Estimating Production Functions with Control Functions When Capital Is Measured With Error

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Abstract

We revisit the production function estimators of Olley and Pakes (1996) and Levinsohn and Petrin (2003). They use control functions to address the simultaneous determination of inputs and productivity. Both assume that input demand is a monotonic function of productivity holding capital constant and then invert this function to condition on productivity during estimation. If the observed capital variable is measured with error, investment will not generally be monotonic in the productivity shock holding observed capital constant. We develop consistent estimators of production function parameters in the face of this measurement error. Our identification and estimation results combine the nonlinear measurement error literature with Wooldridge (2005)'s joint estimation method to construct a proxy for productivity that addresses simultaneity. Our approach directly extends to the case where other inputs like intermediates or labor are observed with error.

Keywords: Production Function, Unobserved Productivity, Measurement Error, Nonparametric Estimation, Control Variate

JEL Classification: C14, C18, D24

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1 Introduction

Production function estimates are a critical input into understanding the sources of economic growth. A major challenge in estimating production function parameters is that output and inputs are simultaneously determined (Marschak and Andrews (1944)). This simultaneity problem results in inputs being correlated with the productivity shock, leading to biased and inconsistent estimates of the parameters.

Historically researchers have used firm-specific fixed effects in panel data to address this problem, relying only on within firm variation in inputs over time to identify the parameters. In their review of the literature Grilliches and Mairesse (1998) remark that fixed effect estimators frequently have led researchers to find point estimates for the capital coefficient that are very low and often not significantly different from zero. They attribute these findings to capital being measured with a significant amount of error coupled with the fact that capital varies very little within firms over the time in a standard panel data set. The fixed effect effectively dispenses with all of the “signal,” leading to a significant amount of attenuation bias in the capital coefficient.

A major contribution of Olley and Pakes (1996; hereafter OP) is their development of an estimation method that is both robust to the simultaneity problem and that does not dispense with all of the between-firm variation in capital. They assume that investment is a function of the capital and the productivity shock, the two state variables. Pakes (1996) provides conditions under which - for any given level of capital - investment is monotonically increasing in the productivity shock. By inverting the investment function they recover the productivity shock, on which they then condition during estimation. Unlike fixed effects the approach has the added advantage that the productivity shock is allowed to vary over time. Since its publication many researchers have confirmed that the OP approach generally leads to higher and more reasonable point estimates for capital than the fixed effect estimator.

One issue that has arisen for the OP estimator is that the inversion is only valid when investment is positive. Levinsohn and Petrin (2003; hereafter LP) note that in panel data many firms report zero investment, which can force researchers to drop a large fraction of their observations from the estimation procedure.¹ They show that it is straightforward to extend the OP logic to intermediate inputs, writing intermediate demand as a function of the productivity shock and capital, and then inverting the intermediate demand function to proxy for the productivity shock.

While both methods add back a significant amount of variation in capital, neither method addresses the issue of measurement error in capital. Capital enters directly as a regressor in the production function and it is an argument in the control function used to condition out the productivity shock. The latter case is more difficult to address econometrically because capital enters the control function in an unknown non-linear way, making standard instrumental variable

¹In the Chilean data they use investment is missing for between 30% and 70% of their observations depending upon which industry was under consideration. The fact that investment is lumpy in firm-level data also may suggest that investment does not fully respond to the productivity shock even when investment is positive.

approaches inconsistent.

In this paper we show how to use recent insights from Hu and Schennach (2008) on nonclassical measurement error to allow for measurement error in capital and other inputs in the LP setting. To achieve identification we recast the conditional moment restrictions used in LP as integrals over conditional density functions of the true inputs. Using lagged input levels as instruments, Hu and Schennach’s method can be used to estimate these conditional density functions when capital and possibly other inputs are measured with error. Once we have estimates of the conditional density functions of the true inputs we can integrate over them to calculate the LP conditional moment restrictions for any candidate parameter value. Our two-step estimator first uses sieve maximum likelihood (hereafter ML) estimation to estimate the conditional density functions. In the second step we use Ai and Chen (2003) to develop a sieve minimum distance (hereafter MD) estimator, integrating the conditional moment restrictions over the measurement-error-corrected conditional density estimates.²

The closest work to our paper is Huang and Hu (2011). They extend the LP setting to the case when intermediate inputs are measured with an additive error. We extend their approach to settings where both the intermediate input and capital can be contaminated by additive or non-additive measurement error.

The remainder of this paper is organized as follows. In Section 2 we review the OP and LP estimators and provide a non-technical overview of our estimation approach. In Section 3 we use the framework from Wooldridge (2005) to develop the general framework for our approach, including the case when both capital and intermediates are measured with error. Section 4 discusses identification and estimation and Sections 5 and 6 provide details on consistency, convergence rates, and asymptotic normality. Section 7 concludes and the technical details are gathered in the Appendix.

2 Allowing for Measurement Error

We review the OP and LP methodologies within the Wooldridge (2005) framework and show where measurement error in capital causes problems. We then show how to use recent results from the nonclassical measurement error literature to correct the problem.

2.1 OP/LP Methodology

The production function is written with the log of output as a function of the log of inputs and shocks

$$y_t = \beta_l l_t + \beta_k k_t + \beta_m m_t + \omega_t + \epsilon_t$$

where l_t denotes labor, k_t denotes capital, and m_t denotes the intermediate input (such as materials or energy). ω_t is the productivity shock, a state variable observed by the firm but unobserved to

²This step extends Ai and Chen (2003) to a setting with unobserved endogenous variables.

the econometrician and assumed to be a first-order Markov. ω_t is the source of the simultaneity problem as freely variable inputs l_t and m_t respond to it. k_t is a state variable and is allowed to be correlated with $E[\omega_t|\omega_{t-1}]$, but it is assumed that $\xi_t = \omega_t - E[\omega_t|\omega_{t-1}]$, the innovation in the productivity shock, is uncorrelated with k_t . ϵ_t denotes an i.i.d. shock that is assumed to be uncorrelated with all of the inputs.

LP write intermediate input demand as a function of the state variables

$$m_t = \mathbf{m}_t(\omega_t, k_t)$$

and provide weak conditions under which $\mathbf{m}_t(\cdot, \cdot)$ is strictly monotonic in ω_t holding k_t constant. The intermediate demand function can then be inverted to obtain the control function for ω_t as a function of observed m_t and k_t , written as $\omega_t = h_t(m_t, k_t)$.³ Wooldridge (2005) uses a single index restriction to approximate unobserved productivity, so in the LP setting one has

$$\omega_t = h_t(m_t, k_t) = \mathbf{c}(m_t, k_t)' \beta_\omega$$

where $\mathbf{c}(m_t, k_t)$ is a known vector function of (m_t, k_t) chosen by researchers. He also writes the nonparametric conditional mean function $E[\omega_t|\omega_{t-1}]$ as

$$E[\omega_t|\omega_{t-1}] = q(\mathbf{c}(m_{t-1}, k_{t-1})' \beta_\omega)$$

for some unknown function $q(\cdot)$.⁴

Rewriting the production function as

$$y_t = \beta_l l_t + \beta_k k_t + \beta_m m_t + E[\omega_t|\omega_{t-1}] + \xi_t + \epsilon_t \quad (1)$$

yields

$$[\xi_t + \epsilon_t](\theta) = y_t - \beta_l l_t - \beta_k k_t - \beta_m m_t - q(\mathbf{c}(m_{t-1}, k_{t-1})' \beta_\omega)$$

with $\beta = (\beta_l, \beta_k, \beta_m, \beta_\omega)$, $\theta = (\beta, q)$. Let the set of conditioning variables be $x_t = (k_t, k_{t-1}, m_{t-1}, l_{t-1})$ and let θ_0 denote the true parameter value. Wooldridge shows that the conditional moment restriction

$$g(x_t; \theta) \equiv E[(\xi_t + \epsilon_t)(\theta)|x_t] \text{ and } g(x_t; \theta_0) = 0$$

is sufficient for identification of $(\beta_l, \beta_k, \beta_m)$ and $E[\omega_t|\omega_{t-1}]$.⁵ It is also robust to the Akerberg,

³OP write investment as a function of the two state variables $i_t = \mathbf{i}_t(\omega_t, k_t)$ and Pakes (1996) provides conditions under which investment is strictly monotonic in ω_t holding k_t constant. OP then invert this function to get the control function with arguments i_t and k_t .

⁴OP use i_t and i_{t-1} instead of m_t and m_{t-1} , respectively for ω_t and $E[\omega_t|\omega_{t-1}]$.

⁵In Section 3 we show how to identify q and β_ω using an additional moment condition from LP.

Caves, and Frazer (2006) criticism of LP (and OP). In equation (1) a function of m_{t-1} and k_{t-1} conditions out $E[\omega_t|\omega_{t-1}]$. ξ_t is not correlated with k_t , so k_t can serve as an instrument for itself. Lagged labor l_{t-1} and lagged materials m_{t-1} serve as instruments for l_t and m_t .⁶ It is also clear from this moment why measurement error in k_t is problematic for the LP (and OP) estimators as both current and lagged values of capital are used in the estimation, and the lagged values enter the estimation problem in a nonlinear way.

2.2 Solution Overview

Our approach is based on rewriting the estimation problem in terms of the conditional densities upon which it is based. For $g(x_t; \theta)$ we have:

$$\begin{aligned} g(x_t; \theta) \equiv & \int y_t f_{y_t|x_t} dy_t - \beta_l \int l_t f_{l_t|x_t} dl_t - \beta_k k_t \\ & - \beta_m \int m_t f_{m_t|x_t} dm_t - q(\mathbf{c}(m_{t-1}, k_{t-1})' \beta_\omega) \end{aligned}$$

and for the population objective function we have e.g.,

$$\begin{aligned} Q(\theta) &= E[g(x_t; \theta)^2] \\ &= \int g(x_t; \theta)^2 f_{k_t, k_{t-1}|m_{t-1}, l_{t-1}} f_{m_{t-1}, l_{t-1}} dx_t \end{aligned} \tag{2}$$

(using Bayes Rule). Written in this way the OP/LP estimation problem can be viewed as one that requires estimates of the conditional densities $f_{y_t|x_t}$, $f_{l_t|x_t}$, and $f_{m_t|x_t}$ to estimate $g(x_t; \theta)$ and then estimates of $f_{k_t, k_{t-1}|m_{t-1}, l_{t-1}}$ and $f_{m_{t-1}, l_{t-1}}$ to integrate to get the sample analog of $Q(\theta)$. Letting capital that is measured with error be denoted as k_t^* , and $x_t^* = (k_t^*, k_{t-1}^*, m_{t-1}, l_{t-1})$, our main challenge is recovering the true densities given that the observed densities are $f_{y_t|x_t^*}$, $f_{l_t|x_t^*}$, and $f_{m_t|x_t^*}$ and $f_{k_t^*, k_{t-1}^*|m_{t-1}, l_{t-1}}$ ($f_{m_{t-1}, l_{t-1}}$ is directly observable from the data).

If we use (k_{t-2}^*, k_{t-3}^*) as instruments for the current and lagged mismeasured capital measurements (k_t^*, k_{t-1}^*) , the true conditional densities can be recovered using recent developments in the nonclassical measurement error literature. In the data we observe the conditional density functions $f_{r_t, k_t^*, k_{t-1}^*|k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t}$ for either (i) $r_t = y_t$ and $\mathbf{s}_t = (m_t, l_t, m_{t-1}, l_{t-1})$ or (ii) $r_t \in \{l_t, m_t\}$ and $\mathbf{s}_t = (m_{t-1}, l_{t-1})$. We can express these observed densities $f_{r_t, k_t^*, k_{t-1}^*|k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t}$ without loss of generality as

⁶As long as $E[m_t|k_t, k_{t-1}, m_{t-1}, l_{t-1}]$ is not a function of (m_{t-1}, k_{t-1}) only we do not need an additional excluded instrument for m_t beyond m_{t-1} . Adding further lagged variables like m_{t-2} as an excluded instrument for m_t can improve efficiency.

$$\begin{aligned}
& f_{r_t, k_t^*, k_{t-1}^* | k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t} \\
&= \int f_{r_t, k_t, k_{t-1}, k_t^*, k_{t-1}^* | k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t} dk_t dk_{t-1} \\
&= \int f_{r_t | k_t, k_{t-1}, k_t^*, k_{t-1}^*, k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t} f_{k_t, k_{t-1}, k_t^*, k_{t-1}^* | k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t} dk_t dk_{t-1} \\
&= \int f_{r_t | k_t, k_{t-1}, k_t^*, k_{t-1}^*, k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t} f_{k_t^*, k_{t-1}^* | k_t, k_{t-1}, k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t} f_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t} dk_t dk_{t-1}.
\end{aligned} \tag{3}$$

where the first equality follows from the properties of densities and the second and third equalities follow from Bayes Rule.

Our key identification assumption is that current and lagged mismeasured capital inputs do not provide additional information on output and other inputs beyond what the true capital inputs do. This implies that the first term under the integral in the last line of equation (3) can be written as $f_{r_t | k_t, k_{t-1}, \mathbf{s}_t}$ and similarly the second term can be written as $f_{k_t^*, k_{t-1}^* | k_t, k_{t-1}, \mathbf{s}_t}$. Thus the exclusion restriction allows us to re-express the last line of equation (3) as

$$f_{r_t, k_t^*, k_{t-1}^* | k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t} = \int f_{r_t | k_t, k_{t-1}, \mathbf{s}_t} f_{k_t^*, k_{t-1}^* | k_t, k_{t-1}, \mathbf{s}_t} f_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t} dk_t dk_{t-1}.$$

Under weak regularity conditions and a standard rank condition for nonparametric instrumental variables we show in Section 4 that this equation has unique solutions for $f_{r_t | k_t, k_{t-1}, \mathbf{s}_t}$, $f_{k_t^*, k_{t-1}^* | k_t, k_{t-1}, \mathbf{s}_t}$, and $f_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t}$ following similar arguments to Hu and Schennach (2008). Given uniqueness a ML estimator can be used in a first stage to recover the conditional densities and a second stage uses these estimated conditional densities to construct the sample analog of the objective function (2), which is then used to recover estimates of θ_0 . The estimator extends directly to cases with measurement error in both capital and other inputs.

3 A Generalized Approach in the Wooldridge (2005) Setting

We first develop the method for the case when only capital is measured with error. We then generalize the setup to allow for measurement error in other inputs and capital.

3.1 Measurement Error in Capital Only

Let $\beta = (\beta_l, \beta_k, \beta_m, \beta_\omega)$, $\theta = (\beta, q)$ and (β_0, θ_0) denote the true values. Following Wooldridge we define the residual functions as

$$\begin{aligned}
\epsilon_t(\beta) &\equiv \rho_1(\mathbf{y}_t; \beta) = y_t - \beta_l l_t - \beta_k k_t - \beta_m m_t - \mathbf{c}(m_t, k_t)' \beta_\omega \\
\xi_t + \epsilon_t(\theta) &\equiv \rho_2(\mathbf{y}_{t,t-1}; \theta) = y_t - \beta_l l_t - \beta_k k_t - \beta_m m_t - q(\mathbf{c}(m_{t-1}, k_{t-1})' \beta_\omega)
\end{aligned}$$

where $\mathbf{y}_t \equiv (y_t, l_t, k_t, m_t)$ and $\mathbf{y}_{t,t-1} \equiv (\mathbf{y}_t, k_{t-1}, m_{t-1})$. The first residual function corresponds to the i.i.d. shock and the second function corresponds to the composite error.

