Robust Econometric Inference with Mixed Integrated and Mildly Explosive Regressors

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Abstract

This paper explores in several prototypical models a convenient inference procedure for non-stationary variable regression that enables robust chi-square testing for a wide class of persistent and endogenous regressors. The approach uses the mechanism of self-generated instruments called IVX instrumentation developed by Magdalinos and Phillips (2009b). We first show that these methods remain valid for regressors with local unit roots in the explosive direction and mildly explosive roots, where the roots are further from unity in the explosive direction than $O(n^{-1})$. It is also shown that Wald testing procedures remain robust for multivariate regressors with certain forms of mixed degrees of persistence. These robustifications are useful in econometric inference, for example, when there are periods of mildly explosive trends in some or all of time series employed in the analysis but the exact knowledge on the regressor persistence is unavailable. Some aspects of the choice of the IVX instruments are investigated and practical guidance is provided but the issue of optimal IVX instrument choice remains unresolved. The methods are straightforward to apply in practical work such as predictive regression applications in finance.

Keywords: Chi-square, Instrumentation, IVX methods, Local to unity, Mild integration, Mild explosiveness, Predictive regression, Robustness.

JEL classification: C22

1 Introduction

Many economic and financial time series exhibit characteristics that include temporary periods of explosive behavior. For macroeconomic series Stock (1991, Table 2) showed that 90% confidence intervals for the autoregressive (AR) roots of the Nelson-Plosser data set contain explosive parameter regions in all but one series (the unemployment rate). For financial series Campbell and Yogo (2006, Table 4) found that 95% confidence intervals for the AR coefficient of the S&P 500

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dividend-price ratio and other series over long historical periods do not rule out explosive roots. In addition to these empirical findings, periodically occurring booms and episodes of financial exuberance support at least temporary explosive trends in economic and financial data. The idea that there are subperiods of explosive roots in economic and financial series is formally analyzed and empirically confirmed in several recent papers (Phillips et al., 2011, 2015a, 2015b; Phillips and Yu, 2011).

There has been growing interest in predictive regressions of the type used in the study by Campbell and Yogo. When the regressors in such models display some degree of persistence, inference procedures need to be robust to the value of the persistence parameter to ensure validity even asymptotically. The robustness requirements become more demanding in cases where there are many regressors with possibly different degrees of persistence. To address some of these complexities in inference in regressions with persistence regressors, Magdalinos and Phillips (2009b, hereafter MP) recently introduced a novel IV procedure (called IVX regression) that provides robust chi-square inference in a much wider vicinity of unity than existing studies that have typically only considered the near integrated (local to unity) single regressor case. In particular, MP showed that self-normalized IVX test statistics have an asymptotically pivotal chi-square distribution for multivariate regressors that may be integrated, near integrated, or mildly integrated and which thereby fall within a vector autoregressive framework (Lutkepohl, 2005) while allowing for more general time series inputs than martingale differences. The tests have been successfully used in applied work on predictive regressions (Kostakis et al., 2014; Gonzalo and Pitarakis, 2012). IVX methods have been also studied in long-horizon regression applications (Phillips and Lee, 2013) and in quantile regression (Lee, 2014) contexts.

The present paper is an exploratory study in prototypical cases that show how the IVX methodology extends to a wider range of potential regressors that includes local unit root (LUR) explosive and mildly explosive roots, thereby covering periods of exuberance in economic and financial data. The limit theory involves some novel developments in the mildly explosive case, wherein the latent IVX instrument which depends on the true values of the localizing coefficients may no longer dominate the asymptotics. The chi-square limit theory of the same self-normalized test statistics of MP is shown to continue to be valid in this wider setting. We also confirm that the limit theory is robust under mixed degrees of persistence, allowing for the simultaneous presence of local to unity (or mildly integrated) roots and mildly explosive roots. These extensions show that IVX regression provides a framework of test procedures that can cover a large class of persistent regressors whose individual characteristics may differ from each other. While the results in the present paper show validity in a prototypical system, the wider practical implication is that empirical researchers may use this framework without having to be specific about the particular properties of individual regressors. Further research that is presently underway aims to provide a unified treatment that justifies the use of this approach for a comparably wide class of regressors in a general multivariate setting.

The paper is organized as follows. Section 2 develops the limit theory for IVX regression under

locally and mildly explosive roots and demonstrates robustness. Section 3 extends these robustness results to cases where the regressors have mixed degrees of persistence or explosive behavior. Section 4 discusses issues associated with the choice of the IVX tuning parameter and provides simulation results. Section 5 concludes. Some technical derivations, supporting lemmas, and proofs of the main results in the paper are contained in the Appendix. An online supplement (Phillips and Lee, 2015c) provides an empirical illustration as well as further lemmas and proofs that are useful in establishing the results of the paper.

2 IVX Regression with Explosive Roots

In what follows, we use the spectral norm $\|M\| = \max_i \{\lambda_i^{1/2} : \lambda_i = \text{an eigenvalue of } M'M\}$. Other norms, such as the L_1 and L_2 norms, are specified in what follows as needed using the notation $\|\cdot\|_{L_i}$ ($i = 1, 2$).

2.1 Framework

We use the framework of MP for the following system:

$$\begin{aligned} y_t &= Ax_t + u_{0t}, \\ x_t &= R_n x_{t-1} + u_{xt}, \\ R_n &= I_K + \frac{C}{n^\alpha}, \text{ for some } \alpha > 0, \end{aligned} \tag{2.1}$$

where A is an $m \times K$ coefficient matrix and $C = \text{diag}(c_1, c_2, \dots, c_K)$ represents the localizing coefficients in the multivariate regressors. Use of a diagonal matrix C in (2.1) has become standard practice in the literature because it helps to distinguish individual components of x_t in terms of their integration properties.¹

The IVX approach of MP allowed the regressors x_t to be (I1) integrated ($C = 0$), (I2) near integrated ($C < 0, \alpha = 1$) and (I3) mildly integrated ($C < 0, \alpha \in (0, 1)$), and developed an inference procedure that is robust to the precise degree of integration². We will show these results are robust under the same framework but with (I4) local to unity on the explosive side of unity ($C > 0, \alpha = 1$) and (I5) mildly explosive roots ($C > 0, \alpha \in (0, 1)$), as well as possibly mixed versions (I6) of these roots.

As in MP or Kostakis et al. (2014), the initial conditions for the driver regressor x_t in (I1)-(I2) and (I3) could be $x_0 = o_p(n^{1/2})$ or $x_0 = o_p(n^{\alpha/2})$, respectively. Indeed, $x_0 = o_p(n^{\alpha/2})$ also applies in the (I4) case. For (I5), in view of Lemma 2.1 and 2.2, we could impose $x_0 = o_p(n^{\alpha/2} \|R_n\|^n) =$

¹The general case of matrix C was considered in the original work on near integrated vector processes (Phillips, 1988) with several examples that illustrate the impact of off diagonal elements on the limit theory. These typically introduce additional components in the limit that take the form of smoothed diffusions.

²Some other cases of neglected nonlinearities (such as structural breaks) and linkages to unit root or explosive behavior may be included in the set up. For instance, Diebold and Inoue (2001) have shown that stochastic regime switching is easily confused empirically with long memory; and certain nonlinear functions of $I(1)$ processes are well-known to behave like stationary long memory processes (Miller and Park, 2010; Kasparis et al., 2014).

$o_p\left(n^{\alpha/2}e^{\|C\|n^{1-\alpha}}\right)$. Other possibilities, where the initial conditions have larger orders may also be analyzed and, as in Phillips and Magdalinos (2009) such cases typically impact the asymptotic theory.

The system (2.1) is prototypical and may readily be extended to cases that suit particular empirical applications. For example, intercepts and deterministic regressors are commonly present in practice in predictive systems like (2.1). Also, the generating mechanism for the predictors x_t may also involve deterministic components or triangular system versions of such components that upon standardization lead to functional (or other) stochastic process limits with comparable deterministic components (e.g. Phillips, Shi and Yu, 2014). Theory and implementation of the methods discussed here under some extensions of this type are considered later in the paper.

For the time series structure of the innovations that appear in (2.1) it is convenient to maintain one of the following two assumptions, which allow for weak dependence and conditional heterogeneity.

Assumption 2.1 (LP)

$$u_t := \begin{bmatrix} u_{0t} \\ u_{xt} \end{bmatrix} = \sum_{j=0}^{\infty} F_j \varepsilon_{t-j}, \quad \varepsilon_t \sim iid(0, \Sigma), \quad \Sigma > 0, \quad E \|\varepsilon_1\|^4 < \infty, \quad (2.2)$$

$$F_0 = I_{m+K}, \quad \sum_{j=0}^{\infty} j \|F_j\| < \infty, \quad F(z) = \sum_{j=0}^{\infty} F_j z^j \text{ and } F(1) = \sum_{j=0}^{\infty} F_j > 0.$$

Assumption 2.2 (NLP)

$$u_t = g(\xi_t) \quad (2.3)$$

where the process $\xi_t = (\dots, \varepsilon_{t-1}, \varepsilon_t)$ consists of time series innovations $\varepsilon_t \sim iid(0, \Sigma)$ as given Assumption LP above, and where $g(\cdot)$ is a measurable function such that u_t is well-defined. Let $\{\varepsilon'_i\}_{i \in \mathbb{Z}}$ be an iid copy of $\{\varepsilon_i\}_{i \in \mathbb{Z}}$. The following geometric moment contraction (GMC) condition holds, for all $t \in N$ with $M > 0$ and $r \in (0, 1)$,

$$E [\|u_t - u_t^*\|^q] \leq M r^t, \text{ with } q \geq 4$$

where $u_t^* = g(\dots, \varepsilon'_{-1}, \varepsilon'_0, \varepsilon_1, \dots, \varepsilon_{t-1}, \varepsilon_t)$.

Assumption LP, which follows MP, imposes conditional homoscedasticity, which may be restrictive in some financial applications. Assumption NLP is more general, includes assumption LP as a specific case, and allows various forms of stationary dependent processes such as GARCH/ARCH and threshold AR processes. As discussed below both LP and NLP conveniently allow for a Beveridge Nelson (BN; Phillips and Solo, 1972) decomposition of u_t . The general NLP case includes, in particular, GARCH processes but surprisingly does not incur any substantial cost in terms of the main derivations compared to proofs that rely directly on the LP conditions. The GMC condition imposed above is a sufficient condition enabling a systematic asymptotic theory for processes

of the form (2.3) - see Wu (2011), where the asymptotics of many nonlinear time series under this condition are provided. The summability condition $\sum \|u_t - u_t^*\| < \infty$ is clearly met under the given GMC condition and this is analogous to the condition $\sum_{j=0}^{\infty} j \|F_j\| < \infty$ under LP. For the (I5) case, the LP assumption is needed for the proof of Lemma 2.2, where the same uniform approximation under NLP is also provided.

Under Assumption LP it is well known that there exists a valid BN decomposition, which leads to the following component-wise expressions (Phillips and Solo, 1992)

$$u_t = F(1)\varepsilon_t - \Delta\tilde{\varepsilon}_t, \quad \tilde{\varepsilon}_t = \sum_{j=0}^{\infty} \tilde{F}_j \varepsilon_{t-j}, \quad \tilde{F}_j = \sum_{s=j+1}^{\infty} F_s, \quad (2.4)$$

$$u_{0t} = F_0(1)\varepsilon_t - \Delta\tilde{\varepsilon}_{0t}, \quad u_{xt} = F_x(1)\varepsilon_t - \Delta\tilde{\varepsilon}_{xt}. \quad (2.5)$$

Under Assumption NLP, we have the related nonlinear BN decomposition (Lee, 2015)

$$u_t = D_t - \Delta\tilde{D}_t = \begin{bmatrix} D_{0t} \\ D_{xt} \end{bmatrix} - \Delta \begin{bmatrix} \tilde{D}_{0t} \\ \tilde{D}_{xt} \end{bmatrix},$$

$$D_t = \sum_{i=t}^{\infty} \mathcal{P}_t u_i, \quad \tilde{D}_t = \sum_{k=1}^{\infty} (E[u_{t+k} | \mathcal{F}_t])$$

with $\mathcal{P}_t X := E[X | \mathcal{F}_t] - E[X | \mathcal{F}_{t-1}]$, which follows the general martingale decomposition of a stationary time series (Hall and Heyde, 1980, chapter 5).

The long run covariance matrices associated with u_t of LP are denoted as

$$\Omega = \sum_{h=-\infty}^{\infty} E(u_t u'_{t-h}) = F(1)\Sigma F(1)', \quad \Delta = \sum_{h=0}^{\infty} E(u_t u'_{t-h}), \quad (2.6)$$

$$\Omega = \begin{bmatrix} \Omega_{00} & \Omega_{0x} \\ \Omega_{x0} & \Omega_{xx} \end{bmatrix}, \quad \Delta = \begin{bmatrix} \Delta_{00} & \Delta_{0x} \\ \Delta_{x0} & \Delta_{xx} \end{bmatrix}.$$

Correspondingly, for NLP u_t we define (c.f. Hall and Heyde, 1980, theorem 5.3)

$$\Omega = \left\| \sum_{t=0}^{\infty} \mathcal{P}_0 u_t \right\|_2^2 = \begin{bmatrix} \Omega_{00} & \Omega_{0x} \\ \Omega_{x0} & \Omega_{xx} \end{bmatrix}.$$

Under (2.2) or (2.3) we have the functional law (Phillips and Solo, 1992):

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{\lfloor ns \rfloor} u_j := \frac{1}{\sqrt{n}} \sum_{j=1}^{\lfloor ns \rfloor} \begin{bmatrix} u_{0t} \\ u_{xt} \end{bmatrix} =: \begin{bmatrix} B_{0n}(s) \\ B_{xn}(s) \end{bmatrix} \Rightarrow \begin{bmatrix} B_0(s) \\ B_x(s) \end{bmatrix} = BM \begin{bmatrix} \Omega_{00} & \Omega_{0x} \\ \Omega_{x0} & \Omega_{xx} \end{bmatrix}, \quad (2.7)$$

where $B = (B'_0, B'_x)'$ is vector Brownian motion (BM) ; and the following local to unity limit law

for cases (I1), (I2) and (I4) also holds (Phillips, 1987):

$$\frac{x_{[nr]}}{\sqrt{n}} \Rightarrow J_x^c(r), \text{ where } J_x^c(r) = \int_0^r e^{(r-s)C} dB_x(s), \quad (2.8)$$

encompassing the (I1) case, where $J_x^c(r) = B_x(r)$ if $C = 0$.

2.2 IVX Construction

Under rather general conditions, MP showed that the limit theory of the IVX estimator of A is mixed normal and that suitably self-normalized test statistics have an asymptotic chi-square distribution. Since the limit theory is pivotal and therefore free of the nuisance parameter C , no information on the degree of regressor persistence is needed to execute tests as long as the regressors fall into the categories (I1), (I2) or (I3). The key step in IVX is the construction of an instrument using only endogenous information available in the regressors $\{x_t\}$ - hence the terminology IVX:

$$\tilde{z}_t = \sum_{j=1}^t R_{nz}^{t-j} \Delta x_j, \text{ where } R_{nz} = I_K + \frac{C_z}{n^\beta}, \beta \in (0, 1), C_z < 0. \quad (2.9)$$

Since $\Delta x_j = \frac{C}{n^\alpha} x_{j-1} + u_{xj}$, we have the decomposition $\tilde{z}_t = \sum_{j=1}^t R_{nz}^{t-j} u_{xj} + \frac{C}{n^\alpha} \sum_{j=1}^t R_{nz}^{t-j} x_{j-1}$ denoted as

$$\tilde{z}_t = z_t + \frac{C}{n^\alpha} \psi_{nt}. \quad (2.10)$$

The rate parameter β and (diagonal) matrix C_z in (2.9) are selected to ensure that z_t is mildly integrated, which assures in turn that z_t is less persistent than a unit root or local to unity regressor x_t . Using conventional observation matrix notation, the bias corrected IVX estimator of A suggested by MP and its estimation error have the form

$$\tilde{A}_n = (Y' \tilde{Z} - n \hat{\Delta}_{0x})(X' \tilde{Z})^{-1}, \quad (2.11)$$

$$\tilde{A}_n - A = (U_0' \tilde{Z} - n \hat{\Delta}_{0x})(X' \tilde{Z})^{-1}, \quad (2.12)$$

where $\hat{\Delta}_{0x}$ is some consistent estimate of the one sided long run covariance matrix Δ_{0x} . The estimator $\tilde{A}_n$ is a simple adjusted version of the conventional IV estimator $\hat{A} = (Y' \tilde{Z})(X' \tilde{Z})^{-1}$ using instruments $\tilde{z}_t$. In view of the decomposition in (2.10), z_t plays the role of a latent mildly integrated instrument and the remainder $\frac{C}{n^\alpha} \psi_{nt}$ is eliminated asymptotically due to its scaling coefficient $\frac{C}{n^\alpha}$. As a result we have nuisance parameter (C) free inference using $\tilde{A}_n$. Martingale limit theory applies to the numerator matrix $U_0' \tilde{Z} - n \hat{\Delta}_{0x}$ in (2.12), and this leads to a mixed normal limit theory that is well suited to inference.

