Uncertainty analysis for the net-section-collapse failure criterion of circumferentially cracked cylinders for multiple arbitrary-shaped circumferential cracks

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Abstract

In this manuscript, a generalized net-section-collapse (NSC) failure criterion of circumferentially cracked pipes with multiple arbitrary-shaped cracks is presented. This generalized NSC formulation is capable of predicting the NSC moment of a pipe with multiple arbitrary-shaped cracks distributed around the circumference of the pipe, regardless of whether they are distributed symmetrically or not. The case in which internal cracks straddle the compressive zone is accounted for in the present formulation. Closed form solutions are provided for the maximum moments of pipes containing multiple cracks with idealized shapes, namely constant depth, semi-elliptical and parabolic crack profiles. Through a series of examples, the results show the effectiveness and accuracy of the method. Using this method, quantification of the effect of the crack profile uncertainty on the maximum bending moment sustained by a cracked pipe is evaluated. It is demonstrated that while the uncertainty associated with the surface roughness of the crack profile has little to no effect on the maximum bending moment, irregular shape profiles have, not surprisingly, a large effect on the estimation of the maximum bending moment. In fact, it is mathematically shown that the uncertainty associated with the maximum bending moment

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is proportional to the magnitude of the crack profile uncertainty and inversely proportional to the square root of the uncertainty sampling size (i.e, surface roughness vs. irregular crack profile).

Keywords: Surface cracks, net section collapse, stability criterion, corrosion, ASME Section XI.

1. Introduction

Although the historical performance of piping systems in nuclear power plants, offshore drilling platforms, gas transmission lines and other applications has been exceptional, their increasing age has led to concerns regarding the unavoidable occurrence and growth of internal corrosion induced cracks (Office of Pipeline Safety, US Department of Transportation, 2004). As piping systems age, internal circumferential cracks and other degradation mechanisms can incur a reduction and loss of structural strength and an eventual failure (Kanninen et al., 1976; Cramlington, 1996). Clearly, regular inspections of piping systems and compliance with standards guidelines, such as the ASME Boiler and Pressure Vessel Code Section XI (1997) in the United States, the RSE-M Code (1997) in France, or the rules for fitness-for-service provided by the Japan Society of Mechanical Engineers (2008) in Japan, can reduce the risk of any undue failure caused by a lack of awareness regarding the integrity of the pipes.

Numerous models and computational analysis have been proposed for predicting the failure load of a pipe containing circumferential cracks. Among them, the net-section-collapse (NSC) analysis (Rahman and Wilkowski, 1998; Rahman, 1998; Lei and Budden, 2004; Kim et al., 2006; Staat and Vu, 2006; Li et al., 2010) is a simple and straightforward method for calculating the maximum load a cracked pipe can sustain assuming that failure is governed by the strength of the materials (generally based on an average between the yield stress and ultimate strength). Rahman and Wilkowski (1998) NSC analysis is

21 however limited to a single symmetrical circumferential crack centered at the top of the
22 pipe circumference. Recently, Li et al. (2010) extended the approach by Rahman and
23 Wilkowski (1998) analysis to an analysis of a cracked pipe with multiple circumferen-
24 tial cracks with constant depths located above the neutral axis (i.e. all in tension). More
25 detailed analyses (Han et al., 1999; Kim et al., 2002, 2003) are based on Finite Element
26 Analysis (FEA) and explicitly model complex crack morphology and capture the three-
27 dimensional nature of the stress field. For example Han et al. (1999) computed the limit
28 moment of local wall thinning pipe under pure bending by employing 3D elastic-plastic
29 finite element analysis. Their results showed that the limit moment of pipe is affected not
30 only by the width of a defect but also by the longitudinal length of a defect. For a given
31 longitudinal length of a defect overpassing some critical value, the authors showed that
32 results from the NSC criterion are in good agreement with the results from finite element
33 analysis.

34 The above mentioned analyses, both analytical and computational, are limited to ide-
35 alized crack profiles (either constant depth, semi-elliptical depth, or parabolic depth) and
36 generally apply to a single circumferential crack. However actual cracks present in piping
37 systems have variable profiles around the pipe circumference (Rahman and Wilkowski,
38 1998). Choosing an idealized crack constitutes a conservative approach. Furthermore,
39 API579 (Anderson and Osage, 2000), BS7910 (British Standards Institution, 2000) and
40 other codes have defects interaction rules that can require multiple defects to be character-
41 ized by a circumscribing ellipse for example reinforcing the conservatism for estimating
42 plastic collapse. Few studies in the literature have looked at the impact of the variability
43 and stochastic nature of cracks on the reduction and loss of structural strength of piping
44 systems. Most notably, Rahman (1997) described the development of a probabilistic frac-
45 ture mechanics model for analyzing circumferential through-walled cracked pipes sub-
46 jected to bending loads. First- and second-order reliability methods (FORM/SORM) as

well as importance sampling and Monte Carlo Simulations were used for conducting a structural reliability analysis and for determining the characteristics of the J-integral for a circumferential through wall cracked pipe as a function of the applied bending moment.

The present work proposes a generalized method for NSC analysis of cracked pipes with multiple arbitrary crack shapes. This method can be used to calculate the maximum load of a circumferentially cracked pipe containing multiple non-interacting internal surface cracks with variable depth subjected to combined bending and tension (pressure induced) loads. Note that since the present formulation is a NSC analysis, the hoop stress is not considered. The effect of hoop stresses on net-section-collapse load for single circumferential cracked pipes has been considered for example in Oh et al. (2008). This manuscript will focus on answering one key question: *Can we quantify how the uncertainty on the NSC moment is affected by the uncertainty on the cracks profiles?* Examples illustrating the idealized cases for a pipe containing multiple cracks with constant depth profiles, semi-elliptical profiles and parabolic profiles will be used as a starting point for discussion on the impact of the uncertainty of the crack profile on the NSC analysis.

The manuscript is organized as follows. Section 2 introduces the generalized net-section-collapse method for a pipe containing multiple surface cracks with arbitrary distribution and discusses the closed-form solutions for constant depth, semi-elliptical and parabolic cracks. Section 3 discusses the treatment and characterization of crack profile uncertainty. Section 4 provides a discussion on the main findings of the work while section 5 provides recommendations for future investigations.

2. The generalized net-section-collapse method for a pipe containing multiple surface cracks with arbitrary distribution

Let us consider a pipe containing multiple independent and dissimilar internal circumferential surface cracks with variable depth profiles as shown in Fig. 1. The pipe has a

mean radius, R_m , an average wall thickness t (consequently the internal radius R_i is simply defined as $R_i = R_m - t/2$) and is subjected to an internal pressure p and an externally applied bending moment M about the x -axis. Let N be the number of independent dissimilar internal surface cracks. Each surface crack “ j ” is characterized by (i) its angular position γ_j (center of the crack) measured from the y -axis, (ii) its arbitrary crack depth profile $a_j(\xi)$ as a function of an angular coordinate measured from the y -axis, and (iii) its total angular length $2\theta_j$. For now, the following analysis assumes that the cracks are distributed symmetrically about the bending plane (y -axis). The asymmetric case will be treated later in this manuscript. Having symmetry about the y -axis ensures a symmetrical distribution of the stresses. In the asymmetric case, the internal bending moment may have components about both the x - and y -axes, violating the equilibrium condition. As previously noted by Rahman and Wilkowski (1998), this does not mean that a real pipe cannot have an asymmetric distribution of circumferential cracks subjected to an external bending moment, rather the NSC criterion may not be the most appropriate method.