Let $\tilde{\mathbf{x}}_t = (k_t, k_{t-1}, m_{t-1}, l_{t-1})$ and let $\mathbf{x}_t = (\tilde{\mathbf{x}}_t, l_t, m_t)$.⁷ Then the conditional moment restrictions take the form

$$\begin{aligned} g_1(\mathbf{x}_t; \beta) &\equiv E[\rho_1(\mathbf{y}_t; \beta) | \mathbf{x}_t] \text{ and } g_1(\mathbf{x}_t; \beta_0) = 0, \\ g_2(\tilde{\mathbf{x}}_t; \theta) &\equiv E[\rho_2(\mathbf{y}_{t,t-1}; \theta) | \tilde{\mathbf{x}}_t] \text{ and } g_2(\tilde{\mathbf{x}}_t; \theta_0) = 0. \end{aligned}$$

As noted by Wooldridge the second moment condition is sufficient for identification production function parameters.

Letting

$$g(\mathbf{x}_t; \theta) = \begin{pmatrix} g_1(\mathbf{x}_t; \beta) \\ g_2(\tilde{\mathbf{x}}_t; \theta) \end{pmatrix}$$

and defining $A = A(\mathbf{x}_t)$ to be some positive-definite weight matrix, the population objective function is given as

$$\begin{aligned} Q(\theta) &= E[g(\mathbf{x}_t; \theta)' A g(\mathbf{x}_t; \theta)] \\ &= \int g(\mathbf{x}_t; \theta)' A g(\mathbf{x}_t; \theta) f_{\mathbf{x}_t} d\mathbf{x}_t \\ &= \int g(\mathbf{x}_t; \theta)' A g(\mathbf{x}_t; \theta) f_{k_t, k_{t-1} | m_t, l_t, m_{t-1}, l_{t-1}} f_{m_t, l_t, m_{t-1}, l_{t-1}} d\mathbf{x}_t. \end{aligned} \tag{4}$$

Let $\mathcal{H}_n = \{q : q = p^{\kappa_{2n}}(\mathbf{c}(m_{t-1}, k_{t-1})' \beta_\omega)' \pi_q\}$ denote a sieve space to approximate the space of functions q where $p^{\kappa_{2n}}(\cdot)$ denotes approximating basis functions with length equal to κ_{2n} (e.g. polynomial approximations or spline approximations). In the first stage we recover estimates of $\hat{f}_{y_t | k_t, k_{t-1}, m_t, l_t, m_{t-1}, l_{t-1}}$, $\hat{f}_{y_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}}$, $\hat{f}_{l_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}}$, and $\hat{f}_{m_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}}$. We also estimate $\hat{f}_{k_t, k_{t-1} | m_t, l_t, m_{t-1}, l_{t-1}}$ in the first stage, which we need to integrate out the unobserved variables (k_t, k_{t-1}) in sample objective function analog to (4). For any parameter value θ we use the first four densities to evaluate the conditional moment functions $\hat{g}(\mathbf{x}_t; \theta)$ as

$$\begin{aligned} \hat{g}_1(\mathbf{x}_t; \beta) &= \int y_t \hat{f}_{y_t | k_t, k_{t-1}, m_t, l_t, m_{t-1}, l_{t-1}} dy_t - \beta_l l_t - \beta_k k_t - \beta_m m_t - \mathbf{c}(m_t, k_t)' \beta_\omega \\ \hat{g}_2(\tilde{\mathbf{x}}_t; \theta) &= \int y_t \hat{f}_{y_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}} dy_t - \beta_l \int l_t \hat{f}_{l_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}} dl_t - \beta_k k_t \\ &\quad - \beta_m \int m_t \hat{f}_{m_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}} dm_t - q(\mathbf{c}(m_{t-1}, k_{t-1})' \beta_\omega). \end{aligned}$$

⁷Under the assumptions maintained in Wooldridge's setup higher-order lagged values of inputs are also available as instruments.

The sieve MD estimator is given as

$$\hat{\theta}_n = \operatorname{argmin}_{\theta=(\beta,q), q \in \mathcal{H}_n} \hat{Q}_n(\theta) \quad (5)$$

where the sample objective function is

$$\hat{Q}_n(\theta) = \frac{1}{n} \sum_{i=1}^n \int \hat{g}(\mathbf{x}_{ti}; \theta)' \hat{A} \hat{g}(\mathbf{x}_{ti}; \theta) \hat{f}_{k_t, k_{t-1} | m_t, i, l_t, i, m_{t-1}, i, l_{t-1}, i} dk_t dk_{t-1}$$

with $\hat{A}$ denoting a consistent estimator of A (e.g. Ai and Chen 2003).

3.2 Measurement Error in Capital and Intermediate Inputs

Our method can be extended to allow both capital and intermediate inputs to be measured with errors. Consider the production function

$$y_t = \beta_l l_t + \beta_k k_t + \beta_m m_t + \omega_t + \epsilon_t$$

where (k_t, m_t) are not directly observable, but instead mismeasured inputs (k_t^*, m_t^*) are observed. The population objective function is rewritten as

$$Q(\theta) = \int g(\mathbf{x}_t; \theta)' A g(\mathbf{x}_t; \theta) f_{k_t, k_{t-1}, m_t, m_{t-1} | l_t, l_{t-1}} f_{l_t, l_{t-1}} d\mathbf{x}_t$$

where $f_{l_t, l_{t-1}}$ is observable from the data. The sieve MD estimator is defined as

$$\hat{\theta}_n = \operatorname{argmin}_{\theta=(\beta,q), q \in \mathcal{H}_n} \hat{Q}_n(\theta)$$

where the sample objective function is given by

$$\hat{Q}_n(\theta) = \frac{1}{n} \sum_{i=1}^n \int \hat{g}(\mathbf{x}_{ti}; \theta)' \hat{A} \hat{g}(\mathbf{x}_{ti}; \theta) \hat{f}_{k_t, k_{t-1}, m_t, m_{t-1} | l_t, i, l_{t-1}, i} dk_t dk_{t-1} dm_t dm_{t-1}.$$

By similar arguments to the results in Section 4 below we can identify the conditional density functions associated with latent true inputs, $(k_t, k_{t-1}, m_t, m_{t-1})$ to construct the moment functions, given mismeasured inputs $(k_t^*, k_{t-1}^*, m_t^*, m_{t-1}^*)$. The identification problem in this case is only about adding more instruments because of the unobserved true intermediate inputs. The set of additional instruments required other than (k_{t-2}^*, k_{t-3}^*) would be lags of mismeasured intermediate input (m_{t-2}^*, m_{t-3}^*) or lags of other (possibly mismeasured) intermediate inputs.

To construct the moment functions we need to identify $f_{y_t | k_t, k_{t-1}, m_t, l_t, m_{t-1}, l_{t-1}}$ in $g_1(\mathbf{x}_t; \beta)$, $f_{y_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}}$, $f_{l_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}}$, and $f_{m_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}}$ in $g_2(\tilde{\mathbf{x}}_t; \theta)$. With the true intermediate input m_t being not observed, the only added complication is to identify the density $f_{m_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}}$ because both m_t and m_{t-1} are not observed, which means both the regressor and

the regressand are unobserved in a regression context. But given identification of $f_{k_t, k_{t-1}, m_t, m_{t-1} | l_{t-1}}$, by Bayes Rule, it is trivial to obtain

$$f_{m_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}} = \frac{f_{k_t, k_{t-1}, m_t, m_{t-1} | l_{t-1}}}{f_{k_t, k_{t-1}, m_{t-1} | l_{t-1}}} = \frac{f_{k_t, k_{t-1}, m_t, m_{t-1} | l_{t-1}}}{\int f_{k_t, k_{t-1}, m_t, m_{t-1} | l_{t-1}} dm_t}. \quad (6)$$

The conditional density functions, $f_{k_t, k_{t-1}, m_t, m_{t-1} | l_t, l_{t-1}}$ in $Q(\theta)$ and $f_{k_t, k_{t-1}, m_t, m_{t-1} | l_{t-1}}$ in (6) associated with latent true inputs, $(k_t, k_{t-1}, m_t, m_{t-1})$, can be also identified by similar arguments to the results in Section 4. Given identification results we then estimate $\hat{g}(\mathbf{x}_t; \theta)$ and $\hat{f}_{k_t, k_{t-1}, m_t, m_{t-1} | l_t, l_{t-1}}$ in $\hat{Q}_n(\theta)$ where $(k_t, k_{t-1}, m_t, m_{t-1})$ are unobserved, but instead $(k_t^*, k_{t-1}^*, m_t^*, m_{t-1}^*)$ are observed using (e.g.) lagged values $(k_{t-2}^*, k_{t-3}^*, m_{t-2}^*, m_{t-3}^*)$ as instruments.

4 Identification and Estimation

In this section we develop identification and estimation of conditional density functions that appear in the moment functions.

4.1 Identification of Density Functions

In order to implement the sieve MD estimation (5) we need to obtain consistent estimators of $f_{y_t | k_t, k_{t-1}, m_t, l_t, m_{t-1}, l_{t-1}}$, $f_{y_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}}$, $f_{l_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}}$, $f_{m_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}}$, and $f_{k_t, k_{t-1} | m_t, l_t, m_{t-1}, l_{t-1}}$. Let either (i) $r_t = y_t$ and $\mathbf{s}_t = (m_t, l_t, m_{t-1}, l_{t-1})$ or (ii) $r_t \in \{l_t, m_t\}$ and $\mathbf{s}_t = (m_{t-1}, l_{t-1})$, depending on problems of interest.

4.1.1 Identification of $f_{r_t | k_t, k_{t-1}, \mathbf{s}_t}$

We make the following assumptions for identification and discussions follow.⁸

Assumption 4.1.

1. (i) $f_{r_t | k_t, k_{t-1}, k_t^*, k_{t-1}^*, k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t} = f_{r_t | k_t, k_{t-1}, \mathbf{s}_t}$;
(ii) $f_{k_t^*, k_{t-1}^* | k_t, k_{t-1}, k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t} = f_{k_t^*, k_{t-1}^* | k_t, k_{t-1}, \mathbf{s}_t}$
2. The operators $L_{k_t^*, k_{t-1}^* | k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t}$ and $L_{k_t^*, k_{t-1}^* | k_t, k_{t-1}, \mathbf{s}_t}$ (defined in the Appendix A) are one-to-one
3. For any $\mathbf{s}_t \in \text{Supp}(\mathbf{s}_t)$ and any $(\tilde{k}_t, \tilde{k}_{t-1}) \neq (\bar{k}_t, \bar{k}_{t-1}) \in \text{Supp}(k_t, k_{t-1})$, the set

$$\{r_t : f_{r_t | k_t, k_{t-1}, \mathbf{s}_t}(r_t | \tilde{k}_t, \tilde{k}_{t-1}, \mathbf{s}_t) \neq f_{r_t | k_t, k_{t-1}, \mathbf{s}_t}(r_t | \bar{k}_t, \bar{k}_{t-1}, \mathbf{s}_t)\}$$

⁸We focus on a decomposition-based approach as in Hu and Schennach (2008). It is worth noting that our identification strategy to recover production function parameters is not only restricted to their approach. Other nonlinear methods to recover true densities from observed data (e.g., repeated measurement or auxiliary data approach) can be also used for this purpose.

has positive probability

4. For any given $\mathbf{s}_t \in \text{Supp}(\mathbf{s}_t)$, there exists a known functional M such that for all $(k_t, k_{t-1}) \in \text{Supp}(k_t, k_{t-1})$

$$M[f_{k_t^*, k_{t-1}^* | k_t, k_{t-1}, \mathbf{s}_t}(\cdot | k_t, k_{t-1}, \mathbf{s}_t)] = (k_t, k_{t-1}).$$

Assumption 4.1.1 (i) is an exclusion restriction that $(k_t^*, k_{t-1}^*, k_{t-2}^*, k_{t-3}^*)$ do not provide additional information on r_t beyond what is known conditional on $(k_t, k_{t-1}, \mathbf{s}_t)$. Assumption 4.1.1 (ii) is a similar exclusion restriction that (k_{t-2}^*, k_{t-3}^*) do not provide additional information on (k_t^*, k_{t-1}^*) beyond what is known conditional on $(k_t, k_{t-1}, \mathbf{s}_t)$. Assumption 4.1.2 is a nonparametric rank condition of the instruments and is important for the invertibility of the integral operators we employ for our identification proof. This identification condition appears in several different forms in the literature. For example it has been phrased as singular value decompositions with nonzero singular values (Darolles, Fan, Florens, and Renault 2011), nonsingularity (Hall and Horowitz 2005, Horowitz 2006), or completeness condition (or bounded completeness) (Newey and Powell 2003, Blundell, Chen, and Kristensen 2007, Andrews 2011).

Assumption 4.1.3 states the conditional density functions of the output and other inputs are distinguishable at two distinct values of capital. Assumption 4.1.3 ensures the uniqueness of the decomposition of an integral operator associated with $f_{\cdot | k_t, k_{t-1}, \mathbf{s}_t}$. This condition is violated only if the distribution of r_t conditional on $(k_t, k_{t-1}, \mathbf{s}_t)$ is identical at different values of (k_t, k_{t-1}) . For output this condition holds as long as $\beta_k > 0$.

Assumption 4.1.4 places restrictions on some measure of the location of a density, denoted by $M[\cdot]$ such as the mean, mode, and quantiles of the distribution. For example the classical measurement error imposes $E[k_t^* | k_t] = k_t$ while the nonclassical measurement error extends to the case e.g. $\text{median}(k_t^* | k_t) = k_t$, which ensures the uniqueness of the decomposition of an integral operator associated with $f_{\cdot | k_t, k_{t-1}, \mathbf{s}_t}$.

Assumption 4.1 is sufficient for identification of the density functions $f_{r_t | k_t, k_{t-1}, \mathbf{s}_t}$, $f_{k_t^*, k_{t-1}^* | k_t, k_{t-1}, \mathbf{s}_t}$, and $f_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t}$.