2.3 Limit Theory for Regressors with (Explosive) Local Unit Roots

It turns out that IVX regression limit theory may be extended to explosive cases. The following result holds for the IVX estimator (2.11) with (I4) and local unit roots on the explosive side of

unity. The limit theory under this (I4) case remains exactly the same as for the near integrated case (I2) or case N(ii) of MP.

Theorem 2.1 (Local Unit Roots on the Explosive Side) *With $\beta \in (1/2, 1)$,*

$$vec \left[n^{\frac{1+\beta}{2}} (\tilde{A}_n - A) \right] \Rightarrow MN \left(0, \left(\tilde{\Psi}_{xx}^{-1} \right)' C_z V_{zz}^x C_z \tilde{\Psi}_{xx}^{-1} \otimes \Omega_{00} \right),$$

where

$$\tilde{\Psi}_{xx} = \Omega_{xx} + \int_0^1 J_x^c(r) dJ_x^c(r)' \text{ and } V_{zz}^x = \int_0^\infty e^{pC_z} \Omega_{xx} e^{pC_z} dp.$$

2.4 Limit Theory for Regressors with Mildly Explosive Roots

The asymptotics in the (I5) case turn out to be more complicated because the remainder term $\frac{C}{n^\alpha} \psi_{nt}$ now dominates the latent instrument z_t in (2.10) and this has a substantial effect on the derivations, as discussed below. Moreover, it is not necessary to include the bias adjustment in this case and the limit theory for $\tilde{A}_n$ is identical to that of the unadjusted $\hat{A} = (Y' \tilde{Z})(X' \tilde{Z})^{-1}$. To present the limit theory concisely, we omit the bias correction term in this Section. In practice the bias corrected version (2.11) would be used since no precise knowledge about the various forms (I1)-(I5) cases is assumed. The limit theory remains unchanged as explained in Remark 2.5. Interestingly, the limit theory for $\hat{A}$ then coincides with that of the OLS estimator (see Theorem 4.1 in Magdalinos and Phillips (2009a)).

We consider case (I5) for mildly explosive regressors with common localizing exponent $\alpha \in (0, 1)$ and localizing coefficient matrix $C > 0$, combined with the moment condition $E \|\varepsilon_1\|^q < \infty$, $q \geq 4$. The following lemmas provide limit theory for standardized versions of the regressor x_t that are needed for the asymptotic development. A rate condition on α ($\alpha > \frac{2}{q}$) is imposed in the second lemma to ensure a uniform strong approximation, ensuring that a wider range of α is admissible when higher moment conditions apply (see the discussion in the Proof of Lemma 2.2).

We start with the following lemma from Magdalinos and Phillips (2009a, lemma 4.1):

Lemma 2.1 *Define $Y_{Cn} := \frac{1}{n^{\alpha/2}} \sum_{j=1}^{k_n} R_n^{-j} F_x(1) \varepsilon_j$ for $k_n \rightarrow \infty$ such that $\|R_n\|^{-k_n} \rightarrow 0$. Then*

$$n^{-\alpha/2} R_n^{-n} x_n = \frac{1}{n^{\alpha/2}} \sum_{j=1}^n R_n^{-j} u_{xj} = \frac{1}{n^{\alpha/2}} \sum_{j=1}^{k_n} R_n^{-j} u_{xj} + o_p(1) =: Y_{Cn} + o_p(1),$$

and

$$Y_{Cn} \Rightarrow Y_C \equiv N \left(0, \int_0^\infty e^{-pC} \Omega_{xx} e^{-pC} dp \right).$$

The requirement $\|R_n\|^{-k_n} \rightarrow 0$ in lemma 2.1 implies that $\frac{k_n}{n^\alpha} \rightarrow \infty$. As in MP (2009a), the quantities Y_{Cn} and Y_C play an important role in the asymptotic theory. In addition, the following uniform approximation holds, which helps to simplify proofs. A similar approximation was shown for the scalar case in Phillips and Magdalinos (2007b).

Lemma 2.2 (Uniform Approximation) *With the same k_n defined in lemma 2.1 and $j_n = j - 1$, there exist random variables $\tilde{Y}_{Cn,j_n}$ and $\tilde{Y}_C$ in a suitably expanded common probability space such that*

$$\sup_{j_n \text{ such that } k_n \leq j_n \leq n} \left\| \tilde{Y}_{Cn,j_n} - \tilde{Y}_C \right\| = o_p(1),$$

where $\tilde{Y}_{Cn,j_n}$ and $\tilde{Y}_C$ are distributionally equivalent versions of $Y_{Cn,j_n} = \frac{1}{n^{\alpha/2}} R_n^{-j_n} x_{j_n}$ and Y_C , so that $\tilde{Y}_{Cn,j_n} =_d Y_{Cn,j_n}$ and $\tilde{Y}_C =_d Y_C$.

Remark 2.1 *The above uniformly strong approximation of the normalized process of a vector of mildly explosive regressors is particularly useful in developing limit theory in mildly explosive regression. But for the purpose of our proof here convergence in probability is enough and for that result somewhat weaker moment conditions may be used (see lemma 3.1 of Phillips, 2007).*

For the sequence k_n used in lemmas 2.1 and 2.2, we have $\frac{k_n}{n^\alpha} \rightarrow \infty$ and it follows that when $t < k_n$, $\|n^{-\alpha/2} R_n^{-t} x_t\|_{L_2}$ either degenerates to zero (when $t = o(n^\alpha)$) or is bounded (when $t = O(k_n)$). Combined with lemma 2.2, we have

$$\sup_{1 \leq t \leq n} \left\| n^{-\alpha/2} R_n^{-t} x_t \right\|_{L_2} < \infty. \quad (2.13)$$

A similar uniform approximation to that of lemma 2.2 holds for ψ_{nt} in (2.10), with a slight modification to the index range of t as shown in the following lemma.

Lemma 2.3 *For k_n and k'_n satisfying the rate condition $\frac{n^{\alpha \vee \beta}}{k_n} + \frac{n^{\alpha \wedge \beta}}{k'_n} \rightarrow 0$ where $\alpha \vee \beta = \max(\alpha, \beta)$ and $\alpha \wedge \beta = \min(\alpha, \beta)$, we have*

$$\frac{1}{n^{\frac{\alpha}{2} + \alpha \wedge \beta}} R_n^{-t} \psi_{nt} = C_{z\alpha\beta} \tilde{Y}_C + o_p(1), \quad (2.14)$$

for all $t \in [k_n + k'_n, n]$ where

$$C_{z\alpha\beta} := \begin{cases} -C_z^{-1}, & \text{if } \beta < \alpha \\ C^{-1}, & \text{if } \alpha < \beta \\ (C - C_z)^{-1}, & \text{if } \alpha = \beta \end{cases}.$$

Remark 2.2 *As the proof of the lemma in the Appendix reveals, the index set $[k_n + k'_n, n]$ for t ensures the negligibility of certain frontal sums involving the standardized components of ψ_{nt} . As will become clear, the condition is used only in the proofs of the intermediate lemmas and does not appear in the main results because the full sums are dominated by the tail summation. Note that even for moderate sample sizes n , the set is well defined. For example, when $n = 30$, $\alpha \vee \beta = 0.8$ and $\alpha \wedge \beta = 0.6$, we have $n - n^{0.85} - n^{0.65} > 0$ so the index set is not empty and evidently has $O(n)$ observations as $n \rightarrow \infty$.*

Remark 2.3 *In (2.14) the standardization by $n^{\frac{\alpha}{2} + \alpha \wedge \beta}$ involves (potentially) both localizing coefficients α and β (depending on their respective magnitudes). The intuition for this standardization*

is that the quantity $\psi_{nt} = \sum_{j=1}^t R_{nz}^{t-j} x_{j-1} = \sum_{j=1}^t \left\{ R_{nz}^{t-j} R_n^{(j-1)} \right\} R_n^{-(j-1)} x_{j-1}$ involves the weighting matrices R_{nz}^{t-j} which downweight the components x_{j-1} in the sum, whereas these components themselves, being mildly explosive, are weighted by $R_n^{-(j-1)}$, which leads to an interacting weighting system involving the matrices $R_{nz}^{t-j} R_n^{(j-1)}$. Upon summation these weights may be dominated by the near stationary components that involve R_{nz} (and β) or the mildly explosive components that involve R_n (and α).

Remark 2.4 In consequence, both localizing coefficients α and β appear in the component limit theory for the numerator and denominator matrices of the IVX estimator (see lemma 2.4 below). Intuitively, if β is smaller than α , then the instruments z_t are near stationary and these near-stationary instruments tend to attenuate the mildly explosive behavior of x_t in the IV regression, thereby leading to the factor $n^{\alpha \wedge \beta}$ in the normalization and slowing down the rate of convergences in the components.

As in (2.13), when $t < k_n + k'_n$, it can be shown that $\|n^{-(\alpha/2+(\alpha \wedge \beta))} R_n^{-t} \psi_{nt}\|_{L_2}$ is either degenerate or bounded. So we have

$$\sup_{1 \leq t \leq n} \left\| n^{-(\alpha/2+(\alpha \wedge \beta))} R_n^{-t} \psi_{nt} \right\|_{L_2} < \infty. \quad (2.15)$$

With these results in hand, the limit theory follows for the numerator and denominator of the IVX estimator under (I5) mildly explosive regressors. The stronger signal in the remainder terms results in a new limit theory that involves the nuisance parameters C and $C_{z\alpha\beta}$.

Lemma 2.4 (IVX Numerator and Denominator) Define $\tilde{\Psi}_{yy} := \int_0^\infty e^{-pC} Y_C Y_C' e^{-pC} dp$.

$$(1) \text{ vec } \left(\frac{1}{n^{(\alpha \wedge \beta)}} \sum_{t=1}^n u_{0t} z_t' R_n^{-n} \right) \implies (CC_{z\alpha\beta} \otimes I_m) \times MN \left(0, \tilde{\Psi}_{yy} \otimes \Omega_{00} \right).$$

$$(2) \frac{1}{n^{\alpha+(\alpha \wedge \beta)}} \sum R_n^{-n} x_t z_t' R_n^{-n} \implies \tilde{\Psi}_{yy} \times (CC_{z\alpha\beta}).$$

Remark 2.5 The limit theory of the numerator given in lemma 2.4-(1) shows that $\sum_{t=1}^n u_{0t} z_t'$ becomes asymptotically negligible in relation to the term $\sum_{t=1}^n u_{0t} \psi_{nt}'$, which is usually a (negligible) remainder, and it is this term that dominates the asymptotics. Moreover, unlike the other cases (I1)-(I4), we no longer need the built-in serial correlation bias correction associated with the $O_p(n)$ sample covariance term $\sum_{t=1}^n u_{0t} z_t'$ and an estimate of the corresponding one sided long run covariance matrix. A similar property arises in the case of OLS estimation under mildly explosive regressors (Magdalinos and Phillips, 2009a).

The limit theory of the IVX estimator (2.11) under (I5) is therefore equivalent to that of the OLS estimator in Magdalinos and Phillips (2009a, theorem 4.1).

Theorem 2.2 (Mildly Explosive Regressor) With $\alpha \in (2/q, 1)$ and $\beta \in (1/2, 1)$,

$$vec \left[n^\alpha \left(\tilde{A}_n - A \right) R_n^n \right] \Rightarrow MN \left(0, \tilde{\Psi}_{yy}^{-1} \otimes \Omega_{00} \right),$$

where

$$\tilde{\Psi}_{yy} = \int_0^\infty e^{-pC} Y_C Y_C' e^{-pC} dp, \text{ and } Y_C \equiv N \left(0, \int_0^\infty e^{-pC} \Omega_{xx} e^{-pC} dp \right).$$

The results of the limit theory given in theorem 2.1 and theorem 2.2 for the LUR explosive and mildly explosive cases depend on the unknown localizing coefficient matrix C and the unknown rate parameter α . So these results are not feasible for practical work as they stand. However, self normalized versions of the statistics have a chi-square limit theory in both the (I4) and (I5) cases, providing a convenient basis for inference.

To illustrate, we follow the usual (linear) hypothesis testing framework in which H is a known $r \times mK$ matrix of rank r and h is a known vector. Obvious extensions hold for general analytic restrictions.

Theorem 2.3 (LUR Explosive or Mildly Explosive Regressor) Under $H_0 : Hvec(A) = h$ with $\alpha \in (2/q, 1)$ and $\beta \in (1/2, 1)$,

$$\left(Hvec \left(\tilde{A}_n \right) - h \right)' \left[H \left\{ \left(X' P_{\tilde{Z}} X \right)^{-1} \otimes \hat{\Omega}_{00} \right\} H' \right]^{-1} \left(Hvec \left(\tilde{A}_n \right) - h \right) \Rightarrow \chi^2(r),$$

where

$$\left(X' P_{\tilde{Z}} X \right)^{-1} = \left\{ \left(\sum_{t=1}^n x_t \tilde{z}_t' \right) \left(\sum_{t=1}^n \tilde{z}_t \tilde{z}_t' \right)^{-1} \left(\sum_{t=1}^n x_t \tilde{z}_t' \right)' \right\}^{-1}.$$

Although the limit theory of the IVX estimator under (I5) mildly explosive regressors differs from the other (I1)-(I4) cases, the usual chi-square limit theory for the self-normalized test statistic still holds. Theorem 2.3, taken together with theorem 3.8 of MP, therefore shows that IVX regression leads to a single inference procedure in all cases (I1)-(I5). The broad range of this limit theory is helpful in empirical work where there is inevitable uncertainty about the degree of persistence in the regressors. Importantly, however, the limit theory in Theorem 2.3 requires x_t to share the same localizing exponent $\alpha \in (0, 1)$. Some relaxation of this condition is explored in Section 3 where we allow for both mildly explosive and mildly integrated regressors.

2.5 Regressions with Intercept

Practical implementations of regressions such as (2.1) commonly involved deterministic components such as intercepts or trends. In predictive regression versions of (2.1), it is especially common to include an intercept in empirical applications. In their recent study of such regressions, Kostakis et al. (2014) allow for the presence of an intercept and show that the first order IVX limit theory

remains the same as MP for cases (I1)-(I3). The (I4) case is essentially the same as the (I2) case considered in Kostakis et al (2014) upon working with a demeaned OU process. This leaves case (I5) which we now consider.