Fig. 2 shows the internal stress distribution in a cracked pipe wall containing N dissimilar cracks. The stress distribution has to be in equilibrium with the applied load, i.e. the internal pressure p and the bending moment M . The hoop stress from the pressure and its resulting multi-axial state of stress were not considered. In Fig. 2, the neutral axis β_N corresponds to the location of stress inversion.

2.1. Case for which all the cracks are in tension

This case corresponds to a configuration of cracks for which $\gamma_j + \theta_j < \pi - \beta_N$ and $\gamma_j - \theta_j > \beta_N - \pi$ for all cracks “ j ”, $\forall j \in [1, N]$. The equilibrium of the axial forces yields,

$$\sigma_f R_m t \left[\sum_{j=1}^N \int_{\gamma_j - \theta_j}^{\gamma_j + \theta_j} -\frac{a_j(\xi)}{t} d\xi \right] + 2\sigma_f R_m t (\pi - \beta_N) - 2\sigma_f R_m t \beta_N = p\pi R_i^2, \quad (1)$$

94 where σ_f is the flow stress. Equation (1) can be used to deduce the neutral axis β_N ,

$$\beta_N = \frac{1}{2} \left[\pi - \frac{p\pi R_i^2}{2\sigma_f R_m t} - \frac{1}{2} \sum_{j=1}^N \int_{\gamma_j - \theta_j}^{\gamma_j + \theta_j} \frac{a_j(\xi)}{t} d\xi \right]. \quad (2)$$

95 From the moment equilibrium, the net section collapse moment M is given by,

$$\begin{aligned} M &= 2\sigma_f R_m^2 t \left[\int_0^{\beta_N} \cos \xi d\xi + \int_0^{\pi - \beta_N} \cos \xi d\xi - \frac{1}{2} \sum_{j=1}^N \int_{\gamma_j - \theta_j}^{\gamma_j + \theta_j} \frac{a_j(\xi)}{t} \cos \xi d\xi \right] \\ &= 2\sigma_f R_m^2 t \left[2 \sin \beta_N - \frac{1}{2} \sum_{j=1}^N \int_{\gamma_j - \theta_j}^{\gamma_j + \theta_j} \frac{a_j(\xi)}{t} \cos \xi d\xi \right], \end{aligned} \quad (3)$$

96 for which the neutral axis β_N is given by equation (2). Note that both equations (2) and (3)
97 involve integral formulations of crack profiles $a_j(\xi)$ requiring in most cases a numerical
98 evaluation of these expressions.

99 It can be easily shown that equations (2) and (3) are consistent with those of Rahman
100 and Wilkowski (1998) for a cracked pipe containing a single symmetrical crack ($N = 1$
101 and $\gamma_j = 0$) and those of Li et al. (2010) for multiple cracks with a constant depth profile
102 symmetrically distributed about the circumference.

103 2.2. Case for which some cracks are in compression or straddle the compressive zone

104 When some cracks are completely in compression or straddle the compressive zone,
105 equations (2)–(3) need to be adjusted to account for crack closure and to allow the trans-
106 mission of any compressive forces. As shown in Fig. 3, among the N cracks present in the
107 circumference of the pipe, several configurations need to be differentiated:

- 108 • N_1 ($0 \leq N_1 \leq N$) cracks “ j ” for which if $\gamma_j + \theta_j < \pi - \beta_N$ and $\gamma_j - \theta_j > \beta_N - \pi$, the
109 cracks are completely in tension (see Fig. 3 (a)).
- 110 • N_2 ($0 \leq N_2 \leq 1$) cracks on the right of the centerline for which if $\beta_N - \pi < \gamma_j + \theta_j \leq$
111 $\pi - \beta_N$ and $-\pi - \beta_N \leq \gamma_j - \theta_j < \beta_N - \pi$, the cracks are partially in compression (see
112 Fig. 3 (b)).

- N_3 ($0 \leq N_3 \leq 1$) cracks on the left of the centerline for which if $\pi - \beta_N < \gamma_j + \theta_j \leq \pi + \beta_N$ and $\pi - \beta_N > \gamma_j - \theta_j \geq \beta_N - \pi$, the cracks are partially in compression (see Fig. 3 (c)).
- N_4 ($0 \leq N_4 \leq 1$) cracks for which if $\gamma_j + \theta_j > \pi - \beta_N$ and $\gamma_j - \theta_j < \beta_N - \pi$, both crack tips are straddling the compressive zone from above the neutral axis (see Fig. 3 (d)).
- N_5 ($0 \leq N_5 \leq 1$) cracks for which if $\gamma_j > 0$ then $\gamma_j + \theta_j > \pi + \beta_N$ and $\gamma_j - \theta_j < \pi - \beta_N$, and if $\gamma_j < 0$ then $\gamma_j + \theta_j > \beta_N - \pi$ and $\gamma_j - \theta_j < -\pi - \beta_N$, the cracks straddle the tensile zone from below the neutral axis (see Fig. 3 (e)).
- N_6 cracks ($N_6 = N - \sum_{j=1}^5 N_i$) are completely in compression (see Fig. 3 (f)).

From the force equilibrium, the calculation of the neutral axis becomes a self-consistent formulation in which the compressive zone implicitly self adjusts such that,

$$4\beta_N = 2\pi - \frac{p\pi R_i^2}{\sigma_f R_m t} - \sum_{j=1}^{N_1} \int_{\gamma_j - \theta_j}^{\gamma_j + \theta_j} \frac{a_j(\xi)}{t} d\xi - \sum_{j=1}^{N_2} \int_{\beta_N - \pi}^{\gamma_j + \theta_j} \frac{a_j(\xi)}{t} d\xi - \sum_{j=1}^{N_3} \int_{\gamma_j - \theta_j}^{\pi - \beta_N} \frac{a_j(\xi)}{t} d\xi - \sum_{j=1}^{N_4} \int_{\beta_N - \pi}^{\pi - \beta_N} \frac{a_j(\xi)}{t} d\xi - \sum_{j=1}^{N_5} \left[\int_{\gamma_j - \theta_j}^{\text{sgn}(\gamma_j)\pi - \beta_N} \frac{a_j(\xi)}{t} d\xi + \int_{\text{sgn}(\gamma_j)\pi + \beta_N}^{\gamma_j + \theta_j} \frac{a_j(\xi)}{t} d\xi \right], \quad (4)$$

Due to physical consideration, note that equation (4) has a converged solution. This can easily be proven by transforming the integral formulation into a series. The convergence is proven by recurrence.