Theorem 4.2. *Suppose Assumption 4.1 holds. Then the observations of $(r_t, k_t^*, k_{t-1}^*, k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t)$ uniquely identify unobserved densities*

$$f_{r_t | k_t, k_{t-1}, \mathbf{s}_t}, f_{k_t^*, k_{t-1}^* | k_t, k_{t-1}, \mathbf{s}_t}, \text{ and } f_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t}.$$

4.1.2 Identification of $f_{k_t, k_{t-1} | m_t, l_t, m_{t-1}, l_{t-1}}$ and $f_{y_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}}$

Next we show that other conditional density functions that appear in the moment functions and the objective function are obtained from both observables and identified objects from the previous section. Using the results in Theorem 4.2 we also identify $f_{k_t, k_{t-1} | m_t, l_t, m_{t-1}, l_{t-1}}$ as

$$f_{k_t, k_{t-1} | m_t, l_t, m_{t-1}, l_{t-1}} = \int f_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, m_t, l_t, m_{t-1}, l_{t-1}} \frac{f_{k_{t-2}^*, k_{t-3}^*, m_t, l_t, m_{t-1}, l_{t-1}}}{f_{m_t, l_t, m_{t-1}, l_{t-1}}} dk_{t-2}^* dk_{t-3}^*$$

because $f_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, m_t, l_t, m_{t-1}, l_{t-1}}$ is identified from Theorem 4.2 and $f_{k_{t-2}^*, k_{t-3}^*, m_t, l_t, m_{t-1}, l_{t-1}}$ and $f_{m_t, l_t, m_{t-1}, l_{t-1}}$ are observed from the data. The following Corollary summarizes the result.

Corollary 4.3. *Suppose Assumption 4.1 holds. Then we identify $f_{k_t, k_{t-1} | m_t, l_t, m_{t-1}, l_{t-1}}$ from the observations of $(y_t, k_t^*, k_{t-1}^*, k_{t-2}^*, k_{t-3}^*, m_t, l_t, m_{t-1}, l_{t-1})$.*

We next identify $f_{y_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}}$. Note that

$$f_{y_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}} = \int f_{y_t | k_t, k_{t-1}, m_t, l_t, m_{t-1}, l_{t-1}} f_{m_t, l_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}} dm_t dl_t$$

and

$$\begin{aligned} f_{m_t, l_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}} &= f_{m_t, l_t, k_t, k_{t-1} | m_{t-1}, l_{t-1}} \frac{f_{m_{t-1}, l_{t-1}}}{f_{k_t, k_{t-1}, m_{t-1}, l_{t-1}}} \\ &= f_{k_t, k_{t-1} | m_t, l_t, m_{t-1}, l_{t-1}} \frac{f_{m_t, l_t, m_{t-1}, l_{t-1}}}{f_{m_{t-1}, l_{t-1}}} \frac{f_{m_{t-1}, l_{t-1}}}{f_{k_t, k_{t-1}, m_{t-1}, l_{t-1}}} \\ &= f_{k_t, k_{t-1} | m_t, l_t, m_{t-1}, l_{t-1}} \frac{f_{m_t, l_t, m_{t-1}, l_{t-1}}}{f_{k_t, k_{t-1}, m_{t-1}, l_{t-1}}}. \end{aligned}$$

Then

$$f_{y_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}} = \int f_{y_t | k_t, k_{t-1}, m_t, l_t, m_{t-1}, l_{t-1}} f_{k_t, k_{t-1} | m_t, l_t, m_{t-1}, l_{t-1}} \frac{f_{m_t, l_t, m_{t-1}, l_{t-1}}}{f_{k_t, k_{t-1}, m_{t-1}, l_{t-1}}} dm_t dl_t. \quad (7)$$

Also note that

$$\begin{aligned} f_{k_t, k_{t-1}, m_{t-1}, l_{t-1}} &= \int f_{k_t, k_{t-1}, k_{t-2}^*, k_{t-3}^*, m_t, l_t, m_{t-1}, l_{t-1}} dk_{t-2}^* dk_{t-3}^* dm_t dl_t \\ &= \int f_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, m_t, l_t, m_{t-1}, l_{t-1}} f_{k_{t-2}^*, k_{t-3}^*, m_t, l_t, m_{t-1}, l_{t-1}} dk_{t-2}^* dk_{t-3}^* dm_t dl_t. \end{aligned} \quad (8)$$

Because (i) $f_{y_t | k_t, k_{t-1}, m_t, l_t, m_{t-1}, l_{t-1}}$ and $f_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, m_t, l_t, m_{t-1}, l_{t-1}}$ are identified from Theorem 4.2 and (ii) $f_{k_t, k_{t-1} | m_t, l_t, m_{t-1}, l_{t-1}}$ is identified from Corollary 4.3 and because $f_{m_t, l_t, m_{t-1}, l_{t-1}}$ and $f_{k_{t-2}^*, k_{t-3}^*, m_t, l_t, m_{t-1}, l_{t-1}}$ are observed from the data (by a nonparametric density estimation), we conclude $f_{y_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}}$ is also identified from (7) and (8). The following Corollary summarizes the result.

Corollary 4.4. *Suppose Assumption 4.1 holds. Then we identify $f_{y_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}}$ from the observations of $(y_t, k_t^*, k_{t-1}^*, k_{t-2}^*, k_{t-3}^*, m_t, l_t, m_{t-1}, l_{t-1})$.*

4.2 Estimation of Conditional Densities

In the sieve MD estimation (5), we first estimate conditional densities that enter $g_1(\mathbf{x}_t; \beta)$ and $g_2(\tilde{\mathbf{x}}_t; \theta)$ for given $\theta \equiv (\beta, q)$ with $\beta \equiv (\beta_l, \beta_k, \beta_m, \beta_\omega)$ and the conditional density that is required to integrate out unobserved variables, (k_t, k_{t-1}) , in $Q(\theta)$. Then we estimate θ in the second stage. So in the first stage, we need to estimate conditional densities associated with unobserved variables, (k_t, k_{t-1}) , such as $f_{k_t, k_{t-1} | m_t, l_t, m_{t-1}, l_{t-1}}$, $f_{y_t | k_t, k_{t-1}, m_t, l_t, m_{t-1}, l_{t-1}}$, $f_{y_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}}$, $f_{l_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}}$, and $f_{m_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}}$. The first conditional density is required to integrate out unobserved variables and the second one is needed to estimate $g_1(\mathbf{x}_t; \beta)$. Last three conditional densities are needed to estimate $g_2(\tilde{\mathbf{x}}_t; \theta)$. Recall that $\tilde{\mathbf{x}}_t = (k_t, k_{t-1}, m_{t-1}, l_{t-1})$ and $\mathbf{x}_t = (\tilde{\mathbf{x}}_t, l_t, m_t)$. Also recall that when $r_t = y_t$, we take $\mathbf{s}_t = (m_t, l_t, m_{t-1}, l_{t-1})$ and when $r_t \in \{l_t, m_t\}$, we take $\mathbf{s}_t = (m_{t-1}, l_{t-1})$. Similarly, we let $\mathbf{z}_t = (k_t, k_{t-1}, \mathbf{s}_t)$ and $\mathbf{z}_t^* = (k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t)$. We now discuss how to estimate those conditional densities.

4.2.1 Estimation of $f_{r_t | k_t, k_{t-1}, \mathbf{s}_t}$

Let

$$\begin{aligned} & f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t}(r_t, k_t, k_{t-1} | \mathbf{z}_t; \alpha_0) \\ &= \int f_{r_t | \mathbf{z}_t}(r_t | \mathbf{z}_t; \psi_0) f_{k_t^*, k_{t-1}^* | \mathbf{z}_t}(k_t^*, k_{t-1}^* | \mathbf{z}_t) f_{k_t, k_{t-1} | \mathbf{z}_t^*}(k_t, k_{t-1} | \mathbf{z}_t^*) dk_t dk_{t-1}, \end{aligned} \quad (9)$$

where $\alpha_0 \equiv (\psi_0, f_1, f_2) \in \mathcal{A} \equiv \Psi \times \mathcal{F}_1 \times \mathcal{F}_2$, $\psi_0 \equiv (\phi_0, \eta_0) \in \Psi \equiv \Phi \times \mathcal{M}$, $f_1 \equiv f_{k_t^*, k_{t-1}^* | \mathbf{z}_t}(k_t^*, k_{t-1}^* | \mathbf{z}_t)$, and $f_2 \equiv f_{k_t, k_{t-1} | \mathbf{z}_t^*}(k_t, k_{t-1} | \mathbf{z}_t^*)$. Here ϕ_0 denotes a finite-dimensional parameter while η_0 denotes an infinite-dimensional parameter that determines the density $f_{r_t | \mathbf{z}_t}$. Therefore in this specification we use a semi-nonparametric density for $f_{r_t | \mathbf{z}_t}$ while we use nonparametric densities for $f_{k_t^*, k_{t-1}^* | \mathbf{z}_t}$ and $f_{k_t, k_{t-1} | \mathbf{z}_t^*}$. We can also consider fully nonparametric densities with additional notation. We choose to use the current specification as (9) for flexibility.

From Theorem 4.2, the parameters can be recovered by solving the maximization problems:

$$\begin{aligned} \alpha_0 &= \arg \max_{\alpha \equiv (\psi, f_1, f_2) \in \mathcal{A}} \\ & E[\ln \int f_{r_t | \mathbf{z}_t}(r_t | \mathbf{z}_t; \psi_0) f_1(k_t^*, k_{t-1}^* | \mathbf{z}_t) f_2(k_t, k_{t-1} | \mathbf{z}_t^*) dk_t dk_{t-1}]. \end{aligned}$$

Note that Φ is finite-dimensional parameter space. But we need to impose some restrictions on the function spaces $(\mathcal{M}, \mathcal{F}_1, \mathcal{F}_2)$ for infinite-dimensional parameters. Let $\xi \in \mathcal{V} \subset \mathcal{R}^{d_\xi}$, $\|\cdot\|_E$ denote the Euclidean norm, and

$$\nabla^{\mathbf{a}} g(\xi) \equiv \frac{\partial^{a_1 + a_2 + \dots + a_{d_\xi}} g(\xi)}{\partial \xi_1^{a_1} \dots \partial \xi_{d_\xi}^{a_{d_\xi}}}$$

denote the $\sum_{i=1}^{d_\xi} a_i$ -th derivative where $\mathbf{a} = (a_1, a_2, \dots, a_{d_\xi})'$ is a vector of nonnegative integers.

Let $\underline{\gamma}$ denote the largest integer satisfying $\underline{\gamma} < \gamma$. The Hölder space $\Lambda^\gamma(\mathcal{V})$ of order $\gamma > 0$ is a space of functions $g : \mathcal{V} \mapsto \mathcal{R}$ such that the first $\underline{\gamma}$ derivatives are bounded and the $\underline{\gamma}$ -th derivative is Hölder continuous with the exponent $\gamma - \underline{\gamma} \in (0, 1]$, i.e., for all $\xi, \xi' \in \mathcal{V}$ and some constant c

$$\max_{\sum_{i=1}^{d_\xi} a_i = \underline{\gamma}} |\nabla^{\mathbf{a}} g(\xi) - \nabla^{\mathbf{a}} g(\xi')| \leq c(\|\xi - \xi'\|_E)^{\gamma - \underline{\gamma}}.$$

The space $\Lambda^\gamma(\mathcal{V})$ becomes a Banach space under the Hölder norm:

$$\|g\|_{\Lambda^\gamma} = \sup_{\xi} |g(\xi)| + \max_{\sum_{i=1}^{d_\xi} a_i = \underline{\gamma}} \sup_{\xi \neq \xi'} \frac{|\nabla^{\mathbf{a}} g(\xi) - \nabla^{\mathbf{a}} g(\xi')|}{(\|\xi - \xi'\|_E)^{\gamma - \underline{\gamma}}} < \infty.$$

A Hölder ball (of radius c) is defined as $\Lambda_c^\gamma(\mathcal{V}) \equiv \{g \in \Lambda^\gamma(\mathcal{V}) : \|g\|_{\Lambda^\gamma} \leq c < \infty\}$. Let $\nu(\cdot)$ be a positive continuous weight function on $\mathcal{V}$ where $\nu(\xi) = (1 + \|\xi\|_E^2)^{-\varsigma/2}$, $\varsigma > \gamma > 0$. Denote $\Lambda_c^{\gamma, \nu}(\mathcal{V})$ as the weighted Hölder space with a weighted Hölder norm $\|g\|_{\Lambda^{\gamma, \nu}} \equiv \|\tilde{g}\|_{\Lambda^\gamma}$ for $\tilde{g}(\xi) \equiv g(\xi)\nu(\xi)$. Then a weighted Hölder ball is defined as $\Lambda_c^{\gamma, \nu}(\mathcal{V}) \equiv \{g \in \Lambda^{\gamma, \nu}(\mathcal{V}) : \|g\|_{\Lambda^{\gamma, \nu}} \leq c < \infty\}$.

Assumption 4.5.

1. $\eta \in \Lambda_c^{\gamma_1, \nu}(\text{Supp}(\cdot))$ where $\gamma_1 > 1$
2. $f_1 \in \Lambda_c^{\gamma_1, \nu}(\text{Supp}(k_t^*, k_{t-1}^*, \mathbf{z}_t))$ where $\gamma_1 > 1$ and $\int f_1(k_t^*, k_{t-1}^* | \mathbf{z}_t) dk_t^* dk_{t-1}^* = 1$ for all $\mathbf{z}_t \in \text{Supp}(\mathbf{z}_t)$
3. $f_2 \in \Lambda_c^{\gamma_1, \nu}(\text{Supp}(k_t, k_{t-1}, \mathbf{z}_t^*))$ where $\gamma_1 > 1$ and $\int f_2(k_t, k_{t-1} | \mathbf{z}_t^*) dk_t dk_{t-1} = 1$ for all $\mathbf{z}_t^* \in \text{Supp}(\mathbf{z}_t^*)$.