We follow the same procedure as in Kostakis et al (2014) of including an intercept in (2.1) and using the undemeaned IVX instruments, so that (2.1) is replaced by the system

$$\begin{aligned} y_t &= \mu + Ax_t + u_{0t}, \\ x_t &= R_n x_{t-1} + u_{xt}, \end{aligned} \tag{2.16}$$

Defining $x_t^\mu = x_t - n^{-1} \sum x_t$, $y_t^\mu = y_t - n^{-1} \sum y_t$ and $u_{0t}^\mu = u_{0t} - n^{-1} \sum u_{0t}$, demeaning (2.16) leads to $y_t^\mu = Ax_t^\mu + u_{0t}^\mu$. Denote the corresponding observation matrices as Y^μ , X^μ and U_0^μ , respectively. Then IVX estimation in the (I5) case, where bias correction does not play any role and is therefore omitted for ease of exposition, leads to the IVX estimator

$$\tilde{A}_n^\mu = (Y^{\mu'} \tilde{Z})(X^{\mu'} \tilde{Z})^{-1} = A + (U_0^{\mu'} \tilde{Z})(X^{\mu'} \tilde{Z})^{-1} \tag{2.17}$$

The effect of demeaning, or estimating an intercept in the regression, becomes relevant only in the limit theory of the sample covariance component $X^{\mu'} \tilde{Z} = \sum x_t \tilde{z}_t'$ (Lemma 2.4-(2)). Upon standardization, this component behaves to the first order just as it does in the case where an intercept is not fitted, as shown in the following proposition.

Proposition 2.4 *For the (I5) case,*

$$\frac{1}{n^{\alpha+(\alpha \wedge \beta)}} \sum R_n^{-n} x_t^\mu \tilde{z}_t' R_n^{-n} = \frac{1}{n^{\alpha+(\alpha \wedge \beta)}} \sum R_n^{-n} x_t \tilde{z}_t' R_n^{-n} + o_p(1).$$

In the same way, the first order limit theory of this sample covariance remains unchanged in the (I4) case. Hence, the effects of including an intercept in the system regression is innocuous, at least in first order asymptotics. Simulations conducted by Kostakis et al (2014) show that the inclusion of an intercept can produce finite sample size distortion in Wald statistic tests in some cases. They also show that a simple correction to the $\tilde{Z}' \tilde{Z}$ signal matrix that appears in the Wald statistic estimated variance matrix of the IVX estimator is successful in reducing this distortion. Readers are referred to their paper for details and further discussion.

3 IVX Regression with Mixed Roots

The results above combined with those of MP show that the IVX approach is applicable for regressors with common roots in a wide vicinity of unity including all (I1) - (I5) cases. We might also expect the same method to be valid when there are mixed degrees of persistence in the regressors, which is likely in some empirical work (e.g. Campbell and Yogo, 2006). This section confirms that conjecture, showing the same limit theory given in theorem 2.2 holds when there are multiple regressors with mixed roots in the vicinity of unity.

For exposition, it is convenient to consider the prototypical model with $m = 1$ and $K = 2$ in (2.1) so that y_t is scalar and x_t is a bivariate AR(1) process with mixed roots. The remaining restrictive element implied by this bivariate regressor system for the general case where there may be many regressors is commonality of the mildly explosive and mildly integrated roots. The simplified system has the form

$$y_t = a'x_t + u_{0t}, \quad a' = \begin{bmatrix} a_1 & a_2 \end{bmatrix}, \quad x_t = \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix}, \quad (3.1)$$

$$x_t = R_n x_{t-1} + u_{xt}, \quad R_n = \begin{bmatrix} \rho_n & 0 \\ 0 & \theta_n \end{bmatrix}, \quad u_{xt} = \begin{bmatrix} u_{x1t} \\ u_{x2t} \end{bmatrix}. \quad (3.2)$$

We impose different degrees of persistence in the two regressors with the following specification:

$$\begin{aligned} \rho_n &= 1 + \frac{c_1}{n^{\alpha_1}}, \quad \text{where } c_1 \in \begin{cases} (-\infty, 0), & \text{if } \alpha_1 \in (0, 1) \\ (-\infty, \infty), & \text{if } \alpha_1 = 1 \end{cases}, \\ \theta_n &= 1 + \frac{c_2}{n^{\alpha_2}}, \quad \text{where } c_2 \in (0, \infty) \text{ and } \alpha_2 \in (0, 1). \end{aligned}$$

Accordingly, x_{1t} falls under one of the specifications (I1)-(I4), while x_{2t} is a mildly explosive regressor corresponding to (I5)³. Here we demonstrate the robustness of IVX estimation for the system parameter a in (3.1) in a mixed root regressor environment.

From the decomposition (2.4) we may express u_t in component form as

$$u_t = \begin{bmatrix} u_{0t} \\ u_{x1t} \\ u_{x2t} \end{bmatrix} = \begin{bmatrix} F_0(1)_{1 \times 3} \\ F_{x1}(1)_{1 \times 3} \\ F_{x2}(1)_{1 \times 3} \end{bmatrix} \begin{bmatrix} \varepsilon_{0t} \\ \varepsilon_{x1t} \\ \varepsilon_{x2t} \end{bmatrix} - \Delta \begin{bmatrix} \tilde{\varepsilon}_{0t} \\ \tilde{\varepsilon}_{x1t} \\ \tilde{\varepsilon}_{x2t} \end{bmatrix},$$

under the same assumptions as (2.2). The long run variance matrices and the limit theory are the same as in (2.6), (2.7) and (2.8) except that the subscripts 0, 1 and 2 now signify u_0 , u_{x1} and u_{x2} , respectively. The IVX instrument is constructed in the same way as (2.9) with $C_z = \text{diag}(c_{z1}, c_{z2}) < 0$ and $\beta \in (1/2, 1)$.

The bias corrected IVX estimator of a has estimation error

$$\begin{aligned} \tilde{a} - a &= \begin{bmatrix} \tilde{a}_1 - a_1 \\ \tilde{a}_2 - a_2 \end{bmatrix} = \left(\tilde{Z}' X \right)^{-1} \left(\tilde{Z}' U_0 - n \hat{\Delta}_{x0} \right) \\ &= \begin{bmatrix} \sum_{t=1}^n \tilde{z}_{1t} x_{1t} & \sum_{t=1}^n \tilde{z}_{1t} x_{2t} \\ \sum_{t=1}^n \tilde{z}_{2t} x_{1t} & \sum_{t=1}^n \tilde{z}_{2t} x_{2t} \end{bmatrix}^{-1} \begin{bmatrix} \sum_{t=1}^n \tilde{z}_{1t} u_{0t} - n \hat{\Delta}_{01} \\ \sum_{t=1}^n \tilde{z}_{2t} u_{0t} - n \hat{\Delta}_{02} \end{bmatrix}. \end{aligned}$$

³Dual manifestations of nonstationarity with different roots of this type were first considered in a 2010 Yale Take Home Examination by the first author. The idea developed later into Phillips and Lee (2015b) who considered inference about the roots in a vector autoregression of the type (3.2).

We employ the following normalizing matrices to accommodate the different orders of magnitude

$$D_n = \begin{bmatrix} n^{\frac{1+(\alpha_1 \wedge \beta)}{2}} & 0 \\ 0 & \theta_n^n n^{\alpha_2 \wedge \beta} \end{bmatrix}, \quad C_n = \begin{bmatrix} n^{\frac{1+(\alpha_1 \wedge \beta)}{2}} & 0 \\ 0 & \theta_n^n n^{\alpha_2} \end{bmatrix}.$$

Then

$$C_n (\tilde{a} - a) = \left(D_n^{-1} \tilde{Z}' X C_n^{-1} \right)^{-1} D_n^{-1} \left(\tilde{Z}' U_0 - n \hat{\Delta}_{x0} \right).$$

To show the joint convergence of $n^{\frac{1+(\alpha_1 \wedge \beta)}{2}} (\tilde{a}_1 - a_1)$ and $\theta_n^n n^{\alpha_2} (\tilde{a}_2 - a_2)$, we first show the asymptotic independence of these two components. As in Magdalinos and Phillips (2009a), the asymptotic behavior of $\theta_n^n n^{\alpha_2} (\tilde{a}_2 - a_2)$ is determined by two independent asymptotic Gaussian processes

$$Y_n = \begin{bmatrix} Y_{c_0 n} \\ Y_{c_2 n} \end{bmatrix} := \begin{bmatrix} \frac{1}{n^{\frac{\alpha_2}{2}}} \sum_{t=k_n+1}^n \theta_n^{-(n-t)} F_0(1) \varepsilon_t \\ \frac{1}{n^{\frac{\alpha_2}{2}}} \sum_{j=1}^{k_n} \theta_n^{-j} F_{x2}(1) \varepsilon_j \end{bmatrix} \Rightarrow \begin{bmatrix} Y_{c_0} \\ Y_{c_2} \end{bmatrix} \equiv N \left(0_{2 \times 1}, \begin{bmatrix} \frac{\Omega_{00}}{2c_2} & 0 \\ 0 & \frac{\Omega_{22}}{2c_2} \end{bmatrix} \right).$$

From MP, the vector martingale $\sum_{t=1}^n \xi_{nt}$ with

$$\xi_{nt} = \begin{bmatrix} \xi_{n1t} \\ \xi_{n2t} \end{bmatrix} := \begin{cases} \begin{bmatrix} \frac{1}{n^{\frac{1+\beta}{2}}} z_{1t-1} F_0(1) \varepsilon_t \\ \frac{1}{\sqrt{n}} F_{x1}(1) \varepsilon_t \end{bmatrix}, & \text{when } (\alpha_1 \wedge \beta) = \beta, \\ & \text{i.e., } x_{1t} \text{ is I(1), I(2), I(4) or I(3) with } \beta < \alpha_1, \\ \begin{bmatrix} \frac{1}{n^{\frac{1+\alpha_1}{2}}} x_{1t-1} F_0(1) \varepsilon_t \\ \frac{1}{\sqrt{n}} F_{x1}(1) \varepsilon_t \end{bmatrix}, & \text{when } (\alpha_1 \wedge \beta) = \alpha_1, \text{ i.e., } x_{1t} \text{ is I(3) with } \alpha_1 < \beta, \end{cases}$$

determines the Gaussian limit theory of $n^{\frac{1+(\alpha_1 \wedge \beta)}{2}} (\tilde{a}_1 - a_1)$. When $(\alpha_1 \wedge \beta) = \beta$, the asymptotic independence between $\sum_{t=1}^n \xi_{n1t}$ and $\sum_{t=1}^n \xi_{n2t}$ is shown in proposition A1 in MP. The same proof also holds for the $(\alpha_1 \wedge \beta) = \alpha_1$ case. The joint convergence of $n^{\frac{1+(\alpha_1 \wedge \beta)}{2}} (\tilde{a}_1 - a_1)$ and $\theta_n^n n^{\alpha_2} (\tilde{a}_2 - a_2)$ is therefore achieved by showing the asymptotic independence between Y_n and $\sum_{t=1}^n \xi_{nt}$, which is done in the following lemma.

Lemma 3.1 *Y_n is asymptotically independent of the vector martingale $\sum_{t=1}^n \xi_{nt}$.*

The normalized denominator matrix is asymptotically diagonal under some mild rate conditions on β and α_i ($i = 1, 2$), which enhances the development of the joint limit theory.

Lemma 3.2 *Under the rate condition $\alpha_1 + \alpha_2 < 1 + \beta$, we have*

$$D_n^{-1} \tilde{Z}' X C_n^{-1} \Rightarrow \begin{bmatrix} \tilde{\Psi}_{11} & 0 \\ 0 & c_{xz2}(\alpha, \beta) c_2 \tilde{\Psi}_{22} \end{bmatrix},$$

where $\tilde{\Psi}_{22} = \frac{Y_{c2}^2}{2c_2}$, and

$$\tilde{\Psi}_{11} = \begin{cases} \frac{1}{-c_{z1}} \left\{ \Omega_{11} + \int_0^1 B_1 dB_1 \right\} & \text{if } x_{1t} \text{ is unit root : } I(1), \\ \frac{1}{-c_{z1}} \left\{ \Omega_{11} + \int_0^1 J_x^{c1} dJ_x^{c1} \right\} & \text{if } x_{1t} \text{ is local to unity : } (I2) \text{ or } (I4), \\ \frac{1}{-2c_{z1}} \Omega_{11} & \text{if } x_{1t} \text{ is mildly integrated - (I3) with } \beta < \alpha_1 \\ \frac{1}{-2(c_1+c_{z1})} \Omega_{11} & \text{if } x_{1t} \text{ is mildly integrated - (I3) with } \beta = \alpha_1 \\ \frac{1}{-2c_1} \Omega_{11} & \text{if } x_{1t} \text{ is mildly integrated - (I3) with } \alpha_1 < \beta \end{cases}.$$

Remark 3.1 Note that $\alpha_1 + \alpha_2 < 1 + \beta$ may not hold in some cases. In particular, if we have a local-to-unity regressor x_{1t} ($\alpha_1 = 1$) and a mildly explosive regressor x_{2t} whose root is very close to being local to unity (i.e., α_2 is close to unity), then $\alpha_2 < \beta$ may not hold and we may not have a diagonal limit for the moment matrix $\tilde{Z}'X$ upon standardization. For example, if $\alpha_1 = 1$ and $\beta < \alpha_2$, then $\alpha_1 + \alpha_2 < 1 + \beta$ fails. Intuitively, even though x_{1t} and x_{2t} have different orders of magnitude in this case, their asymptotic behavior is not distinct enough to ensure negligibility of the off diagonal elements when IVX leads to an instrument $\tilde{z}_{2t}$ that is close to stationarity (β close to 0.5). In such cases, the range of β for which $\tilde{Z}'X$ is diagonal asymptotically is restricted to the smaller region $\beta \in (\alpha_2, 1)$.

The limit theory of $C_n(\tilde{a} - a)$ is therefore obtained from the independent marginal convergence of the two components and the mixed roots affect each of these components separately in the limit.

Theorem 3.1 Under the rate conditions $\alpha_1 \in (1/3, 1)$ and $\beta \in ((\alpha_2 \vee 2/3), 1)$,

$$C_n(\tilde{a} - a) \Rightarrow MN \left(0_{2 \times 1}, \begin{bmatrix} \tilde{\Phi}_{11}^{-1} \Omega_{00} & 0 \\ 0 & \tilde{\Psi}_{22}^{-1} \Omega_{00} \end{bmatrix} \right),$$

where $\tilde{\Psi}_{22} = \frac{Y_{c2}^2}{2c_2}$, $Y_{c2} \equiv N(0, \frac{\Omega_{22}}{2c_2})$ as in theorem 2.2, and $\tilde{\Psi}_{11}$ is given as

$$\tilde{\Phi}_{11}^{-1} = \begin{cases} \left\{ \frac{1}{-c_{z1}} \left(\Omega_{11} + \int_0^1 B_1 dB_1 \right) \right\}^{-2} \left(\frac{1}{-2c_{z1}} \Omega_{11} \right) & \text{if } x_{1t} \text{ is unit root - } I(1) \\ \left\{ \frac{1}{-c_{z1}} \left\{ \Omega_{11} + \int_0^1 J_x^{c1} dJ_x^{c1} \right\} \right\}^{-2} \left(\frac{1}{-2c_{z1}} \Omega_{11} \right) & \text{if } x_{1t} \text{ is local to unity - (I2) or (I4)} \\ \left(\frac{1}{-2c_{z1}} \Omega_{11} \right)^{-1} & \text{if } x_{1t} \text{ is mildly integrated - (I3) with } \beta < \alpha_1 \\ \frac{(c_1+c_{z1})^2}{c_1^2} \left(\frac{1}{-2c_1} \Omega_{11} \right)^{-1} & \text{if } x_{1t} \text{ is mildly integrated - (I3) with } \beta = \alpha_1 \\ \left(\frac{1}{-2c_1} \Omega_{11} \right)^{-1} & \text{if } x_{1t} \text{ is mildly integrated - (I3) with } \alpha_1 < \beta \end{cases}.$$

Remark 3.2 The rate condition $\alpha_1 \in (1/3, 1)$ ensures that the numerator of IVX estimator of the first component $\sum_{t=1}^n \tilde{z}_{1t} u_{0t} - n\hat{\Delta}_{01}$ has an asymptotic normal distribution. If we replace the weakly dependent structure of u_{0t} with an iid or mds structure for u_{0t} , as is common in predictive regressions, a wider region of α_1 is possible (see Kostakis et al., 2014). The condition that $\beta \in$

$((\alpha_2 \vee 2/3), 1)$ is explained as follows. First, the requirement that the choice parameter β exceeds $2/3$ accommodates consistent estimation of the long run covariance using $\hat{\Delta}_{x0}$ (see Lemma A0 in MP). The condition that β exceeds α_2 ensures asymptotic diagonality of the denominator matrix of IVX (the rate condition in Lemma 3.2). Section 4 shows that reliable choices of β are contained in this region.