From the moment equilibrium, the net section collapse moment M is given by,

$$M = 2\sigma_f R_m^2 t \left[2 \sin \beta_N - \frac{1}{2} \sum_{j=1}^{N_r} \int_{\gamma'_j - \theta'_j}^{\gamma'_j + \theta'_j} \frac{a_j(\xi)}{t} \cos \xi d\xi \right], \quad (5)$$

where β_N is given by (4), and $N_r = N_1 + N_2 + N_3 + N_4 + 2N_5$ (note that cracks totally in compression are omitted from the summation due to crack closure) and the adjusted crack position γ'_j and crack half-length θ'_j are defined such that:

- 131 • If all the cracks are in tension, $\gamma'_j = \gamma_j$ and $\theta'_j = \theta_j$.
- 132 • If a crack is straddling the compressive zone on the right side of the centerline,
133 $\gamma'_j = \beta_N + \theta'_j - \pi$ and $\theta'_j = \frac{1}{2} [\gamma_j + \theta_j + \pi - \beta_N]$.
- 134 • If a crack is straddling the compressive zone on the left side of the centerline, $\gamma'_j =$
135 $\pi - \beta_N - \theta'_j$ and $\theta'_j = \frac{1}{2} [\pi - \beta_N - \gamma_j + \theta_j]$.
- 136 • If both crack tips are straddling the compressive zone from above the neutral axis,
137 $\gamma'_j = 0$ and $\theta'_j = \pi - \beta_N$.
- 138 • If both crack tips are straddling the tensile zone from below the neutral axis, the
139 crack needs to be split into two cracks such that $\gamma'_{j,1} = \pi - \beta_N - \theta'_{j,1}$ and $\theta'_{j,1} =$
140 $\frac{1}{2} [\text{sgn}(\gamma_j) \pi - \beta_N - \gamma_j + \theta_j]$; and $\gamma'_{j,2} = \beta_N + \theta'_{j,2} - \pi$ and $\theta'_{j,2} = \frac{1}{2} [\gamma_j + \theta_j - \text{sgn}(\gamma_j) \pi - \beta_N]$.

141 Equations (1)–(5) are valid for circumferential surface cracks with any arbitrary shape
142 provided that they are symmetrically distributed about the pipe circumference with respect
143 to the pipe centerline. In the event that surface cracks are not symmetrically distributed, as
144 was mentioned in Li et al. (2010), the position of the coordinate axes have to be established
145 in order to minimize the bending moment to obtain a conservative estimation.

146 2.3. Asymmetric distribution of cracks around the pipe circumference

147 If the distribution of surface cracks around the circumference of the pipe is asymmetric
148 with respect to its centerline, new positions of the coordinate axis have to be defined in
149 order to minimize the failure bending moment for conservative estimation. As shown in
150 Fig. 4, if we denote by φ the rotation angle that minimizes the bending moment, the net
151 section collapse moment after rotation M' is simply given by,

$$M' = 2\sigma_f R_m^2 t \left[2 \sin \beta_N - \frac{1}{2} \sum_{j=1}^N \int_{\gamma_j - \theta_j - \varphi}^{\gamma_j + \theta_j - \varphi} \frac{a_j(\xi)}{t} \cos \xi d\xi \right]. \quad (6)$$

152 The positions of the coordinate axes for which the net section collapse moment is at a
 153 minimum satisfy the following condition,

$$M' = \min M, \forall \varphi \in [-\pi, \pi] . \quad (7)$$

154 This rotated configuration can be assimilated as the weighed equivalent symmetric case.
 155 Note that, in some cases for which all the cracks are idealized and in tension, and assuming
 156 that M' is a smooth function (i.e. at least C^1), the solution to equation (7) can be reduced
 157 to,

$$\frac{\partial M'}{\partial \varphi} = 0 . \quad (8)$$

158 For both of these conditions (C^1 function and all cracks in tension), equation (6) reduces
 159 to only one harmonic and, as such, this equation will lead to the global minimum or max-
 160 imum for $\varphi \in [-\pi, \pi]$. We will show in section 4.1 that the condition in equation (8) may
 161 not necessarily be a satisfactory condition for finding the minimum bending moment.

162 2.4. Discretization of crack profiles

163 In the most general case, equations (4)–(5) do not possess a closed-form solution,
 164 rather a discretization of the crack profiles is necessary. The crack profiles are discretized
 165 around the circumference of the pipe with a constant angular resolution of $\Delta\theta$ independent
 166 of the crack length θ_j and regardless of the crack “ j ”. As such the number of angular
 167 discretized coordinates for crack “ j ” is simply given by,

$$\eta_j = \lceil \frac{2\theta_j R_i}{\Delta\theta} \rceil , \quad (9)$$

168 where $\lceil \cdot \rceil$ denotes the ceiling function (i.e. smallest integer greater than the actual value).
 169 Note that in this case, the size of discretized steps may vary slightly from crack to crack,
 170 but this approach allows exactly the entire length of each crack to be covered with roughly
 171 the same angular discretized resolution. As illustrated in Fig. 5, the integration of a given

172 crack profile $a_j(\xi)$ over some range around the circumference of the pipe can be evaluated
 173 using discrete values of $a_j(\xi)$ by numerical quadrature such that,

$$\int a_j(\xi) d\xi = \sum_{k=1}^{\eta_j} a_j(\xi_k) \frac{2\theta_j R_i}{\eta_j}, \quad (10)$$

174 where ξ_k is the k^{th} discrete value of the angular coordinate ξ , $a(\xi_k)$ is the crack depth at
 175 angular location ξ_k .

176 Similarly to the generalized NSC formulation, some logical requirements for cracks in
 177 tension and compression need to be considered based on the previously defined cases for
 178 crack location:

- 179 • Trivially, for all cracks that are entirely in tension, the steps that are in tension are k
 180 such that $1 \leq k \leq \eta_j$.
- 181 • For cracks that are in compression on the left with respect to the polar coordinates,
 182 we must define a left bound, LB_1 , the step defining the approximate boundary be-
 183 tween the compressive and tensile zones. LB_1 satisfies the following relationship:

$$LB_1 = \lfloor \eta_j \frac{\beta_N - \pi - \gamma_j + \theta_j + \varphi}{2\theta_j} \rfloor, \quad (11)$$

185 where $\lfloor \cdot \rfloor$ denotes the floor function (i.e. the largest integer smaller than the actual
 186 value). Thus the discrete steps k are in tension for all k such that $LB_1 \leq k \leq \eta_j$.

- 187 • For cracks that are in compression on the right with respect to the polar coordinates,
 188 we must define a right bound, RB_1 , the step defining the approximate boundary
 189 between the compressive and tensile zones. RB_1 satisfies the following relationship:

$$RB_1 = \lceil \eta_j \frac{\pi - \beta_N - \gamma_j + \theta_j + \varphi}{2\theta_j} \rceil. \quad (12)$$

190 Thus the discrete steps k are in tension for all k such that $1 \leq k < RB_1$.

- For cracks for which both crack tips are straddling the compressive zone above the neutral axis, we can use the same right bound, RB_1 , and left bound, LB_1 , defined above. Then discrete steps k are in tension for all k such that $LB_1 \leq k < RB_1$.
- For cracks that straddle the tensile zone from below the neutral axis, we have two sub-cases, $\gamma_j > 0$ and $\gamma_j < 0$:

- For $\gamma_j > 0$ we must define a new bound on the left with respect to the polar coordinates, LB_2 . LB_2 satisfies the following relationship:

$$LB_2 = \lfloor \eta_j \frac{\beta_N + \pi - \gamma_j + \theta_j + \varphi}{2\theta_j} \rfloor. \quad (13)$$

If RB_1 remains defined as above, then the discrete steps k are in tension for all k such that $1 \leq k < RB_1$ and $LB_2 \leq k \leq \eta_j$.

- For $\gamma_j < 0$, we must define a new bound on the right with respect to the polar coordinates, RB_2 . RB_2 satisfies the following relationship:

$$RB_2 = \lceil \eta_j \frac{-\pi - \beta_N - \gamma_j + \theta_j + \varphi}{2\theta_j} \rceil. \quad (14)$$

If LB_1 remains defined as above, then the discrete steps k are in tension for all k such that $1 \leq k < RB_2$ and $LB_1 \leq k \leq \eta_j$.