We define the spaces which satisfy relevant restrictions:

$$\begin{aligned} \mathcal{M} &= \{\eta(\cdot) : \text{Assumption 4.5.1 holds}\}, \\ \mathcal{F}_1 &= \{f_1(\cdot, \cdot | \cdot) : \text{Assumption 4.1.2, 4.1.4, and 4.5.2 hold}\}, \\ \mathcal{F}_2 &= \{f_2(\cdot, \cdot | \cdot) : \text{Assumption 4.1.2 and 4.5.3 hold}\}. \end{aligned}$$

Let $\{p_j, j = 1, 2, \dots\}$ denote a sequence of known univariate basis functions (e.g., Fourier series, power series, spline, wavelets, etc.). For $i \in \{\eta, 1, 2\}$, we define $p^{k_{n_i}} = (p_1, \dots, p_{k_{n_i}})$ as a tensor-product linear sieve basis. For the consistent estimation of α_0 , we replace the parameter spaces with finite-dimensional compact sieve spaces:

$$\begin{aligned} \mathcal{M}_n &= \{\eta(\cdot) = p^{k_{n_\eta}}(\cdot)' \pi_\eta \text{ for all } \pi_\eta \text{ s.t. Assumption 4.5.1 holds}\}, \\ \mathcal{F}_{1n} &= \{f_1(k_t^*, k_{t-1}^* | \mathbf{z}_t) = p^{k_{n_1}}(k_t^*, k_{t-1}^*, \mathbf{z}_t)' \pi_1 \text{ for all } \pi_1 \text{ s.t. Assumption 4.1.2, 4.1.4, and 4.5.2 hold}\}, \\ \mathcal{F}_{2n} &= \{f_2(k_t, k_{t-1} | \mathbf{z}_t^*) = p^{k_{n_2}}(k_t, k_{t-1}, \mathbf{z}_t^*)' \pi_2 \text{ for all } \pi_2 \text{ s.t. Assumption 4.1.2 and 4.5.3 hold}\}. \end{aligned}$$

We then can replace the densities, f_1 and f_2 with functions in the sieve spaces:

$$\begin{aligned} f_1(k_t^*, k_{t-1}^* | \mathbf{z}_t) &= \sum_{j_1=0}^{j_{1n}} \sum_{j_2=0}^{j_{2n}} \sum_{j_3=0}^{j_{3n}} \varphi_{j_1 j_2 j_3} p_{j_1}(k_t^* - k_{t-1}^*) p_{j_2}(k_{t-1}^*) p_{j_3}(\mathbf{z}_t), \\ f_2(k_t, k_{t-1} | \mathbf{z}_t^*) &= \sum_{j_1=0}^{j_{1n}} \sum_{j_2=0}^{j_{2n}} \sum_{j_3=0}^{j_{3n}} \gamma_{j_1 j_2 j_3} p_{j_1}(k_t - k_{t-1}) p_{j_2}(k_{t-1}) p_{j_3}(\mathbf{z}_t^*). \end{aligned}$$

Let the projection of the true parameter α_0 onto the space $\mathcal{A}_n$ where $\mathcal{A}_n = \Psi_n \times \mathcal{F}_{1n} \times \mathcal{F}_{2n}$ with $\Psi_n = \Phi \times \mathcal{M}_n$:

$$\begin{aligned} \Pi_n \alpha &\equiv \alpha_n \\ &= \arg \max_{\alpha_n \equiv (\psi, f_1, f_2) \in \mathcal{A}_n} E[\ln \int f_{r_t | \mathbf{z}_t}(r_t | \mathbf{z}_t; \psi) f_1(k_t^*, k_{t-1}^* | \mathbf{z}_t) f_2(k_t, k_{t-1} | \mathbf{z}_t^*) dk_t dk_{t-1}]. \end{aligned}$$

Then the proposed sieve ML estimators of α_0 is defined as follows:

$$\begin{aligned} \hat{\alpha}_n &\equiv (\hat{\psi}_n, \hat{f}_{1n}, \hat{f}_{2n}) \\ &= \arg \max_{(\psi, f_1, f_2) \in \mathcal{A}_n} \frac{1}{n} \sum_{i=1}^n \ln \int f_{r_t | \mathbf{z}_t}(r_{t,i} | \mathbf{z}_{t,i}; \psi) f_1(k_{t,i}^*, k_{t-1,i}^* | \mathbf{z}_{t,i}) f_2(k_t, k_{t-1} | \mathbf{z}_{t,i}^*) dk_t dk_{t-1}. \end{aligned} \quad (10)$$

4.2.2 Estimation of $f_{k_t, k_{t-1} | m_t, l_t, m_{t-1}, l_{t-1}}$ and $f_{y_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}}$

Based on Corollary 4.3, we propose the following estimator of $f_{k_t, k_{t-1} | m_t, l_t, m_{t-1}, l_{t-1}}$

$$\hat{f}_{k_t, k_{t-1} | m_t, l_t, m_{t-1}, l_{t-1}} = \frac{\int \hat{f}_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, m_t, l_t, m_{t-1}, l_{t-1}} \hat{f}_{k_{t-2}^*, k_{t-3}^*, m_t, l_t, m_{t-1}, l_{t-1}} dk_{t-2}^* dk_{t-3}^*}{\hat{f}_{m_t, l_t, m_{t-1}, l_{t-1}}} \quad (11)$$

where $\hat{f}_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, m_t, l_t, m_{t-1}, l_{t-1}}$ is $\hat{f}_2(k_t, k_{t-1} | \mathbf{z}_t^*)$ from the sieve ML estimation in the previous section. $(\hat{f}_{k_{t-2}^*, k_{t-3}^*, m_t, l_t, m_{t-1}, l_{t-1}}, \hat{f}_{m_t, l_t, m_{t-1}, l_{t-1}})$ can be estimated by any consistent nonparametric density estimators of observables. Similarly, based on Corollary 4.4, we can estimate $f_{y_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}}$ as

$$\begin{aligned} &\hat{f}_{y_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}} \\ &= \frac{\int \hat{f}_{y_t | k_t, k_{t-1}, m_t, l_t, m_{t-1}, l_{t-1}} \hat{f}_{k_t, k_{t-1} | m_t, l_t, m_{t-1}, l_{t-1}} \hat{f}_{m_t, l_t, m_{t-1}, l_{t-1}} dm_t dl_t}{\int \hat{f}_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, m_t, l_t, m_{t-1}, l_{t-1}} \hat{f}_{k_{t-2}^*, k_{t-3}^*, m_t, l_t, m_{t-1}, l_{t-1}} dk_{t-2}^* dk_{t-3}^* dm_t dl_t} \end{aligned} \quad (12)$$

where $\hat{f}_{y_t | k_t, k_{t-1}, m_t, l_t, m_{t-1}, l_{t-1}}$ is $\hat{f}_{r_t | \mathbf{z}_t}(r_t | \mathbf{z}_t; \psi)$ for $r_t = y_t$ from the sieve ML estimation. To summarize, $(\hat{f}_{k_t, k_{t-1} | m_t, l_t, m_{t-1}, l_{t-1}}, \hat{f}_{y_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}})$ can be estimated by the sieve ML estimation obtained from (10) and consistent estimators of $(\hat{f}_{k_{t-2}^*, k_{t-3}^*, m_t, l_t, m_{t-1}, l_{t-1}}, \hat{f}_{m_t, l_t, m_{t-1}, l_{t-1}})$ from nonparametric density estimation with observable data.

5 Consistency and Convergence rate

Our estimation approach is a two-step method such that in the first stage we estimate conditional density functions associated with moment functions and in the second stage we estimate production function parameters. Our first stage to estimate conditional density functions follows Hu and Schennach (2008). Recall that α denote collections of different kinds of conditional density functions and parameters that determine the conditional density functions, which we need to estimate in the first stage. Also recall that $\beta = (\beta_l, \beta_k, \beta_m, \beta_\omega)$, $\theta = (\beta, q)$ and (β_0, θ_0) denote the true values. First we obtain consistency of $\hat{\alpha}$ (and $\hat{\theta}$) under a strong metric, $\|\cdot\|_{s,\alpha}$ (and $\|\cdot\|_{s,\theta}$), and then establish that $\hat{\alpha}$ (and $\hat{\theta}$) converge to α_0 (and θ_0) at a rate faster than $n^{-1/4}$ under a weak metric, $\|\cdot\|_\alpha$ (and $\|\cdot\|_\theta$), which is necessary to obtain $\sqrt{n}$ -asymptotic normality for $\hat{\beta}$. Our notations and results are built on Hu and Schennach (2008) and Ai and Chen (2003) (also see Chen 2007, Blundell, Chen, and Kristensen (2007), and Song 2012).

5.1 Consistency

Let $D_t \equiv (r_t, k_t^*, k_{t-1}^*, \mathbf{z}_t^*)$. Let the first smoothing parameter $\kappa_{1n} = d_{\phi_1} + \kappa_{1n_\eta} + \kappa_{1n_1} + \kappa_{1n_2}$ from the sieve approximation for f_1 and f_2 where $d_{\phi_1} = \dim(\Phi_1)$, $\kappa_{1n_\eta} = \dim(\mathcal{M}_n)$, $\kappa_{1n_1} = \dim(\mathcal{F}_{1n})$, and $\kappa_{1n_2} = \dim(\mathcal{F}_{2n})$ and also let the second smoothing parameter $\kappa_{2n} = \dim(\mathcal{H}_n)$ from the sieve approximation for $q(\cdot)$. Define $\|\theta\|_{s,\theta} \equiv \|\beta\|_E + \|q\|_{\infty,\nu}$, $\|\alpha\|_{s,\alpha} \equiv \|\phi\|_E + \|\eta\|_{\infty,\nu} + \|f_1\|_{\infty,\nu} + \|f_2\|_{\infty,\nu}$, where $\|g\|_{\infty,\nu} \equiv \sup_\xi |g(\xi)\nu(\xi)|$ with a weight function $\nu(\xi) = (1 + \|\xi\|_E^2)^{-\varsigma/2}$, $\varsigma > \gamma_1 > 0$. Because the supports of the mismeasured variables could be unbounded, we use the weighted sup-norm metric.

Assumption 5.1.

1. The data $\{D_{ti} = (r_{t,i}, k_{t,i}^*, k_{t-1,i}^*, \mathbf{z}_{t,i}^*) : i = 1, \dots, n\}$ are i.i.d. for all $t = 1, \dots, T$
2. The density of D_t , f_{D_t} , satisfies $\int \nu(D_t)^{-2} f_{D_t}(D_t) d(D_t) < \infty$.

This assumption is about the nature of data. Next we assume α belong to a class of function that is well approximated by sieves.

Assumption 5.2.

1. There is a metric $\|\cdot\|_{s,\alpha}$ such that $\mathcal{A} \equiv \Psi \times \mathcal{F}_1 \times \mathcal{F}_2$ is compact under $\|\cdot\|_{s,\alpha}$
2. For any $\alpha \in \mathcal{A}$, there exists $\Pi_n \alpha \in \mathcal{A}_n \equiv \Psi_n \times \mathcal{F}_{1n} \times \mathcal{F}_{2n}$ with $\Psi_n \equiv \Phi \times \mathcal{M}_n$ such that $\|\Pi_n \alpha - \alpha\|_{s,\alpha} = o(1)$.

We now impose an envelop condition and Hölder continuity on the log likelihood.

Assumption 5.3.

1. $E[|\ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*)|^2]$ is bounded
2. There exists a measurable function $h_1(D_t)$ with $E[|h_1(D_t)|^2] < \infty$ such that for any $\bar{\alpha} = (\bar{\psi}, \bar{f}_1, \bar{f}_2)' \in \mathcal{A}$,

$$\left| \frac{f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}^{[1]}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \bar{\alpha}, \nu)}{f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \bar{\alpha})} \right| \leq h_1(D_t),$$

where the pathwise first derivative $f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}^{[1]}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \bar{\alpha}, \nu)$ is defined in the proof of Theorem 5.10.1.

Assumption 5.4. $\kappa_{1n}/n \rightarrow 0, \kappa_{2n}/n \rightarrow 0$ and $\kappa_{1n}, \kappa_{2n} \rightarrow \infty$.

This is about rate conditions on the smoothing parameters for consistency. We next assume the parameter of interest θ is identified, which is satisfied under standard nonparametric rank conditions (e.g., completeness condition in Newey and Powell 2003).

Assumption 5.5. $\theta_0 \in \Theta$ is the only $\theta \in \Theta$ satisfying $g(\mathbf{x}_t; \theta) = 0$.

We now assume a consistent estimator of the weighting function is available.

Assumption 5.6. 1. $\hat{A}(\mathbf{x}_t) = A(\mathbf{x}_t) + o_p(1)$ uniformly over $\mathbf{x}_t \in \mathcal{X}_t \equiv \text{Supp}(\mathbf{x}_t)$; 2. $A(\mathbf{x}_t)$ is finite and strictly positive uniformly over $\mathbf{x}_t \in \mathcal{X}_t$.

Next we assume the moment function has bounded second moment and is continuous.

Assumption 5.7. 1. $E[|g(\mathbf{x}_t; \theta_0)|^2]$ is bounded; 2. $g(\mathbf{x}_t; \theta)$ is Hölder continuous in $\theta \in \Theta$.

We next restrict the parameter space, Θ , and a metric, $\|\cdot\|_{s, \theta}$. We also define a sieve space used for approximating the parameter space.

Assumption 5.8.

1. There is a metric $\|\cdot\|_{s, \theta}$ such that $\Theta \equiv \mathcal{B} \times \mathcal{H}$ is compact under $\|\cdot\|_{s, \theta}$
2. For any $\theta \in \Theta$, there exists $\Pi_n \theta \in \Theta_n \equiv \mathcal{B} \times \mathcal{H}_n$ such that $\|\Pi_n \theta - \theta\|_{s, \theta} = o(1)$.

We also impose consistency of $\hat{f}_{k_{t-2}^*, k_{t-3}^*, m_t, l_t, m_{t-1}, l_{t-1}}$ and $\hat{f}_{m_t, l_t, m_{t-1}, l_{t-1}}$, which is well established in standard nonparametric density estimations (e.g. kernel or sieve estimation), required in steps to estimate $f_{k_t, k_{t-1} | m_t, l_t, m_{t-1}, l_{t-1}}$ and $f_{y_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}}$.

Assumption 5.9.

1. $\hat{f}_{k_{t-2}^*, k_{t-3}^*, m_t, l_t, m_{t-1}, l_{t-1}} = f_{k_{t-2}^*, k_{t-3}^*, m_t, l_t, m_{t-1}, l_{t-1}} + o_p(1)$ uniformly over all $\mathbf{z}_t^* = (k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t) \in \mathcal{Z}_t^* \equiv \text{Supp}(\mathbf{z}_t^*)$

2. $\hat{f}_{m_t, l_t, m_{t-1}, l_{t-1}} = f_{m_t, l_t, m_{t-1}, l_{t-1}} + o_p(1)$ uniformly over all $\mathbf{s}_t = (m_t, l_t, m_{t-1}, l_{t-1}) \in \mathcal{S}_t \equiv \text{Supp}(\mathbf{s}_t)$.

We state consistency of the proposed estimators of α_0 and θ_0 in the following theorem.

Theorem 5.10.

1. Suppose Assumptions 4.1, 4.5 and 5.1-5.4 hold. Then $\|\hat{\alpha}_n - \alpha_0\|_{s, \alpha} = o_p(1)$.
2. Suppose Assumptions 4.1, 4.5 and 5.1-5.9 hold. Then $\|\hat{\theta}_n - \theta_0\|_{s, \theta} = o_p(1)$.

5.2 Convergence Rates

In this section, we establish $n^{-1/4}$ convergence rate of $\hat{\alpha}_n$ (and $\hat{\theta}_n$) under a weaker metric, $\|\cdot\|_\alpha$ (and $\|\cdot\|_\theta$), which is sufficient to obtain $\sqrt{n}$ -asymptotic normality of $\hat{\beta}_n$ (and also $\hat{\phi}_n$). We need to introduce additional notations. Again our notations and results are closely related to Hu and Schennach (2008), Ai and Chen (2003), and Song (2012).

