Lattice Boltzmann method for solving non-equilibrium radiative transport problems

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Abstract

Recently developed lattice Boltzmann equation (LBE) framework [1, 2] for radiation transport is extended to solve time-dependent non-equilibrium radiative and neutron transport problems. Dynamics of radiation and material energy exchange is modeled by coupling radiation transport equation with the material energy equation in a one-dimensional isotropically scattering homogenous medium. The LBE solutions are compared against the existing semi-analytical P_1 benchmark and show good agreement for different times. Furthermore, a two-dimensional D_2Q_8 lattice Boltzmann method is proposed for solving the neutron transport equation in a heterogeneous media (e.g., checkerboard lattice with pure scattering and absorbing cells). The results obtained with LBE are in good agreement with the existing discrete ordinate method results for the benchmark problem.

Keywords:

Lattice Boltzmann, Radiation Transport, Non-equilibrium, Radiation-Material interactions

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Nomenclature

Ω	Angular direction	r	Spatial location
α	Radiative loss coefficient	S	Energy source
ϕ_0	Total radiative flux	s_0	Neutron source
ψ	Non dimensional radiation intensity	T	Temperature
σ_a	Absorption cross-section	t	Time
σ_s	Scattering cross-section	T_H	Reference temperature
σ_t	Total cross-section	v	Speed of radiative transport
θ	Non dimensional temperature	W_i	Directional weights
C_v	Specific heat	I	Radiation intensity

1. Introduction

Most of the leading computer codes for solving radiation and neutron transport physics are based on probabilistic Monte Carlo methods [3, 4] (e.g., MCNP, KENO) or the deterministic discrete ordinate methods (DOM - S_N) (e.g., Denovo [5], DeCART [6]) or methods of characteristics (MOC). In Monte Carlo codes, the computational domain is composed of an assembly of several primitive geometries (e.g., cubes, cylinders, spheres), and particle populations are sampled and tracked in those subdomains for their probabilistic interactions with matter. Whereas, the deterministic methods for radiation transport usually rely on a spatial mesh with angular phase space and energy discretizations. Although these highly specialized discretizations are extremely useful for transport only applications, they are very difficult to use in a multi-physics setup for coupling to fluid flow (single and two

phase laminar or turbulent flows), conjugate heat transfer, and structural mechanics continuum codes. One reason being the entirely different spatial and temporal discretization and convergence approaches (e.g., finite volume, finite element and finite difference methods on finely-resolved unstructured tetrahedral, hexahedral or polyhedral grids) used in the continuum mechanics codes, which makes the grid transformation and interpolated field mapping very cumbersome and may even lead to a loss of accuracy. Therefore, there is a clear need to explore new methods that can allow tightly coupled solutions of both the transport and continuum physics in the same spatial and temporal discretization framework. The lattice Boltzmann equation (LBE) method is a relatively new mesoscopic kinetic approach that can provide such a single solver framework. Recently, a LBE approach was developed [1] to solve the steady state radiation transport problems in one and two dimensions. Results for a D_2Q_4 lattice—equivalent to discrete ordinates S_2 method for time-independent problems—were presented in that work [1]. In this paper, we further extend the LBE approach to solve nonequilibrium radiation and neutron transport problems in one and two dimensions and demonstrate agreements with the existing numerical benchmarks.

Nonequilibrium radiation transport with coupled dynamic material properties has always been a problem of significant importance for astrophysics, solar energy, nuclear engineering, and radiology applications. This coupled problem requires the solution to the linear Boltzmann transport equation in conjunction with linear or nonlinear equations for material dynamics. Consequently most of the practically significant cases do not have analytical solutions and specialized numerical methods are therefore required to solve them. However, for some one-dimensional problems, analytical and semi-analytical P_n solutions do exist which can be used

for validating the numerical methods. One such classical problem in radiation-material energy exchange is known as the *Marshak wave problem*. The general form of that equation is nonlinear due to the temperature term [7] and have to be linearized [8] to produce analytical, P_n and diffusion-approximation based solutions. Ganapol and Pomraning [9] have solved a particular nonequilibrium radiative transfer problem in which a free surface is irradiated with a constant radiation flux. They used a method of multiple collisions and provided solution in the form of an infinite series of integrals. Su and Olson [10] later developed a complete analytical solution in one dimension to a nonequilibrium transport problem with a source term.