Using the discretization scheme described above, we can now numerically estimate the position of the neutral axis such that,

$$4\beta_N = 2\pi - \frac{p\pi R_i^2}{\sigma_f R_m t} - \sum_{j=1}^N \sum_{k=1}^{\eta_j} \frac{D[a_j(\xi_k)]}{t} \frac{2\theta_j}{\eta_j}. \quad (15)$$

where

$$D[a_j(\xi_k)] = \begin{cases} a_j(\xi_k) & \text{if } \xi_k \text{ is in the tensile zone.} \\ 0 & \text{if } \xi_k \text{ is in the compressive zone.} \end{cases} \quad (16)$$

209 In order to find the bending moment following this discretization, we must first define the
 210 center location for each discrete step, $\gamma_{j,k}$ such that,

$$\gamma_{j,k} = \gamma_j - \theta_j + \left(k - \frac{1}{2}\right) \frac{2\theta_j}{\eta_j} . \quad (17)$$

211 The numerical estimate of the bending moment is subsequently given by,

$$M = 2\sigma_f R_m^2 t \left[2 \sin \beta_N - \sum_{j=1}^N \sum_{k=1}^{\eta_j} \frac{D[a_j(\xi_k)]}{t} \frac{\theta_j}{\eta_j} \cos(\gamma_{j,k} - \varphi) \right] . \quad (18)$$

212 In the case of the asymmetric distribution of cracks around the circumference of the pipe,
 213 we can simply vary the value of φ in a discretized manner between $-\pi$ and π , and sub-
 214 sequently estimate the value of β_N and the bending moment M . Based on our previously
 215 stated condition in equation (7) for the position of the coordinate axes for which the net
 216 section collapse is at a minimum, we retain the solution of φ that leads to the minimum
 217 value of the bending moment over the discrete variation of φ .

218 2.5. *Special cases*

219 As shown in Fig. 6, three distinct idealized crack geometries were considered: constant-
 220 depth crack, elliptical crack, and parabolic crack. The mathematical formulation for each
 221 profile is as follows,

$$a_j(\xi) = \begin{cases} a_{0,j} & \text{for constant-depth cracks.} \\ a_{0,j} \sqrt{1 - \left(\frac{\xi - \gamma_j}{\theta_j}\right)^2} & \text{for semi-elliptical cracks.} \\ a_{0,j} \left[1 - \left(\frac{\xi - \gamma_j}{\theta_j}\right)^2 \right] & \text{for parabolic cracks.} \end{cases} \quad (19)$$

222 Note the difference in formulation for parabolic cracks between the present formulation
 223 and that of Rahman (1998). A typo in the formulation by Rahman (1998) leads $a_j(\xi)$ to
 224 reach its maximum value at $\xi = -\theta_j$ and its minimum at θ_j , rather than being centered

around γ_j . The present formulation corrects this minor misrepresentation of the crack profile, resulting in slightly different results for the parabolic case (in the case of one crack) compared to those of Rahman (1998) but more in line with the results obtained for the two other crack profiles. For these three cases, analytical solutions to equations (2)–(5) can be easily obtained.

2.5.1. Constant-depth cracks

For a cracked pipe containing constant-depth cracks, the crack depth profile $a_j(\xi)$ is simply represented by $a_j(\xi) = a_{0,j}$, where $a_{0,j}$ is the depth of crack “ j ” at the center of the crack. The generalized NSC equations (2)–(3) reduce in this case to simple closed-form expressions:

$$\beta_N = \frac{1}{2} \left[\pi - \frac{p\pi R_i^2}{2\sigma_f R_m t} - \sum_{j=1}^N \frac{a_{0,j} \theta_j}{t} \right], \quad (20)$$

for the neutral axis and,

$$M = 2\sigma_f R_m^2 t \left[2 \sin \beta_N - \sum_{j=1}^N \frac{a_{0,j}}{t} \cos \gamma_j \sin \theta_j \right], \quad (21)$$

for the net-section-collapse moment.

2.5.2. Parabolic cracks

For a cracked pipe containing cracks with a parabolic crack depth profile, the crack depth profile $a_j(\xi)$ is simply represented by $a_j(\xi) = a_{0,j} \left[1 - ((\xi - \gamma_j)/\theta_j)^2 \right]$, where $a_{0,j}$ is the nominal depth of crack “ j ”. The generalized NSC equations (2)–(3) reduce in this case to closed-form expressions,

$$\beta_N = \frac{\pi}{2} - \frac{p\pi R_i^2}{4\sigma_f R_m t} - \sum_{j=1}^N \frac{a_{0,j} \theta_j}{3t}, \quad (22)$$

for the neutral axis and,

$$M = 4\sigma_f R_m^2 t \left[\sin \beta_N - \sum_{j=1}^N \frac{2a_{0,j} \cos \gamma_j (\sin \theta_j - \theta_j \cos \theta_j)}{t\theta_j^2} \right], \quad (23)$$

243 for the net-section collapse moment.

244 2.5.3. Semi-elliptical cracks

245 For a cracked pipe containing cracks with a semi-elliptical crack depth profile, the
 246 crack depth profile $a_j(\xi)$ is simply represented by $a_j(\xi) = a_{0,j}[1 - ((\xi - \gamma_j)/\theta_j)^2]^{1/2}$, where
 247 $a_{0,j}$ is the nominal depth of crack “j”. The generalized NSC equations (2)–(3) reduce in
 248 this case to closed-form expressions,

$$\beta_N = \frac{\pi}{2} - \frac{p\pi R_i^2}{4\sigma_f R_m t} - \frac{1}{8} \sum_{j=1}^N \frac{a_{0,j}\pi\theta_j}{t}, \quad (24)$$

249 for the neutral axis and,

$$M = \sigma_f R_m^2 t \left[4 \sin \beta_N - \frac{a_{0,j}}{t} \sum_{m=1}^{\infty} \theta_j^{2m+1} \frac{\Gamma\left(\frac{2m+1}{2}\right) \Gamma\left(\frac{3}{2}\right)}{\Gamma(m+2)} \sum_{l=m}^{\infty} \frac{(-1)^l}{(2m)!(2l-2m)!} \gamma_j^{2l-2m} \right], \quad (25)$$

250 for the net-section collapse moment, with the function $\Gamma(m)$ defined by,

$$\Gamma(m) = \int_0^{\infty} x^m \exp(-x) dx, \text{ for } x > 0. \quad (26)$$

251 3. Treatment and characterization of crack profiles uncertainty

252 As illustrated in the above, cracks are usually represented with an idealized shape, the
 253 most common profiles being constant depth, parabolic or semi-elliptical. In reality, due
 254 to material heterogeneities and the inherent stochastic character of surface cracking and
 255 corrosion processes, the shape of cracks can substantially deviate from these idealized
 256 conceptions. Furthermore, crack growth mechanisms such as fatigue do not generally lead
 257 to the often assumed idealized defect shapes. These deviations may manifest themselves at
 258 different length scales with respect to the crack length: (i) at the smaller scale in the form

of surface roughness or (ii) at larger scale in the form of irregular crack profiles. Fig. 7 illustrates these deviations over the idealized cases. These variations can be represented as a correction toward the idealized shape that is characterized with an uncertainty distribution. The corrective term accounting for the crack profile uncertainty can be considered either as a sum (reflecting how much the true shape deviates from the reference) or a product (representing the ratio of deviation from the reference). In this section we describe the treatment of the uncertainty and its characterization in the context of the Net Section Collapse method for multiple circumferential surface cracks by characterizing the uncertainty on the crack profiles either as an additive contribution or a multiplicative contribution over the nominal crack profile and demonstrate the equivalence that can be made between the two.