Denote the first pathwise derivative of $\ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha)$ at the direction $[\alpha - \alpha_0]$ evaluated at α_0 by:

$$\frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{d\alpha} [\alpha - \alpha_0] \equiv \left. \frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; (1 - \tau)\alpha_0 + \tau\alpha)}{d\tau} \right|_{\tau=0}$$

almost everywhere. Also for $\alpha_1, \alpha_2 \in \mathcal{A}$ denote

$$\begin{aligned} \frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{d\alpha} [\alpha_1 - \alpha_2] &= \frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{d\alpha} [\alpha_1 - \alpha_0] \\ &\quad - \frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{d\alpha} [\alpha_2 - \alpha_0]. \end{aligned}$$

Then the pathwise derivative is written by:

$$\begin{aligned} &\frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{d\alpha} [\alpha - \alpha_0] \\ &= \frac{1}{f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)} \\ &\quad \times \left\{ \int \frac{d}{d\psi} f_{r_t | \mathbf{z}_t}(r_t | \mathbf{z}_t; \psi_0) [\psi - \psi_0] f_{k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(k_t^*, k_{t-1}^* | \mathbf{z}_t^*) f_{k_t, k_{t-1} | \mathbf{z}_t^*}(k_t, k_{t-1} | \mathbf{z}_t^*) dk_t dk_{t-1} \right. \\ &\quad + \int f_{r_t | \mathbf{z}_t}(r_t | \mathbf{z}_t; \psi_0) [f_1(k_t^*, k_{t-1}^* | \mathbf{z}_t^*) - f_{k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(k_t^*, k_{t-1}^* | \mathbf{z}_t^*)] f_{k_t, k_{t-1} | \mathbf{z}_t^*}(k_t, k_{t-1} | \mathbf{z}_t^*) dk_t dk_{t-1} \\ &\quad \left. + \int f_{r_t | \mathbf{z}_t}(r_t | \mathbf{z}_t; \psi_0) f_{k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(k_t^*, k_{t-1}^* | \mathbf{z}_t^*) [f_2(k_t, k_{t-1} | \mathbf{z}_t^*) - f_{k_t, k_{t-1} | \mathbf{z}_t^*}(k_t, k_{t-1} | \mathbf{z}_t^*)] dk_t dk_{t-1} \right\}. \end{aligned}$$

For any $\alpha_1, \alpha_2 \in \mathcal{A}$, the metric $\|\cdot\|_\alpha$ is defined as

$$\|\alpha_1 - \alpha_2\|_\alpha \equiv \sqrt{E \left\{ \left(\frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{d\alpha} [\alpha_1 - \alpha_2] \right)^2 \right\}}.$$

We state similar notations for θ . Denote the first pathwise derivative of $\rho(\mathbf{y}_{t,t-1}; \theta)$ at the direction $[\theta - \theta_0]$ evaluated at θ_0 by:

$$\frac{d\rho(\mathbf{y}_{t,t-1}; \theta_0)}{d\theta} [\theta - \theta_0] \equiv \frac{d\rho(\mathbf{y}_{t,t-1}; (1-\tau)\theta_0 + \tau\theta)}{d\tau} \Big|_{\tau=0}$$

almost everywhere and for any $\theta_1, \theta_2 \in \Theta$ denote

$$\begin{aligned} \frac{d\rho(\mathbf{y}_{t,t-1}; \theta_0)}{d\theta} [\theta_1 - \theta_2] &\equiv \frac{d\rho(\mathbf{y}_{t,t-1}; \theta_0)}{d\theta} [\theta_1 - \theta_0] - \frac{d\rho(\mathbf{y}_{t,t-1}; \theta_0)}{d\theta} [\theta_2 - \theta_0], \\ \frac{dg(\mathbf{x}_t; \theta_0)}{d\theta} [\theta_1 - \theta_2] &\equiv \begin{pmatrix} E \left\{ \frac{d\rho_1(\mathbf{y}_t; \beta_0)}{d\beta} [\beta_1 - \beta_2] \Big| \mathbf{x}_t \right\} \\ E \left\{ \frac{d\rho_2(\mathbf{y}_{t,t-1}; \theta_0)}{d\theta} [\theta_1 - \theta_2] \Big| \tilde{\mathbf{x}}_t \right\} \end{pmatrix}. \end{aligned}$$

Similarly, for any $\theta_1, \theta_2 \in \Theta$, the metric $\|\cdot\|_\theta$ is defined as:

$$\|\theta_1 - \theta_2\|_\theta \equiv \sqrt{E \left\{ \left(\frac{dg(\mathbf{x}_t; \theta_0)}{d\theta} [\theta_1 - \theta_2] \right)' A(\mathbf{x}_t) \frac{dg(\mathbf{x}_t; \theta_0)}{d\theta} [\theta_1 - \theta_2] \right\}}.$$

We now state sufficient conditions for the convergence rate of the proposed estimators of α_0 and θ_0 . First we strengthen the requirement for the approximation order enough to obtain the convergence rate of estimators up to $o_p(n^{-1/4})$. This is also well known to be satisfied when α and θ belong to a class of bounded and smooth functions.

Assumption 5.11.

1. $\ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha)$ satisfies an envelope condition in $\alpha \in \mathcal{A}_n$
2. $\ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha) \in \Lambda_c^{\mu_1, \nu}(\text{Supp}(r_t, k_t^*, k_{t-1}^*, \mathbf{z}_t^*))$ for some constant $c > 0$ with $\mu_1 > d_{D_t}/2$, for all $\alpha \in \mathcal{A}_n$, where d_{D_t} is the dimension of D_t .

This states an envelop condition and a smoothness condition on $\ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha)$. The next assumption quantifies the approximation error of $\Pi_n \alpha$ to α , by the sieve.

Assumption 5.12. *There is a constant $\mu_1 > 0$ such that for any $\alpha \in \mathcal{A}$, there exists $\Pi_n \alpha \in \mathcal{A}_n$ satisfying $\|\Pi_n \alpha - \alpha\|_\alpha = O(\kappa_{1n}^{-\mu_1})$, and $\kappa_{1n}^{-\mu_1} = o(n^{-1/4})$.*

Similarly, next assumptions are imposed for estimating θ in the second stage. Assumption 5.14 below is well known to be satisfied for sieve approximations when Θ is a Hölder class.

Assumption 5.13.

1. Each element of $\rho(\mathbf{y}_{t,t-1}; \theta)$ satisfies an envelope condition in $\theta \in \Theta_n$
2. Each element of $g(\mathbf{x}_t; \theta) \in \Lambda_c^{\mu_2, \nu}(\mathcal{X}_t)$ for some constant $c > 0$ with $\mu_2 > d_{\mathbf{x}_t}/2$, for all $\theta \in \Theta_n$, where $d_{\mathbf{x}_t}$ is the dimension of $\mathbf{x}_t$.

Assumption 5.14. There is a constant $\mu_2 > 0$ such that for any $\theta \in \Theta$, there exists $\Pi_n \theta \in \Theta_n$ satisfying $\|\Pi_n \theta - \theta\|_\theta = O(\kappa_{2n}^{-\mu_2})$, and $\kappa_{2n}^{-\mu_2} = o(n^{-1/4})$.

Next assumption restricts the estimation error of the weight matrix.

Assumption 5.15. $\hat{A}(\mathbf{x}_t) = A(\mathbf{x}_t) + o_p(n^{-1/4})$ uniformly over $\mathbf{x}_t \in \mathcal{X}_t$.

Define $\xi_{0n} \equiv \sup_{(\xi_{11}, \xi_{12}, \xi_{13}) \in \text{Supp}(\eta) \cup \text{Supp}(k_t^*, k_{t-1}^*, \mathbf{z}_t) \cup \text{Supp}(k_t, k_{t-1}, \mathbf{z}_t^*)} \|p^{\kappa_{n\eta}}(\xi_{11}), p^{\kappa_{n1}}(\xi_{12}), p^{\kappa_{n2}}(\xi_{13})\|_E$, which is non-decreasing in κ_{1n} . Also $N(\delta_1, \mathcal{A}_n, \|\cdot\|_{s,\alpha})$ and $N(\delta_2, \Theta_n, \|\cdot\|_{s,\theta})$ define the minimal number of radius δ_1 covering balls of $\mathcal{A}_n$ under the $\|\cdot\|_{s,\alpha}$ metric, and the minimal number of radius δ_2 covering balls of Θ_n under the $\|\cdot\|_{s,\theta}$ metric, respectively. Two assumptions below impose conditions related to the sieve approximation in the first stage, such as size of sieve space, well-definedness of the norm, and the norm equivalence.

Assumption 5.16.

1. $\kappa_{1n} \times \ln n \times \xi_{0n}^2 \times n^{-1/2} = o(1)$
2. $\ln[N(\delta_1, \mathcal{A}_n, \|\cdot\|_{s,\alpha})] \leq \text{const.} \times \kappa_{1n} \times \ln(\kappa_{1n}/\delta_1)$.

Assumption 5.17.

1. $\mathcal{A}$ is convex in α_0 and $f_{r_t|\mathbf{z}_t}(r_t|\mathbf{z}_t; \psi)$ is pathwise differentiable at ψ_0
2. For some $c_{11}, c_{12} > 0$,

$$c_{11}E \left\{ \ln \frac{f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha)} \right\} \leq \|\alpha - \alpha_0\|_\alpha^2 \leq c_{12}E \left\{ \ln \frac{f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha)} \right\}$$

holds for all $\alpha_1 \in \mathcal{A}_n$ with $\|\alpha - \alpha_0\|_{s,\alpha} = o(1)$.

Similarly, conditions associated with the sieve approximation in the second stage are imposed in the next two assumptions.

Assumption 5.18.

1. $\kappa_{2n} \times \ln n \times \xi_{0n}^2 \times n^{-1/2} = o(1)$
2. $\ln[N(\delta_2, \Theta_n, \|\cdot\|_{s,\theta})] \leq \text{const.} \times \kappa_{2n} \times \ln(\kappa_{2n}/\delta_2)$.

Assumption 5.19.

1. Θ is convex in θ_0 and $\rho(\mathbf{y}_{t,t-1}; \theta)$ is pathwise differentiable at θ_0
2. For some $c_{21}, c_{22} > 0$,

$$c_{21} E \{g(\mathbf{x}_t; \theta)' A(\mathbf{x}_t) g(\mathbf{x}_t; \theta)\} \leq \|\theta - \theta_0\|_\theta^2 \leq c_{22} E \{g(\mathbf{x}_t; \theta)' A(\mathbf{x}_t) g(\mathbf{x}_t; \theta)\}$$

holds for all $\theta \in \Theta_n$ with $\|\theta - \theta_0\|_{s,\theta} = o(1)$.

We next impose convergence rates of $\hat{f}_{k_{t-2}^*, k_{t-3}^*, m_t, l_t, m_{t-1}, l_{t-1}}$ and $\hat{f}_{m_t, l_t, m_{t-1}, l_{t-1}}$, which appear in (11) and (12).

Assumption 5.20.

1. $\hat{f}_{k_{t-2}^*, k_{t-3}^*, m_t, l_t, m_{t-1}, l_{t-1}} = f_{k_{t-2}^*, k_{t-3}^*, m_t, l_t, m_{t-1}, l_{t-1}} + o_p(n^{-1/4})$ uniformly over all $\mathbf{z}_t^* = (k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t) \in \mathcal{Z}_t^*$
2. $\hat{f}_{m_t, l_t, m_{t-1}, l_{t-1}} = f_{m_t, l_t, m_{t-1}, l_{t-1}} + o_p(n^{-1/4})$ uniformly over all $\mathbf{s}_t = (m_t, l_t, m_{t-1}, l_{t-1}) \in \mathcal{S}_t$.

Under these Assumptions we obtain the following convergence rate results in the weaker metrics.

Theorem 5.21.

1. Suppose Assumptions 4.1, 4.5, 5.1-5.4, 5.11-5.12, and 5.16-5.17 hold. Then $\|\hat{\alpha}_n - \alpha_0\|_\alpha = o_p(n^{-1/4})$.
2. Suppose Assumptions 4.1, 4.5, 5.1-5.9, and 5.11-5.20 hold. Then $\|\hat{\theta}_n - \theta_0\|_\theta = o_p(n^{-1/4})$.

6 Asymptotic Normality

We now establish the asymptotic normality and $\sqrt{n}$ -consistency of $\hat{\phi}_n$ and $\hat{\beta}_n$. We adopt useful notations introduced in Ai and Chen (2003). Let $\overline{\mathbf{V}}_1, \overline{\mathbf{V}}_2$ denote the closure of the linear span of $\mathcal{A} - \alpha_0$ under the metric $\|\cdot\|_\alpha$ and the closure of the linear span of $\Theta - \theta_0$ under the metric $\|\cdot\|_\theta$, respectively. Note $\overline{\mathbf{V}}_1 = \mathcal{R}^{d_\phi} \times \overline{\mathcal{W}}_1$ with $\overline{\mathcal{W}}_1 \equiv \overline{\mathcal{M} \times \mathcal{F}_1 \times \mathcal{F}_2} - \{(\eta_0, f_{k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(k_t^*, k_{t-1}^* | \mathbf{z}_t^*), f_{k_t, k_{t-1} | \mathbf{z}_t^*}(k_t, k_{t-1} | \mathbf{z}_t^*))'\}$ and $\overline{\mathbf{V}}_2 = \mathcal{R}^{d_\beta} \times \overline{\mathcal{W}}_2$ with $\overline{\mathcal{W}}_2 \equiv \overline{\mathcal{H}} - \{q_0\}$. Then $(\overline{\mathbf{V}}_1, \|\cdot\|_\alpha)$ becomes a Hilbert space with the inner product $\langle v_{11}, v_{12} \rangle_\alpha$ induced by $\|\cdot\|_\alpha$:

$$\langle v_{11}, v_{12} \rangle_\alpha = E \left\{ \frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{d\alpha} [v_{11}] \cdot \frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{d\alpha} [v_{12}] \right\}.$$

Also $(\overline{\mathbf{V}}_2, \|\cdot\|_\theta)$ is a Hilbert space with the inner product:

$$\langle v_{21}, v_{22} \rangle_\theta = E \left\{ \left(\frac{dg(\mathbf{x}_t; \theta_0)}{d\theta} [v_{21}] \right)' A(\mathbf{x}_t) \frac{dg(\mathbf{x}_t; \theta_0)}{d\theta} [v_{22}] \right\}.$$

The pathwise derivative at α_0 is defined as

$$\begin{aligned}
& \frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{d\alpha} [\alpha - \alpha_0] \\
= & \frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{d\phi'} (\phi - \phi_0) + \frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{d\eta} [\eta - \eta_0] \\
& + \frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{df_1} [f_1 - f_{k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(k_t^*, k_{t-1}^* | \mathbf{z}_t^*)] \\
& + \frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{df_2} [f_2 - f_{k_t, k_{t-1} | \mathbf{z}_t^*}(k_t, k_{t-1} | \mathbf{z}_t^*)]
\end{aligned}$$

and the pathwise derivative at θ_0 is defined as

$$\frac{dg(\mathbf{x}_t; \theta_0)}{d\theta} [\theta - \theta_0] \equiv \frac{dg(\mathbf{x}_t; \theta_0)}{d\beta'} [\beta - \beta_0] + \frac{dg(\mathbf{x}_t; \theta_0)}{dq} [q - q_0].$$