In this paper, we propose an extended LBE to solve the coupled nonequilibrium problem in one-dimension and compare D_1Q_2 LBE results with the analytical P_1 results from McClarren et al. [11]. The same LBE approach is then shown to be extensible for problems in two dimensions by solving time dependent transport problems on D_2Q_8 and D_2Q_{16} LBE lattices. The results are presented for a benchmark problem having a heterogenous checkerboard lattice of absorbing and scattering media. This two-dimensional checkerboard problem was also used by Brunner [12] as a test case to compare different radiation transport methods (diffusion, flux limited diffusion, P_1 , P_7 , S_6 and Implicit Monte Carlo). The following section describes the classical equations, assumptions, and approximations used in this paper to model the radiation transport physics with the LBE.

2. Mathematical formulation

The interaction of radiation and material energies is modeled by coupling the radiation transport equation (RTE) with the material energy balance equation. Assum-

ing homogenous materials and isotropic scattering, the set of governing equations for non-equilibrium radiative transfer can be written as

$$\begin{aligned} \frac{1}{v} \frac{\partial I(r, \Omega, t)}{\partial t} + \Omega \cdot \nabla I(r, \Omega, t) = \sigma_a \left[\frac{1}{4\pi} \alpha T(r, t)^4 - I(r, \Omega, t) \right] \\ + \sigma_s \left[\frac{1}{4\pi} \int_{4\pi} I(r, \Omega', t) d\Omega' - I(r, \Omega, t) \right] + S(r, \Omega, t) \end{aligned} \quad (1)$$

and

$$\frac{1}{v} C_v \frac{\partial T(r, t)}{\partial t} = \sigma_a \left[\int_{4\pi} I(r, \Omega, t) d\Omega - \alpha T(r, t)^4 \right] \quad (2)$$

where $I(r, \Omega, t)$ is the radiation intensity at any spatial location r and time t in the direction Ω ; v is the speed of radiative transport; $T(r, t)$ is the temperature; σ_a is the absorption cross-section; σ_s is the scattering cross-section; $S(r, \Omega, t)$ is the volumetric energy source; C_v is the specific heat; and α is the radiative loss coefficient. These equations can also be written using total cross-section $\sigma_t = \sigma_a + \sigma_s$ as a parameter.

By assuming σ_a and σ_s to be constant [10] and C_v to be proportional to the cube of temperature T , as modeled by Pomraning [8],

$$C_v = \alpha T^3, \quad (3)$$

Eqs. (1) and (2) can be linearized. Following non-dimensionalization can be carried out with respect to αT_H^4 , where T_H is a reference temperature:

$$\psi(r, \Omega, t) = \frac{I(r, \Omega, t)}{\alpha T_H^4}, \quad (4)$$

$$s(r, \Omega, t) = \frac{S(r, \Omega, t)}{\alpha T_H^4}, \quad (5)$$

$$\theta(r, t) = \frac{T(r, t)^4}{T_H^4}, \quad (6)$$

and which results in the following set of governing equations:

$$\begin{aligned} \frac{1}{v} \frac{\partial \psi(r, \Omega, t)}{\partial t} + \Omega \cdot \nabla \psi(r, \Omega, t) = \sigma_a \left[\frac{1}{4\pi} \theta(r, t) - \psi(r, \Omega, t) \right] \\ + \sigma_s \left[\frac{1}{4\pi} \int_{4\pi} \psi(r, \Omega', t) d\Omega' - \psi(r, \Omega, t) \right] + s(r, \Omega, t), \end{aligned} \quad (7)$$

$$\frac{1}{v} \frac{\partial \theta(r, t)}{\partial t} = \sigma_a \left[\int_{4\pi} \psi(r, \Omega', t) d\Omega' - \theta(r, t) \right]. \quad (8)$$

After introducing a finite set of directions and associated angular distribution functions or radiation intensity for each lattice direction, above equations (7) and (8) can be discretized to yield the equivalent LBE, and are provided below:

$$\begin{aligned} \psi_\Omega(r + v_\Omega \Delta t, t + \Delta t) - \psi_\Omega(r, t) = \\ \sigma_a \left[W_\Omega \theta(r, t) - \psi_\Omega(r, t) \right] \Delta t \\ + \sigma_s \left[W_\Omega \sum_{\Omega'} W_{\Omega'} \psi_{\Omega'}(r, t) - \psi_\Omega(r, t) \right] \Delta t \\ + s_\Omega(r, t) \Delta t, \end{aligned} \quad (9)$$

$$\theta(r, t + \Delta t) - \theta(r, t) = \sigma_a \left[\sum_{\Omega'} W_{\Omega'} \psi_{\Omega'}(r, t) - \theta(r, t) \right] \Delta t. \quad (10)$$