Remark 3.3 Since $\tilde{\Psi}_{22} = \frac{Y_{c2}^2}{2c_2}$ and $Y_{c0} \equiv N\left(0, \frac{\Omega_{00}}{2c_2}\right)$ is independent of Y_{c2} , we have

$$\theta_n^{\alpha_2} (\tilde{a}_2 - a_2) \implies 2c_2 \frac{Y_{c0}}{Y_{c2}} \equiv 2c_2 \left(\frac{\Omega_{00}}{\Omega_{22}} \right)^{1/2} \mathcal{C}$$

where $\mathcal{C}$ is a standard Cauchy variate, giving the same result as that of Magdalinos and Phillips (2009a, remark 4.1a). This result again confirms the asymptotic equivalence of IVX and OLS estimation with mildly explosive regressors.

As anticipated, the same self-normalized test statistic continues to deliver an asymptotic chi-square test free of any nuisance parameters, at least for full vector tests.

Theorem 3.2 Under $H_0 : a = a_0$ and the rate condition in Theorem 3.2,

$$(\tilde{a} - a_0)' \left[(X' P_{\tilde{Z}} X)^{-1} \hat{\Omega}_{00} \right]^{-1} (\tilde{a} - a_0) \implies \chi^2(2),$$

where $(X' P_{\tilde{Z}} X)^{-1}$ is defined in Theorem 2.3.

This result demonstrates considerable robustness of the IVX approach to mixed degrees of persistence in the regressors, thereby widening the ambit of empirical research covered by this procedure and providing a step towards a single valid procedure for inference that allows for a large class of persistent, but differently behaved regressors. More general cases that allow for multivariate regressors with multiple mixed roots are treated in the same way. For example, the process $x_t = R_n x_{t-1} + u_{xt}$ may have a coefficient matrix of the form

$$R_n = \begin{bmatrix} (R_{1n})_{K_1 \times 1} & 0 \\ 0 & (R_{2n})_{K_2 \times 1} \end{bmatrix},$$

where R_{1n} has roots in the (I1)-(I4) class with common localizing exponent and R_{2n} involves mildly explosive coefficients of the form (I5) again with common localizing exponent. Analogous arguments to those in this section lead to asymptotic independence between suitably standardized versions of the processes for each group of regressors and the sample moment matrices $X' \tilde{Z}$ and $\tilde{Z}' \tilde{Z}$ will be asymptotically block diagonal after similar normalizations. Hence, in the same way as Theorem 3.2 we end up with a chi-square test that applies for a general class of mixed regressors.

While the theory presented here includes mixed regressors, it does not provide a fully general treatment that admits regressors with variable localizing exponents, either for the mildly integrated

or the mildly explosive regressors. Similarly it does not provide a general limit theory of Wald testing with an arbitrary restriction matrix H when there are mixed regressors and variable localizing exponents. Limit theory under completely agnostic conditions where there are multiple localizing exponents $\{\alpha_k : k = 1, \dots, K\}$ in R_n and general restrictions to be tested is a much more challenging task and is the subject of ongoing research by Magdalinos and Phillips. The partial results provided by Theorem 3.2 are nonetheless useful in bivariate regression cases with possibly mixed degrees of persistence and show clearly the potential of the IVX methodology to encompass a wide range of persistence in the regressors.

4 On the Choice of IVX Tuning Parameter

Implementation of IVX estimation requires choice of the tuning parameter β that is involved in the generation of the IVX instruments via (2.9). Evidently, larger values of β generally produce higher rates of convergence (c.f. theorem 2.1) and more efficient test procedures may therefore be expected. On the other hand, the central idea of IVX instrumentation – filtering a persistent regressor to generate an instrument of less persistence, thereby ensuring validity of chi-squared test limit theory – suggests that we need to impose an upper bound for β that is less than unity.

To fix ideas and keep the following discussion simple, we use the predictive regression setting that has been widely adopted in empirical finance, whereby the one period ahead dependent variable y_{t+1} is used instead of y_t in (2.1) and the regression error u_{0t+1} is assumed to be a martingale difference. These modifications do not change the limit theory presented earlier and correspond to the standard environment of existing studies, such as Campbell and Yogo (2006) and Jansson and Moreira (2006). The martingale difference structure on u_{0t+1} implies that there is no predictability of y_{t+1} under the null hypothesis $H_0 : a = 0$. With this structure, the following DGP is imposed

$$\begin{aligned} y_{t+1} &= \mu + a'x_t + u_{0t+1}, \quad u_{0t+1} \sim mds(0, \Sigma_{00}), \\ x_{t+1} &= \left(I_K + \frac{C}{n}\right)x_t + u_{xt+1}, \end{aligned} \tag{4.1}$$

where u_{xt+1} is a linear process generated as in (2.2). The AR dynamics of the predictive variables is formulated as in (2.1) with (I1)-(I6) specifications, describing the various degrees of persistence⁴.

4.1 Failure of Standard MSE Choice Criteria

Since larger values of β improve convergence rates whereas bias effects enter and the IVX mixed normal limit theory fails when $\beta = 1$, conventional asymptotic mean squared error (MSE) criteria (and associated cross validation approaches) might be expected to lead to suitable empirical choice criteria for β . As we now show, the asymptotic MSE criterion monotonically decreases as β increases,

⁴Standard present value models in financial theory do not usually admit explosive process solutions, since these are typically ruled out by transversality conditions and zero initial conditions on the submartingale solution. On the other hand, finite horizon models, where transversality conditions are not needed, can admit solutions with transient periods of explosive bubbles - see Phillips and Lee (2015a) for this possibility.

encouraging the choice of an upper bound of unity for β which raises the convergence rate to the optimal level but invalidates inference.

To fix ideas let $m = 1$, $K = 1$ and $\mu = 0$ in (4.1). Then

$$y_{t+1} = ax_t + u_{0t+1}, \quad (4.2)$$

$$x_{t+1} = \rho x_t + u_{xt+1}, \quad \rho = 1 + \frac{c}{n}, \quad c \in (-\infty, \infty) \quad (4.3)$$

Our analysis focuses on the (I2)-(I4) cases. Based on the IVX construction (2.9) with $\rho_z = 1 + \frac{c_z}{n^\beta}$, the generated AR(1) IVX series is

$$\tilde{z}_{t+1} = \rho_z \tilde{z}_t + \Delta x_t \sim \rho_z \tilde{z}_t + u_{xt+1},$$

and it is straightforward to show the decomposition

$$(\hat{a}_{IVX} - a) = \frac{\sum_{t=1}^n \tilde{z}_t u_{0.xt+1}}{\sum_{t=1}^n \tilde{z}_t x_t} + \left(\frac{\Sigma_{0x}}{\Omega_{xx}} \right) \frac{\sum_{t=1}^n \tilde{z}_t u_{xt+1}}{\sum_{t=1}^n \tilde{z}_t x_t},$$

where $u_{0.xt+1} = u_{0t+1} - \frac{\Sigma_{0x}}{\Omega_{xx}} u_{xt+1}$ and Ω_{xx} and Σ_{0x} are the corresponding (long-run) covariances. This decomposition is frequently used in the predictive regression literature and is adapted here to the IVX regression framework to investigate possible choices of β .

By IVX limit theory we may use

$$\frac{\sum_{t=1}^n \tilde{z}_t u_{xt+1}}{\sum_{t=1}^n \tilde{z}_t x_t} = \frac{\sum_{t=1}^n \tilde{z}_t u_{xt+1}}{\sum_{t=1}^n \tilde{z}_t^2} + o_p(1) = (\hat{\rho}_z - \rho_z) + o_p(1)$$

since the difference between $\sum_{t=1}^n \tilde{z}_t x_t$ and $\sum_{t=1}^n \tilde{z}_t^2$ is negligible for the (I2)-(I4) cases, as shown in MP. Then

$$\begin{aligned} (\hat{a}_{IVX} - a) &\sim \frac{\sum_{t=1}^n \tilde{z}_t u_{0.xt+1}}{\sum_{t=1}^n \tilde{z}_t x_t} + \left(\frac{\Sigma_{0x}}{\Omega_{xx}} \right) (\hat{\rho}_z - \rho_z) \\ &= \frac{\sum_{t=1}^n \tilde{z}_t u_{0.xt+1}}{\sum_{t=1}^n \tilde{z}_t x_t} + \left(\frac{\Sigma_{0x}}{\Omega_{xx}} \right) \left(\hat{\rho}_z - \rho_z - \frac{1}{n^\beta} \text{Bias}(\hat{\rho}_z) \right) + \frac{1}{n^\beta} \left(\frac{\Sigma_{0x}}{\Omega_{xx}} \right) \text{Bias}(\hat{\rho}_z) \end{aligned}$$

where the expression for the bias, $\text{Bias}(\hat{\rho}_z)$, can be found in Theorem 4.2 in Phillips and Magdalinos (2007b). Next observe that

$$n^{\frac{1+\beta}{2}} (\hat{a}_{IVX} - a) = \frac{n^{-\frac{1+\beta}{2}} \sum_{t=1}^n \tilde{z}_t u_{0.xt+1}}{n^{-(1+\beta)} \sum_{t=1}^n \tilde{z}_t x_t} + \left(\frac{\Sigma_{0x}}{\Omega_{xx}} \right) n^{\frac{1+\beta}{2}} \left(\hat{\rho}_z - \rho_z - \frac{1}{n^\beta} \text{Bias}(\hat{\rho}_z) \right) + n^{\frac{1-\beta}{2}} \left(\frac{\Sigma_{0x}}{\Omega_{xx}} \right) \text{Bias}(\hat{\rho}_z)$$

from which we deduce that

$$AMSE(\hat{a}_{IVX}) = \frac{1}{n^{1+\beta}} V + \frac{1}{n^{2\beta}} B^2,$$

where $V = \text{var}(\hat{a}_{IVX})$ and $B = \text{bias}(\hat{a}_{IVX})$ symbolically. Since $AMSE(\hat{a}_{IVX})$ is strictly decreases

ing in $\beta \in (0, 1)$ this criterion suggests that β be chosen as close to unity as possible. This analytical approach therefore provides no informative guidance on an upper bound $\bar{\beta} < 1$ for use in practical work. Simulations with cross-validation methods (not reported here) show that these methods encounter the same difficulty. On the other hand, our own simulation exercises investigating different choices for β in IVX regression broadly corroborate findings reported in Kostakis et al (2014) that a choice of $\beta = 0.95$ works well in terms of size control and power performance for a wide variety of regressors.

4.2 Simulation-based guidance

This section reports simulations performed to assess the performance of IVX and provide some practical guidance on suitable IVX persistence (β or R_{nz}) choices in finite sample sizes. We follow the same DGP as in (4.1) with innovations

$$u_t = \begin{pmatrix} u_{0t} \\ u_{xt} \end{pmatrix} \sim iid N(0, \Sigma). \quad (4.4)$$

The IVX instruments are constructed as in (2.9) and inference is based on the IVX estimator (2.17). The intercept in (4.1) is handled by demeaning, as explained in Section 2.5. We set $C_z = -5$ and vary β to explore size and power properties according to different degrees of IVX persistence. In finite samples of size $n = 100, 250$, the value $C_z = -5$ and various choices of $\beta \in (1/2, 1)$ (or $\beta \in (\alpha_2, 1)$) deliver a suitably wide range of autoregressive coefficients R_{nz} for the generation of the IVX instruments for investigation.

4.2.1 Single regressor

We first run simulations with a single local to unity regressor (i.e. with localizing exponent $\alpha = 1$). Although some of the main results in the paper relate to locally and mildly explosive regressors, we include results for stationary-side local to unity and unit root cases as well. Accordingly, localization coefficients for the regressor are selected from $c \in \{-50, -20, -10, -5, 0, 1, 3\}$. The first five cases $\{-50, -25, -5, -2, 0\}$ have been commonly studied in earlier works. The explosive choices $\{1, 3\}$ are new. The variance matrix of the innovations in (4.4) is parameterized as

$$\Sigma = \begin{pmatrix} 1 & \delta \\ \delta & 1 \end{pmatrix}$$

with $\delta = -0.95$. This value reflects realistic error correlation in predictive regressions and is commonly used in simulations reported in the literature. Since existing studies typically focus on a single predictor and a one-sided test, we similarly consider here tests of the null $H_0 : a = 0$ against $H_1 : a > 0$.

Figure 1⁵ summarizes the size and power performance of IVX tests using $\beta = 0.95$, and includes

⁵This Figure 1 is comparable to Figure 1 of Kostakis et al. (2014), although we report results for a one sided test

the Campbell-Yogo Q (CY-Q) test and its modified version for comparison⁶. In view of the extensive simulation results reported in Kostakis et al. (2014), we include a few representative cases here that highlight some of the new aspects of the limit theory.

- Figure 1 Here -

The invalidity of the CY-Q test for stationary and certain mildly integrated regressors was demonstrated recently in Phillips (2014) and that failure in this test is clearly evident in the first two graphs of Figure 1 ($c = -50$, $c = -25$, with $n = 250$) where CY-Q has zero size and low power. It is therefore particularly interesting to see the performance of the modified version of the CY-Q test that avoids the centering difficulties of the CY-Q test. In fact, the modified version of the CY-Q test shows good performance for these cases but its power function is dominated by IVX power for these cases and for all other values of c except $c = 0$. The modified CY-Q test performs well in the unit root case in terms of both size control and power. Overall, the results show greater reliability for the IVX test than either version of the two CY-Q tests.

These results corroborate the recommendation made in Kostakis et al. (2014) for use of the parameter setting $\beta = 0.95$ in empirical practice with IVX. The results here support the use of this choice for a range of empirically relevant persistent regressors in the I(1)-I(5) class.

4.2.2 Multiple regressors

We next consider the empirically relevant case of multiple predictive regressors. We take the bivariate case $K = 2$ in (4.1) to illustrate and consider four scenarios: (i) stationary local to unity and unit root cases ($c_1 = -25$ or -5 and $c_2 = 0$); (ii) bivariate unit root predictors ($c_1 = c_2 = 0$); (iii) stationary local to unity and mildly explosive root cases ($c_1 = -25$ or -5 and $c_2 = 3$); and (iv) unit root and mildly explosive root case ($c_1 = 0$ and $c_2 = 3$). These scenarios mimic several common empirical situations, including the practical example considered later of exchange rate predictive regression with two persistent predictors. The innovation structure follows (4.4) and uses the specific correlation matrix estimated in the empirical application in Section 5⁷

$$\Sigma = \begin{pmatrix} 1 & 0.10 & -0.29 \\ 0.10 & 1 & -0.03 \\ -0.29 & -0.03 & 1 \end{pmatrix}.$$

Figure 2 reports size and power performance in testing the joint predictive regression hypothesis $H_0 : a_1 = a_2 = 0$ using the χ^2 test of Theorem 3.2 and the same demeaning procedure from Section

while Kostakis et al. use a two sided test. Comparisons are given for explosive cases with $c = 1, 3$ for CY-Q test, in view of the limited availability of critical values for the Modified CY-Q test.

