A multidimensional discontinuous Galerkin modeling framework for overland flow and channel routing

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Abstract

In this paper, we present the development and application of a new multi-dimensional, unstructured-mesh model for simulating coupled overland/open-channel flows in the kinematic wave approximation regime. The modeling approach makes use of discontinuous Galerkin (DG) finite element spatial discretizations of variable polynomial degree p , paired with explicit Runge–Kutta time steppers, and is supported by advancements made to an automatic mesh generation tool, ADMESH+, that is used to construct constrained triangulations for channel routing. The developed modeling framework is applied to a set of four test cases, where numerical results are found to compare well with known analytic solutions, experimental data and results from another well-established (structured, finite difference) model within the area of application. The numerical results obtained demonstrate the accuracy and robustness of the developed modeling framework and highlight the potential benefits of using p (polynomial) refinement for hydrological simulations.

Keywords: overland flow, open-channel flow, discontinuous Galerkin, finite element, mesh generation

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1. Introduction

Physically-based (or synonymously process-based) watershed modeling in hydrology has a history of roughly five decades; see the recent review articles [1, 2]. At a very broad level, it seeks to describe the movement of water throughout the hydrologic cycle by means of the basic conservation laws of fluid mechanics and thermodynamics [3]. Comprehensively (and more specifically), it involves modeling the flow and transport of water through the atmosphere, over the (land) surface and through the subsurface using conservation (or balance) equations of mass, momentum and energy, with relevant atmosphere-land and surface-subsurface interface and boundary conditions. Given the multi-scale and multi-physics nature of these varied hydrological components, the complex coupling mechanisms that exist between them and a number of other factors (see, for example, [1]), development of robust and efficient physically-based models for watershed hydrology remains a challenge [2].

This challenge exists at both the system- and component-level development of hydrological models; e.g., the surface flow component of the hydrological system alone, which is the focus of the present work, exhibits many of the challenging features found at the system level, such as widely varying scales, multiple physical processes and coupled (sub)components. This leads directly to the consideration of a number of (interrelated) model development questions regarding, for example, governing equations and related parameterizations used, geometric representations employed (e.g., structured or unstructured domain discretization, dimensionality of the various watershed features) and numerical methods implemented. Ideally, the developed modeling framework should offer a certain degree of flexibility in addressing these questions and be in a position to continue to develop and evolve, allowing model developers and users, for example, to accommodate new high-resolution data sets with relative ease, take advantage of ever-evolving computational platforms and be able to couple their models to other model components of the larger hydrological system.

In this paper, we present the development and application of a new sur-

31 face flow hydrological modeling framework for simulating coupled overland and
 32 open-channel flow that is designed to address these points and meet some of
 33 the challenges posed by hydrological modeling. The modeling framework makes
 34 use of discontinuous Galerkin (DG) methods and mesh generation procedures
 35 designed specifically for the developed approach. DG methods appear to be
 36 an especially judicious choice as the numerical basis for hydrological models on
 37 several fronts. First, as a class of finite element methods, DG methods can
 38 make use of unstructured meshes to represent the complex geometry typical of
 39 watersheds in an accurate and computationally efficient manner. This is par-
 40 ticularly the case for the specific multidimensional approach considered in this
 41 work (and discussed in more detail below). Second, with numerical solutions
 42 based on piecewise polynomial spaces that allow discontinuities to occur be-
 43 tween elements, DG methods can also easily support the use of polynomial (p)
 44 refinement or enrichment as an efficient means to obtain highly accurate solu-
 45 tions, as is demonstrated in the numerical results presented in this paper, and
 46 to support the use of high-order interpolants in data-rich settings. Additionally,
 47 the methods have been shown to be highly parallelizable in implementations on
 48 both traditional parallel platforms [4, 5] and graphics-processing units (GPUs)
 49 [6, 7, 8]. And finally, DG methods possess the important property of local
 50 (element-wise) conservation, which is of *fundamental* importance in hydrologi-
 51 cal modeling, especially when coupling various model components [2, 9].

52 In developing this modeling framework, we took, as a simple starting point,
 53 the conceptual model of a drainage basin (equivalently watershed or catchment)
 54 presented in [10] described as consisting of “two geomorphological components:
 55 a set of hillslopes dominated by unconfined overland flow and a branching net-
 56 work of channels conveying concentrated flow.” This conceptual model is actu-
 57 ally very useful from a computational modeling perspective in that the division
 58 of the drainage basin into two distinct (albeit coupled) geomorphological com-
 59 ponents correspondingly points to the use of two distinct modeling approaches
 60 to represent them, in terms of describing both their geometry and the associated
 61 flow.

62 With respect to geometric representations, the latter geomorphological com-
 63 ponent (the channel network) is most naturally represented as a one-dimensional,
 64 tree-like structure, while the “unconfined” nature of the overland flow compo-
 65 nent points to the use of a two-dimensional representation that allows for a
 66 certain degree of flexibility in representing the geometry given both the geo-
 67 metric complexity of a typical drainage basin and the coupled nature of the
 68 two geomorphological components. In the model developed here, we make use
 69 of *constrained* meshes or triangulations, wherein the line segments composing
 70 the channel network are forced into the triangulation as element edges (but
 71 are otherwise unstructured); see, for example, Figures 5 and 6. (We note that
 72 the one-dimensional domain discretizations are also “unstructured” in the sense
 73 that variable-length one-dimensional elements can be used along the channel
 74 networks.) Part of the challenge in implementing and applying such an ap-
 75 proach is the construction of high-quality, constrained unstructured meshes,
 76 which is by no means trivial. This challenge is addressed in the present work
 77 through the continued development of an automatic mesh generation platform,
 78 called ADMESH+, developed by the first three authors [11] and discussed in
 79 Section 3.3 of this paper.

80 With respect to the associated flows, we first note that the “unconfined”
 81 and “concentrated” surface flows of these two geomorphological components
 82 are both well described by the (depth-averaged) two-dimensional shallow water
 83 equations but can, in many situations, be modeled with sufficient accuracy using
 84 a number of simplified equations derived from them. First, for example, given
 85 the decision to represent the channel network component in one-dimension, the
 86 associated flow can be described adequately using the section-averaged, one-
 87 dimensional shallow water (or Saint-Venant) equations. Moreover, based on the
 88 relative magnitude of the terms appearing in either the one- or two-dimensional
 89 shallow water equations in a given flow scenario, various approximations of a
 90 simpler nature can be applied. These approximations include, for example, the
 91 so-called gravity, diffusion and kinematic wave equations, the solutions of which
 92 have been shown to adequately describe the flow under proper sets of conditions

93 (see, for example, [12], which provides a nice summary and characterization of
94 these conditions).