For each component ϕ_j of ϕ , $j = 1, 2, \dots, d_\phi$, we define $w_{1j}^* \in \overline{\mathcal{W}}_1$ as

$$\begin{aligned}
& w_{1j}^* \equiv (\eta_j^*, f_{1j}^*, f_{2j}^*)' \\
= & \arg \min_{(\eta, f_1, f_2)' \in \overline{\mathcal{W}}_1} E \left\{ \left(\frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{d\phi_j} - \frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{d\eta} [\eta_j] \right. \right. \\
& \left. \left. - \frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{df_1} [f_{1j}] - \frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{df_2} [f_{2j}] \right)^2 \right\}.
\end{aligned}$$

Also for each component β_j of β , $j = 1, 2, \dots, d_\beta$, we define $w_{2j}^* \in \overline{\mathcal{W}}_2$ as

$$\begin{aligned}
& w_{2j}^* \\
= & \arg \min_{w_{2j} \in \overline{\mathcal{W}}_2} E \left\{ \left(\frac{dg(\mathbf{x}_t; \theta_0)}{d\beta_j} - \frac{dg(\mathbf{x}_t; \theta_0)}{dq} [w_{2j}] \right)' A(\mathbf{x}_t) \left(\frac{dg(\mathbf{x}_t; \theta_0)}{d\beta_j} - \frac{dg(\mathbf{x}_t; \theta_0)}{dq} [w_{2j}] \right) \right\}.
\end{aligned}$$

We let $w_1^* = (w_{11}^*, w_{12}^*, \dots, w_{1d_\phi}^*)$ and define

$$\begin{aligned}
& \frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{df} [w_{1j}^*] \\
= & \frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{d\eta} [\eta_j^*] + \frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{df_1} [f_{1j}^*] \\
& + \frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{df_2} [f_{2j}^*]
\end{aligned}$$

and

$$\begin{aligned} & \frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{df} [w_1^*] \\ &= \left(\frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{df} [w_{11}^*], \dots, \frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{df} [w_{1d_\phi}^*] \right). \end{aligned}$$

Also let $w_2^* = (w_{21}^*, w_{22}^*, \dots, w_{2d_\beta}^*)$ and

$$\frac{dg(\mathbf{x}_t; \theta_0)}{dq} [w_2^*] = \left(\frac{dg(\mathbf{x}_t; \theta_0)}{dq} [w_{21}^*], \dots, \frac{dg(\mathbf{x}_t; \theta_0)}{dq} [w_{2d_\beta}^*] \right).$$

We also define the row vectors

$$\begin{aligned} G_{w_1^*}(D_t, \alpha_0) &\equiv \frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{d\phi'} - \frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{df} [w_1^*], \\ G_{w_2^*}(\mathbf{x}_t, \theta_0) &\equiv \frac{dg(\mathbf{x}_t; \theta_0)}{d\beta'} - \frac{dg(\mathbf{x}_t; \theta_0)}{dq} [w_2^*]. \end{aligned}$$

Define $s_1(\alpha) \equiv \lambda_1' \phi$ for $\lambda_1 \in \mathcal{R}^{d_\phi}$, $\lambda_1 \neq 0$ and $s_2(\theta) \equiv \lambda_2' \theta$ for $\lambda_2 \in \mathcal{R}^{d_\beta}$, $\lambda_2 \neq 0$. Let $J_1 = E[G_{w_1^*}(D_t, \alpha_0)' G_{w_1^*}(D_t, \alpha_0)]$ and $J_2 = E[G_{w_2^*}(\mathbf{x}_t, \theta_0)' G_{w_2^*}(\mathbf{x}_t, \theta_0)]$. Then the Riesz representation theorem implies that there exist representers v_1^* and v_2^* such that

$$s_1(\alpha) - s_1(\alpha_0) \equiv \lambda_1'(\phi - \phi_0) = \langle v_1^*, \alpha - \alpha_0 \rangle_\alpha,$$

for all $\alpha \in \mathcal{A}$ where $v_1^* \equiv (v_\phi^*, v_f^*) \in \overline{\mathbf{V}}_1$, $v_\phi^* = J_1^{-1} \lambda_1$, $v_f^* = -w_1^* \times v_\phi^*$ and that

$$s_2(\theta) - s_2(\theta_0) \equiv \lambda_2'(\theta - \theta_0) = \langle v_2^*, \theta - \theta_0 \rangle_\theta,$$

for all $\theta \in \Theta$ where $v_2^* \equiv (v_\beta^*, v_q^*) \in \overline{\mathbf{V}}_2$, $v_\beta^* = J_2^{-1} \lambda_2$, $v_q^* = -w_2^* \times v_\beta^*$.

We denote, for any $v_1 \in \overline{\mathbf{V}}_1$,

$$\frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha)}{d\alpha} [v_1] \equiv \frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha + \tau v_1)}{d\tau} \Big|_{\tau=0} \quad \text{a.s. in } D_t.$$

We also denote

$$\frac{d\rho(\mathbf{y}_{t,t-1}; \theta)}{d\theta} [v_2] \equiv \frac{d\rho(\mathbf{y}_{t,t-1}; \theta + \tau v_2)}{d\tau} \Big|_{\tau=0} \quad \text{a.s. in } \mathbf{y}_{t,t-1},$$

and

$$\frac{dg(\mathbf{x}_t; \theta)}{d\theta} [v_2] \equiv E \left\{ \frac{d\rho(\mathbf{y}_{t,t-1}; \theta)}{d\theta} [v_2] \Big| \mathbf{x}_t \right\} \quad \text{a.s. in } \mathbf{x}_t,$$

for any $v_2 \in \overline{\mathbf{V}}_2$.

Define $\mathcal{N}_{01n} = \{\alpha : \alpha \in \mathcal{A}_n, \|\alpha - \alpha_0\|_\alpha = o(n^{-1/4}), \|\alpha - \alpha_0\|_{s,\alpha} = o(1)\}$ and define $\mathcal{N}_{01}$ the same except $\mathcal{A}_n$ replaced with $\mathcal{A}$. We denote a local alternative $\alpha^*(\alpha, \varepsilon_n) = (1 - \varepsilon_n)\alpha + \varepsilon_n(v_1^* + \alpha_0)$ with $\varepsilon_n = o(n^{-1/2})$, for $\alpha \in \mathcal{N}_{01n}$. Let $\Pi_n \alpha^*(\alpha, \varepsilon_n)$ be the projection of $\alpha^*(\alpha, \varepsilon_n)$ onto $\mathcal{A}_n$. Also define $\mathcal{N}_{02n} = \{\theta : \theta \in \Theta_n, \|\theta - \theta_0\|_\theta = o(n^{-1/4}), \|\theta - \theta_0\|_{s,\theta} = o(1)\}$ and define $\mathcal{N}_{02}$ the same except Θ_n replaced with Θ .

We state the sufficient conditions for the $\sqrt{n}$ -normality of $\hat{\phi}_n$ and $\hat{\beta}_n$.

Assumption 6.1.

1. $E[G_{w_1^*}(D_t, \alpha_0)' G_{w_1^*}(D_t, \alpha_0)]$ exists, is bounded, and is positive-definite
2. $\phi_0 \in \text{int}(\Phi)$.

Assumption 6.2. There is $v_{1n}^* = (v_\phi^*, -\Pi_n w_1^* \times v_\phi^*) \in \mathcal{A}_n - \{\alpha_0\}$ such that $\|v_{1n}^* - v_1^*\|_\alpha = o(n^{-1/4})$.

Assumption 6.1 contains a standard rank condition and the bounded second moment condition that ensure the existence of the Riesz representation. Assumption 6.2 states that the Riesz representer is also well approximated by the sieves. Next we impose an envelope condition on the second derivative of the log likelihood, which is associated with the stochastic equicontinuity.

Assumption 6.3. There exists measurable function $h_2(D_t)$ with $E[(h_2(D_t))^2] < \infty$ such that for any $\bar{\alpha} = (\bar{\psi}, \bar{f}_1, \bar{f}_2)' \in \mathcal{N}_{01}$,

$$\left| \frac{f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}^{[1]}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \bar{\alpha}, \nu)}{f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \bar{\alpha})} \right|^2 + \left| \frac{f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}^{[2]}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \bar{\alpha}, \nu)}{f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \bar{\alpha})} \right| \leq h_2(D_t)$$

where the pathwise first derivative $f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}^{[1]}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \bar{\alpha}, \nu)$ and the pathwise second derivative $f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}^{[2]}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \bar{\alpha}, \nu)$ are defined in the proofs of Theorem 5.10.1 and Theorem 6.12.1, respectively.

We now introduce some notations for the assumptions below. Let

$$\begin{aligned} & \frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{d\alpha} [p^{\kappa_{1n}}] \\ = & \left(\left(\frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{d\phi} \right)' , \left(\frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{d\eta} [p^{\kappa_{1n}}] \right)' , \right. \\ & \left. \left(\frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{df_1} [p^{\kappa_{1n}}] \right)' , \left(\frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{df_2} [p^{\kappa_{1n}}] \right)' \right)' , \end{aligned}$$

where for $\tilde{f} = \eta, f_1$, or f_2 ,

$$\begin{aligned}
& \frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{d\tilde{f}} [p^{\kappa_{1n}}] \\
&= \left(\frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{d\tilde{f}} [p_1^{\kappa_{1n}}], \dots, \frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{d\tilde{f}} [p_{\kappa_{1n}}^{\kappa_{1n}}] \right)', \text{ and} \\
& \frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{d\phi} \\
&= \left(\frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{d\phi_1}, \dots, \frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{d\phi_{d_\phi}} \right)'.
\end{aligned}$$

Let

$$\Omega_{\kappa_{1n}} = E \left\{ \left(\frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{d\alpha} [p^{\kappa_{1n}}] \right) \left(\frac{d \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \alpha_0)}{d\alpha} [p^{\kappa_{1n}}] \right)' \right\}.$$

We now impose a normalization on the approximating functions, which trivially holds for e.g. orthonormalized approximating basis functions.

Assumption 6.4. *The smallest eigenvalue of the matrices $\Omega_{\kappa_{1n}}$ is bounded away from zero, and $\|p_j^{\kappa_{1n}}\|_{\infty, \nu} < \infty$ for $j = 1, 2, \dots, \kappa_{1n}$ uniformly in κ_{1n} .*

Next we impose Lipschitz conditions on the fourth derivatives of the log densities.

Assumption 6.5. *For all $\alpha \in \mathcal{N}_{01n}$, there exists a measurable function $h_3(D_t)$ with $E[|h_3(D_t)|] < \infty$ such that*

$$\left| \frac{d^4}{d\tau^4} \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \bar{\alpha} + \tau(\alpha - \alpha_0)) \right|_{\tau=0} \leq h_3(D_t) \|\alpha - \alpha_0\|_{s, \alpha}^4.$$

Now we add conditions on the moment function. First, we impose a local identification condition for β_0 . This also ensures the existence of the Riesz representer.

Assumption 6.6.

1. $E[G_{w_2^*}(\mathbf{x}_t, \theta_0)' A(\mathbf{x}_t) G_{w_2^*}(\mathbf{x}_t, \theta_0)]$ exists, is bounded, and is positive-definite
2. $\beta_0 \in \text{int}(\mathcal{B})$
3. $\Sigma_0(\mathbf{x}_t) \equiv \text{var}[\rho(\mathbf{y}_{t, t-1}; \theta_0) | \mathbf{x}_t]$ is positive-definite for all $\mathbf{x}_t \in \mathcal{X}_t$.

Assumption 6.7. *There is $v_{2n}^* = (v_\beta^*, -\Pi_n w_2^* \times v_\beta^*) \in \Theta_n - \{\theta_0\}$ such that $\|v_{2n}^* - v_2^*\|_\theta = o(n^{-1/4})$.*

This condition is required to ensure that the sieve space $\mathcal{H}_n$ approximates the space $\bar{\mathcal{H}} - \{q_0\}$, so the Riesz representer is also well approximated by the sieves. Now we impose an envelope condition for $\rho(\mathbf{y}_{t, t-1}; \theta)$.

Assumption 6.8.

1. For all $\theta \in \mathcal{N}_{02}$, the pathwise first derivative $(d\rho(\mathbf{y}_{t,t-1}; \theta)/d\theta)[v_2]$ exists a.s. $\mathbf{y}_{t,t-1} \in \mathcal{Y}_t$
2. Each element of $(d\rho(\mathbf{y}_{t,t-1}; \theta)/d\theta)[v_{2n}^*]$ satisfies an envelope condition and is Hölder continuous in $\theta \in \mathcal{N}_{02n}$
3. Each element of $(dg(\mathbf{x}_t; \theta)/d\theta)[v_{2n}^*]$ is in $\Lambda_c^\gamma(\mathcal{X}_t)$, $\gamma > d_x/2$ for all $\alpha \in \mathcal{N}_{02}$.

Next two assumptions are necessary to control the asymptotic bias when ρ is a highly nonlinear function of θ .

Assumption 6.9. Uniformly over $\theta \in \mathcal{N}_{02n}$, we have

$$E \left(\left\| \frac{dg(\mathbf{x}_t; \theta)}{d\theta}[v_{2n}^*] - \frac{dg(\mathbf{x}_t; \theta_0)}{d\theta}[v_{2n}^*] \right\|_E^2 \right) = o(n^{-1/2}).$$

Additionally, we impose the following assumption.

Assumption 6.10. Uniformly over $\theta \in \mathcal{N}_{02}$, $\bar{\theta} \in \mathcal{N}_{02n}$, we have

$$E \left(\left\{ \frac{dg(\mathbf{x}_t; \theta_0)}{d\theta}[v_2^*] \right\} A(\mathbf{x}_t) \left\{ \frac{dg(\mathbf{x}_t; \theta)}{d\theta}[\bar{\theta} - \theta_0] - \frac{dg(\mathbf{x}_t; \theta_0)}{d\theta}[\bar{\theta} - \theta_0] \right\} \right) = o(n^{-1/2}).$$

The following condition is required to have the higher order terms asymptotically negligible in the stochastic expansions to obtain the asymptotic normality.

Assumption 6.11. For all $\theta \in \mathcal{N}_{02n}$, the pathwise second derivative $d^2\rho(\mathbf{y}_{t,t-1}; \theta + \tau v_{2n}^*)/d\tau^2|_{\tau=0}$ exists a.s. $\mathbf{y}_{t,t-1} \in \mathcal{Y}_t$, and is bounded by a measurable function $c(\mathbf{y}_{t,t-1})$ with $E[c(\mathbf{y}_{t,t-1})^2] < \infty$.

We now conclude the $\sqrt{n}$ -normality of $(\hat{\beta}_n, \hat{\phi}_n)$ in the following theorem.

Theorem 6.12.