⁶The modified CY-Q test uses a uniformly valid confidence interval (CI) for ρ , rather than Stock's CI (1991) to avoid the invalidity issue with stationary predictor. In particular, we use the critical values from Andrews and Guggenberger (2014, Table 2).

⁷We selectively report here results with bivariate predictors that mimic the scenario in the empirical section below. Other scenarios considered in the simulations are available upon request.

2.5. The IVX persistence parameter is selected from $\beta \in \{0.85, 0.90, 0.95\}$ and the localizing coefficients are set to $C_z = \text{diag}(-5, -5)$. The nominal (asymptotic) test size is 0.05, the sample size $n = 100$, and the number of replications is 2,500. Again, finite sample test size is evidently well controlled, with increasing local power produced as β increases. Local power reaches unity quickly for the mixed root cases with a mildly explosive regressor.

- Figure 2 Here -

These results reveal that finite sample size and power of IVX predictability tests seem reliable across a range of different cases. Flexible choices of $\beta \in \{0.85, 0.90, 0.95\}$ with $C_z = -5$ seem to work well for finite sample sizes as low as $n = 100$, even for multiple regressor cases, and for regressors in classes (I1)-(I6). The findings generally corroborate those of Kostakis et al (2014).

5 Conclusion

This paper shows that the IVX method of Magdalinos and Phillips (2009b) is robust under locally and mildly explosive regressors as well as certain combinations of mixed integrated regressors. The models we have dealt with in the present paper are prototypical with features that allow for mixed regressors, giving a limit theory that shows the easy amenability of IVX methodology to standard testing procedures. However, while suggestive, these models do not provide a fully general environment that admits multiple regressors with variable localizing exponents, either for the mildly integrated or the mildly explosive regressors. Similarly, it does not provide a general limit theory of Wald testing with arbitrary restrictions for mixed regressors. Analytic research on the properties of IVX in such general environments is ongoing work by Magdalinos and Phillips. But the partial results provided here, combined with those in Kostakis et al (2014) are highly suggestive of the potential of these method to deal with mixed degrees of persistence, uncertainty over persistence degrees, and multiple regressors with different time series memory characteristics and long run properties.

A limitation of the IVX approach is that the localizing parameter β used in the construction of the instruments must be chosen by the empirical investigator. Asymptotic theory justifies a wide range of flexibility in the choice of β provided some general restrictions on this parameters are observed. But since the localizing rate α that controls the degree of persistence in the regressors is unknown these restrictions are imperfectly known. A convenient approach for much practical work is for investigators to simply assume that the regressors have (local) unit roots, or mildly explosive. This framework covers most practical situations including those that are commonly used in predictive regression, giving an assumed rate of $\alpha = 1$ for the regressors. For this general setting, theory indicates that the procedure offers a wide degree of flexibility in the choice of β and the construction of IVX instruments. Simulations confirm that this flexibility continues to hold in finite samples and that good size and power properties hold for various choices of $\beta \in [0.85, 0.95]$, with $\beta = 0.95$ typically producing greatest discriminatory power while still controlling size.

Figure 1: Empirical size/power performances of IVX, CY-Q and Modified CY-Q tests

Figure 2: Empirical size/power of IVX tests with bivariate regressors

6 Technical Appendix

This Appendix provides some useful preliminary lemmas and their proofs as well as proofs of the main theorems in the paper. The LUR explosive and mildly explosive cases are investigated in the following sections and main proofs for the case of mixed roots follow.

6.1 LUR explosive Regressors: (I4)

We consider the (I4) LUR explosive case ($C > 0, \alpha = 1$) with $\frac{1}{2} < \beta < 1$. Here, the same limit theory as MP continues to hold.

Lemma 6.1 (*Lemma 3.1 in PM with $C > 0$ and $\alpha = 1$*)

1. $n^{-\frac{1+\beta}{2}} \sum_{t=1}^n u_{0t} \tilde{z}'_t = n^{-\frac{1+\beta}{2}} \sum_{t=1}^n u_{0t} z'_t + o_p(1)$
2. $n^{-(1+\beta)} \sum_{t=1}^n x_t \tilde{z}'_t = n^{-(1+\beta)} \sum_{t=1}^n x_t z'_t - n^{-2} \sum_{t=1}^n x_{t-1} x'_{t-1} C C_z^{-1} + o_p(1)$
3. $n^{-(1+\beta)} \sum_{t=1}^n \tilde{z}_t \tilde{z}'_t = n^{-(1+\beta)} \sum_{t=1}^n z_t z'_t + o_p(1)$

Proof. 1. Propositions A1 and A2 in MP hold with $C > 0$ and $\alpha = 1$ without any substantial change in the proofs. To get the uniform bound for $E \|\psi_{nt}\|^2$, instead of using $\|R_n\|^{i-l} \leq 1$ for $l \leq i$ when $C < 0$, we can use when $C > 0$,

$$\|R_n\|^{i-l} \leq \exp(C) + o(1), \text{ for } i - l \leq n,$$

and we still have the same order of magnitude $\sup_{1 \leq t \leq n} E \|\psi_{nt}\|^2 = O(n^{1+2\beta})$ implying

$$n^{-(\frac{1+\beta}{2}+1)} \sum_{t=1}^n u_{0t} \psi'_{nt} = o_p(1), \text{ and } n^{-\frac{1+\beta}{2}} \sum_{t=1}^n u_{0t} \tilde{z}'_t = n^{-\frac{1+\beta}{2}} \sum_{t=1}^n u_{0t} z'_t + o_p(1).$$

Consequently, the limit theory of the numerator $n^{-\frac{1+\beta}{2}} \sum_{t=1}^n u_{0t} \tilde{z}'_t$ does not involve C at all. The proofs of 2 and 3 do not depend on the sign of C and use the distributional limit result

$$n^{-1/2} x_{[ns]} \implies J_x^c(r)$$

for $C > 0$ from Phillips (1987). ■

The limit theory of the IVX estimator (2.11) is therefore the same as the case under (I2), or N(ii) in MP, proving Theorem 3.1-1 and 3.2 for (I4) LUR explosive regressors.

6.2 Mildly Explosive Regressors: (I5)

We collect proofs for the (I5) mildly explosive regressor case here. As in MP, we consider two possible cases $\frac{2}{q} < \beta < \alpha < 1$ and $\frac{2}{q} < \alpha < \beta < 1$. In both cases, IVX asymptotics with mildly explosive regressors is developed and the pivotal chi-square limit theory is shown to be valid.

Proof of Lemma 2.2 (LP case). By back substitution and from the linear BN decomposition of u_t ,

$$\frac{1}{n^{\alpha/2}} R_n^{-(j-1)} x_{j-1} = \frac{1}{n^{\alpha/2}} \sum_{i=1}^{j-1} R_n^{-i} u_{xi} = \frac{1}{n^{\alpha/2}} \sum_{i=1}^{j-1} R_n^{-i} F_x(1) \varepsilon_i - \frac{1}{n^{\alpha/2}} \sum_{i=1}^{j-1} R_n^{-i} \Delta \tilde{\varepsilon}_{xi}, \text{ under LP.}$$

Consider the partial sum process $\eta_n(r) = \frac{1}{\sqrt{n}}S_{[nr]}$ with $S_j = \sum_{i=1}^j \varepsilon_i$. Using a strong approximation, we can enlarge the original probability space and construct a vector Brownian motion $\omega = BM(\Sigma)$ on this space with the property that for $\varepsilon > 0$, $E[\|\varepsilon_i\|^q] < \infty$,

$$\sup_{0 \leq r \leq 1} \|\eta_n(r) - \omega(r)\| = o_{a.s.} \left(n^{-1/2+1/q+\varepsilon} \right).$$

Define $\frac{e_i}{\sqrt{n}} = \omega\left(\frac{i}{n}\right) - \omega\left(\frac{i-1}{n}\right)$ whose distribution is iid normal with $E(e_i e_i') = \Sigma$. We generate another mildly explosive AR(1) process, $z_i = R_n z_{i-1} + v_{xi}$, where, under LP, $v_{xi} = F_x(1)e_i - \Delta \tilde{e}_{xi}$ is a linear process defined in a similar way to u_{xi} but using e_i instead of ε_i . Then

$$\frac{1}{n^{\alpha/2}} R_n^{-(j-1)} z_{j-1} = \frac{1}{n^{\alpha/2}} \sum_{i=1}^{j-1} R_n^{-i} F_x(1) e_i - \frac{1}{n^{\alpha/2}} \sum_{i=1}^{j-1} R_n^{-i} \Delta \tilde{e}_{xi}, \quad \text{under LP}.$$

Letting $j_n = j - 1$ be such that $k_n \leq j_n \leq n$ we have, under LP,

$$\frac{1}{n^{\alpha/2}} \sum_{i=1}^{j_n} R_n^{-i} F_x(1) e_i := \tilde{Y}_{Cn, j_n} \equiv N \left(0, \frac{1}{n^{\alpha}} \sum_{i=1}^{j_n} R_n^{-i} \Omega_{xx} R_n^{-i} \right),$$

by construction. Define the limit process $\lim_n \tilde{Y}_{Cn, j_n} = \tilde{Y}_C \equiv N(0, \int_0^\infty e^{-pC} \Omega_{xx} e^{-pC} dp)$. The strong approximation between these two processes will be shown below in (iii). Observe that

$$\begin{aligned} \left\| \frac{1}{n^{\alpha/2}} R_n^{-(j-1)} x_{j_n} - \tilde{Y}_C \right\| &\leq \left\| \frac{1}{n^{\alpha/2}} \sum_{i=1}^{j-1} R_n^{-i} F_x(1) \varepsilon_i - \frac{1}{n^{\alpha/2}} \sum_{i=1}^{j-1} R_n^{-i} F_x(1) e_i \right\| \\ &+ \left\| \tilde{Y}_{Cn} - \tilde{Y}_C \right\| + \left\| \frac{1}{n^{\alpha/2}} \sum_{i=1}^{j-1} R_n^{-i} \Delta \tilde{e}_{xi} \right\| + \left\| \frac{1}{n^{\alpha/2}} \sum_{i=1}^{j-1} R_n^{-i} \Delta \tilde{e}_{xi} \right\|. \end{aligned} \quad (6.1)$$

We now show that each term in (6.1) is $o_{a.s.}(1)$ uniformly over $j - 1 = j_n \in [k_n, n]$.

$$(i) \sup_{k_n \leq j-1 \leq n} \left\| \frac{1}{n^{\alpha/2}} \sum_{i=1}^{j-1} R_n^{-i} F_x(1) \varepsilon_i - \frac{1}{n^{\alpha/2}} \sum_{i=1}^{j-1} R_n^{-i} F_x(1) e_i \right\| = o_{a.s.}(1).$$

$$\begin{aligned} \frac{1}{n^{\alpha/2}} \sum_{i=1}^{j-1} R_n^{-i} F_x(1) \varepsilon_i &= \frac{n^{1/2}}{n^{\alpha/2}} \sum_{i=1}^{j-1} R_n^{-i} F_x(1) \left(\eta_n \left(\frac{i}{n} \right) - \eta_n \left(\frac{i-1}{n} \right) \right) \\ &= \frac{n^{1/2}}{n^{\alpha/2}} \left\{ \sum_{i=1}^{j-1} R_n^{-i} F_x(1) \eta_n \left(\frac{i}{n} \right) - \sum_{i=1}^{j-1} R_n^{-i} F_x(1) \eta_n \left(\frac{i-1}{n} \right) \right\} \\ &= \frac{n^{1/2}}{n^{\alpha/2}} \left\{ \sum_{i=1}^{j-1} R_n^{-i} F_x(1) \eta_n \left(\frac{i}{n} \right) - \sum_{s=0}^{j-2} R_n^{-s-1} F_x(1) \eta_n \left(\frac{s}{n} \right) \right\} \\ &= \frac{n^{1/2}}{n^{\alpha/2}} \left\{ \sum_{i=1}^{j-1} R_n^{-i} F_x(1) \eta_n \left(\frac{i}{n} \right) - \sum_{i=1}^{j-1} R_n^{-i-1} F_x(1) \eta_n \left(\frac{i}{n} \right) + R_n^{-j-2} F_x(1) \eta_n \left(\frac{j-1}{n} \right) \right\} \\ &= \frac{n^{1/2}}{n^{\alpha/2}} \left\{ (1 - R_n^{-1}) \sum_{i=1}^{j-1} R_n^{-i} F_x(1) \eta_n \left(\frac{i}{n} \right) + R_n^{-j-2} F_x(1) \eta_n \left(\frac{j-1}{n} \right) \right\}. \end{aligned}$$

Similarly,

$$\frac{1}{n^{\alpha/2}} \sum_{i=1}^{j-1} R_n^{-i} F_x(1) e_i = \frac{n^{1/2}}{n^{\alpha/2}} \left\{ (1 - R_n^{-1}) \sum_{i=1}^{j-1} R_n^{-i} F_x(1) \omega \left(\frac{i}{n} \right) + R_n^{-j-2} F_x(1) \omega \left(\frac{j-1}{n} \right) \right\}.$$

Therefore,

$$\begin{aligned} & \left\| \frac{1}{n^{\alpha/2}} \sum_{i=1}^{j-1} R_n^{-i} F_x(1) \varepsilon_i - \frac{1}{n^{\alpha/2}} \sum_{i=1}^{j-1} R_n^{-i} F_x(1) e_i \right\| \\ & \leq \frac{n^{1/2}}{n^{\alpha/2}} \|F_x(1)\| \left(\sup_{0 \leq r \leq 1} \|\eta_n(r) - \omega(r)\| \right) \|1 - R_n^{-1}\| \sum_{i=1}^{j-1} \|R_n\|^{-i} \\ & \quad + \frac{n^{1/2}}{n^{\alpha/2}} \|R_n\|^{-j-2} \|F_x(1)\| \left(\sup_{0 \leq r \leq 1} \|\eta_n(r) - \omega(r)\| \right) \\ & = \frac{n^{1/2}}{n^{\alpha/2}} \left(\sup_{0 \leq r \leq 1} \|\eta_n(r) - \omega(r)\| \right) \|F_x(1)\| \left(\|1 - R_n^{-1}\| \sum_{i=1}^{j-1} \|R_n\|^{-i} + \|R_n\|^{-j-2} \right), \end{aligned}$$

and for $k_n \leq j \leq n$, $\left(\|1 - R_n^{-1}\| \sum_{i=1}^{j-1} \|R_n\|^{-i} + \|R_n\|^{-j-2} \right)$ is $O(1)$ since $\|R_n\|^{-j} \leq \exp(-\min(c_i) \frac{j}{n^\alpha}) + o(1) = o(1)$ with $\frac{n^\alpha}{k_n} + \frac{n^\alpha}{n} \rightarrow 0$.