3.1. Crack profiles uncertainty as an additive variation over the nominal crack profile

First let us look at the uncertainty of the crack profile as an additive correction over the idealized crack profile. In this case, the crack depth $a_j(\xi)$ for any given crack “ j ” is represented as the sum of the idealized crack depth $a_{0,j}(\xi)$ and a deviation term $\epsilon_j(\xi)$ such that

$$a_j(\xi) = a_{0,j}(\xi) + \epsilon_j(\xi) , \quad (27)$$

where $\epsilon_j(\xi)$ is a random variable whose distribution reflects the type and strength of the uncertainty. Note that $\epsilon_j(\xi)$ is sampled from a distribution whose expected value $E(\epsilon_j)$ is 0.

The uncertainty of the crack profile from the idealized crack can be described by its magnitude and its frequency. In terms of $\epsilon_j(\xi)$, the magnitude can be represented by the standard deviation $\sigma(\epsilon_j)$, while the frequency can be represented by the sample size $n_{s,j}$ for crack “ j ” (i.e. how many times the value deviates from the idealized crack profile). In other words, for a high sampling size and a low standard deviation, one would consider the

uncertainty of the crack profile to be due to surface roughness, while for a small sampling size and a high standard deviation, one would consider the uncertainty of the crack profile to be due to irregular profiles over the idealized profile. Note that the sampling size can vary from crack to crack, however for convenience and clarity in the rest of this manuscript, we assume a constant sampling size (i.e. $n_{s,j} = n_s, \forall j \in [1, N]$).

In this case, the change ΔF in the force equilibrium equation (1) can simply be expressed by,

$$\Delta F = -\frac{1}{2t} \sum_{j=1}^N \int_{\gamma_j - \theta_j}^{\gamma_j + \theta_j} \epsilon_j(\xi) d\xi \quad \text{with} \quad E(\epsilon_j) = 0. \quad (28)$$

Given the fact that an integral formulation is used to estimate the effect of ϵ_j on ΔF , one can sort each sampled value of ϵ_j and represent their variation as a Cumulative Distribution Function (CDF). The mean of the uncertainty is zero, causing the effect of the uncertainty on the force equilibrium to decrease when the standard deviation tends to zero or when the sample size increases. As a consequence, it is now clear that, based on the aforementioned remark, the effect of the surface roughness (for a given amplitude) is minimal. Going back to the analysis of the force equilibrium, note that the location of the crack is irrelevant in equation (28), such that the integration can be carried over the interval $[0, 2\theta_j]$ and,

$$\int_{\gamma_j - \theta_j}^{\gamma_j + \theta_j} \epsilon_j(\xi) d\xi = \int_0^{2\theta_j} \epsilon_j(\xi) d\xi = \int_0^{2\theta_j} \sum_{m=1}^{n_s} I_m[\epsilon_{j,m}(\xi)] d\xi, \quad (29)$$

with,

$$I_m[\epsilon_{j,m}(\xi)] = \begin{cases} \epsilon_{j,m} & \text{if } \xi \in \left[\frac{2(m-1)}{n_s} \theta_j, \frac{2m}{n_s} \theta_j \right] \\ 0 & \text{otherwise} \end{cases}, \quad (30)$$

After some calculation, the integration of equation (29) reduces to,

$$\int_{\gamma_j - \theta_j}^{\gamma_j + \theta_j} \epsilon_j(\xi) d\xi = \frac{2\theta_j}{n_s} \sum_{m=1}^{n_s} \epsilon_{j,m}. \quad (31)$$

Clearly from equation (31), by discretizing the randomness in the surface crack profile and sampling it, the change in force equilibrium ΔF can simply be defined by the uncertainty in the average estimate of the distribution representing the deviation from an idealized crack using a sample of size n_s . The standard error of the mean is estimated as the standard deviation of the population divided by the square root of the sample size. As a result, the change ΔF in force equilibrium resulting from the uncertainty on the crack profile can be estimated as being equal to,

$$\sum_{j=1}^N \int_{\gamma_j - \theta_j}^{\gamma_j + \theta_j} \epsilon_j(\xi) d\xi \approx \pm \sum_{j=1}^N 2\theta_j \frac{\sigma(\epsilon_j)}{\sqrt{n_s}} . \quad (32)$$

If the uncertainty in ϵ_j is represented using a normal distribution (i.e. $\epsilon_j \sim \mathcal{N}(0, \sigma(\epsilon_j))$) then a t-distribution with $n_s - 1$ degrees of freedom can be used to represent the uncertainty in the force equilibrium equation. The result is what would be expected intuitively: the force equilibrium will be more affected by the uncertainty when the amplitude of the uncertainty is increasing and/or the frequency is reduced (i.e. irregular crack profile). One interesting consequence of equation (32) lies in the fact that one can expect the force equilibrium to be affected linearly by the standard deviation (i.e. amplitude) and to be inversely proportional to the square root of the sampling size (i.e. period).

The change ΔM in the moment equilibrium equation (3) is twofold. The first of the changes mentioned above affect β_N in a similar fashion (with variation coming from the change in force equilibrium). The second change affects the terms in the cracked region of the circumference of the pipe. Concerning the second of these changes, it can be expressed as,

$$\Delta M_1 = -\sigma_f R_m^2 t \sum_{j=1}^N \int_{\gamma_j - \theta_j}^{\gamma_j + \theta_j} \epsilon_j(\xi) \cos \xi d\xi . \quad (33)$$

A similar discretization used for the force equilibrium term can be applied to decompose

321 this set of integrals, leading to the following estimation,

$$\begin{aligned} \Delta M_1 = -\sigma_f R_m^2 t \sum_{j=1}^N \sum_{m=1}^{n_s} \epsilon_{j,k} & \left[-\sin \left(\gamma_j + \frac{2(m-1) - n_s}{n_s} \theta_j \right) \right. \\ & \left. + \sin \left(\gamma_j + \frac{2m - n_s}{n_s} \theta_j \right) \right] . \end{aligned} \quad (34)$$

322 A first-order Taylor series expansion of the sine function can be used to obtain an approx-
323 imation of this sum such that,

$$\Delta M_1 \simeq \mp \sigma_f R_m^2 t \sum_{j=1}^N 2\theta_j \frac{\sigma(\epsilon_j)}{\sqrt{n_s}} . \quad (35)$$

324 Once again the result is as expected intuitively with the moment equilibrium being more
325 affected by the uncertainty when the amplitude of the uncertainty is growing and/or the
326 frequency is reduced.

327 3.2. Crack profiles uncertainty as an multiplicative variation over the nominal crack pro- 328 file

329 When the uncertainty is applied as a multiplicative correction over the nominal ideal-
330 ized crack profile, the crack depth $a_j(\xi)$ for any given crack “ j ” can be represented as the
331 product of the idealized crack depth $a_{0,j}(\xi)$ and a deviation term $\epsilon_j(\xi)$ such that

$$a_j(\xi) = a_{0,j}(\xi) \epsilon_j(\xi) , \quad (36)$$

332 where again $\epsilon_j(\xi)$ is a random variable whose distribution reflects the type and strength
333 of the uncertainty. Note that $\epsilon_j(\xi)$ in this case now has to be sampled from a distribution
334 that needs to be strictly positive and with an expected value of 1. When using a multi-
335 plier, the equations leading to a change in force equilibrium and moment equilibrium are
336 more complex and simplify only when the idealized crack shape is constant. As such, for
337 the case of constant depth, the results presented for uncertainty as a multiplier are then

equivalent to the results presented for uncertainty as a sum. For other variation of crack shape, the equations lead to an integral of two convoluted functions (one representing the crack shape and the other the uncertainty), whose solution would be specific to the distribution selected and the crack shape considered. No simple derivations can be found and a numerical approach is preferred.