For a two dimensional Cartesian LBE lattice, equation (9) reduces to

$$\begin{aligned} \psi_i(x + v_{i,x} \Delta t, y + v_{i,y} \Delta t, t + \Delta t) = \psi_i(x, y, t) \\ - \Delta t \left(\sigma_t \psi_i(x, y, t) + \sigma_s W_i \sum_j W_j \psi_j(x, y, t) + W_i s(x, y, t) \right), \end{aligned} \quad (11)$$

where $\psi_i(x, y, t)$ is the discrete angular neutron flux in the i^{th} lattice direction at (x, y) location at time t . Lattice velocities $v_{i,x}$ and $v_{i,y}$ are chosen such that the distance traveled by particles in a discrete time interval Δt i.e., $v_{i,x}\Delta t$ and $v_{i,y}\Delta t$ equals to the distance between the neighboring lattice points. Furthermore, W_i represents directional weighting factor for each direction of travel.

While approximating the integrals in the equations (7) and (8) numerically, weights W_j were set to '1'.

Figure 1: Lattice directions for two-dimensional D_2Q_8 system.

In 2D, the most commonly used lattice framework for LBE is D_2Q_8 typically used to solve fluid mechanics problems, which has 8 directions = 1 – 8 (Fig. 1), and the standard corresponding notation for v_i can be written as:

$$v_i = \begin{cases} v \left(\cos \frac{(i-1)\pi}{2}, \sin \frac{(i-1)\pi}{2} \right) & i = 1 - 4 \\ \sqrt{2}v \left(\cos \left(\frac{(i-5)\pi}{2} + \frac{\pi}{4} \right), \sin \left(\frac{(i-5)\pi}{2} + \frac{\pi}{4} \right) \right) & i = 5 - 8 \end{cases} \quad (12)$$

The effective speed of the first four cardinal directions $i = 1 - 4$ is equal to v and for diagonal directions $i = 5 - 8$ are equal to $\sqrt{2}v$. The weights W_i for directions with the same speeds are equal because of symmetry. The weights for

each cardinal and diagonal direction lead to isotropic generalized lattice tensors of rank 2 and 4 [13, 14]. This is achieved by conservation of lattice angular moments up to the fourth order over scattering or any other radiation interaction. In case of an isotropic function $w(\mu)$, the odd moments vanish,

$$\sum_{i=1}^8 v_{i\alpha} W_i = \int_{-1}^1 w(\mu) \mu d\mu = 0 \quad (13)$$

$$\sum_{i=1}^8 v_{i\alpha} v_{i\beta} v_{i\gamma} W_i = \int_{-1}^1 w(\mu) \mu_\alpha \mu_\beta \mu_\gamma d\mu = 0, \quad (14)$$

and the even moments can be expressed as

$$\sum_{i=1}^8 W_i = \int_{-1}^1 w(\mu) d\mu = 1 \quad (15)$$

$$\sum_{i=1}^8 v_{i\alpha} v_{i\beta} W_i = \int_{-1}^1 w(\mu) \mu_\alpha \mu_\beta d\mu = \delta_{\alpha\beta} \quad (16)$$

$$\begin{aligned} \sum_{i=1}^8 v_{i\alpha} v_{i\beta} v_{i\gamma} v_{i\eta} W_i &= \int_{-1}^1 w(\mu) \mu_\alpha \mu_\beta \mu_\gamma \mu_\eta d\mu \\ &= (\delta_{\alpha\beta} \delta_{\gamma\eta} + \delta_{\alpha\gamma} \delta_{\beta\eta} + \delta_{\alpha\eta} \delta_{\beta\gamma}), \end{aligned} \quad (17)$$

where, α, β, γ , and η are the indices for representative dimensions. These moment equations are solved for the D_2Q_8 LBE system, and the weights for cardinal and diagonal directions are obtained as $\frac{1}{5}$ and $\frac{1}{20}$, respectively.