95 The simplest of these approximations, the kinematic wave approxima-
96 tion, has been found to be “accurate enough” or “reasonably accurate” (see [13]
97 and also [14] and the references therein) under the natural conditions occurring
98 in many drainage basins and is chosen as the basis for the developed model.
99 Use of the kinematic wave approximations is appealing from both a conceptual
100 and numerical perspective, in that both the overland and open-channel flows
101 can be described using single *hyperbolic* equations (rather than systems of equa-
102 tions). Thus, DG methods are a natural choice given the great success they have
103 demonstrated for the solution of hyperbolic conservation laws in a wide variety
104 of fields (see, for example, [15, 16, 17, 18, 19]), though not, it is to be noted, in
105 the present context of kinematic approximations for open-channel and overland
106 flow. We also note that our decision to employ kinematic wave approximations
107 is dictated, in part, by the viewpoint that this work represents the first step
108 in the development of such a (DG-based) model, starting with (essentially) the
109 simplest approximation in order to investigate both the merits and limitations
110 of such an approach (which, to a certain extent, are revealed in the test cases
111 considered in Section 4.1 of this paper).

112 The rest of this paper is organized as follows. In the next section, we first
113 present the (depth-averaged) two- and (section-averaged) one-dimensional shal-
114 low water equations, inasmuch that their corresponding kinematic wave approxi-
115 mations — the governing equations of the model — can be introduced in context.
116 This is followed by a description of the discretization techniques employed by the
117 model, which consist of the multidimensional domain discretization approach in-
118 troduced above, the one- and two-dimensional DG spatial discretizations and a
119 set of explicit Runge–Kutta (RK) discretizations to advance the model in time.
120 In Section 3.3, we outline the specific steps involved in and the computational
121 tools employed for generating constrained computational meshes (i.e., domain
122 discretizations) for the developed modeling approach. Next, results from a set
123 of four test cases are presented in Section 4.1, where numerical solutions are

124 compared to known analytic solutions, experimental data and numerical results
 125 from another (well-established) model in the field. And finally, to conclude, we
 126 provide a brief summary of the work and discuss our ongoing and future work
 127 in this area.

128 2. Governing equations

129 2.1. Overland flow

130 Using a Cartesian coordinate system (x, y, z) , consider a land surface $z =$
 131 $f(x, y)$ that describes the topography of a given watershed. When this land
 132 surface is subjected to a rainfall rate $r(x, y, t)$, which (in general) will be a
 133 function of both the horizontal (x, y) coordinates and time t , that exceeds its
 134 infiltration rate $i(x, y, t)$, flow over the land surface — known as surface runoff
 135 or overland flow (the latter term will be used here) — can occur. This type of
 136 flow can be described by the two-dimensional shallow water equations, which
 137 consist of (i) a depth-integrated continuity equation

$$\frac{\partial H}{\partial t} + \nabla \cdot H\mathbf{U} = r - i,$$

138 where $H(x, y, t)$ is the depth of water and $\mathbf{U} = (u(x, y, t), v(x, y, t))$ is the depth-
 139 averaged, horizontal velocity vector; and (ii) momentum balance equations in
 140 the horizontal directions, which can be written in the form

$$\frac{\partial}{\partial t} (H\mathbf{U}) + \nabla \cdot \mathbf{F} = g H (\mathbf{S}_0 - \mathbf{S}_f),$$

141 where

$$\mathbf{F} = \begin{pmatrix} Hu^2 + \frac{1}{2}gh^2 & Huv \\ Huv & Hv^2 + \frac{1}{2}gH^2 \end{pmatrix}, \quad \mathbf{S}_0 = \begin{pmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial y} \end{pmatrix}, \quad \mathbf{S}_f = \begin{pmatrix} S_{f_x} \\ S_{f_y} \end{pmatrix},$$

142 with g being gravitational acceleration and the vectors $\mathbf{S}_0$ and $\mathbf{S}_f$ being the
 143 bed (land) slope and friction slope terms, respectively. The latter is typically
 144 computed using a Manning-type resistance law,

$$\mathbf{S}_f = \frac{n^2 \|\mathbf{U}\|}{H^{4/3}} \mathbf{U},$$

where n is the Manning resistance coefficient, which is dependent on the land surface roughness and can be obtained from standard references, see, for example, [20].

2.2. Open-channel flow

Overland flow often collects into natural and man-made channels that act to route the water within a given watershed to a single converging point. The resulting (open-channel) flow can be described using one-dimensional, section-averaged versions of the equations given above (also known as the Saint-Venant equations). These equations can be written in the form

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial s} = q_L \quad (1)$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial s} \left(\frac{Q^2}{A} + g I_1 \right) = g A (S_0 - S_f) + g I_2,$$

where s is a curvilinear coordinate along the river or channel centerline, A denotes the cross-sectional area of flow, $Q = \bar{u}A$ is the mean cross-sectional volumetric discharge (note: $\bar{u}$ is the section-averaged velocity), q_L is a lateral inflow rate per unit length of channel, and I_1 and I_2 are integral terms accounting for pressure forces (hydrostatic and wall pressure terms, respectively).

In the case of open-channel flow, a Manning formulation for the friction slope S_f takes the form

$$S_f = \frac{n \bar{u}^2}{R_h^{2/3}},$$

where $R_h = A/P$ is the hydraulic radius, which is a ratio of the cross-sectional area of flow A and the wetted perimeter P .

2.3. The kinematic wave approximation

In many cases of both open-channel and overland flow, the friction and bed-slope terms are dominant in the momentum equations (i.e., the remaining terms are negligibly small). This leads to, for example, in the case of open-channel flow, an approximate relationship of the form

$$S_0 \approx S_f = \frac{n \bar{u}^2}{R_h^{2/3}}.$$

169 Solving this expression for $\bar{u}$ gives an approximation for the velocity of the form

170

$$\bar{u} \approx \alpha \sqrt{S_0} R_h^\beta, \quad (2)$$

171 where $\alpha = 1/n$ and $\beta = 2/3$. (Note: setting $\alpha = C$, the Chezy coefficient, and
172 $\beta = 1/2$ in (2) gives the well-known Chezy formula).

173 Substitution of (2) into (1) results in the so-called kinematic wave approx-
174 imation for open-channel flow, which reduces the problem to a single equation
175 in terms of a single unknown A ; that is,

Kinematic wave approximation: 1D open-channel flow

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial s} \underbrace{\left(\alpha \sqrt{S_0} R_h^\beta A \right)}_{= Q} = q_L. \quad (3)$$

176 Similar considerations in the case of overland flow result in the following
177 kinematic wave approximation (when using a Manning-type resistance law)

Kinematic wave approximation: 2D overland flow

$$\frac{\partial H}{\partial t} + \nabla \cdot \mathbf{q} = r - i, \quad (4)$$

where $\mathbf{q} = (q_x, q_y)$ with

$$q_x = \frac{S_{0x}}{n \sqrt{S_0}} H^{5/3}, \quad q_y = \frac{S_{0y}}{n \sqrt{S_0}} H^{5/3},$$

178 where S_{0x} and S_{0y} are the x and y components, respectively, of the slope vector
179 $\mathbf{S}_0$; see, for example, [21] for a more detailed exposition on this equation. Note
180 that the equations are coupled via the discharge terms q_L and $\mathbf{q}$ of Equations
181 (3) and (4), respectively, which is described in more detail for our modeling
182 approach in subsection 3.2.1.