Therefore,

$$\sup_{k_n \leq j-1 \leq n} \left\| \frac{1}{n^{\alpha/2}} \sum_{i=1}^{j-1} R_n^{-i} F_x(1) \varepsilon_i - \frac{1}{n^{\alpha/2}} \sum_{i=1}^{j-1} R_n^{-i} F_x(1) e_i \right\| = \frac{n^{1/2}}{n^{\alpha/2}} o_{a.s.} \left(n^{-1/2+1/q+\varepsilon} \right) = o_{a.s.} \left(n^{\frac{1}{q}-\frac{\alpha}{2}+\varepsilon} \right),$$

and the exponent $\frac{1}{q} - \frac{\alpha}{2} + \varepsilon < 0$ for small enough ε because $\alpha > \frac{2}{q}$ and $q \geq 4$. The strong approximation is therefore sharper under higher moment conditions and when the signal strength of the mildly explosive regressors is closer to the local to unity region. Conversely, if the signal strength is closer to the purely explosive case ($\alpha \leq \frac{1}{2}$) higher moment conditions are needed to ensure that $\frac{1}{q} - \frac{\alpha}{2} + \varepsilon < 0$. The approximation is not possible in the pure explosive case ($\alpha = 0$).

$$(ii) \sup_{k_n \leq j-1 \leq n} \left\| \frac{1}{n^{\alpha/2}} \sum_{i=1}^{j-1} R_n^{-i} \Delta \tilde{\varepsilon}_{xi} \right\| = o_{a.s.}(1) = \sup_{k_n \leq j-1 \leq n} \left\| \frac{1}{n^{\alpha/2}} \sum_{i=1}^{j-1} R_n^{-i} \Delta \tilde{e}_{xi} \right\|.$$

Using summation by parts

$$\frac{1}{n^{\alpha/2}} \sum_{i=1}^{j-1} R_n^{-i} \Delta \tilde{\varepsilon}_{xi} = -\frac{1}{n^{\alpha/2}} \sum_{i=1}^{j-2} \Delta R_n^{-i-1} \tilde{\varepsilon}_{xi} - \frac{R_n^{-1}}{n^{\alpha/2}} \tilde{\varepsilon}_{x0} + \frac{R_n^{-(j-1)}}{n^{\alpha/2}} \tilde{\varepsilon}_{xj-1},$$

and from the given moment condition

$$\begin{aligned} P \left(\left\| \frac{1}{n^{\alpha/2}} \tilde{\varepsilon}_{xn} \right\| > \epsilon \right) & \leq \frac{E \|\tilde{\varepsilon}_{xn}\|^q}{n^{\frac{q\alpha}{2}} \epsilon^q} = O \left(\frac{1}{n^{\frac{q\alpha}{2}}} \right), \\ \sum_{n=1}^{\infty} P \left(\left\| \frac{1}{n^{\alpha/2}} \tilde{\varepsilon}_{xn} \right\| \right) & \leq O \left(\sum_{n=1}^{\infty} \frac{1}{n^{\frac{q\alpha}{2}}} \right) < \infty, \end{aligned}$$

as long as $q\alpha > 2$ which is satisfied by the condition $\alpha > \frac{2}{q}$. Thus, $n^{-\alpha/2}R_n^{-1}\tilde{\varepsilon}_{x0} = o_{a.s.}(1)$ by the Borel-Cantelli lemma. Using the fact that with $\|R_n\|^{-j} = o(1)$ when $j-1 \in [k_n, n]$, we also have $n^{-\alpha/2}R_n^{-(j-1)}\tilde{\varepsilon}_{xj-1} = o_{a.s.}(1)$. Note further that

$$\frac{1}{n^{\alpha/2}} \sum_{i=1}^{j-2} \Delta R_n^{-i-1} \tilde{\varepsilon}_{xi} = \frac{R_n^{-1}C}{n^{\frac{3\alpha}{2}}} \sum_{i=1}^{j-2} R_n^{-i} \tilde{\varepsilon}_{xi},$$

and $R_n^{-1}C = O(1)$. For $k_n \leq j \leq n$

$$E \left(\left\| \frac{1}{n^{\frac{3\alpha}{2}}} \sum_{i=1}^{j-2} R_n^{-i} \tilde{\varepsilon}_{xi} \right\|^q \right)^{\frac{1}{q}} = \left\| \frac{1}{n^{\frac{3\alpha}{2}}} \sum_{i=1}^{j-2} R_n^{-i} \tilde{\varepsilon}_{xi} \right\|_{L_q} \leq \frac{\|\tilde{\varepsilon}_{xi}\|_{L_q}}{n^{\frac{3\alpha}{2}}} \sum_{i=1}^{j-2} \|R_n\|^{-i} = O\left(\frac{1}{n^{\frac{\alpha}{2}}}\right),$$

since $\tilde{\varepsilon}_{xi}$ is stationary and $n^{-\alpha} \sum_{i=1}^{j-2} \|R_n\|^{-i} = O(1)$ for $k_n \leq j \leq n$. Hence

$$E \left(\left\| \frac{1}{n^{\frac{3\alpha}{2}}} \sum_{i=1}^{j-2} R_n^{-i} \tilde{\varepsilon}_{xi} \right\|^q \right) = O\left(\frac{1}{n^{\frac{q\alpha}{2}}}\right),$$

and

$$P \left(\left\| \frac{1}{n^{\frac{3\alpha}{2}}} \sum_{i=1}^{j-2} R_n^{-i} \tilde{\varepsilon}_{xi} \right\| > \epsilon \right) \leq \frac{E \left(\left\| \frac{1}{n^{\frac{3\alpha}{2}}} \sum_{i=1}^{j-2} R_n^{-i} \tilde{\varepsilon}_{xi} \right\|^q \right)}{\epsilon^{\frac{q}{2}}} = O\left(\frac{1}{n^{\frac{q\alpha}{2}}}\right).$$

We then have $n^{-3\alpha/2}R_n^{-1}C \sum_{i=1}^{j-2} R_n^{-i} \tilde{\varepsilon}_{xi} = o_{a.s.}(1)$ under the given moment conditions just as before.

The proof for $n^{-\alpha/2} \sum_{i=1}^{j-1} R_n^{-i} \Delta \tilde{c}_{xi}$ is exactly same.

(iii) $\|\tilde{Y}_{Cn} - \tilde{Y}_C\| = o_{a.s.}(1)$

For clarity, we denote $n^{-\alpha/2} \sum_{i=1}^{j-1} R_n^{-i} F_x(1)e_i = \tilde{Y}_{Cn} := \tilde{Y}_{Cn,j-1}$, and define the martingale array $\{\mathcal{F}_{n,j-1}, \tilde{Y}_{Cn,j-1} : j-1 \geq k_n\}$ with natural filtration $\mathcal{F}_{n,j-1}$. We have $E_{\mathcal{F}_{n,j-1}}[\tilde{Y}_{Cn,j}] = \tilde{Y}_{Cn,j-1}$ and

$$E[\tilde{Y}_{Cn,j} \tilde{Y}_{Cn,j}'] = \frac{1}{n^\alpha} \sum_{i=1}^{j-1} R_n^{-i} \Omega_{xx} R_n^{-i} \rightarrow \int_0^\infty e^{-pC} \Omega_{xx} e^{-pC} dp < \infty \text{ for } j-1 \geq k_n.$$

By the martingale convergence theorem for L_2 -bounded martingales (e.g., Hall and Heyde, 1980), $\tilde{Y}_{Cn,j-1} \rightarrow_{a.s.} \tilde{Y}_C \equiv N(0, \int_0^\infty e^{-pC} \Omega_{xx} e^{-pC} dp)$, which is a distributionally equivalent copy of Y_C . Combining (i)-(iii) gives the required result. ■

Proof of Lemma 2.2 (NLP). From the nonlinear BN decomposition of u_t we have

$$\begin{aligned} \frac{1}{n^{\alpha/2}} R_n^{-(j-1)} x_{j-1} &= \frac{1}{n^{\alpha/2}} \sum_{i=1}^{j-1} R_n^{-i} u_{xi} \\ &= \frac{1}{n^{\alpha/2}} \sum_{i=1}^{j-1} R_n^{-i} D_{xi} - \frac{1}{n^{\alpha/2}} \sum_{i=1}^{j-1} R_n^{-i} \Delta \tilde{D}_{xi}, \quad \text{under NLP} \end{aligned} \quad (6.2)$$

We modify the proof of Theorem 3 in Wu (2007), where a strong invariance principle for a non-weighted partial sum $(\sum_i u_{xi})$ was given. The conditions in Theorem 3 of Wu (2007) are met under

the GMC condition imposed in Assumption NLP. We proceed in two steps. **(i)** First, we show the remainder term is $o_{a.s.}(1)$ in (6.2).

$$\sup_{j_n} \left\| \frac{1}{n^{\alpha/2}} \sum_{i=1}^{j_n} R_n^{-i} \Delta \tilde{D}_{xi} \right\| \leq \sup_{i \leq j_n} \left\| \frac{R_n^{-i}}{n^{\alpha/2}} \right\| \sup_{j_n} \left\| \sum_{i=1}^{j_n} \Delta \tilde{D}_{xi} \right\| = \frac{1}{n^{\alpha/2}} \sup_{j_n} \left\| \sum_{i=1}^{j_n} \Delta \tilde{D}_{xi} \right\|,$$

and from Theorem 3-(ii) in Wu (2007), it is known that

$$\sup_{j_n} \left\| \sum_{i=1}^{j_n} \Delta \tilde{D}_{xi} \right\| = o_{a.s.} \left(n^{1/q} (\log n)^{1/2+1/q} (\log \log)^{2/q} \right),$$

so that

$$\sup_{j_n} \left\| \frac{1}{n^{\alpha/2}} \sum_{i=1}^{j_n} R_n^{-i} \Delta \tilde{D}_{xi} \right\| = o_{a.s.} \left(n^{\frac{1}{q} - \frac{\alpha}{2}} (\log n)^{1/2+1/q} (\log \log)^{2/q} \right) = o_{a.s.}(1)$$

when $\frac{1}{q} - \frac{\alpha}{2} < 0$, as in the proof of LP case. **(ii)** The strong approximation now follows from

$$\sup_{j_n} \left\| \frac{1}{n^{\alpha/2}} \sum_{i=1}^{j_n} R_n^{-i} u_{xi} - \frac{1}{n^{\alpha/2}} \sum_{i=1}^{j_n} R_n^{-i} D_{xi} \right\| = o_{a.s.}(1),$$

by a similar argument, viz.,

$$\begin{aligned} \sup_{j_n} \left\| \frac{1}{n^{\alpha/2}} \sum_{i=1}^{j_n} R_n^{-i} D_{xi} - \tilde{Y}_C \right\| &\leq \frac{1}{n^{\alpha/2}} \sup_{j_n \leq n} \|B_x(T_{j_n}) - \omega(j_n)\| \\ &= o_{a.s.} \left(n^{\frac{1}{q} - \frac{\alpha}{2}} (\log n)^{1/2+1/q} (\log \log)^{2/q} \right) = o_{a.s.}(1), \end{aligned}$$

in which we enlarge the original probability space and construct a vector Brownian motion $\omega = BM(\Omega_{xx})$ and partial sums $T_{j_n} = \sum_{i=1}^{j_n} \tau_i$ that have the same conditional mean and variance with $\sum_{i=1}^{j_n} D_{xi}$ almost surely (see Hall and Heyde (1980) or page 2314 of Wu (2007)). As in LP case, the strong approximation rate is sharper under higher moment conditions and when the signal strength of the mildly explosive regressors is closer to the local to unity region. Higher moment conditions are also needed when the signal strength is closer to the purely explosive case ($\alpha \leq \frac{1}{2}$). ■

Proof of Lemma 2.3. Since k_n satisfies $\frac{n^{\alpha \vee \beta}}{k_n} \rightarrow 0$, we have both $\|R_n\|^{-k_n} \rightarrow 0$ and $\|R_{nz}\|^{k_n} \rightarrow 0$, so the condition for k_n in lemma 2.2 holds. In addition $\|R_{nz}\|^t \rightarrow 0$ and $\|R_n\|^{-t} \rightarrow 0$ since $t > k_n$. We have

$$\frac{1}{n^{\frac{\alpha}{2} + \alpha \wedge \beta}} R_n^{-t} \psi_{nt} = \frac{1}{n^{\frac{\alpha}{2} + \alpha \wedge \beta}} R_n^{-t} \sum_{j=1}^t R_{nz}^{t-j} x_{j-1} = \frac{1}{n^{\frac{\alpha}{2} + \alpha \wedge \beta}} R_n^{-t} \sum_{j=k_n+1}^t R_{nz}^{t-j} x_{j-1} + o_p(1),$$

because the frontal summation over $1 \leq j \leq k_n$ is negligible as we now show. In particular, using

(2.13), we have

$$\begin{aligned}
\left\| \frac{1}{n^{\frac{\alpha}{2} + \alpha \wedge \beta}} R_n^{-t} \sum_{j=1}^{k_n} R_{nz}^{t-j} x_{j-1} \right\|_{L_1} &= \left\| \frac{1}{n^{\alpha \wedge \beta}} R_n^{-t} \sum_{j=1}^{k_n} R_{nz}^{t-j} R_n^{(j-1)} \left(\frac{R_n^{-(j-1)} x_{j-1}}{n^{\alpha/2}} \right) \right\|_{L_1} \\
&\leq \sup_{j=1} \left\| \frac{R_n^{-(j-1)} x_{j-1}}{n^{\alpha/2}} \right\|_{L_2} \frac{1}{n^{\alpha \wedge \beta}} \|R_n\|^{-t} \|R_{nz}\|^t \sum_{j=1}^{k_n} \|R_{nz}\|^{-j} \|R_n\|^{j-1} \\
&= O \left(\frac{1}{n^{\alpha \wedge \beta}} \|R_{nz}\|^{t-1} \|R_n\|^{-t} \frac{\|R_{nz}\|^{-k_n} \|R_n\|^{k_n} - 1}{\|R_{nz}\|^{-1} \|R_n\| - 1} \right) \\
&= O \left(\frac{\|R_{nz}\|^{t-k_n-1} \|R_n\|^{k_n-t} - \|R_n\|^{-t} \|R_{nz}\|^{t-1}}{n^{\alpha \wedge \beta} (\|R_{nz}\|^{-1} \|R_n\| - 1)} \right) = o(1),
\end{aligned}$$

since

$$\begin{aligned}
n^{\alpha \wedge \beta} (\|R_{nz}\|^{-1} \|R_n\| - 1) &= n^{\alpha \wedge \beta} \left(\frac{1 + \frac{\max(c_i)}{n^\alpha}}{1 + \frac{\max(c_{zi})}{n^\beta}} - 1 \right) = n^{\alpha \wedge \beta} \left(\frac{\frac{\max(c_i)}{n^\alpha} - \frac{\max(c_{zi})}{n^\beta}}{1 + \frac{\max(c_{zi})}{n^\beta}} \right) \\
&\rightarrow \begin{cases} -\max(c_{zi}), & \text{if } \beta < \alpha \\ \max(c_i), & \text{if } \alpha < \beta \\ \max(c_i) - \max(c_{zi}), & \text{if } \alpha = \beta \end{cases},
\end{aligned}$$

which is non-zero and finite in all cases (recall $c_{zi} < 0$, $c_i > 0$ for all i). Also $\|R_n\|^{-t} \|R_{nz}\|^t = o(1)$ and $\|R_{nz}\|^{t-k_n} \|R_n\|^{k_n-t} = o(1)$ where the second equality holds because

$$\|R_{nz}\|^{t-k_n} \|R_n\|^{k_n-t} = O \left(e^{\max(c_{zi}) \frac{t-k_n}{n^\beta}} e^{-\max(c_i) \frac{t-k_n}{n^\alpha}} \right),$$

and $\frac{k'_n}{n^{\alpha \wedge \beta}} \leq \frac{t-k_n}{n^{\alpha \wedge \beta}} \rightarrow \infty$ by the given rate condition. Hence $n^{-(\frac{\alpha}{2} + \alpha \wedge \beta)} R_n^{-t} \sum_{j=1}^{k_n} R_{nz}^{t-j} x_{j-1} = o_p(1)$.