3. Numerical Examples

3.1. Su-Olson benchmark problem for nonequilibrium radiative transfer

Let us consider a nonequilibrium radiative transfer problem, proposed by Su and Olson [10], in an initially cold (zero temperature), homogeneous, infinite, and

Table 1: Weights for different lattices

Lattice	Direction, i	Weight, W_i
D_1Q_2	1-2	0.50
D_2Q_4	1-4	0.25
D_2Q_8	1-4	0.20
	5-8	0.05

isotropically scattering slab with an internal radiation source. An analytical solution was developed in [10] for the coupled radiation transport and material energy balance equation for a one dimensional planar geometry $x \in (-\infty, \infty)$ with negligible effect from any other mode of energy exchange. The governing one dimensional transport equations were linearized by assuming temperature-independent absorption and scattering cross-sections and assuming a cubic temperature dependence of C_v . The same system of equations with source terms, as used by [10] and described in sec. II, are used here to verify our LBM solution method.

For a 1D slab, the variable $\psi(r, \Omega, t) \equiv \psi(x, \mu, t)$ has only two directions, with direction cosines $\mu = \pm \frac{1}{\sqrt{3}}$, and governing Eqs. (7) and (8) reduce to

$$\begin{aligned} \frac{1}{v} \frac{\partial \psi(x, \mu, t)}{\partial t} + \mu \frac{\partial \psi(x, \mu, t)}{\partial x} = \sigma_a \left[\frac{1}{2} \theta(x, t) - \psi(x, \mu, t) \right] + \\ \sigma_s \left[\frac{1}{2} \int_{-1}^1 \psi(x, \mu', t) d\mu' - \psi(x, \mu, t) \right] + s(x, \mu, t) \end{aligned} \quad (18)$$

and

$$\frac{1}{v} \frac{\partial \theta(x, t)}{\partial t} = \sigma_a \left[\int_{-1}^1 \psi(x, \mu', t) d\mu' - \theta(x, t) \right], \quad (19)$$

where $x \in (-\infty, \infty)$ is the spatial variable, μ is the cosine of the radiation direction measured with respect to the x-axis, $t \in (0, \infty)$ is the time variable, ψ is the radiation intensity, θ is proportional to the fourth power of the local material

temperature [Eq. (6)], v is the radiation speed, s is the radiation source, and σ_a and σ_s are the absorption and scattering cross sections of the material. An isotropic radiation source s that is uniformly distributed in a finite space $[-x_0, x_0]$ is turned on at $t = 0$ and remains active only for a finite period of time $[0, t_0]$.

The source term

$$s(x, \mu, t) = \frac{1}{2} s_x s_t \quad (20)$$

has direct dependence on space,

$$s_x = \frac{1}{2x_0} [H(x + x_0) - H(x - x_0)], \quad (21)$$

and time,

$$s_t = [H(t) - H(t - t_0)], \quad (22)$$

where H is a unit Heaviside function, x_0 is the length of half-domain where source is active, and t_0 is the time duration for which the source is active. The boundary conditions and initial condition are described below:

$$\begin{aligned} \psi_\mu(x = -\infty, t) &= 0, \\ \psi_{-\mu}(x = +\infty, t) &= 0, \\ \psi_{\pm\mu}(x, t = 0) &= 0. \end{aligned} \quad (23)$$

For this 1D problem, the LBM numerical solution can only be obtained for a computational domain of finite size and is chosen to be of length $200x_0$. In 1D there are only two angular directions which have direction cosines $\pm \frac{1}{\sqrt{3}}$, and each having weight fraction equal to 1. In Fig. [2] a-b, the LBM solution is compared against the semi-analytic P_1 solution [11] obtained in an infinite domain. Results provided in the figure are only for the half-domain due to symmetry of the problem. Except for the limitation of numerical method to simulate infinite domain,

Table 2: Parameter values used from Su and Olson [10]

σ_a	1
σ_s	0
v	1
x_0	0.5
t_0	10

all the other parameters and conditions are kept same as used in the analytical solution and are provided in Table [2] below.

The LBM results shown in Fig. [2] are in good agreement with the P_1 solution and thus verify the accuracy and applicability of the proposed method for such problems. It can be proved with the help of the following analysis that the two representative solutions of this problem (corresponding to both angular directions) can be used to retrieve the P_1 approximate form of the coupled integro-differential equations. Consider ψ_+ and ψ_- to be the solutions corresponding to directions $+\frac{1}{\sqrt{3}}$ and $-\frac{1}{\sqrt{3}}$, respectively, which satisfy the Eqs. (18) and (19). Substituting ψ_+ and ψ_- in Eqs. (18) and (19) and approximating the integral operation with a summation over the two directions yield