Figure 1: Simple schematic of a watershed of horizontal extent Ω , with boundary $\partial\Omega$ indicated by the thick black line. In hydrologic modeling, the watershed’s drainage network (the set of dashed blue lines denoted by ω) is typically represented as a set of connected line segments ω_i . The inset (*right*) shows a simple example of how junctions are handled in the numerical model; see discussion in Section 3.2.

3. Discretization

3.1. Domain discretization

Consider a watershed, or drainage basin, with horizontal extent Ω as illustrated in Figure 1. By definition (of a watershed), all surface water flow within Ω will converge to a single outlet point P . The outlet point is the terminus of the watershed’s drainage network — a tree-like structure of “concentrated” (open-channel) flow pathways that collect the “unconfined” overland flow of neighboring hillslopes. This description gives rise to the conceptual model of a drainage basin presented earlier that consists of “two geomorphological components: a set of hillslopes dominated by unconfined overland flow and a branching network of channels conveying concentrated flow” [10].

Given the relative scales of these two components, in hydrological modeling

the latter component (the channel network) is typically represented simply as a set of connected line segments, which we denote by ω_i ; see inset of Figure 1. This indicates that the set of hillslopes that constitute (what we will refer to as) the overland flow component completely covers the watershed. Therefore, in our modeling approach, the kinematic wave approximation for overland flow, i.e., Eq. (4), is solved over the entire domain Ω , while the kinematic wave approximation for open-channel flow, i.e., Eq. (3), is solved over the (one-dimensional) domain ω .

Discretization of the watershed needs to take into account the coupled nature of the two geomorphological components. This is accomplished in the present modeling approach through the use of a finite element partition, or mesh, $\mathcal{T}_h$ of Ω , consisting of elements Ω_j with edges aligned along ω ; see, as examples, Figures 5 and 6. The collection of edges aligned along ω form a one-dimensional finite element partition $\mathcal{E}_h$ of ω , consisting of channel elements ω_i . The construction of such a mesh, starting from a given digital elevation model (DEM), is described in detail in Section 3.3.

3.2. The discontinuous Galerkin spatial discretizations

Given finite element partitions as described above, we obtain weak forms of (3) and (4) by multiplying the equations by suitably smooth test functions $v(s)$ and $w(x, y)$ and integrating over each one-dimensional ($\omega_i \in \mathcal{E}_h$) and two-dimensional ($\Omega_j \in \mathcal{T}_h$) element, respectively, where the flux terms are integrated by parts. This yields

$$\int_{\omega_i} \frac{dA}{dt} v ds - \int_{\omega_i} Q \frac{dv}{ds} ds + Q v \Big|_{\partial\omega_i} = \int_{\omega_i} q_L v ds,$$

$$\int_{\Omega_j} \frac{dH}{dt} w d\mathbf{x} - \int_{\Omega_j} \mathbf{q} \cdot \nabla w d\mathbf{x} + \int_{\partial\Omega_j} \mathbf{q} \cdot \mathbf{n} w ds = \int_{\Omega_j} (r - i) w d\mathbf{x},$$

where $d\mathbf{x} = dx dy$, $\partial\omega_i$ and $\partial\Omega_j$ indicate the boundaries of ω_i and Ω_j , respectively, and $\mathbf{n}$ is the outward unit normal to the element boundary $\partial\Omega_j$.

Next, we seek approximate (trial) solutions (A_h, H_h) to (A, H) that belong

221 to the finite-dimensional spaces

$$\begin{aligned}\mathbb{V}_{hp}(\omega) &= \left\{ v : v|_{\omega_i} \in \mathbb{P}_p(\omega_i), \forall \omega_i \in \mathcal{E}_h \right\}, \\ \mathbb{W}_{hp}(\Omega) &= \left\{ w : w|_{\Omega_j} \in \mathbb{Q}_p(\Omega_j), \forall \Omega_j \in \mathcal{T}_h \right\},\end{aligned}$$

222 respectively, where $\mathbb{P}_p$ denotes the space of polynomials of degree (at most) p
223 and $\mathbb{Q}_p$ denotes the space of bipoynomials complete up to (at most) degree p .

224 Note that the spaces defined above do not enforce global continuity of the
225 functions. Therefore, the trial solutions may be discontinuous at element bound-
226 aries. Consider, for example, a discontinuity in A_h at a (non-junction) point s_j
227 with values

$$A_j^- = \lim_{s_j \rightarrow s_j^-} A_h(s_j, t) \text{ and } A_j^+ = \lim_{s_j \rightarrow s_j^+} A_h(s_j, t)$$

228 as s_j is approached from the upstream and downstream element, respectively.
229 This discontinuity in A_h , in turn, introduces, dual values of the flux $Q_j^- =$
230 $Q(A_j^-)$ and $Q_j^+ = Q(A_j^+)$ at point s_j . In the DG formulation, we replace these
231 (possibly) dual-valued fluxes (again, at non-junction points) with single-valued
232 numerical fluxes $\widehat{Q}_j$. Given that kinematic waves propagate downstream only,
233 we can simply employ an upstream numerical flux, that is, $\widehat{Q}_j \equiv Q_j^-$. Similarly,
234 along a given flow edge Γ_i in the two-dimensional formulation, we replace $\mathbf{q}|_{\Gamma_i}$
235 by $\widehat{\mathbf{q}} = \mathbf{q}(H_i^-)$, where H_i^- is the upstream trace of H_h along edge Γ_i .

236 In the one-dimensional formulation, note that there may be multiple up-
237 stream elements at junction points s_j , leading to multiple flux values $Q_{i,j}^-$ at
238 point s_j from the multiple upstream elements ω_i . Therefore, in the downstream
239 element, we take the *incoming* numerical flux to be

$$\widehat{Q}_j^{(\text{in})} = \sum_i Q_{i,j}^- \quad (\text{In downstream elements of junctions}),$$

240 where the sum is over all upstream elements ω_i connected at point j . (Note,
241 however, that the outgoing numerical flux used in a given upstream element ω_i
242 is simply $\widehat{Q}_j = Q_{i,j}^-$ as before.) The inset of Figure 1 presents a simple example
243 of this approach for a junction with two upstream elements. (We note that the

channel network is an arborescence, rooted at the outlet point of the drainage basin; see, for example, [22]. In the present context, this indicates that there would not be more than one downstream element at a junction point.)