Now we have

$$\begin{aligned}
\frac{1}{n^{\frac{\alpha}{2} + \alpha \wedge \beta}} R_n^{-t} \psi_{nt} &= \frac{1}{n^{\frac{\alpha}{2} + \alpha \wedge \beta}} R_n^{-t} \sum_{j=k_n+1}^t R_{nz}^{t-j} x_{j-1} + o_p(1) \\
&= \frac{1}{n^{\alpha \wedge \beta}} R_n^{-t} \sum_{j=k_n+1}^t R_{nz}^{t-j} R_n^{(j-1)} \left(R_n^{-(j-1)} \frac{x_{j-1}}{n^{\alpha/2}} \right) + o_p(1) \\
&= \left(\frac{1}{n^{\alpha \wedge \beta}} R_n^{-t} \sum_{j=k_n+1}^t R_{nz}^{t-j} R_n^{(j-1)} \right) (\tilde{Y}_C + o_p(1)) \quad (\text{using lemma 2.2}) \\
&= \left(\frac{1}{n^{\alpha \wedge \beta}} R_n^{-t} \sum_{j=1}^t R_{nz}^{t-j} R_n^{(j-1)} \right) (\tilde{Y}_C + o_p(1)), \quad (\text{putting back the negligible front sum}).
\end{aligned}$$

Note that

$$\begin{aligned} & \frac{1}{n^{(\alpha \wedge \beta)}} \sum_{j=1}^t R_{nz}^{t-j} R_n^{j-1-t} \\ &= \frac{1}{n^{(\alpha \wedge \beta)}} \text{diag} \left(\sum_{j=1}^t \left(1 + \frac{c_{z1}}{n^\beta}\right)^{t-j} \left(1 + \frac{c_1}{n^\alpha}\right)^{j-1-t}, \dots, \sum_{j=1}^n \left(1 + \frac{c_{zK}}{n^\beta}\right)^{t-j} \left(1 + \frac{c_K}{n^\alpha}\right)^{j-1-t} \right). \end{aligned}$$

For all i

$$\begin{aligned} & \frac{1}{n^{(\alpha \wedge \beta)}} \sum_{j=1}^t \left(1 + \frac{c_{zi}}{n^\beta}\right)^{t-j} \left(1 + \frac{c_i}{n^\alpha}\right)^{j-1-t} = \frac{1}{n^{(\alpha \wedge \beta)}} \left(1 + \frac{c_{zi}}{n^\beta}\right)^{t-1} \left(1 + \frac{c_i}{n^\alpha}\right)^{-t} \frac{\left\{ \left(1 + \frac{c_{zi}}{n^\beta}\right)^{-t} \left(1 + \frac{c_i}{n^\alpha}\right)^t - 1 \right\}}{\left(1 + \frac{c_{zi}}{n^\beta}\right)^{-1} \left(1 + \frac{c_i}{n^\alpha}\right) - 1} \\ &= \frac{1}{n^{(\alpha \wedge \beta)}} \frac{\left\{ \left(1 + \frac{c_{zi}}{n^\beta}\right)^{-1} - \left(1 + \frac{c_{zi}}{n^\beta}\right)^{t-1} \left(1 + \frac{c_i}{n^\alpha}\right)^{-t} \right\}}{\left(1 + \frac{c_{zi}}{n^\beta}\right)^{-1} \left(1 + \frac{c_i}{n^\alpha}\right) - 1} = \frac{1 + o(1)}{n^{(\alpha \wedge \beta)} \left(\frac{c_i}{n^\alpha} - \frac{c_{zi}}{n^\beta} - \frac{cc_{zi}}{n^{\alpha+\beta}} \right)} \\ &\rightarrow -\frac{1}{c_{zi}} \text{ if } \beta < \alpha, \quad \frac{1}{c_i} \text{ if } \alpha < \beta \text{ and } \frac{1}{c_i - c_{zi}} \text{ if } \alpha = \beta. \end{aligned}$$

Hence,

$$\frac{1}{n^{(\alpha \wedge \beta)}} R_n^{-t} \sum_{j=1}^t R_{nz}^{t-j} R_n^{(j-1)} \rightarrow \begin{cases} -C_z^{-1}, & \text{if } \beta < \alpha \\ C^{-1}, & \text{if } \alpha < \beta \\ (C - C_z)^{-1}, & \text{if } \alpha = \beta \end{cases} =: C_{z\alpha\beta}, \quad (6.3)$$

giving the required result. ■

Proof of Lemma 2.4. Part (1). From (2.10), we have

$$\begin{aligned} \frac{1}{n^{(\alpha \wedge \beta)}} \sum_{t=1}^n u_{0t} z'_t R_n^{-n} &= \frac{1}{n^{(\alpha \wedge \beta)}} \sum_{t=1}^n u_{0t} z'_t R_n^{-n} + \frac{1}{n^{\alpha + (\alpha \wedge \beta)}} \sum_{t=1}^n u_{0t} \psi'_{nt} R_n^{-n} C \\ &= \frac{1}{n^{\alpha + (\alpha \wedge \beta)}} \sum_{t=1}^n u_{0t} \psi'_{nt} R_n^{-n} C + o_p(1), \end{aligned}$$

where the last equality holds because R_n^{-n} dominates the order of magnitude of $\sum_{t=1}^n u_{0t} z'_t = O_p(n)$ (c.f., Magdalinos and Phillips, 2009a, equation (10)).

To use lemma 2.3 we first show that $n^{-(\alpha + (\alpha \wedge \beta))} \sum_{t=1}^{k_n + k'_n - 1} u_{0t} \psi'_{nt} R_n^{-n} = o_p(1)$ for k_n and k'_n satisfying the conditions of lemma 2.3. Note that

$$\begin{aligned} & \left\| \frac{1}{n^{\alpha + (\alpha \wedge \beta)}} \sum_{t=1}^{k_n + k'_n - 1} u_{0t} \psi'_{nt} R_n^{-n} \right\|_{L_1} = \left\| \frac{1}{n^{\frac{\alpha}{2}}} \sum_{t=1}^{k_n + k'_n - 1} u_{0t} \left(\frac{1}{n^{\frac{\alpha}{2} + (\alpha \wedge \beta)}} R_n^{-t} \psi_{nt} \right)' R_n^{t-n} \right\|_{L_1} \\ &\leq \left(\sup_t \|u_{0t}\|_{L_2} \right) \left(\sup_t \left\| \frac{1}{n^{\frac{\alpha}{2} + (\alpha \wedge \beta)}} R_n^{-t} \psi_{nt} \right\|_{L_2} \right) \frac{\|R_n\|^{-n}}{n^{\frac{\alpha}{2}}} \sum_{t=1}^{k_n + k'_n - 1} \|R_n\|^t \text{ and using (2.15),} \end{aligned}$$

$$= O\left(\frac{n^{\frac{\alpha}{2}} \|R_n\|^{k_n+k'_n-n}}{n^\alpha (1 - \|R_n\|)}\right) = O\left(n^{\frac{\alpha}{2}} \|R_n\|^{k_n+k'_n-n}\right) = o(1),$$

because of the exponentially fast convergence of $\|R_n\|^{k_n+k'_n-n} \rightarrow 0$.

Hence, we have $n^{-(\alpha+\alpha\wedge\beta)} \sum_{t=1}^{k_n+k'_n-1} u_{0t} \psi'_{nt} R_n^{-n} = o_p(1)$.

Using the same sum splitting argument as in lemma 2.3,

$$\begin{aligned} & \text{vec}\left(\frac{1}{n^{\alpha+(\alpha\wedge\beta)}} \sum_{t=1}^n u_{0t} \psi'_{nt} R_n^{-n}\right) = \text{vec}\left(\frac{1}{n^{\alpha+(\alpha\wedge\beta)}} \sum_{t=k_n+k'_n}^n u_{0t} \psi'_{nt} R_n^{-n}\right) + o_p(1) \\ &= \text{vec}\left(\frac{1}{n^{\frac{\alpha}{2}}} \sum_{t=k_n+k'_n}^n u_{0t} \left(\frac{1}{n^{\frac{\alpha}{2}+(\alpha\wedge\beta)}} R_n^{-t} \psi_{nt}\right)' (R_n^{t-n})'\right) + o_p(1) \\ &= \text{vec}\left(\frac{1}{n^{\frac{\alpha}{2}}} \sum_{t=k_n+k'_n}^n u_{0t} (C_{z\alpha\beta} \tilde{Y}_C + o_p(1))' (R_n^{t-n})'\right) + o_p(1) \quad (\text{using lemma 2.3}) \\ &= \text{vec}\left(\frac{1}{n^{\frac{\alpha}{2}}} \sum_{t=k_n+k'_n}^n u_{0t} (\tilde{Y}_C)' (R_n^{t-n} C_{z\alpha\beta})'\right) + o_p(1) \\ &= \left(\frac{1}{n^{\frac{\alpha}{2}}} \sum_{t=k_n+k'_n}^n R_n^{-(n-t)} C_{z\alpha\beta} \otimes u_{0t}\right) (\tilde{Y}_C) + o_p(1) \\ &= (C_{z\alpha\beta} \otimes I_m) \left(\frac{1}{n^{\frac{\alpha}{2}}} \sum_{t=1}^n R_n^{-(n-t)} \otimes u_{0t}\right) (\tilde{Y}_C) + o_p(1) \quad (\text{putting back the front sum}) \\ &\Rightarrow (C_{z\alpha\beta} \otimes I_m) \times MN \left(0, \int_0^\infty e^{-pC} \tilde{Y}_C \tilde{Y}_C' e^{-pC} dp \otimes \Omega_{00}\right), \end{aligned}$$

where the last step comes from the same procedure as in equations (22)-(26) of Magdalinos and Phillips (2009a). Finally,

$$\begin{aligned} \text{vec}\left(\frac{1}{n^{(\alpha\wedge\beta)}} \sum_{t=1}^n u_{0t} z'_t R_n^{-n}\right) &= (C \otimes I_m) \text{vec}\left(\frac{1}{n^{\alpha+(\alpha\wedge\beta)}} \sum_{t=1}^n u_{0t} \psi'_{nt} R_n^{-n}\right) + o_p(1) \\ &\Rightarrow (CC_{z\alpha\beta} \otimes I_m) \times MN \left(0, \int_0^\infty e^{-pC} Y_C Y_C' e^{-pC} dp \otimes \Omega_{00}\right), \end{aligned}$$

where we have used Y_C instead of its distributional copy $\tilde{Y}_C$ since we are concerned with weak convergence in the original probability space from this point onwards.

Part (2). From $\sum x_t \tilde{z}'_t = \sum x_t z'_t + n^{-\alpha} \sum_{t=1}^n x_t \psi'_{nt} C$, we have

$$\begin{aligned} \frac{1}{n^{\alpha+(\alpha \wedge \beta)}} \sum R_n^{-n} x_t \tilde{z}'_t R_n^{-n} &= \frac{1}{n^{\alpha+(\alpha \wedge \beta)}} \sum R_n^{-n} x_t z'_t R_n^{-n} + \frac{1}{n^{2\alpha+(\alpha \wedge \beta)}} \sum_{t=1}^n R_n^{-n} x_t \psi'_{nt} R_n^{-n} C \\ &= \frac{1}{n^{2\alpha+(\alpha \wedge \beta)}} \sum_{t=1}^n R_n^{-n} x_t \psi'_{nt} R_n^{-n} C + o_p(1), \end{aligned}$$

as in Part (1) and using the same sum splitting argument again with lemma 2.2 and 2.3,

$$\begin{aligned} \frac{1}{n^{2\alpha+(\alpha \wedge \beta)}} \sum_{t=1}^n R_n^{-n} x_t \psi'_{nt} R_n^{-n} &= \frac{1}{n^{2\alpha+(\alpha \wedge \beta)}} \sum_{t=k_n+k'_n}^n R_n^{-n} x_t \psi'_{nt} R_n^{-n} + o_p(1) \\ &= \frac{1}{n^\alpha} \sum_{t=k_n+k'_n}^n R_n^{-(n-t)} \left(\frac{R_n^{-t}}{n^{\frac{\alpha}{2}}} x_t \right) \left(\frac{R_n^{-t}}{n^{\frac{\alpha}{2}+(\alpha \wedge \beta)}} \psi_{nt} \right)' R_n^{-(n-t)} + o_p(1) \\ &= \frac{1}{n^\alpha} \sum_{t=k_n+k'_n}^n R_n^{-(n-t)} \left(\tilde{Y}_C \right) \left(\tilde{Y}_C \right)' R_n^{-(n-t)} C_{z\alpha\beta} + o_p(1) \\ &= \frac{1}{n^\alpha} \sum_{t=1}^n R_n^{-(n-t)} \left(\tilde{Y}_C \right) \left(\tilde{Y}_C \right)' R_n^{-(n-t)} C_{z\alpha\beta} + o_p(1) \\ &\Rightarrow \int_0^\infty e^{-pC} Y_C Y_C' e^{-pC} dp C_{z\alpha\beta}. \end{aligned}$$

It follows that

$$\frac{1}{n^{\alpha+(\alpha \wedge \beta)}} \sum R_n^{-n} x_t \tilde{z}'_t R_n^{-n} \Rightarrow \int_0^\infty e^{-pC} Y_C Y_C' e^{-pC} dp C C_{z\alpha\beta}.$$

■

Proof of Theorem 2.2. We have the following limit theory for the IVX estimator $\hat{A}_n = \left(Y' \tilde{Z} \right) \left(X' \tilde{Z} \right)^{-1}$ without the bias correction shown in (2.12). Joint convergence of the numerator and denominator in lemma 2.4 parts (1) and (2) is established as in Magdalinos and Phillips (2009a, Proof of Theorem 4.1). Using these results with $\hat{A}_n - A = \left(U'_0 \tilde{Z} \right) \left(X' \tilde{Z} \right)^{-1}$ we have

$$\begin{aligned} \text{vec} \left[n^\alpha \left(\hat{A}_n - A \right) R_n^n \right] &= \left[\left\{ \left(\frac{1}{n^{\alpha+(\alpha \wedge \beta)}} \sum R_n^{-n} x_t \tilde{z}'_t R_n^{-n} \right)' \right\}^{-1} \otimes I_m \right] \text{vec} \left(\frac{1}{n^{(\alpha \wedge \beta)}} \sum_{t=1}^n u_{0t} \tilde{z}'_t R_n^{-n} \right) \\ &\Rightarrow \left[\left(\left\{ \tilde{\Psi}_{yy} \times (C C_{z\alpha\beta}) \right\}' \right)^{-1} \otimes I_m \right] \\ &\quad \times (C C_{z\alpha\beta} \otimes I_m) \times MN \left(0, \tilde{\Psi}_{yy} \otimes \Omega_{00} \right) \\ &\equiv MN \left(0, \left(\tilde{\Psi}_{yy} \right)^{-1} \otimes \Omega_{00} \right). \end{aligned}$$

The bias corrected IVX estimator given in (2.12) is asymptotically equivalent to the uncorrected estimator $\hat{A}_n$ due to the signal strength of the $X'\tilde{Z}$ matrix, i.e.,

$$\hat{A}_n - \tilde{A}_n = n\hat{\Delta}_{0x}(X'\tilde{Z})^{-1} = O_p\left(\frac{n}{n^{\alpha+(\alpha\wedge\beta)}\|R_n\|^{2n}}\right) = o_p(1),$$

and

$$\begin{aligned} n^\alpha (\tilde{A}_n - A) R_n^n &= n^\alpha (\hat{A}_n - A) R_n^n + O_p\left(\frac{n^{1-(\alpha\wedge\beta)}}{\|R_n\|^n}\right) \\ &= n^\alpha (\hat{A}_n - A) R_n^n + o_p(1). \end{aligned}$$

As a result, the same limit theory holds regardless of the bias correction, proving theorem 2.2. ■

The following lemma helps in characterizing the variance matrix asymptotics for (2.11).