$$\begin{aligned} \frac{1}{v} \frac{\partial \psi_+(x, t)}{\partial t} + \frac{1}{\sqrt{3}} \frac{\partial \psi_+(x, t)}{\partial x} &= \sigma_a \left[\frac{1}{2} \theta(x, t) - \psi_+(x, t) \right] + \\ &\sigma_s \left[\frac{1}{2} (\psi_+(x, t) + \psi_-(x, t)) - \psi_+(x, t) \right] + \frac{1}{2} s_x s_t, \end{aligned} \quad (24)$$

$$\begin{aligned} \frac{1}{v} \frac{\partial \psi_-(x, t)}{\partial t} - \frac{1}{\sqrt{3}} \frac{\partial \psi_-(x, t)}{\partial x} &= \sigma_a \left[\frac{1}{2} \theta(x, t) - \psi_-(x, t) \right] + \\ &\sigma_s \left[\frac{1}{2} (\psi_+(x, t) + \psi_-(x, t)) - \psi_-(x, t) \right] + \frac{1}{2} s_x s_t, \end{aligned} \quad (25)$$

and

$$\frac{1}{v} \frac{\partial \theta(x, t)}{\partial t} = \sigma_a \left[(\psi_+(x, t) + \psi_-(x, t)) - \theta(x, t) \right]. \quad (26)$$

The approximate forms of analytical expressions for the zeroth and first moments can be written as

$$\phi_0(x, t) = \int_{-1}^1 \psi(x, \mu', t) d\mu' = [(\psi_+(x, t) + \psi_-(x, t))] \quad (27)$$

and

$$\phi_1(x, t) = \int_{-1}^1 \mu' \psi(x, \mu', t) d\mu' = \frac{1}{\sqrt{3}} (\psi_+(x, t) - \psi_-(x, t)). \quad (28)$$

Adding Eqs. (24) and (25) and then substituting the above moment expressions yield

$$\frac{1}{v} \frac{\partial \phi_0(x, t)}{\partial t} + \frac{\partial \phi_1(x, t)}{\partial x} = \sigma_a \left[\theta(x, t) - \phi_0(x, t) \right] + s_x s_t. \quad (29)$$

Now multiplying Eqs. (24) and (25) with their respective directional cosines $+\frac{1}{\sqrt{3}}$ and $-\frac{1}{\sqrt{3}}$ and adding the resultant equations give

$$\frac{1}{v} \frac{\partial \phi_1(x, t)}{\partial t} + \frac{1}{3} \frac{\partial \phi_0(x, t)}{\partial x} = -(\sigma_a + \sigma_s) \phi_1(x, t). \quad (30)$$

Equation (26) can be rewritten as

$$\frac{1}{v} \frac{\partial \theta(x, t)}{\partial t} = \sigma_a \left[\phi_0(x, t) - \theta(x, t) \right]. \quad (31)$$

Above Eqs. (29–31) are equivalent to the P_1 approximate form of the nonequilibrium radiative transport equations and can readily be solved [11].

3.2. Two-dimensional nuclear checkerboard benchmark problem

Let us consider a 2D checkerboard problem that was identified by Brunner [12] as one of the benchmarks to test and compare different radiation transport methods (diffusion, flux limited diffusion, P_1 , P_7 , S_6 and Implicit Monte Carlo) and their underlying approximations. In this benchmark, a 7 cm x 7 cm 2D square region (Fig. [3]) is considered which is made of a purely-scattering bulk material ($\sigma_t = \sigma_s = 1 \text{ cm}^{-1}$) with eleven purely-absorbing 1 cm x 1 cm checkerboards ($\sigma_t = \sigma_a = 10 \text{ cm}^{-1}$) and surrounded by vacuum on the outside of domain boundaries, while continuity of angular flux is assumed at the interfacial boundaries within the domain. There is a particle source present in the central square (colored white in Fig. [3]) of the domain which also acts as a scattering region. Furthermore, it is assumed that the particles, once absorbed, are removed from the system without heating it up; therefore, eliminating the need to model the material energy equation. In addition, the particles are assumed to travel at a constant speed of 1 cm/s inside the domain.