Replacing (A, H) by their trial solutions (A_h, H_h) , using the numerical fluxes $(\widehat{Q}, \widehat{\mathbf{q}})$ for $(Q, \mathbf{q})$ at the element boundaries as defined above, and taking the test functions $(v, w) = (v_h, w_h)$, we arrive at the following discrete weak form of the problem: find $A_h \in \mathbb{V}_{hp}$ and $H_h \in \mathbb{W}_{hp}$ such that for all test functions $v_h \in \mathbb{V}_{hp}$ and $w_h \in \mathbb{W}_{hp}$

$$\begin{aligned} \int_{\omega_i} \frac{dA_h}{dt} v_h ds - \int_{\omega_i} Q(A_h) \frac{dv_h}{ds} ds + \widehat{Q} v_h^{(\text{int})} \Big|_{\partial\omega_i} &= \int_{\omega_i} q_L v_h ds, \\ \int_{\Omega_j} \frac{dH_h}{dt} w_h d\mathbf{x} - \int_{\Omega_j} \mathbf{q}(H_h) \cdot \nabla w_h d\mathbf{x} + \int_{\partial\Omega_j} \widehat{\mathbf{q}} \cdot \mathbf{n} w_h^{(\text{int})} ds &= \int_{\Omega_j} (r - f) w_h d\mathbf{x}, \end{aligned}$$

over each $\omega_i \in \mathcal{E}_h$ and each $\Omega_j \in \mathcal{T}_h$. (Note: $v_h^{(\text{int})}$ and $w_h^{(\text{int})}$ denote the interior traces of the test functions.)

Upon choosing bases for the finite element spaces $\mathbb{V}_{hp}$ and $\mathbb{W}_{hp}$, the above discrete weak forms reduce to systems of time-dependent ordinary differential equations (ODEs) of the form

$$\frac{d\mathbf{A}}{dt} = \mathcal{R}_{hp}^{(1D)}(\mathbf{A}), \quad \frac{d\mathbf{H}}{dt} = \mathcal{R}_{hp}^{(2D)}(\mathbf{H}), \quad (5)$$

where $\mathbf{A}$ and $\mathbf{H}$ are vectors of time-dependent degrees of freedom associated with A_h and H_h , respectively, and $\mathcal{R}_{hp}^{(nD)}$ is the n -dimensional DG spatial operator.

As an example, in the one-dimensional case, given a set of basis functions $\Phi = [\phi_0, \phi_1, \dots, \phi_p]^T$ for the space $\mathbb{V}_{hp}$, the trial solution A_h over ω_i can be expressed as

$$A_h|_{\omega_i} = \Phi^T \mathbf{A}_i$$

where $\mathbf{A}_i$ is a (column) vector of the time-dependent degrees of freedom of the DG solution over ω_i . The discrete weak form for each ω_i can then be written as a system of ODEs in the form

$$\mathbf{M}_i \frac{d\mathbf{A}_i}{dt} = \mathbf{F}_i,$$

266 where the components, M_{kl} , of the element mass matrix $\mathbf{M}_i$ are given by $M_{kl} =$
 267 $\int_{\omega_i} \phi_k \phi_l dx$ and

$$\mathbf{F}_i = \left[F_i(\phi_0), F_i(\phi_1), \dots, F_i(\phi_p) \right]^T$$

268 with

$$F_i(\phi_k) = \int_{\Omega_i} Q \frac{d\phi_k}{ds} ds - \widehat{Q} \phi_k^{(\text{int})} \Big|_{\partial\omega_i}$$

269 We note that $\mathbf{M}_i$ is a local (element) matrix and that the elements are coupled
 270 through the numerical flux terms appearing in $\mathbf{F}_i$.

271 Inverting the element mass matrices, the full set of one-dimensional semidis-
 272 crete equations over a set of N elements can be written compactly as

$$\frac{d\mathbf{A}}{dt} = \mathcal{R}_{hp}^{(1D)}(\mathbf{A})$$

273 where

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_N \end{pmatrix}, \quad \mathcal{R}_{hp}^{(1D)}(\mathbf{A}) = \begin{pmatrix} \mathbf{M}_1^{-1} \mathbf{F}_1 \\ \mathbf{M}_2^{-1} \mathbf{F}_2 \\ \vdots \\ \mathbf{M}_N^{-1} \mathbf{F}_N \end{pmatrix}.$$

274 A similar set of equations is obtained in the two-dimensional case.

275 In our implementation, we use the set of Legendre polynomials of degree p
 276 as our basis for $\mathbb{V}_{hp}$ and either a tensor product of Legendre polynomials, in
 277 the case of quadrilateral elements, or the so-called Dubiner basis, in the case of
 278 triangular elements, for $\mathbb{W}_{hp}$; see, for example, [23], for more details on these
 279 bases. We note that these are so-called modal bases that are hierarchical; for
 280 example, with the Dubiner basis, a first-order approximation is obtained by
 281 using one basis function, $\phi_0 = 1$, a second-order approximation by adding two
 282 additional basis functions, $\phi_1 = (3y + 1)/2$ and $\phi_2 = x + (y + 1)/2$, etc. All
 283 integrals of the DG spatial operators are computed using Gauss quadrature of
 284 sufficient degree.

285 3.2.1. Multidimensional coupling of the spatial discretizations

286 The one- and two-dimensional DG equations are coupled through the lateral
 287 flux terms q_L . Specifically, for a given pair of two-dimensional elements Ω_j and

Figure 2: An illustration of the multidimensional coupling that is accomplished by simply setting $q_L = \hat{q} \cdot \mathbf{n}$ at the common 1D and 2D (edge) Gauss-Lobatto integration points.

288 $\Omega_{j'}$ with “left” (l) and “right” (r) edges, Γ_l and Γ_r , that are coincident with
 289 channel element ω_i , we must have

$$\int_{\omega_i} q_L ds = \int_{\Gamma_l} \hat{q} \cdot \mathbf{n} ds + \int_{\Gamma_r} \hat{q} \cdot \mathbf{n} ds.$$

290 This condition is enforced numerically by using the same set of Gauss quadrature
 291 points for numerically integrating the one-dimensional channel elements ω_i and
 292 the edge integrals of the the two-dimensional overland flow elements. Then,
 293 we simply set $q_L = \hat{q}_l \cdot \mathbf{n} + \hat{q}_r \cdot \mathbf{n}$ at a given Gauss point. Here, we make use
 294 of Gauss-Lobatto rules, which include the endpoints of the interval, of degree
 295 $2p+1$ for a DG spatial discretization of degree p , as suggested in [24]; see Figure
 296 2 for an illustration of this idea for $p = 1$ (linear) elements.

297 3.3. Runge-Kutta Time Discretization

298 The systems of ODEs given by (5) are solved in time using explicit, strong-
 299 stability-preserving (SSP) Runge-Kutta (RK) time steppers of order $p+1$, where
 300 p is the degree of the DG spatial approximation. Application of an s -stage SSP
 301 RK method to, for example, the one-dimensional system of ODEs using a time

step of Δt can be written in the following Shu–Osher representation

$$\begin{aligned}\mathbf{A}^{(0)} &= \mathbf{A}^n \\ \mathbf{A}^{(i)} &= \sum_{l=0}^{i-1} \alpha_{il} \mathbf{A}^{(l)} + \Delta t \beta_{il} \mathcal{R}_{hp}^{(1D)}(\mathbf{A}^{(l)}), \quad i = 1, 2, \dots, s \\ \mathbf{A}^{n+1} &= \mathbf{A}^{(s)},\end{aligned}$$

where $\mathbf{A}^n \equiv \mathbf{A}(n\Delta t)$, $n \in \mathbb{N}$, and where α_{il} and β_{il} are coefficients satisfying certain order, consistency and SSP constraints; see, for example, [25]. We note that minmod-based slope limiters are applied at each RK stage to prevent spurious oscillations; see, for example, [24].