1. Suppose Assumptions 4.1, 4.5, 5.1-5.4.1, 5.11-5.12, 5.16-5.17, and 6.1- 6.5 hold. Then we have $\sqrt{n}(\hat{\phi}_n - \phi_0) \xrightarrow{d} N(0, J_1^{-1})$, where $J_1 = E[G_{w_1^*}(D_t, \alpha_0)' G_{w_1^*}(D_t, \alpha_0)]$.
2. Suppose Assumptions 4.1, 4.5, 5.1-5.9, 5.11-5.20, and 6.1- 6.11 hold. Then $\sqrt{n}(\hat{\beta}_n - \beta_0) \xrightarrow{d} N(0, J_2^{-1})$, where

$$J_2 = E[G_{w_2^*}(\mathbf{x}_t, \theta_0)' A(\mathbf{x}_t) G_{w_2^*}(\mathbf{x}_t, \theta_0)] \times (E[G_{w_2^*}(\mathbf{x}_t, \theta_0)' A(\mathbf{x}_t) \Sigma_0(\mathbf{x}_t) A(\mathbf{x}_t) G_{w_2^*}(\mathbf{x}_t, \theta_0)])^{-1} \\ \times E[G_{w_2^*}(\mathbf{x}_t, \theta_0)' A(\mathbf{x}_t) G_{w_2^*}(\mathbf{x}_t, \theta_0)].$$

As in Ai and Chen (2003) one can also obtain an efficient two-step estimator of β_0 by using the optimal weight matrix as $A(\mathbf{x}_t) = \Sigma_0(\mathbf{x}_t)^{-1}$. In this case the asymptotic variance reduces to $J_2^{-1} = \{E[G_{w_2^*}(\mathbf{x}_t, \theta_0)' \Sigma_0(\mathbf{x}_t)^{-1} G_{w_2^*}(\mathbf{x}_t, \theta_0)]\}^{-1}$.

This asymptotic normality results show that in our estimation pre-stage estimations of the conditional mean functions via the conditional density functions do not contribute to the asymptotic distribution of $\hat{\beta}_n$. Thus estimation of the conditional mean functions or the conditional density functions satisfies the asymptotic independence with $\hat{\beta}_n$. This is indeed the same insight we find in Ai and Chen (2003) that the estimation of the conditional moment functions does not contribute to the asymptotic distribution of their sieve MD estimator.

7 Conclusion

The control function estimators for production functions of Olley and Pakes (1996) and Levinsohn and Petrin (2003) are not consistent if the arguments that enter the control function for productivity are measured with error. The problem is complicated by the fact that the unobserved productivity control function is an unknown non-linear function of capital and other inputs, so standard instrumental variable techniques cannot be employed. We combine results from the nonclassical measurement error literature (e.g. Hu and Schennach 2008) with a sieve minimum distance estimation techniques to recover the production function parameters (e.g. Ai and Chen 2003, Chen 2007, Song 2012) . We develop the asymptotic distribution for our estimator, which accounts for the multiple stages and the infinite-dimensional nuisance parameters.

Appendix

A Identification of Conditional Densities: Proof of Theorem 4.2

We first define integral operators. $L_{R_1, R_2 | R_3, R_4, R_5}$ denote an integral operator mapping $g \in \text{Supp}(r_3, r_4)$ to $L_{R_1, R_2 | R_3, R_4, R_5} g \in \text{Supp}(r_1, r_2)$ for a given $R_5 = r_5$:

$$[L_{R_1, R_2 | R_3, R_4, R_5} g](r_1, r_2) \equiv \int f_{R_1, R_2 | R_3, R_4, R_5}(r_1, r_2 \mid r_3, r_4, r_5) g(r_3, r_4) dr_3 dr_4.$$

In addition, $\Delta_{R_1 | R_2, R_3, R_4}$ denotes a diagonal operator mapping $g \in \text{Supp}(r_2, r_3)$ to $\Delta_{R_1 | R_2, R_3, R_4} g \in \text{Supp}(r_2, r_3)$ for a given $R_4 = r_4$:

$$\Delta_{R_1 | R_2, R_3, R_4} g \equiv f_{R_1 | R_2, R_3, R_4}(r_1 \mid r_2, r_3, r_4) g(r_2, r_3).$$

Proof. Recall that $r_t = y_t$ and $\mathbf{s}_t = (m_t, l_t, m_{t-1}, l_{t-1})$ or $r_t \in \{l_t, m_t\}$ and $\mathbf{s}_t = (m_{t-1}, l_{t-1})$. We have

$$\begin{aligned} f_{r_t, k_t^*, k_{t-1}^* | k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t} &= \int f_{r_t, k_t, k_{t-1}, k_t^*, k_{t-1}^* | k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t} dk_t dk_{t-1} \\ &= \int f_{r_t | k_t, k_{t-1}, k_t^*, k_{t-1}^* | k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t} f_{k_t, k_{t-1}, k_t^*, k_{t-1}^* | k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t} dk_t dk_{t-1} \\ &= \int f_{r_t | k_t, k_{t-1}, \mathbf{s}_t} f_{k_t^*, k_{t-1}^* | k_t, k_{t-1}, k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t} f_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t} dk_t dk_{t-1} \\ &= \int f_{r_t | k_t, k_{t-1}, \mathbf{s}_t} f_{k_t^*, k_{t-1}^* | k_t, k_{t-1}, \mathbf{s}_t} f_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t} dk_t dk_{t-1} \end{aligned}$$

by Assumption 4.1.1. Now we need to show uniqueness of the decomposition. Using operator notation, we have

$$\begin{aligned} &[L_{r_t, k_t^*, k_{t-1}^* | k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t} g](k_t^*, k_{t-1}^*) \\ &= \int f_{r_t, k_t^*, k_{t-1}^* | k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t} g(k_{t-2}^*, k_{t-3}^*) dk_{t-2}^* dk_{t-3}^* \\ &= \int \int f_{r_t | k_t, k_{t-1}, \mathbf{s}_t} f_{k_t^*, k_{t-1}^* | k_t, k_{t-1}, \mathbf{s}_t} f_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t} dk_t dk_{t-1} g(k_{t-2}^*, k_{t-3}^*) dk_{t-2}^* dk_{t-3}^* \\ &= \int f_{k_t^*, k_{t-1}^* | k_t, k_{t-1}, \mathbf{s}_t} f_{r_t | k_t, k_{t-1}, \mathbf{s}_t} \int f_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t} g(k_{t-2}^*, k_{t-3}^*) dk_{t-2}^* dk_{t-3}^* dk_t dk_{t-1} \\ &= \int f_{k_t^*, k_{t-1}^* | k_t, k_{t-1}, \mathbf{s}_t} f_{r_t | k_t, k_{t-1}, \mathbf{s}_t} [L_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t} g](k_t, k_{t-1}) dk_t dk_{t-1} \\ &= \int f_{k_t^*, k_{t-1}^* | k_t, k_{t-1}, \mathbf{s}_t} [\Delta_{r_t | k_t, k_{t-1}, \mathbf{s}_t} L_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t} g](k_t, k_{t-1}) dk_t dk_{t-1} \\ &= [L_{k_t^*, k_{t-1}^* | k_t, k_{t-1}, \mathbf{s}_t} \Delta_{r_t | k_t, k_{t-1}, \mathbf{s}_t} L_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t} g](k_t^*, k_{t-1}^*). \end{aligned}$$

Thus we have

$$L_{r_t, k_t^*, k_{t-1}^* | k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t} = L_{k_t^*, k_{t-1}^* | k_t, k_{t-1}, \mathbf{s}_t} \triangle_{r_t | k_t, k_{t-1}, \mathbf{s}_t} L_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t}.$$

From the integration of the above equation over all r_t , we have

$$L_{k_t^*, k_{t-1}^* | k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t} = L_{k_t^*, k_{t-1}^* | k_t, k_{t-1}, \mathbf{s}_t} L_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t}.$$

Then

$$L_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t} = L_{k_t^*, k_{t-1}^* | k_t, k_{t-1}, \mathbf{s}_t}^{-1} L_{k_t^*, k_{t-1}^* | k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t},$$

since $L_{k_t^*, k_{t-1}^* | k_t, k_{t-1}, \mathbf{s}_t}$ is one-to-one. Since $L_{k_t^*, k_{t-1}^* | k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t}$ is also one-to-one, rearranging two equations above gives

$$L_{r_t, k_t^*, k_{t-1}^* | k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t} L_{k_t^*, k_{t-1}^* | k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t}^{-1} = L_{k_t^*, k_{t-1}^* | k_t, k_{t-1}, \mathbf{s}_t} \triangle_{r_t | k_t, k_{t-1}, \mathbf{s}_t} L_{k_t^*, k_{t-1}^* | k_t, k_{t-1}, \mathbf{s}_t}^{-1}.$$

Thus the observed operator on the left hand side has the form of an eigenvalue-eigenfunction decomposition, with the eigenvalues corresponding to the density $f_{r_t | k_t, k_{t-1}, \mathbf{s}_t}$ and the eigenfunctions corresponding to the density $f_{k_t^*, k_{t-1}^* | k_t, k_{t-1}, \mathbf{s}_t}$.

Assumption 4.1 ensures that this decomposition is unique, by a similar argument in Hu and Schennach (2008). Given the identification of $f_{r_t | k_t, k_{t-1}, \mathbf{s}_t}$ and $f_{k_t^*, k_{t-1}^* | k_t, k_{t-1}, \mathbf{s}_t}$, $f_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t}$ is also identified from

$$L_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t} = L_{k_t^*, k_{t-1}^* | k_t, k_{t-1}, \mathbf{s}_t}^{-1} L_{k_t^*, k_{t-1}^* | k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t}$$

where $L_{k_t^*, k_{t-1}^* | k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t}$ is observable. □

B Alternative Identification of $f_{l_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}}$ and $f_{m_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}}$

We consider an alternative way of identifying $f_{l_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}}$ and $f_{m_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}}$. We show that they can be derived from

$$(f_{y_t | k_t, k_{t-1}, m_t, l_t, m_{t-1}, l_{t-1}}, f_{k_t^*, k_{t-1}^* | k_t, k_{t-1}, m_t, l_t, m_{t-1}, l_{t-1}}, f_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, m_t, l_t, m_{t-1}, l_{t-1}}) \quad (13)$$

which are the objects identified as the results of Theorem 4.2. One advantage of this approach is that we need only one sieve MLE for (13). Others can be derived from these densities. On the other hand, it requires density estimation associated with inverse of linear operators. We first assume

Assumption B.1.

1. $f_{l_t | k_t, k_{t-1}, k_{t-2}^*, k_{t-3}^*, m_{t-1}, l_{t-1}} = f_{l_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}}$
2. $f_{m_t | k_t, k_{t-1}, k_{t-2}^*, k_{t-3}^*, m_{t-1}, l_{t-1}} = f_{m_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}}$

3. The operator $L_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, m_{t-1}, l_{t-1}}$ is one-to-one.

We have from Assumption B.1.1 that,

$$\begin{aligned} f_{l_t | k_{t-2}^*, k_{t-3}^*, m_{t-1}, l_{t-1}} &= \int f_{l_t | k_t, k_{t-1}, k_{t-2}^*, k_{t-3}^*, m_{t-1}, l_{t-1}} f_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, m_{t-1}, l_{t-1}} dk_t dk_{t-1} \\ &= \int f_{l_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}} f_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, m_{t-1}, l_{t-1}} dk_t dk_{t-1} \end{aligned}$$

which, in operator notation, is $L_{l_t | k_{t-2}^*, k_{t-3}^*, m_{t-1}, l_{t-1}} = L_{l_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}} L_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, m_{t-1}, l_{t-1}}$. Then, from Assumption B.1.3, we have that

$$L_{l_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}} = L_{l_t | k_{t-2}^*, k_{t-3}^*, m_{t-1}, l_{t-1}} L_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, m_{t-1}, l_{t-1}}^{-1}. \quad (14)$$

Further note we identify

$$f_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, m_{t-1}, l_{t-1}} = \int f_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, m_t, l_t, m_{t-1}, l_{t-1}} f_{m_t, l_t | k_{t-2}^*, k_{t-3}^*, m_{t-1}, l_{t-1}} dm_t dl_t$$

because $f_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, m_t, l_t, m_{t-1}, l_{t-1}}$ is identified from Theorem 4.2 and $f_{m_t, l_t | k_{t-2}^*, k_{t-3}^*, m_{t-1}, l_{t-1}}$ is observed from the data. Then from (14), $f_{l_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}}$ is also identified because $f_{l_t | k_{t-2}^*, k_{t-3}^*, m_{t-1}, l_{t-1}}$ is observed from the data.

Similarly, we can identify $f_{m_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}}$. We have from Assumption B.1.2 that,

$$\begin{aligned} f_{m_t | k_{t-2}^*, k_{t-3}^*, m_{t-1}, l_{t-1}} &= \int f_{m_t | k_t, k_{t-1}, k_{t-2}^*, k_{t-3}^*, m_{t-1}, l_{t-1}} f_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, m_{t-1}, l_{t-1}} dk_t dk_{t-1} \\ &= \int f_{m_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}} f_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, m_{t-1}, l_{t-1}} dk_t dk_{t-1} \end{aligned}$$

which, in operator notation, is $L_{m_t | k_{t-2}^*, k_{t-3}^*, m_{t-1}, l_{t-1}} = L_{m_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}} L_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, m_{t-1}, l_{t-1}}$. Then, from Assumption B.1.3, we have that

$$L_{m_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}} = L_{m_t | k_{t-2}^*, k_{t-3}^*, m_{t-1}, l_{t-1}} L_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, m_{t-1}, l_{t-1}}^{-1}.$$

Since $f_{k_t, k_{t-1} | k_{t-2}^*, k_{t-3}^*, m_{t-1}, l_{t-1}}$ is already shown to be identified above and $f_{m_t | k_{t-2}^*, k_{t-3}^*, m_{t-1}, l_{t-1}}$ is observed from the data, $f_{m_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}}$ is identified. We summarize the results.

Corollary B.2. Suppose Assumptions 4.1 and B.1 hold. Then we identify $f_{l_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}}$ and $f_{m_t | k_t, k_{t-1}, m_{t-1}, l_{t-1}}$ from the observations $(y_t, k_t^*, k_{t-1}^*, k_{t-2}^*, k_{t-3}^*, m_t, l_t, m_{t-1}, l_{t-1})$.