Algorithmically, the use of discretization to estimate the equations simplifies the introduction of uncertainty in the cracks. Whether the sum or multiplication is selected, the uncertainty is sampled according to its frequency (sample size) and amplitude (range of the distribution) and applied along the crack to the crack depth. The equations of force equilibrium and moment equilibrium can be estimated with the newly defined shape.

4. Results and Discussion

In this section several numerical examples based the formulation developed above are presented to show the appropriateness and effectiveness of the proposed NSC estimation for pipes containing multiple circumferential cracks of arbitrary shapes.

4.1. Analytical solutions for idealized cracks

We first consider a pipe with mean radius $R_m = 254$ mm, and a wall thickness of $t = 25.4$ mm, while the material flow stress is assumed to be $\sigma_f = 300$ MPa. The pipe is subjected to a combined bending moment and constant internal pressure of $p = 15.51$ MPa. Comparison between the various crack profiles is shown in Fig. 8. The variation of the normalized net section collapse moment M/M_0 is displayed as a function of the normalized crack depth θ/π for various ratios of the normalized maximum depth a/t . The normalizing moment M_0 is simply obtained from classical mechanics of materials and equation (3) when no cracks are present. As expected, the moment-carrying capacity of the pipe decreases with increasing crack coverage (both in terms of angle and crack depth). It is

362 interesting to note the similarity and consistency in behavior between the various crack
363 profiles. This was not the case in Rahman (1998) which contained a typo in the equation
364 describing the parabolic crack profile (Rahman's equation for the parabolic crack profile
365 reads $a(\xi) = (1 - \xi/\theta)^2$ but should have read $a(\xi) = 1 - (\xi/\theta)^2$).

366 With that in mind, we now compare the present method to reported experimental data
367 to gauge the accuracy of its predictions for various crack profiles. Fig. 9 compares the
368 experimental results from Hasegawa et al. (2007) to the generalized net section collapse
369 formulation presented in this manuscript for a pipe specimen containing two circumferen-
370 tial cracks symmetrically distributed around its circumference with respect to the y-axis.
371 Hasegawa et al. (2007) conducted four-point bending experiments without any pressure on
372 a 4-in diameter pipe containing two cracks fabricated by electric discharge machining. For
373 these experiments, the reported outer diameter of the pipe is 114.8mm with a wall thick-
374 ness t of 8.6mm. The pipe is made out of 304 stainless steel with a measured flow stress
375 of $\sigma_f = 491$ MPa. The measured depth of the surface cracks is reported to be $a/t = 0.74$
376 with the crack angle of $\theta = \pi/6$. Three data points were obtained by varying the angular
377 distance α between the cracks, with $\alpha \in [0, \pi/6]$. From the results shown in Fig. 9, we find
378 a good agreement between the results obtained from our generalized net section collapse
379 methodology and the experimental results. While the formulation assuming cracks with
380 a constant depth profile underestimates the experimental data, the formulations assuming
381 that the crack profile is either parabolic or semi-elliptical overestimate the experimental
382 results. Clearly, with such formulation, the crack profiles could be optimized and cali-
383 brated to perfectly match experimental data points. Such a task is beyond the scope of the
384 present manuscript.

385 As mentioned above, the proposed methodology to estimate the moment-carrying ca-
386 pacity of a pipe containing multiple cracks with arbitrary profiles can be used as an ap-
387 propriate and effective rupture estimation method. In the rest of the manuscript, unless

388 mentioned otherwise, we consider a pipe with the materials and geometry parameters used
 389 to generate Fig. 8. The example illustrated in Fig. 10 and Fig. 11 clarifies the effect of
 390 the number of cracks on the evolution of the net section collapse moment as a function
 391 of the combined circumferential crack coverage for both constant crack depth profile (see
 392 Fig. 11(a)) and semi-elliptical crack profile (see Fig. 11(b)). Cracks are distributed evenly
 393 around the circumference of the pipe. Note that for the cases for which we have two and
 394 four circumferential cracks, for small crack coverage, one crack may be in complete com-
 395 pression. A noticeable difference between the four cases can be clearly observed between
 396 the single crack configuration and the multiple crack configuration. The main explanation
 397 for this difference is due to the fact that in the configuration for which multiple cracks are
 398 present, more pipe ligaments in the tensile zone of the pipe carry the load than in the case
 399 of a single crack with an equivalent crack coverage. Additionally, it should be noted that,
 400 in the case of constant crack depth, when the combined crack coverage reaches the total
 401 circumference of the pipe (i.e. $\theta/\pi = 1$), the estimation of the limit load carrying capac-
 402 ity of the pipe converges towards the same value regardless of the number of cracks. As
 403 expected, this is due to the fact that for that limit, all of the cases (in terms of the number
 404 of cracks) are equivalent to having a single crack along the circumference of the pipe. On
 405 the contrary, in the case of semi-elliptical or parabolic crack profiles, the above mentioned
 406 note is no longer valid since that, for a given combined crack coverage θ/π , the uncracked
 407 cross section of the pipe will differ based on the number of cracks distributed around its
 408 circumference.

409 We now turn our attention to relevance and significance of the formulation in the case
 410 of an asymmetric distribution of cracks around the pipe circumference. Fig. 12(a) illus-
 411 trates the variation of the normalized net section collapse moment with respect to the ro-
 412 tation angle φ in the cases of a pipe containing two, three and four circumferential cracks,
 413 respectively. We notice that, as we rotate the positions of the coordinate axes in order

to minimize the failure bending moment for conservative estimation, an N -fold rotational symmetry is observed for N cracks with a variation of M/M_0 of about 7%. It is interesting to note that the variation decreases as the number of cracks increases. While it seems that in Fig. 12(a) the variation of M/M_0 is perfectly smooth, the variation is not a perfect harmonic. If we look at the contribution to the bending moment from each individual crack in Fig. 12(b), we can see how the compressive zone impacts the perfect harmonic. This effect is accentuated by the number of cracks and their area coverage. As such, this explains the decreases in amplitude of the variation of M/M_0 with respect to the rotation angle φ .

Fig. 13(a) and Fig. 13(b) emphasize the possible existence of local extrema as we rotate the positions of the coordinate axes to minimize the failure bending moment. These local extrema do not necessarily correspond to the global minimum used for the conservative estimation. Fig. 13(a) clearly shows that when no rotation is applied, the distribution of cracks actually corresponds to a local extrema (i.e. $\partial M'/\partial \varphi = 0$). A negative rotation would lead to a local minimum, but a positive rotation gives the true global minimum. Similarly, Fig. 13(b) illustrates similar observations. It is noteworthy to observe that in this case if we were to use equation (8) as a condition to find the minimum, the opposite effect would be found. Both of the examples discussed above serve as a justification for using equation (7) as a necessary and satisfactory condition for finding the minimum failure bending moment as opposed to using equation (8) as it was proposed in Li et al. (2010) for example. .