4. Mesh generation for the multidimensional approach

Starting from a DEM of a given geographical area, the generation of computational meshes for the numerical approach outlined above is carried out in two main steps:

1. A DEM data-extraction step that makes use of a set of open-source Matlab functions called TopoToolbox [26] to pre-process (e.g., pit-fill, smooth, etc) the DEM data and automatically extract watershed boundaries and channel networks. Figure 3, for example, shows the extracted watershed boundary and channel network obtained from a sample DEM (see also Figure 17 of Section 4.1). Various options are available for identifying the extent of the channel network in TopoToolbox; e.g., selection can be made based on the Strahler stream order or calculated flow accumulation; see [26] for details.
2. A mesh-generation step that makes use of the extracted DEM data and a program developed by the first three authors, called ADMESH+ [11], which has been further developed under this work. This step ultimately provides the underlying mesh data structure that is required by the outlined numerical approach and that is described in detail below.

Figure 3: The channel network (blue lines) and watershed boundary (black lines) extracted from a sample DEM using TopoToolbox.

4.1. ADMESH+: *Generating a constrained mesh and the accompanying data structure*

A full description of the ADMESH+ methodology can be found in [11], which incorporates and builds on the ideas and methodology of Persson’s DistMesh program — a simple, open-source mesh generator implemented in Matlab [27]. Briefly, a mesh- or element-size function, $h(\mathbf{x})$, is first constructed that is used to prescribe element sizes h throughout the domain. These sizes are based on a number of geometric factors, such as shoreline/boundary curvature and bathymetric/topographic gradients, as well as user-defined inputs, such as target maximum/minimum element sizes and mesh-grading specifications (i.e., the ratio of neighboring elements should not exceed some specified factor). Given the element-size function, a Delaunay triangulation of an initial set of mesh nodes with a density proportional to $1/h(x)^2$ is then generated and the nodes of this initial mesh are re-positioned by solving for force equilibrium (iteratively) around each node, making use of a spring mechanics’ analogy; see [11] for details.

The main extension of the ADMESH+ methodology for this work is to allow

Figure 4: Insets showing the original representations of the channel network (*left*, blue line) and watershed boundary (*right*, black line) obtained from the DEM and the smooth spline approximations (red lines) used in ADMESH+.

for interior constraints when generating the mesh. That is, we wish to compute a triangulation in which element edges are constrained along the specified channel network and watershed boundaries. Given that DEM data is typically recorded on a structured grid, the extracted channel networks and watershed boundaries will often exhibit a stair-case like appearance on the scale of the original DEM. Rather than constraining element edges to conform exactly to this representation, which can lead to low-quality elements on a scale smaller than desired in the neighborhood of the constraints, we compute and work with a smooth (Hobby) spline approximation [28] of the constraints, as illustrated in Figure 4.

The spline approximation allows for a simple calculation of the “curvature” of the constraints, which can then be taken into account in sizing elements along them. Specifically, the user can specify the number of elements desired

354 per radian of curvature, which acts to place smaller element edge lengths in areas
 355 of high curvature, in order to capture the relevant geometry, but allows larger
 356 element edges to be used in straighter segments. A specific (uniform) element
 357 edge size along the constraints can also be specified by the user. Given these
 358 specified constraints, the ADMESH+ methodology described above was modified
 359 to first create a constrained Delaunay triangulation and then perform the force
 360 equilibrium calculations, where all nodes along the constraints are held fixed.
 361 Figure 5 shows an example of a constrained triangulation created by ADMESH+.
 362 The mesh contains 34,411 elements that range in size from approximately 50
 363 to 400 meters and was generated automatically in 2.46 minutes. The element
 364 quality of the mesh, on a 0–1 scale, where 0 is a completely degenerate triangle
 365 and 1 is an equilateral triangle, averaged 0.98 with a minimum element quality
 366 of 0.70; see [11] for more details on the mesh-quality measure.

367 In addition to accommodating interior constraints, ADMESH+ was also fur-
 368 ther developed to provide the underlying mesh data structure required by the
 369 described numerical approach. A “standard” mesh data structure consists of
 370 a list of nodal point coordinates, each associated with a unique identifier (ID),
 371 and an element connectivity list that contains the nodal point IDs defining a
 372 given element. This type of mesh data structure is part of the standard output
 373 of ADMESH+ (and all mesh generators). For the numerical approach consid-
 374 ered here, the data structure needs to be enriched to include a list identifying
 375 all converging edges of the mesh, i.e., those edges whose attached elements both
 376 have topographic gradients (slope vectors $\mathbf{S}_0$) pointing towards the edge; see
 377 sketch of Figure 6. As flow propagates only in the direction of the slope vector
 378 for the kinematic wave approximation (that is, downhill), this list serves as the
 379 indication that flow into these edges will not be allowed to propagate through
 380 into the neighboring elements. Instead, the flow into the converging edges will
 381 act as lateral inflow, $\mathbf{q}_L$, into a one-dimensional kinematic wave approximation
 382 that is solved along that edge.

383 All edges along the original channel network will be converging edges. In
 384 addition to these, the (interpolated) topography of the triangulation may also

Figure 5: Example of a finite element mesh generated automatically by ADMESH+, where the red lines indicate the constrained channel network.

385 produce converging edges that are not part of the defined channel network (see
 386 red lines of Figure 6). For these types of converging edges, the collected overland
 387 flow is routed to a node on the defined channel network via the steepest path.
 388 A one-dimensional kinematic wave approximation is solved over all edges along
 389 this path; Figure 6 illustrates an example of these concepts, where all converging
 390 edges and flow paths are highlighted.

Figure 6: A three-dimensional profile view of a finite element mesh produced by ADMESH+ for a sample watershed, where blue lines indicate edges that are part of the original constrained channel network, red lines indicate one-dimensional overland flow routing edges and large open arrows indicate flow direction over each element face. The sketch to the right illustrates a converging edge.

391 5. Numerical Results

392 In this section, the developed model is applied to four test cases, comparing
393 the numerical results to both analytic solutions and experimental data. In
394 what follows, the developed model is referred to as the DG-SAKE (= Section-
395 Averaged Kinematic wave Equation) model. In the first two test cases, we
396 verify the DG-SAKE model against known analytic solutions for the kinematic
397 wave approximation equations for both overland and open-channel flow. This
398 is followed by model validation against two sets of experimental results that
399 validate the overland and open-channel flow components of the model. Both
400 the model verification and validation studies examine first the overland flow
401 component of the model system, followed by the testing of the fully coupled
402 overland flow/open-channel multidimensional approach used by the model

403 5.1. Overland flow on an impervious plane

404 In [29, 30, 31], Singh uses the method of characteristics to derive analytic
405 solutions to the one-dimensional kinematic wave equation for flow occurring over
406 a plane due to rainfall under several different scenarios, such as storms moving
407 up or down an impervious or infiltrating plane. Here, we examine the basic
408 case of overland flow occurring from uniform rainfall over an impervious plane
409 (see [32] for an application of the DG-SAKE model to the moving storm test
410 case of Singh and details on the analytic solutions). This is a commonly used
411 verification test case for overland flow models; see, for example, [33, 34, 35]

412 A schematic of the problem is shown in Figure 7, where the specific numerical
413 values that are used here are indicated. An initial condition of $H(x, 0) = 0$ is
414 taken over the entirety of the domain and an upstream boundary condition of
415 $H(0, t) = 0$ is specified for all time t . The uniform rainfall rate of $r_o = 50.8$
416 mm/h is applied for a duration of $T = 30$ minutes at which point the rain stops.

