir} - dm_r$$

$$\dot{m}_{rh} = \dot{m}_{rro} - dm_{ro}$$

$$\dot{m}_{hs} = \dot{m}_{rh} - dm_h$$

$$\dot{m}_{sa} = dm_a$$

Temperatures, T

$$dT_c = T_c \left(\frac{dP}{P} + \frac{dV_c}{V_c} - \frac{dm_c}{m_c} \right)$$

$$dT_s = T_s \left(\frac{dP}{P} + \frac{dV_s}{V_s} - \frac{dm_s}{m_s} \right)$$

$$T_r = \frac{(T_h - T_k)}{\ln(T_h/T_k)}$$

$$T_a = \frac{(T_s - T_c)}{\ln(T_s/T_c)}$$

Conditional Temperatures

$$\text{if } \dot{m}_{ck} > 0 \text{ then } T_{ck} = T_c, \text{ else } T_{ck} = T_k$$

$$\text{if } \dot{m}_{hs} > 0 \text{ then } T_{hs} = T_h, \text{ else } T_{hs} = T_s$$

Energy, Q

$$dQ_k = \frac{C_v V_k dP}{R} - C_p (T_{ck} \dot{m}_{ck} - T_k \dot{m}_{kr})$$

$$dQ_r = \frac{C_v V_r dP}{R} - C_p (T_k \dot{m}_{rir} - T_h \dot{m}_{rro})$$

$$dQ_h = \frac{C_v V_h dP}{R} - C_p (T_h \dot{m}_{rh} - T_{hs} \dot{m}_{hs})$$

$$dQ_a = \frac{C_v V_a dP}{R} - C_p T_s \dot{m}_{sa}$$

Work, W

$$dW = dW_c + dW_s$$

$$W = W_c + W_s$$

Subscripts

- c – compression space
- k – cooler
- ri – cooler-regenerator manifold
- r – regenerator
- ro – regenerator-heater manifold
- h – heater
- e – expansion space
- a – appendix gap
- ck – compression space to cooler
- kr – cooler to cooler-regenerator manifold
- rir – cooler-regenerator manifold to regenerator
- rro – regenerator to regenerator-heater manifold
- rh – regenerator-heater manifold to heater
- he – heater to expansion space
- ea – expansion space to appendix gap

Appendix C

External Component Solution Method

Numbers in () refer to equations in text