4.2. *Effect of crack profile uncertainty on the NSC moment*

As mentioned in section 3, we have have considered the uncertainty associated with crack profiles as a corrective term either as an additive or as a multiplicative variation over the nominal crack profiles. The uncertainty considered in the examples below follow a uniform distribution but other distributions could be used as well. Fig. 14 presents an il-

439 lustration of the effects of the uncertainty associated with both the sampling size n_s (see
 440 Fig. 14(a)), and the standard deviation $\sigma(\epsilon_j)$ (see Fig. 14(b)) in the case of a pipe contain-
 441 ing a single circumferential crack when the uncertainty over the nominal crack profile is
 442 considered as an addition variation. As shown in Fig. 14(c) and (d), through the numerical
 443 discretization scheme implemented, the uncertainty associated with the maximum bend-
 444 ing moment is proportional to the magnitude of the crack profile uncertainty and inversely
 445 proportional to the square root of the uncertainty sampling size (i.e, surface roughness vs.
 446 irregular crack profile) corresponding exactly to the mathematical formulation presented
 447 in equation (35). Obviously, similar results can be obtained in the case of a pipe containing
 448 multiple cracks. It is interesting to note that in the case of a pipe containing cracks with a
 449 constant depth crack profile, a one-to-one equivalence of the amplitude of the uncertainty
 450 between the additive case and the multiplicative case can be derived. Indeed, if we con-
 451 sider the normalized crack depth with respect to the pipe thickness (i.e. $a/t \in [0, 1]$) for
 452 both the additive and multiplicative variation, we have,

$$\frac{a_{j,+}(\xi)}{t} = \frac{a_{0,j}(\xi)}{t} + \epsilon_{j,+}(\xi) , \quad (37)$$

453 in the additive case with the additive profile variation $\epsilon_{j,+} \sim \mathcal{U}(-m_\epsilon, m_\epsilon)$. This additive
 454 variation can be expressed through a normalized uniform distribution $u_+ \sim \mathcal{U}(0, 1)$ such
 455 that $a_{j,+}/t$ reduces to

$$\frac{a_{j,+}(\xi)}{t} = \frac{a_{0,j}(\xi)}{t} - m_\epsilon + 2m_\epsilon u_+ . \quad (38)$$

456 Similarly in the case of the multiplicative variation, the normalized crack depth with re-
 457 spect to the pipe thickness can be expressed as,

$$\frac{a_{j,\times}(\xi)}{t} = \frac{a_{0,j}(\xi)}{t} \epsilon_{j,\times}(\xi) \quad (39)$$

458 with the multiplicative profile variation $\epsilon_{j,\times} \sim \mathcal{U}(1 - r_\epsilon, 1 + r_\epsilon)$. Again, this multiplicative
 459 variation can be expressed through a normalized uniform distribution $u_\times \sim \mathcal{U}(0, 1)$ such

460 that $a_{j,\times}/t$ reduces to

$$\frac{a_{j,\times}(\xi)}{t} = (1 - r_\epsilon + 2r_\epsilon u_\times) \frac{a_{0,j}(\xi)}{t} \quad (40)$$

461 The equivalence between the multiplicative and additive case is satisfied when

$$\frac{a_{j,+}(\xi)}{t} = \frac{a_{j,\times}(\xi)}{t}, \quad (41)$$

462 when $u_+ = u_\times$. The above logic simplifies in the case of a constant crack depth profile
 463 (i.e. $a_{0,j}(\xi) = a_{0,j}$ such that the equivalence between the additive variation case and the
 464 multiplicative case is obtained through the relation,

$$m_\epsilon r_\epsilon = \frac{a_{0,j}}{t}. \quad (42)$$

465 Table 1 list the equivalence of uncertainty when it is treated as an additive variation and a
 466 multiplicative variation over the nominal crack profile for selected amplitudes with a con-
 467 stant crack depth profile of $a/t = 0.7$. Fig. 15 shows the perfect match between both cases.
 468 Looking at the condition expressed in equation (41), we should note that this equivalence
 469 is valid through the whole crack only in the case of a constant crack depth. For other crack
 470 profiles, the ratio $a(\xi)/t$ changes with the depth and therefore changes the equivalence in
 471 equation (42) along the crack. As expected, as shown in Fig. 16, if we look at the relative
 472 error between the uncertainty treated as an additive variation and a multiplicative varia-
 473 tion for the effect of the amplitude of the crack profile on the net section collapse moment
 474 in the case where the nominal crack profile is assumed constant for various uncertainty
 475 amplitudes (see Fig. 16(a)), and in the case where the nominal crack profile is assumed
 476 semi-elliptical for various uncertainty amplitudes (see Fig. 16(b)), we clearly see that in
 477 the case of a constant crack depth, the relative is exactly null, while in the case of semi-
 478 elliptical crack profiles, differences exist depending on how the uncertainty is treated.

479 This exercise shows that, regardless of the way uncertainty on the crack profile is dealt
 480 with, the uncertainty associated with the surface roughness of the crack profile has little to

no effect on the maximum bending moment, while the uncertainty associated with irregular shape profiles has a large effect on the estimation of the maximum bending moment. The idealized case of nominal crack profiles may not necessarily be the most conservative estimation of the failure bending moment.

4.3. General remarks

One of the problems with the application of limit load solutions for circumferential surface cracked pipes, even in the case of very ductile materials such as stainless steel, is that these solutions tend to be non-conservative for short angle cracks or when other failure mechanisms (e.g., ductile fracture or necking) control failure. The reason for the non-conservatism for short angle cracks lies in the fact that the net section for such cracks cannot achieve full plasticity prior to a crack breaking through the wall and becoming a through-wall crack. Furthermore, in the case of the idealized crack shapes, the closest representation of short deep cracks (i.e. cracks with a small θ and large a/t) considered in our analysis would be circular shaped cracks. For that short range of short angles, an elastic-plastic fracture mechanics method is necessary for these small cracks. For example, Kurihara et al. (1988) observed this phenomenon while performing tests on a short surface cracked pipe subjected to bending. Kurihara et al. (1988) proposed a semi-empirical correction to the limit-load criterion on the basis of test results. This limit-load criterion gives a conservative prediction by choosing an optimum parameter to empirically fit the experimental data for short and deep circumferential surface cracks. Miller (1988) notes in his review the distinction between global collapse and local collapse and refers to earlier work on instability of ligaments ahead of deep cracks in plates. Such an approach could be adapted in the present formulation and considered for short deep cracks. Alternatively, similar to the approach used in codes such as API 579 (Anderson and Osage, 2000) or BS 7910 (British Standards Institution, 2000), the present formulation could be augmented by

506 including ductile fracture assessment explicitly in addition to collapse.

507 **5. Conclusion**

508 This manuscript presents a generalized net-section-collapse (NSC) failure criterion of
509 circumferentially cracked pipes with multiple arbitrary-shaped cracks originally developed
510 by Rahman and Wilkowski (1998) and extended to multiple crack formalism. The method-
511 ology uses the equilibrium of axial forces and the moment equilibrium for the cracked pipe
512 to derive the limit-load analysis of cracks with any arbitrary shape. Closed form solutions
513 are provided for the maximum moments of cracked pipes containing multiple cracks with
514 idealized shapes, namely constant depth, semi-elliptical and parabolic crack profiles. In
515 the case of arbitrarily-shaped cracks, it is necessary to use a discretization scheme for the
516 profiles of the cracks in order to come up with numerical estimates of the limit-loads.