417 The problem is solved over a one-dimensional mesh using a uniform element
418 size of 9.25 m (20 elements) and a time step of $\Delta t = 1.5$ s. A simulation time
419 of 1 hour is used, and piecewise polynomial spaces of degree $p = 0$ and $p = 1$

Figure 7: A schematic of the impervious plane test case subjected to a uniform rainfall rate of duration T .

are employed. The computed and analytic outflow discharges as a function of time (that is, the outflow discharge hydrographs) are shown in Figure 8. The numerical solutions compare well with the analytic solution; however, as shown in detail in the zoom-in of Figure 8, the $p = 0$ solution fails to adequately capture the latter part of the rising limb and the peak discharge at this mesh resolution. Note, however, that the $p = 1$ solution at the same mesh resolution provides a much more accurate solution, sharply resolving the rising limb leading up to the peak discharge. Figure 9, shows a second solution using a $p = 0$ approximation, this time using 640 elements. At this mesh resolution, the $p = 0$ approximation provides a solution comparable to the 20-element, $p = 1$ solution but at the expense of using 32 times the mesh resolution. More specifically, the 20-element, $p = 1$ solution has a root-mean-square error at the outflow point (over the duration of the simulation) of approximately 2.2×10^{-11} , which is two orders of magnitude lower than the 20-element, $p = 0$ solution error ($\approx 3.9 \times 10^{-9}$). This result suggests the potential benefits of using p (polynomial) refinement versus standard h (mesh) refinement for modeling overland flow.

Figures 8 and 9 can also be compared to Figure 4 of [34], which compares the analytic solution of this test case to numerical results from two different models using a normalized mesh spacing ($\Delta x/L$) of 0.0025. The DG-SAKE model is providing visually comparable results to these models at a much coarser (by an order of magnitude) normalized mesh spacing of 0.05.

Figure 8: Outflow discharge hydrographs of the impervious plane test case for a uniform rainfall rate of duration $T = 30$ minutes, showing the analytic solution (filled yellow circles), along with the DG-SAKE $p = 0$ (dashed blue line) and $p = 1$ (solid purple line) solutions using 20 elements, for (a) the entire simulation period and (b) a zoom-in at peak discharge, where it can be observed that the $p = 0$ solution fails to adequately capture the peak discharge at the current mesh resolution.

Figure 9: Outflow discharge hydrographs of the impervious plane test case for a uniform rainfall rate of duration $T = 30$ minutes, showing the analytic solution (filled yellow circles), along with the DG-SAKE $p = 0$ (dashed blue line) and $p = 1$ (solid purple line) solutions, using $N = 640$ and $N = 20$ elements, respectively, for (a) the entire simulation period and (b) a zoom-in at peak discharge, where it can be observed that the $p = 0$ and $p = 1$ solutions are now comparable but at the expense of the $p = 0$ solution using 32 times the mesh resolution.

441 5.2. The catchment-stream problem

442 In order to test both the overland and open-channel flow components of
 443 the model, as well as the coupling mechanism between them, we examine the
 444 catchment-stream problem proposed by Wooding [36], which provides analytic
 445 solutions for both the overland and open-channel flow resulting from a uniform
 446 rainfall rate r . A schematic of the problem is shown in Figure 10, which consists
 447 of two identical rectangular planes joined to form a V. The line of intersection
 448 of the planes (the width of the channel shown in Figure 10 is for illustrative
 449 purposes) is taken to be a stream (an open channel) with a slope of S_0 in the y
 450 direction. (*Note:* This slope is neglected on the catchment surfaces in both the
 451 analytic solution provided by Wooding and the mesh representation used here
 452 as shown in Figure 11.)

453 The numerical values we use for this test case are shown in Figure 10. Specif-
 454 ically, the (impervious) overland flow planes are taken to be of dimension 1000
 455 m $\times$ 800 m, with a constant downward slope in the x direction of $S_{0_x} = 0.05$
 456 and, as noted above, no slope in the y direction (denoted by S_{0_y} in the figure).
 457 The width of the channel at the intersection of the planes is taken to be 20
 458 m. The overland flow planes are subjected to a uniform rainfall rate of 10.8
 459 mm/h for a duration of 1.5 hours, and the Manning’s roughness coefficients in
 460 the overland and open-channel flow equations are set to $n = 0.015$ and $n = 0.15$,
 461 respectively. As in the previous test case, both the initial water depths and the
 462 upstream flow boundary conditions are taken to be zero. The problem domain
 463 is discretized using the triangular mesh shown in Figure 11, which consists of
 464 120 two-dimensional overland flow elements and 6 one-dimensional channel ele-
 465 ments of equal length, formed by the edges of the triangular elements along the
 466 line $x = 800$.

467 The problem is solved with the DG-SAKE model using piecewise polynomial
 468 spaces of degree $p = 2$ for both the overland and open-channel flow DG dis-
 469 cretizations. A duration of 3.3 hours is simulated using a time step of $\Delta t = 35.8$
 470 seconds. The outflow discharge hydrograph for one of the overland flow planes
 471 at a point along the bottom of the plane (i.e., along the line $x = 800$) is shown

472 in the top plot of Figure 12, where it can be observed that good agreement is
 473 obtained between the analytic and DG-SAKE solutions. As described in sec-
 474 tion 3.2.1, the computed discharges along the edges of the triangular elements
 475 at the bottom of the planes are fed into the one-dimensional component of the
 476 model as lateral inflow source terms (q_L of Equation 3). The resulting outflow
 477 discharge hydrograph for the open-channel at point $(x, y) = (800, 0)$ is shown in
 478 the bottom plot of Figure 12, where again good agreement between the model
 479 results and the analytic solution can be observed — in this case, up to the point
 480 of the recession limb of the hydrograph. (*Note:* Evaluation of the analytic so-
 481 lution for the recession limb portion of the stream outflow hydrograph proved
 482 problematic due to its complexity, see equations on page 264 of [36].) As with
 483 the previous test case, the DG-SAKE model results can also be compared to
 484 Figure 4 of [34], which visually compares results from several different models
 485 for this problem. Again, the DG-SAKE model is producing (visually) compa-
 486 rable result at a much coarser mesh resolution (compare the meshes of Figure
 487 11 of this work and Figure 3 of [34]).