Lemma 6.2 $\frac{1}{n^{2(\alpha\wedge\beta)}} \sum R_n^{-n} \tilde{z}_t \tilde{z}_t' R_n^{-n} \implies CC_{z\alpha\beta} \tilde{\Psi}_{yy} CC_{z\alpha\beta}.$

Proof. We have the decomposition of the sample moment matrix

$$\begin{aligned} \sum \tilde{z}_t \tilde{z}_t' &= \sum \left(z_t + \frac{C}{n^\alpha} \psi_{nt} \right) \left(z_t + \frac{C}{n^\alpha} \psi_{nt} \right)' \\ &= \sum z_t z_t' + \frac{C}{n^\alpha} \sum \psi_{nt} z_t' + \frac{1}{n^\alpha} \sum z_t \psi_{nt}' C + \frac{1}{n^{2\alpha}} \sum C \psi_{nt} \psi_{nt}' C'. \end{aligned}$$

By the same methods used in lemma 2.3 and 2.4, it is straightforward to show that

$$\begin{aligned} &\frac{1}{n^{2\alpha+2(\alpha\wedge\beta)}} \sum_{t=1}^n R_n^{-n} \psi_{nt} \psi_{nt}' R_n^{-n} \\ &= \frac{1}{n^\alpha} \sum_{t=1}^n R_n^{-(n-t)} \left(\frac{1}{n^{\frac{\alpha}{2}+(\alpha\wedge\beta)}} R_n^{-t} \psi_{nt} \right) \left(\frac{1}{n^{\frac{\alpha}{2}+(\alpha\wedge\beta)}} R_n^{-t} \psi_{nt} \right)' R_n^{-(n-t)} \\ &= \frac{1}{n^\alpha} \sum_{t=k_n+k_n'}^n R_n^{-(n-t)} \left(\frac{1}{n^{\frac{\alpha}{2}+(\alpha\wedge\beta)}} R_n^{-t} \psi_{nt} \right) \left(\frac{1}{n^{\frac{\alpha}{2}+(\alpha\wedge\beta)}} R_n^{-t} \psi_{nt} \right)' R_n^{-(n-t)} + o_p(1) \\ &= \frac{1}{n^\alpha} \sum_{t=k_n+k_n'}^n R_n^{-(n-t)} \left(C_{z\alpha\beta} \tilde{Y}_C \right) \left(C_{z\alpha\beta} \tilde{Y}_C \right)' R_n^{-(n-t)} + o_p(1) \end{aligned}$$

$$\begin{aligned}
&= (C_{z\alpha\beta}) \left(\frac{1}{n^\alpha} \sum_{t=k_n+k'_n}^n R_n^{-(n-t)} (\tilde{Y}_C) (\tilde{Y}_C)' R_n^{-(n-t)} \right) (C_{z\alpha\beta}) + o_p(1) \\
&= (C_{z\alpha\beta}) \left(\frac{1}{n^\alpha} \sum_{t=1}^n R_n^{-(n-t)} (\tilde{Y}_C) (\tilde{Y}_C)' R_n^{-(n-t)} \right) (C_{z\alpha\beta}) + o_p(1) \\
&\Rightarrow (C_{z\alpha\beta}) \int_0^\infty e^{-pC} Y_C Y_C' e^{-pC} dp (C_{z\alpha\beta}).
\end{aligned}$$

This term clearly dominates all the other terms since z_t is mildly integrated. Hence

$$\begin{aligned}
\frac{1}{n^{2(\alpha\wedge\beta)}} \sum R_n^{-n} \tilde{z}_t \tilde{z}_t' R_n^{-n} &= C \left(\frac{1}{n^{2\alpha+2(\alpha\wedge\beta)}} \sum_{t=1}^n R_n^{-n} \psi_{nt} \psi_{nt}' R_n^{-n} \right) C + o_p(1) \\
&\Rightarrow (CC_{z\alpha\beta}) \int_0^\infty e^{-pC} Y_C Y_C' e^{-pC} dp (CC_{z\alpha\beta}) \\
&\equiv CC_{z\alpha\beta} \tilde{\Psi}_{yy} CC_{z\alpha\beta}.
\end{aligned}$$

■

The robust chi-square limit theory of the self-normalized IVX estimator follows and is given in the next proof.

Proof of Theorem 2.3. We have the following limit theory for the variance estimator,

$$\begin{aligned}
&n^{2\alpha} (R_n^n \otimes I_m) \left[(X' P_{\tilde{Z}} X)^{-1} \otimes \hat{\Omega}_{00} \right] (R_n^n \otimes I_m) \\
&= \left[\frac{1}{n^{2\alpha}} R_n^{-n} \left\{ X' \tilde{Z} (\tilde{Z}' \tilde{Z})^{-1} \tilde{Z}' X \right\} R_n^{-n} \right]^{-1} \otimes \hat{\Omega}_{00} \\
&= \left[\left(\frac{1}{n^{\alpha+(\alpha\wedge\beta)}} \sum R_n^{-n} x_t \tilde{z}_t' R_n^{-n} \right) \left(\frac{1}{n^{2(\alpha\wedge\beta)}} \sum R_n^{-n} \tilde{z}_t \tilde{z}_t' R_n^{-n} \right)^{-1} \left(\frac{1}{n^{\alpha+(\alpha\wedge\beta)}} \sum R_n^{-n} x_t \tilde{z}_t' R_n^{-n} \right)' \right]^{-1} \otimes \hat{\Omega}_{00} \\
&\Rightarrow \left[\left(\tilde{\Psi}_{yy} (CC_{z\alpha\beta}) \right) \left((CC_{z\alpha\beta}) \tilde{\Psi}_{yy} (CC_{z\alpha\beta}) \right)^{-1} \left(\tilde{\Psi}_{yy} (CC_{z\alpha\beta}) \right)' \right]^{-1} \otimes \Omega_{00} \\
&\equiv \tilde{\Psi}_{yy}^{-1} \otimes \Omega_{00},
\end{aligned}$$

and again the weak convergence is joint with that of the estimator components. Hence

$$\begin{aligned}
&vec \left(\tilde{A}_n - A \right)' \left[(X' P_{\tilde{Z}} X)^{-1} \otimes \hat{\Omega}_{00} \right]^{-1} vec \left(\tilde{A}_n - A \right) \\
&= vec \left(\tilde{A}_n - A \right)' (R_n^n \otimes I_m) (R_n^n \otimes I_m)^{-1} \left[(X' P_{\tilde{Z}} X)^{-1} \otimes \hat{\Omega}_{00} \right]^{-1} (R_n^n \otimes I_m)^{-1} (R_n^n \otimes I_m) vec \left(\tilde{A}_n - A \right) \\
&= \left\{ vec \left[n^\alpha \left(\tilde{A}_n - A \right) R_n^n \right] \right\}' \left[n^{2\alpha} (R_n^n \otimes I_m) \left[(X' P_{\tilde{Z}} X)^{-1} \otimes \hat{\Omega}_{00} \right] (R_n^n \otimes I_m) \right]^{-1} vec \left[n^\alpha \left(\tilde{A}_n - A \right) R_n^n \right] \\
&\Rightarrow \chi^2(mK),
\end{aligned}$$

proving theorem 2.3. ■

Proof of Proposition 2.4. Note that

$$\frac{1}{n^{\alpha+(\alpha\wedge\beta)}} \sum R_n^{-n} x_t^\mu \tilde{z}_t' R_n^{-n} = \frac{1}{n^{\alpha+(\alpha\wedge\beta)}} \sum R_n^{-n} x_t \tilde{z}_t' R_n^{-n} - \frac{1}{n^{1+\alpha+(\alpha\wedge\beta)}} \sum R_n^{-n} x_t \sum \tilde{z}_t' R_n^{-n}.$$

Using the fact that the IVX remainder is the dominating order in the (I5) case, we have

$$\begin{aligned} \frac{1}{n^{1+\alpha+(\alpha\wedge\beta)}} \sum R_n^{-n} x_t \sum \tilde{z}_t' R_n^{-n} &= \frac{1}{n^{1+\alpha+(\alpha\wedge\beta)}} \sum R_n^{-n} x_t \sum \tilde{z}_t' R_n^{-n} + \frac{1}{n^{1+2\alpha+(\alpha\wedge\beta)}} \sum R_n^{-n} x_t \sum \psi_{nt}' R_n^{-n} C \\ &= \frac{1}{n^{1+2\alpha+(\alpha\wedge\beta)}} \sum R_n^{-n} x_t \sum \psi_{nt}' R_n^{-n} C + o_p(1), \end{aligned}$$

and using Lemma 2.2 and 2.3, we deduce that

$$\begin{aligned} &\frac{1}{n^{1+2\alpha+(\alpha\wedge\beta)}} \sum R_n^{-n} x_t \sum \psi_{nt}' R_n^{-n} C = \\ &\frac{n^{3\alpha+(\alpha\wedge\beta)}}{n^{1+2\alpha+(\alpha\wedge\beta)}} \left\{ n^{-\alpha} \sum R_n^{-n} \left(\frac{x_t}{n^{\alpha/2}} R_n^{-t} \right) R_n^t \right\} \left\{ n^{-\alpha} \sum \left(\frac{\psi_{nt}}{n^{\alpha/2+(\alpha\wedge\beta)}} R_n^{-t} \right)' R_n^{-n} C \right\} = O_p \left(\frac{1}{n^{1-\alpha}} \right) = o_p(1), \end{aligned}$$

which gives the required result. ■

6.3 The Case of Mixed Roots: (I6)

We provided the proofs for Theorem 3.1 and 3.2, while the proofs for Lemma 3.1, 3.2 and 6.3 are available from Phillips and Lee (2015c).

Proof of Theorem 3.1. We write the scaled estimation error as

$$C_n(\tilde{a} - a) = \left(D_n^{-1} \begin{bmatrix} \sum_{t=1}^n \tilde{z}_{1t} x_{1t} & \sum_{t=1}^n \tilde{z}_{1t} x_{2t} \\ \sum_{t=1}^n \tilde{z}_{2t} x_{1t} & \sum_{t=1}^n \tilde{z}_{2t} x_{2t} \end{bmatrix} C_n^{-1} \right)^{-1} D_n^{-1} \begin{bmatrix} \sum_{t=1}^n \tilde{z}_{1t} u_{0t} - n \hat{\Delta}_{0x1} \\ \sum_{t=1}^n \tilde{z}_{2t} u_{0t} - n \hat{\Delta}_{0x2} \end{bmatrix},$$

and using Lemma 3.2,

$$\begin{bmatrix} n^{\frac{1+(\alpha_1\wedge\beta)}{2}} (\tilde{a}_1 - a_1) \\ \theta_n^n n^{\alpha_2} (\tilde{a}_2 - a_2) \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{n^{1+(\alpha_1\wedge\beta)}} \sum_{t=1}^n \tilde{z}_{1t} x_{1t} \right)^{-1} \frac{1}{n^{\frac{1+(\alpha_1\wedge\beta)}{2}}} \sum_{t=1}^n \left(\tilde{z}_{1t} u_{0t} - \hat{\Delta}_{0x1} \right) \\ \left(\frac{1}{\theta_n^{2n} n^{\alpha_2+(\alpha_2\wedge\beta)}} \sum_{t=1}^n \tilde{z}_{2t} x_{2t} \right)^{-1} \frac{1}{\theta_n^n n^{(\alpha_2\wedge\beta)}} \sum_{t=1}^n \left(\tilde{z}_{2t} u_{0t} - \hat{\Delta}_{0x2} \right) \end{bmatrix} + o_p(1)$$

The limit theory of $n^{\frac{1+(\alpha_1\wedge\beta)}{2}} (\tilde{a}_1 - a_1)$ is exactly a scalar version of Theorem 3.4 and 3.7 from MP

so that

$$\begin{aligned}
& n^{\frac{1+(\alpha_1 \wedge \beta)}{2}} (\tilde{a}_1 - a_1) \\
\Rightarrow & \begin{cases} MN(0, \left\{ \frac{1}{-c_{z1}} \left(\Omega_{11} + \int_0^1 B_1 dB_1 \right) \right\}^{-2} \left(\frac{1}{-2c_{z1}} \Omega_{11} \right) \Omega_{00}) & \text{if } x_{1t} \text{ is unit root - I(1)} \\ MN(0, \left\{ \frac{1}{-c_{z1}} \left\{ \Omega_{11} + \int_0^1 J_x^{c1} dJ_x^{c1} \right\} \right\}^{-2} \left(\frac{1}{-2c_{z1}} \Omega_{11} \right) \Omega_{00}) & \text{if } x_{1t} \text{ is local to unity - (I2) or (I4)} \\ N(0, \left(\frac{1}{-2c_{z1}} \Omega_{11} \right)^{-1} \Omega_{00}) & \text{if } x_{1t} \text{ is mildly integrated - (I3) with } \beta < \alpha_1 \\ N(0, \frac{(c_1+c_{z1})^2}{c_1^2} \left(\frac{1}{-2c_1} \Omega_{11} \right)^{-1} \Omega_{00}) & \text{if } x_{1t} \text{ is mildly integrated - (I3) with } \beta = \alpha_1 \\ N(0, \left(\frac{1}{-2c_1} \Omega_{11} \right)^{-1} \Omega_{00}) & \text{if } x_{1t} \text{ is mildly integrated - (I3) with } \alpha_1 < \beta \end{cases}
\end{aligned}$$

By defining

$$\tilde{\Phi}_{11}^{-1} = \begin{cases} \left\{ \frac{1}{-c_{z1}} \left(\Omega_{11} + \int_0^1 B_1 dB_1 \right) \right\}^{-2} \left(\frac{1}{-2c_{z1}} \Omega_{11} \right) & \text{if } x_{1t} \text{ is unit root - I(1)} \\ \left\{ \frac{1}{-c_{z1}} \left\{ \Omega_{11} + \int_0^1 J_x^{c1} dJ_x^{c1} \right\} \right\}^{-2} \left(\frac{1}{-2c_{z1}} \Omega_{11} \right) & \text{if } x_{1t} \text{ is local to unity - (I2) or (I4)} \\ \left(\frac{1}{-2c_{z1}} \Omega_{11} \right)^{-1} & \text{if } x_{1t} \text{ is mildly integrated - (I3) with } \beta < \alpha_1 \\ \frac{(c_1+c_{z1})^2}{c_1^2} \left(\frac{1}{-2c_1} \Omega_{11} \right)^{-1} & \text{if } x_{1t} \text{ is mildly integrated - (I3) with } \beta = \alpha_1 \\ \left(\frac{1}{-2c_1} \Omega_{11} \right)^{-1} & \text{if } x_{1t} \text{ is mildly integrated - (I3) with } \alpha_1 < \beta \end{cases} \quad (6.4)$$

we have

$$n^{\frac{1+(\alpha_1 \wedge \beta)}{2}} (\tilde{a}_1 - a_1) \Rightarrow MN(0, \tilde{\Phi}_{11}^{-1} \Omega_{00}) .$$

From lemma 2.4-(1), we have

$$\frac{1}{\theta_n^n n^{(\alpha_2 \wedge \beta)}} \sum_{t=1}^n \tilde{z}_{2t} u_{0t} \Rightarrow c_{z2} \alpha \beta c_2 MN(0, \tilde{\Psi}_{22} \Omega_{00}) .$$

and together with lemma 3.2 this leads to

$$\theta_n^n n^{\alpha_2} (\tilde{a}_2 - a_2) \Rightarrow MN(0, \tilde{\Psi}_{22}^{-1} \Omega_{00}) ,$$

which is a special case of theorem 2.2. Joint convergence and asymptotic independence follow from lemma 3.1, thereby completing the proof. ■

The same mechanism for variance estimation, as shown in the following lemma, now leads to nuisance parameter free inference in the corresponding self-normalized test statistics.

Lemma 6.3 $\left[C_n (X' P_{\tilde{Z}} X)^{-1} C_n \hat{\Omega}_{00} \right]^{-1} \Rightarrow \left[\begin{bmatrix} \tilde{\Phi}_{11} \Omega_{00}^{-1} & 0 \\ 0 & \tilde{\Psi}_{22} \Omega_{00}^{-1} \end{bmatrix} \right] .$

Proof of Theorem 3.2. Using Lemma 3.2, Theorem 3.1 and lemma 6.3, we have directly by

continuous mapping

$$\begin{aligned} & (\tilde{a} - a_0)' \left[(X' P_{\tilde{Z}} X)^{-1} \hat{\Omega}_{00} \right]^{-1} (\tilde{a} - a_0) \\ = & (\tilde{a} - a_0)' C_n \left[C_n (X' P_{\tilde{Z}} X)^{-1} C_n \hat{\Omega}_{00} \right]^{-1} C_n (\tilde{a} - a) \implies \chi^2(2). \end{aligned}$$

■

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7 References

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