517 Numerical calculations on various crack shapes and configurations were performed.
518 The results showed that significant differences exist in limit-load predictions for pipes
519 when crack profiles are approximated by constant-depth, semi-elliptical, or parabolic crack
520 profiles.

521 In fact through a statistical analysis, it is demonstrated that, while the uncertainty asso-
522 ciated with the surface roughness of the crack profile has little to no effect on the maximum
523 bending moment, irregular crack shape profiles have a large effect on the estimation of the
524 maximum bending moment. Quantification of the uncertainty on the limit-load is directly
525 proportional to the magnitude of the crack profile uncertainty and inversely proportional
526 to the square root of the uncertainty sampling size (i.e surface roughness vs. irregular
527 crack profile), regardless of whether the uncertainty on the crack profile is assumed to be
528 additive or multiplicative.

529 Finally, while the generalized NSC analysis presented in this manuscript shows good
530 agreement with experimental data, it should be noted that the present method may not be

531 well-suited for short surface cracks (especially short deep cracks), because the assumption
532 that the cross section becomes fully plastic cannot be realized before the crack breaks
533 through the wall even in fully ductile, tough materials such as stainless steel.

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Table 1: Equivalence of uncertainty when the it is treated as an additive variation and a multiplicative variation over the nominal crack profile for a constant crack depth profile

$\epsilon_{j,+}$ (%)	Equivalent $\epsilon_{j,\times}$ (%)
1%	$1\%/0.7 \simeq 1.43\%$
5%	$5\%/0.7 \simeq 7.14\%$
10%	$10\%/0.7 \simeq 14.29\%$
20%	$20\%/0.7 \simeq 28.51\%$
30%	$30\%/0.7 \simeq 42.86\%$

Figure 1: Archetype of arbitrary shaped cracks distributed around the circumference of a pipe. The grey-shaded area of the pipe corresponds to the angular location of the neutral axis.

(a)

(b)

Figure 2: (a) All the cracks are in tension, (b) Some cracks are in compression.

Figure 3: (a) Case 1: all the cracks are in tension, (b) Case 2: cracks on the left of the centerline are partially in compression, (c) Case 3: cracks on the right of the centerline are partially in compression, (d) Case 4: cracks for which both crack tips straddle the compression zone, (e) Case 5: cracks for which both crack tips straddle the tensile zone, (f) Case 6: all cracks are in compression.

Figure 4: Convention for an asymmetrical configuration of cracks.

Figure 5: Illustration of the crack discretization.

(a)

(b)

(c)

Figure 6: Idealized crack profiles typically used in pipe fracture analysis: (a) constant depth crack, (b) parabolic crack, (c) semi-elliptical crack.

(a)

(b)

(c)

Figure 7: Illustration of the crack profile uncertainty deviating from the idealized cases: (a) constant depth crack, (b) parabolic crack, (c) semi-elliptical crack.

Figure 8: Variation of the normalized net section collapse moment with respect to crack size for a cracked pipe with a single crack for the idealized cases: (a) constant depth crack, (b) parabolic crack, (c) semi-elliptical crack. Parameters used are the following: $R_m = 254$ mm, $t = 25.4$ mm, $\sigma_f = 300$ MPa, and $p = 15.51$ MPa.

Figure 9: Comparison of the results from the generalized net section collapse analysis with experimental results from Hasegawa et al. (2007) using different idealized crack profiles (constant depth, parabolic and semi-elliptical crack profiles). Results are normalized with respect to the equivalent net section collapse moment for a single crack. Parameters used are the following: $R_m = 53.1$ mm, $t = 8.6$ mm, $\sigma_f = 491$ MPa, $p = 0$ MPa, $N = 2$, $a/t = 0.74$, $\theta = \pi/6$.

Figure 10: Crack configuration used in Fig. 11 and Fig. 12: (a) Single crack, (b) two cracks equally distributed around the circumference ($N = 2$), (c) three cracks equally distributed around the circumference ($N = 3$) and (d) four cracks equally distributed around the circumference ($N = 4$).

(a)

(b)

Figure 11: Effect of the number of cracks on the evolution of the net section collapse moment as a function of the combined circumferential crack coverage for (a) constant crack depth profile and (b) semi-elliptical crack profile. Solid black lines are for a ratio of $a/t = 0.7$, dotted red lines for a ratio of $a/t = 0.3$. Asterix symbol (*) denotes one crack ($N = 1$) distributed around the circumference, solid square symbol (■) denotes two cracks ($N = 2$) distributed symmetrically around the circumference, solid circle symbol (●) denotes three cracks ($N = 3$) distributed symmetrically around the circumference, and solid triangle symbol (▲) denotes four cracks ($N = 4$) distributed symmetrically around the circumference. Crack configurations are presented in Fig. 10. Parameters used are same as the ones used in Fig. 8.

(a)

(b)

Figure 12: (a) Variation of the normalized net section collapse moment with respect to the rotation angle φ for a pipe with multiple semi-elliptical cracks distributed around the circumference. (b) Contribution to the bending moment from each individual crack. Crack configurations are presented in Fig. 10. Parameters used are same as the ones used in Fig. 8. Additionally $a/t = 0.7$ and $\theta/\pi = 0.8$.

Figure 13: Illustration of the rotation angle φ minimizing the bending moment for asymmetric distribution of cracks around the circumference of the pipe: (a) 3 cracks asymmetrically distributed around the circumference (cracks are positioned at $\gamma_j = 0, 1.9$, and -2.0 rad with a crack depth of $a_{0,j} = 23.1$ mm and crack angle $\theta_j = 0.5$ rad, $\forall j \in [1, 3]$), (b) corresponding variation of the net section collapse bending moment as a function of the rotation angle φ , (c) Asymmetric configuration of a single crack straddling the neutral axis (cracks is positioned at $\gamma_j = 3.1$ rad with a crack depth of $a_{0,j} = 17.78$ mm and crack angle $\theta_j = 1.26$ rad), (d) corresponding variation of the net section collapse bending moment as a function of the rotation angle φ . Parameters for both of these cases are $R_m = 289.35$ mm, $t = 30.9$ mm, $p = 9.18$ MPa, and $\sigma_f = 257$ MPa.

(a)

(b)

(c)

(d)

Figure 14: Effect of the uncertainty associated with (a) the sampling size n_s (i.e period), and (b) the standard deviation $\sigma(\epsilon_j)$ (i.e. amplitude) in the case of a pipe containing a single circumferential crack. Linear correlations is observed between (c) the inverse square root of the sampling size on the uncertainty and the standard deviation of the NSC moment and (d) the uncertainty amplitude and the standard deviation of the NSC. Parameters used are same as the ones used in Fig. 8. Additionally $a/t = 0.7$.

Figure 15: Equivalence for the effect of the amplitude of the crack profile on the net section collapse moment between (a) the uncertainty being treated an additive variation over the nominal crack profile, and (b) the uncertainty being treated an multiplicative variation over the nominal crack profile. Crack profile used is a constant depth crack profile. Parameters used are same as the ones used in Fig. 8. Additionally $a/t = 0.7$.

Figure 16: Relative error between the uncertainty treated as an additive variation and a multiplicative variation for the effect of the amplitude of the crack profile on the net section collapse moment: (a) in the case the nominal crack profile is assumed constant for various uncertainty amplitudes, and (b) in the case the nominal crack profile is assumed semi-elliptical for various uncertainty amplitudes. Parameters used are same as the ones used in Fig. 8. Additionally $a/t = 0.7$.