Figure 12: Outflow discharge hydrographs for (a) the two-dimensional (2D) overland flow at a point along $x = 800$ (from one of the planes) and (b) the one-dimensional (1D) open-channel flow at point $(x, y) = (800, 0)$, where it can be observed that the analytic solutions (filled yellow circles) and the DG-SAKE $p = 2$ solutions (solid blue lines) are in good agreement. (See note in text regarding the analytic solution for the recession limb of the stream outflow discharge hydrograph).

488 5.3. Iwagaki's experiment

489 Next, we validate the overland flow component of the DG-SAKE model
 490 against a set of well-known experimental results from Iwagaki [37], which have
 491 been used in a number of validation studies; see, for example, [38, 39, 40]. Fig-
 492 ure 13 shows a schematic of Iwagaki's experimental setup, which consisted of an
 493 aluminum flume made up of three rectangular sections of progressively steeper
 494 slope in moving from top to bottom (the specific dimensions are shown in the
 495 schematic). These three planes were (simultaneously) subjected to rainfall in-
 496 tensities (uniform over a given plane) of rates $r = 0.1080, 0.0638$ and 0.0800
 497 cm/s, respectively from top to bottom. Three different experiments were con-
 498 ducted with rainfall durations of $T = 10, 20$ and 30 seconds being applied over
 499 each plane for a given experiment.

500 We discretize the Iwagaki test domain using 24 rectangular elements, with
 501 each of the 3 planes consisting of eight elements of length of 0.196 m as shown in
 502 Figure 14. The Manning's roughness coefficient is set to $n = 0.009$ (as reported
 503 by Iwagaki), and numerical simulations were run for a total of 70 seconds with a
 504 time step $\Delta t = 0.10$ seconds. Simulations were performed for each of the three
 505 rainfall durations, $t_d = 10, 20$ and 30 seconds, using piecewise polynomial spaces
 506 of degree $p = 1$. Numerical solutions for the outflow discharge hydrographs
 507 along with Iwagaki's experimental data at the outlet of the domain are shown
 508 in Figures 15a–15c.

509 The results show that the DG-SAKE model performs very well for the short
 510 duration test case (plot (a) of Figure 15), with output matching the rising limb,
 511 peak discharge and recession limb of the hydrograph. For cases (b) and (c), the
 512 longer duration cases, the numerical solutions tend to predict the peak runoff of
 513 the discharge too early. Also, in case (b), the peak numerical discharge is slightly
 514 over estimated; however, the numerical results for cases (b) and (c) still pro-
 515 vide reasonable approximations to the experimental data (without calibration),
 516 especially during the final phases of the recession limbs of the hydrographs. Ad-
 517 ditionally, it should be noted that the errors observed are most likely a result
 518 of the simplifying assumptions inherent in the kinematic wave approximation,

519 i.e., the remaining dynamic terms of the momentum equation are not negligibly
 520 small as assumed (see Section 2.3). A number of studies have investigated the
 521 criteria for which the kinematic wave approximation is suitable; see, for exam-
 522 ple, [12, 13, 41] (in particular see the “contour” maps of [12]). These criteria
 523 involve a dimensionless parameter termed the kinematic wave number K . With-
 524 out going into detail here, a simple analysis revealed that the kinematic wave
 525 numbers associated with cases (b) and (c) tended to (low) values of K for which
 526 the kinematic wave approximation is not as valid; see [12]. In these situations,
 527 the K value of the numerical solution could actually be used to monitor the
 528 applicability of the kinematic wave approximation and act at a threshold value
 529 for switching from the kinematic wave approximation to the full shallow water
 530 equations. This idea is discussed in more detail in Section 5.4.2.

Figure 13: A schematic of the experimental flume used in Iwagaki's Experiment.

Figure 14: Finite element mesh of Iwagaki's experimental flume.

Figure 15: Outflow discharge hydrographs for the Iwagaki test case, comparing the experimental results (filled yellow circles) and the DG-SAKE $p = 1$ solutions (solid blue lines), for the experiments of duration (a) $T = 10$, (b) $T = 20$ and (c) $T = 30$ seconds.

531 5.4. Goodwin Creek experimental watershed

532 In this final test case, results from the coupled overland and open-channel
533 flow components of the DG-SAKE model are validated against data from the
534 Goodwin Creek Experimental Watershed (GCEW) established by the U.S. De-
535 partment of Agriculture’s National Sedimentation Laboratory in 1981 [42]. The
536 GCEW, located in Panola County, Mississippi, is extensively monitored and has
537 served as a platform for several hydrology and model validation studies; see, for
538 example, [43, 35, 42, 44, 45, 46]

539 Numerical results from the DG-SAKE model (discussed in detail below)
540 are compared to observational data from on-site stream-flow gauges and to
541 numerical results from a calibration study performed with the CASC2D model
542 — a well-established, finite difference-based model for simulating overland and
543 open-channel flow; see [44, 45, 47]. For consistency in comparison, input data
544 used by DG-SAKE (with the exception of the computational mesh) was the
545 same as that used by the CASC2D model (obtained online from [44]). This
546 includes elevation, land-cover, soil-type and precipitation data as discussed in
547 the next subsection. The infiltration term of Equation (4) was computed using
548 the Green–Ampt model in DG-SAKE; see [32] for more details. No calibration
549 of the DG-SAKE model was performed.

550 5.4.1. Model input data

551 Elevation (DEM), soil-type and land-cover data are provided on 90-m×90-
552 m spaced grids as shown in the plots of Figure 16. The horizontal extent of
553 the GCEW DEM is approximately 21.3 km² with a relief of about 57 m. This
554 DEM serves as the input to construct the finite element mesh used by DG-
555 SAKE as described below. With regard to the soil-type and land-cover data,
556 seven different soil types and four different land-cover classifications have been
557 identified in the GCEW; see middle and bottom plots, respectively, of Figure 16.
558 This data informs the values of the Green–Ampt infiltration model parameters
559 and the Manning’s roughness coefficients used in DG-SAKE for the overland
560 flow model component. (Note: a uniform Manning’s roughness coefficient of

561 $n = 0.035$ was used in the open-channel flow component of the model and cross
562 sections were approximated as being rectangular). Additional information on
563 this data can be found in the extensive documentation provided in [42, 45].

564 From the GCEW DEM, the channel network is first identified and then ap-
565 proximated, along with the boundary of the domain, by a smooth spline using
566 the approach described in Section 3.3; see Figure 17. Given these approxima-
567 tions, a finite element mesh is then generated using ADMESH+, insuring that
568 element edges are aligned along the defined channel network as shown in Figure
569 18. (Note: elevation data at element nodes and at points within an element
570 required for computation, i.e., at numerical integration points, is obtained from
571 a bilinear interpolant constructed from the original DEM data). As demon-
572 strated in the example of Section 3.3, converging edges that are not part of the
573 defined channel network can occur and are identified in the mesh data struc-
574 ture provided by ADMESH+; see red lines of Figure 19. The constructed mesh
575 consists of nearly 4,000 elements, ranging in size from approximately 58 to 187
576 m, exhibits an average mesh quality of 0.97 (see discussion of Section 3.3) and
577 was generated automatically in 20 seconds.