The governing one-speed neutron transport equation for this benchmark problem—with assumed isotropic scattering and without coupling to the material energy equation—can be written as follows:

$$\begin{aligned} \frac{1}{v} \frac{\partial \psi(r, \Omega, t)}{\partial t} + \Omega \cdot \nabla \psi(r, \Omega, t) = & -\sigma_t \psi(r, \Omega, t) \\ & + \frac{\sigma_s}{4\pi} \int_{4\pi} \psi(r, \Omega', t) d\Omega' \\ & + s_0(r, \Omega, t), \end{aligned} \quad (32)$$

where $\psi(r, \Omega, t)$ is a mono-energetic (one-speed), space- and time-dependent, angular neutron flux with the direction of neutron travel characterized by the unit vector $\Omega = \frac{v}{|v|}$; v is the neutron speed; $s_0(r, \Omega, t)$ is a space- and time-dependent

source of mono-energetic neutrons; and σ_s and $\sigma_t = \sigma_s + \sigma_a$ are scattering (isotropic) and total cross-sections, respectively. Table [3] provides values chosen for the absorption σ_a and scattering σ_s cross-sections and other parameters for this benchmark problem.

This described D_2Q_8 scheme is used to solve the lattice problem with grid spacing $\Delta x = \Delta y = 0.025 \text{ cm}$. In Fig. [4], LBM simulation results for $\log_{10} \sum \psi$ are compared with the corresponding discrete ordinate method (DOM) solutions, provided by Brunner [12] for different times ($t = 3.2 \text{ s}$ and steady state $t = 20 \text{ s}$), after the source is turned on at $t = 0$. At $t = 3.2 \text{ s}$, particles get just the right amount of time to reach the principle boundaries but not the corners of the computational domain. The two dimensional contours of total radiation intensity from both methods in Fig. [4] look similar, and are particularly in a very good agreement near the source region.

The line plots along the diagonal direction of checkerboard, shown in Fig. [5], present more clearly this agreement between LBM and DOM in the internal region. However, away from the source and near the boundaries the results from two methods don't show high degree of agreement. This can be attributed to the different number of directions in the S_6 DOM and D_2Q_8 LBM methodology, and due to different orientation of the discrete directions used in the numerical solutions. The aforementioned numerical differences between the two methods can be corrected by adopting the following possible corrective measures (i) Increasing the number of directions in a lattice, or (ii) by transporting particles beyond the first neighbors, or (iii) by introducing P_N corrective terms in the LBM algorithm based on the spherical harmonics approximations conserving certain moments of the Boltzmann equation

Table 3: Parameter values for the two dimensional nuclear checkerboard benchmark problem [12]

Box	Red	Blue	White
$\sigma_a(cm^{-1})$	10	0	0
$\sigma_s(cm^{-1})$	0	1	1
$\sigma_t(cm^{-1})$	10	1	1
$v(cm/sec)$	1	1	1
$s_0(\#/cm^3 - sec)$	0	0	1

4. Conclusions

This paper extends the recently proposed LBE [1] for solving benchmark non-equilibrium radiative and neutron transport problems. Motivation behind this work is to unify the computational framework for solving highly coupled multi-physics problems in nuclear and thermal engineering. The time-dependent non-equilibrium radiative transfer problem is a classical one dimensional problem where radiative and material energy are mutually coupled. The LBE solution of this problem is shown to be equivalent to P_1 form for all times. The most common practical application of linear Boltzmann equation is for simulating neutron population in nuclear reactors. A two dimensional benchmark problem of checkerboard lattice cells with source, scattering and absorbing media is solved using D_2Q_8 LBE. The results closely agree with the solution from S_6 DOM, except near the boundaries. The Lattice Boltzmann method experiences similar problem of ray effects as in case of discrete ordinates method, and further work is necessary to mitigate such unphysical numerical effects. However, results so far show that LBE can be a powerful tool for solving coupled multi-physics problems which can be described by Boltzmann equation. Thus, further efforts should be directed to extend this LBE framework for higher dimensions, higher accuracy with more

discrete directions and multi-energy levels.

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Figure 2: Solution for the Su-Olson problem with Table [2] parameters using LBM: a) Radiative Energy b) Material Energy

Figure 3: Two dimensional nuclear checkerboard benchmark problem. Blue and red colored regions are purely scattering and purely absorbing regions, respectively. White region in the center acts as a combined source and scattering region.

Figure 4: Nuclear checkerboard problem solution for $\log_{10} \sum \psi$ with Table [3] parameters: a) S_6 solution at 3.2 sec, b) LBM D_2Q_8 solution at 3.2 sec, c) S_6 solution at steady state, d) LBM D_2Q_8 solution at steady state

Figure 5: Line plots (D_2Q_8 vs DOM S_6) for nuclear checkerboard problem solution