578 Finally, input data for precipitation was constructed from data obtained
579 from sixteen precipitation gauges located within (or just outside) the GCEW
580 (see Figure 19). The particular storm event used had a rainfall duration of 4.8
581 hours, an average rainfall depth of 73.6 mm and exhibited average and maxi-
582 mum rainfall rates of 14.7 and 51.6 mm/h, respectively, with very little rainfall
583 preceding the event [45]. Rainfall rates used in DG-SAKE were interpolated spa-
584 tially at all computational points using an inverse distance (squared) weighting
585 and interpolated linearly in time.

Figure 16: Contour maps of the 90-meter-spaced elevation (*top*), soil-type (*middle*) and land-cover (*bottom*) data for the GCEW test case as used by DG-SAKE and the CASC2D model.

Figure 17: The smooth spline approximation of the the channel network (blue curves) and the boundary (black curve) obtained from the GCEW DEM (colors indicating elevation).

Figure 18: Finite element mesh of the GCEW domain generated using ADMESH+, with the preserved channel network shown in red.

Figure 19: Location of the precipitation gauges (green triangles) used in the GCEW test case shown on the finite element mesh, where the blue and red lines indicate the 1D channel network and overland flow routing paths, respectively.

5.4.2. Model results and comparison

Two numerical simulations were performed for this test case using DG-SAKE. The first simulation uses piecewise constant, $p = 0$, DG spatial approximations, while the second uses higher-order piecewise quadratic, $p = 2$, approximations. A duration of 13 hours was simulated in each case, with different time steps being used for the $p = 0$ and $p = 2$ cases based on the CFL condition ($\Delta t = 42$ and 32.02 seconds, respectively). Results from both simulations are compared to observational data and CASC2D model results located at six stream discharge gauges distributed throughout the GCEW (numbered 1, 4, 6, 7, 8 and 10 as shown in Figure 20). Hydrograph results shown in Figures 21 and 22 (for $p = 0$ and $p = 2$, respectively) are presented in units of stream velocity V (mm/h), which is computed by dividing the discharge at each station by the sub-basin area contributing flow to that station. Table 1 provides a more quantitative assessment of the results, the specific contents of which are discussed in detail below. The DG-SAKE simulations were performed on a 64-bit DELL Precision M4800 machine with 16-GB RAM and a 2.40-GHz Intel i7

Figure 20: Location of stream gauges (filled orange circles) in the GCEW used in the validation study.

processor. The CPU time for the first simulation, which has 3,944 total degrees of freedom, was 14.47 seconds, and the CPU time for the second, which has 23,664 degrees of freedom, was 6.54 minutes.

First, as can be observed in the plots of Figure 21, the $p = 0$ DG-SAKE simulation results are, in general, comparable to those of the CASC2D model. Both models accurately predict the time to peak flow and capture the general temporal character of the rising and recession limbs of the hydrographs (with the CASC2D model generally providing slightly more accurate representations of the recession limbs). One exception to the second observation is the noticeable lag time in the DG-SAKE simulation of the rising limb at station 1. We comment on this at greater length below. Magnitudes of the peak flow are captured with varying degrees of accuracy at the six gauge locations, with more sizable errors in both models being observed at gauge station numbers 6, 8 and 14. (It is noted in [45] that these gauges correspond to the smallest sub-basins.)

As was observed in the previous test cases, one (numerical) strategy for improving model results is the use of p refinement. Indeed, the $p = 2$ DG-SAKE results provide notably more accurate predictions of the magnitudes of the peak flow as can be observed in the plots of Figure 22 (while continuing to accurately capture the time to peak). Table 1 provides a quantitative measure

621 of these improvements, where the percent decrease in error achieved with p
 622 refinement (i.e., in going from $p = 0$ to $p = 2$) is shown. With all but the first
 623 gauge station, significant decreases in error are observed, ranging from 24 to 91
 624 percent. The average (over all gauge stations) of the absolute errors in peak flow
 625 for the CASC2D and DG-SAKE model simulations are also shown in Table 1,
 626 where it can be observed that the $p = 2$ DG-SAKE solution provides the lowest
 627 average error and the lowest error at four of the six gauge stations (numbers in
 628 bold in the table).

Figure 21: Hydrograph results at six stream gauges in the GCEW, comparing observational (measured) data (filled yellow circles), the CASC2D model results (dashed black lines) and the DG-SAKE $p = 0$ model results (solid blue lines).

Figure 22: Hydrograph results at six stream gauges in the GCEW, comparing observational (measured) data (filled yellow circles), the CASC2D model results (dashed black lines) and the DG-SAKE $p = 2$ model results (solid blue lines).

Table 1: Peak discharge summary of model results.

Gauge	Peak flow rate, V (mm/hr)				Absolute error (mm/hr)			% Decrease in error with p refinement
	DG-SAKE				DG-SAKE			
	Obs.	CASC2D	$p = 0$	$p = 2$	CASC2D	$p = 0$	$p = 2$	
1	7.06	7.01	7.72	7.85	0.05	0.66	0.79	...
4	9.94	10.45	8.45	11.07	0.51	1.49	1.13	24.16
6	11.71	8.62	8.79	11.96	3.09	2.92	0.25	91.44
7	11.61	10.87	8.25	11.01	0.74	3.36	0.60	82.14
8	17.07	8.50	10.57	14.71	8.57	6.50	2.36	63.69
14	15.24	8.31	8.41	11.00	6.93	6.83	4.24	37.92
Average absolute error =					3.32	3.63	1.56	57.02

629 6. Summary

630 In this paper, we have presented the development and application of a multi-
631 dimensional unstructured mesh model, called DG-SAKE, for simulating coupled
632 overland/open-channel flows in the kinematic wave approximation regime. The
633 modeling approach makes use of DG finite element spatial discretizations of
634 variable degree p , paired with explicit RK time steppers, and is supported by
635 advancements made to an automatic mesh generation tool, ADMESH+, previ-
636 ously developed by the first three authors. The developed model was applied
637 to a set of four test cases, where numerical results were found to compare well
638 with known analytic solutions, experimental data and results from a (struc-
639 tured, finite difference) model, CASC2D, that is well-established in this area
640 of application. Of particular note, the numerical results obtained from DG-
641 SAKE demonstrate the potential benefits of using p refinement, over standard
642 h (mesh) refinement, in this setting in order to achieve more accurate solutions
643 in an efficient manner.

644 Future work on this particular model will focus on the development of a
645 multi-physics approach within the model, whereby a numerical estimate of the
646 kinematic wave (and Froude) numbers within each element will be computed
647 and used to determine the appropriate set of governing equations, e.g., the
648 kinematic wave approximations or the full shallow water equations, that should
649 be solved in an element at a particular time. Among other advantages, this
650 would allow, for example, the model to properly simulate so-called backwater
651 effects that can occur in some flow scenarios. Finally, our future work in the
652 general area of hydrological modeling will explore coupling the developed model
653 to a DG-based subsurface flow model.

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