C Proof of Asymptotic Results

Recall that $\tilde{\mathbf{x}}_t = (k_t, k_{t-1}, m_{t-1}, l_{t-1})$, $\mathbf{x}_t = (\tilde{\mathbf{x}}_t, l_t, m_t)$, $\mathbf{s}_t = (m_t, l_t, m_{t-1}, l_{t-1})$, $\mathbf{z}_t = (k_t, k_{t-1}, \mathbf{s}_t)$ and $\mathbf{z}_t^* = (k_{t-2}^*, k_{t-3}^*, \mathbf{s}_t)$. Define

$$\begin{aligned}
Q(\theta) &= E[g(\mathbf{x}_t; \theta)' A(\mathbf{x}_t) g(\mathbf{x}_t; \theta)] \\
&= \int g(\mathbf{x}_t; \theta)' A(\mathbf{x}_t) g(\mathbf{x}_t; \theta) f_{k_t, k_{t-1} | \mathbf{s}_t} f_{\mathbf{s}_t} dx_t \\
Q_n(\theta) &= \frac{1}{n} \sum_{i=1}^n \int g(\mathbf{x}_{ti}; \theta)' A(\mathbf{x}_{ti}) g(\mathbf{x}_{ti}; \theta) f_{k_t, k_{t-1} | \mathbf{s}_{ti}} dk_t dk_{t-1} \\
\hat{Q}_n(\theta) &= \frac{1}{n} \sum_{i=1}^n \int \hat{g}(\mathbf{x}_{ti}; \theta)' \hat{A}(\mathbf{x}_{ti}) \hat{g}(\mathbf{x}_{ti}; \theta) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}} dk_t dk_{t-1}.
\end{aligned}$$

C.1 Consistency: Proof of Theorem 5.10

The first result (Theorem 5.10.1) follows from a similar argument to the proof of Lemma 2 in Hu and Schennach (2008). In the proof we obtain the bound for the pathwise first derivative by

$$\begin{aligned}
& \left| \frac{d}{d\tau} \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \bar{\alpha} + \tau(\alpha_1 - \alpha_2)) \right|_{\tau=0} \\
& \leq \frac{1}{|f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \bar{\alpha})|} \\
& \quad \times \left\{ \int \left| \frac{d}{d\psi} f_{r_t | \mathbf{z}_t}(r_t | \mathbf{z}_t; \bar{\psi}) \nu^{-1}(\cdot) \bar{f}_1(k_t^*, k_{t-1}^* | \mathbf{z}_t) \bar{f}_2(k_t, k_{t-1} | \mathbf{z}_t^*) \right| dk_t dk_{t-1} \right. \\
& \quad + \int \left| f_{r_t | \mathbf{z}_t}(r_t | \mathbf{z}_t; \bar{\psi}) \nu^{-1}(k_t^*, k_{t-1}^* | \mathbf{z}_t) \bar{f}_2(k_t, k_{t-1} | \mathbf{z}_t^*) \right| dk_t dk_{t-1} \\
& \quad \left. + \int \left| f_{r_t | \mathbf{z}_t^*}(r_t | \mathbf{z}_t; \bar{\psi}) \bar{f}_1(k_t^*, k_{t-1}^* | \mathbf{z}_t) \nu^{-1}(k_t, k_{t-1} | \mathbf{z}_t^*) \right| dk_t dk_{t-1} \right\} \|\alpha_1 - \alpha_2\|_{s, \alpha} \\
& \equiv \left| \frac{f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}^{[1]}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \bar{\alpha}, \nu)}{f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \bar{\alpha})} \right| \|\alpha_1 - \alpha_2\|_{s, \alpha}
\end{aligned}$$

where $\bar{\alpha} = (\bar{\psi}, \bar{f}_1, \bar{f}_2) \in \mathcal{A}$.

The second result (Theorem 5.10.2) can be proved by verifying the three conditions i)-iii) of Lemma A.1 of Newey and Powell (2003). Condition i) is explicitly assumed in Assumption 5.5. Condition iii) is satisfied by choosing $\Pi_n \theta \in \Theta_n$ such that $\|\Pi_n \theta - \theta\|_{s, \theta} = o(1)$ in Assumption 5.8.2.

We now verify Condition ii) of their Lemma A.1 by checking Conditions (i)-(iii) in Lemma A.2 of Newey and Powell (2003). First, Assumption 5.8.1 assumes their compactness of a parameter space in Lemma A.2 Condition (i). We verify their Lemma A.2 Condition (ii) about pointwise convergence in θ . Note that for a generic positive constant C ,

$$\begin{aligned}
& \left| \widehat{Q}_n(\theta) - Q_n(\theta) \right| \\
& \leq \frac{C}{n} \cdot \left| \sum_{i=1}^n \int \left(\|\hat{g}(\mathbf{x}_{ti}; \theta) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}}\|_E^2 - \|g(\mathbf{x}_{ti}; \theta) f_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}}\|_E^2 \right) dk_t dk_{t-1} \right| \\
& \leq \frac{C}{n} \cdot \int \left(\sum_{i=1}^n \|\hat{g}(\mathbf{x}_{ti}; \theta) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} - g(\mathbf{x}_{ti}; \theta) f_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}}\|_E^2 \right. \\
& \quad \left. + 2 \sum_{i=1}^n \|g(\mathbf{x}_{ti}; \theta) f_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}}\|_E \sum_{i=1}^n \|\hat{g}(\mathbf{x}_{ti}; \theta) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} - g(\mathbf{x}_{ti}; \theta) f_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}}\|_E \right) dk_t dk_{t-1},
\end{aligned}$$

by Assumption 5.6. Since $\sum_{i=1}^n \|g(\mathbf{x}_{ti}; \theta) f_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}}\|_E^2 / n = O_p(1)$ by the Markov inequality from Assumption 5.7.1, it suffices to show

$$\sum_{i=1}^n \left\| \hat{g}(\mathbf{x}_{ti}; \theta) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} - g(\mathbf{x}_{ti}; \theta) f_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right\|_E^2 / n = o_p(1)$$

to verify $\left| \widehat{Q}_n(\theta) - Q_n(\theta) \right| = o_p(1)$.

We note that for any given (k_t, k_{t-1})

$$\begin{aligned}
& E \left[\sum_{i=1}^n \left\| \hat{g}(\mathbf{x}_{ti}; \theta) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} - g(\mathbf{x}_{ti}; \theta) f_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right\|_E^2 / n \right] \\
& = E \left[\frac{1}{n} \sum_{i=1}^n \left\{ \hat{g}_1(\mathbf{x}_{ti}; \cdot) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} - g_1(\mathbf{x}_{ti}; \cdot) f_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right\}^2 \right] \\
& \quad + E \left[\frac{1}{n} \sum_{i=1}^n \left\{ \hat{g}_2(\tilde{\mathbf{x}}_{ti}; \cdot) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} - g_2(\tilde{\mathbf{x}}_{ti}; \cdot) f_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right\}^2 \right] \\
& = E \left[\frac{1}{n} \sum_{i=1}^n \left\{ \left(\int y_t \hat{f}_{y_t | k_t, k_{t-1}, \mathbf{s}_{ti}} dy_t - \beta_l l_{ti} - \beta_k k_t - \beta_m m_{ti} - \mathbf{c}(m_{ti}, k_t)' \beta_\omega \right) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right. \right. \\
& \quad \left. \left. - \left(\int y_t f_{y_t | k_t, k_{t-1}, \mathbf{s}_{ti}} dy_t - \beta_l l_{ti} - \beta_k k_t - \beta_m m_{ti} - \mathbf{c}(m_{ti}, k_t)' \beta_\omega \right) f_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right\}^2 \right. \\
& \quad \left. + \left(\left(\int y_t \hat{f}_{y_t | k_t, k_{t-1}, m_{t-1, i}, l_{t-1, i}} dy_t - \beta_l \int l_t \hat{f}_{l_t | k_t, k_{t-1}, m_{t-1, i}, l_{t-1, i}} dl_t - \beta_k k_t \right. \right. \right. \\
& \quad \left. \left. - \beta_m \int m_t \hat{f}_{m_t | k_t, k_{t-1}, m_{t-1, i}, l_{t-1, i}} dm_t - q(\mathbf{c}(m_{t-1, i}, k_{t-1})' \beta_\omega) \right) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right. \\
& \quad \left. \left. - \left(\int y_t f_{y_t | k_t, k_{t-1}, m_{t-1, i}, l_{t-1, i}} dy_t - \beta_l \int l_t f_{l_t | k_t, k_{t-1}, m_{t-1, i}, l_{t-1, i}} dl_t - \beta_k k_t \right. \right. \right. \\
& \quad \left. \left. - \beta_m \int m_t f_{m_t | k_t, k_{t-1}, m_{t-1, i}, l_{t-1, i}} dm_t - q(\mathbf{c}(m_{t-1, i}, k_{t-1})' \beta_\omega) \right) f_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right\}^2 \right]
\end{aligned}$$

$$\begin{aligned}
&= E \left[\frac{1}{n} \sum_{i=1}^n \left\{ \left(\int y_t \left(\hat{f}_{y_t|k_t, k_{t-1}, \mathbf{s}_{ti}}^{\frac{1}{2}} - f_{y_t|k_t, k_{t-1}, \mathbf{s}_{ti}}^{\frac{1}{2}} \right) dy_t \right. \right. \right. \\
&\quad - (\beta_l l_{ti} + \beta_k k_t + \beta_m m_{ti} + \mathbf{c}(m_{ti}, k_t)' \beta_\omega) \left(\hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} - f_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right) \Big)^2 \\
&\quad + \left(\int y_t \left(\hat{f}_{y_t|k_t, k_{t-1}, m_{t-1, i}, l_{t-1, i}}^{\frac{1}{2}} - f_{y_t|k_t, k_{t-1}, \mathbf{s}_{ti}}^{\frac{1}{2}} \right) dy_t \right. \\
&\quad - \beta_l \int l_t \left(\hat{f}_{l_t|k_t, k_{t-1}, m_{t-1, i}, l_{t-1, i}}^{\frac{1}{2}} - f_{y_t|k_t, k_{t-1}, \mathbf{s}_{ti}}^{\frac{1}{2}} \right) dl_t \\
&\quad - \beta_m \int m_t \left(\hat{f}_{m_t|k_t, k_{t-1}, m_{t-1, i}, l_{t-1, i}}^{\frac{1}{2}} - f_{m_t|k_t, k_{t-1}, \mathbf{s}_{ti}}^{\frac{1}{2}} \right) dm_t \\
&\quad \left. \left. - (\beta_k k_t + q(\mathbf{c}(m_{t-1, i}, k_{t-1})' \beta_\omega)) \left(\hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} - f_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right) \right)^2 \right\} \right] \\
&= E \left[\frac{1}{n} \sum_{i=1}^n \left\{ \left(\int y_t \hat{f}_{y_t|k_t, k_{t-1}, \mathbf{s}_{ti}} dy_t \left(\hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} - f_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right) \right. \right. \right. \\
&\quad + \int y_t \left(\hat{f}_{y_t|k_t, k_{t-1}, \mathbf{s}_{ti}} - f_{y_t|k_t, k_{t-1}, \mathbf{s}_{ti}} \right) dy_t f_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \\
&\quad - (\beta_l l_{ti} + \beta_k k_t + \beta_m m_{ti} + \mathbf{c}(m_{ti}, k_t)' \beta_\omega) \left(\hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} - f_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right) \Big)^2 \\
&\quad + \left(\int y_t \hat{f}_{y_t|k_t, k_{t-1}, m_{t-1, i}, l_{t-1, i}} dy_t \left(\hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} - f_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right) \right. \\
&\quad + \left(\int y_t \left(\hat{f}_{y_t|k_t, k_{t-1}, m_{t-1, i}, l_{t-1, i}} - f_{y_t|k_t, k_{t-1}, \mathbf{s}_{ti}} \right) dy_t f_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right. \\
&\quad - \beta_l \left(\int l_t \hat{f}_{l_t|k_t, k_{t-1}, m_{t-1, i}, l_{t-1, i}} dl_t \left(\hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} - f_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right) \right. \\
&\quad + \int l_t \left(\hat{f}_{l_t|k_t, k_{t-1}, m_{t-1, i}, l_{t-1, i}} - f_{l_t|k_t, k_{t-1}, \mathbf{s}_{ti}} \right) dl_t f_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \Big) \\
&\quad - \beta_m \left(\int m_t \hat{f}_{m_t|k_t, k_{t-1}, m_{t-1, i}, l_{t-1, i}} dm_t \left(\hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} - f_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right) \right. \\
&\quad + \int m_t \left(\hat{f}_{m_t|k_t, k_{t-1}, m_{t-1, i}, l_{t-1, i}} - f_{m_t|k_t, k_{t-1}, \mathbf{s}_{ti}} \right) dm_t f_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \Big) \\
&\quad \left. \left. - (\beta_k k_t + q(\mathbf{c}(m_{t-1, i}, k_{t-1})' \beta_\omega)) \left(\hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} - f_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right) \right)^2 \right\} \right]
\end{aligned}$$

$$\begin{aligned}
&\leq E \left[\frac{1}{n} \sum_{i=1}^n \left\{ \left(\int \left| y_t \hat{f}_{y_t|k_t, k_{t-1}, \mathbf{s}_{ti}} \right| dy_t \right) \nu^{-1}(k_t, k_{t-1}, \mathbf{z}_t^*) \right\} \left\| \hat{f}_2^{\frac{1}{2}} - f_2^{\frac{1}{2}} \right\|_{\infty, v} \right. \\
&\quad + \int \left| y_t \frac{df_{y_t|k_t, k_{t-1}, \mathbf{s}_{ti}; \psi}}{d\psi} \nu^{-1}(\cdot) \right\| \left\| \hat{\psi} - \psi \right\|_{\infty, v} dy_t \left| f_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right| \\
&\quad + \left| \beta_l l_{ti} + \beta_k k_t + \beta_m m_{ti} + \mathbf{c}(m_{ti}, k_t)' \beta_\omega \right| \left\| \nu^{-1}(k_t, k_{t-1}, \mathbf{z}_t^*) \right\| \left\| \hat{f}_2^{\frac{1}{2}} - f_2^{\frac{1}{2}} \right\|_{\infty, v} \Bigg)^2 \\
&\quad + \left(\left| \int y_t \hat{f}_{y_t|k_t, k_{t-1}, m_{t-1, i}, l_{t-1, i}} dy_t \right| \left\| \nu^{-1}(k_t, k_{t-1}, \mathbf{z}_t^*) \right\| \left\| \hat{f}_2^{\frac{1}{2}} - f_2^{\frac{1}{2}} \right\|_{\infty, v} \right. \\
&\quad + \int \left| y_t \frac{df_{y_t|k_t, k_{t-1}, m_{t-1, i}, l_{t-1, i}; \psi}}{d\psi} \nu^{-1}(\cdot) \right\| \left\| \hat{\psi} - \psi \right\|_{\infty, v} dy_t \left| f_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right| \\
&\quad + \left| \beta_l \int l_t \hat{f}_{l_t|k_t, k_{t-1}, m_{t-1, i}, l_{t-1, i}} dl_t \right\| \left\| \nu^{-1}(k_t, k_{t-1}, \mathbf{z}_t^*) \right\| \left\| \hat{f}_2^{\frac{1}{2}} - f_2^{\frac{1}{2}} \right\|_{\infty, v} \\
&\quad + |\beta_l| \int \left| l_t \frac{df_{l_t|k_t, k_{t-1}, m_{t-1, i}, l_{t-1, i}; \psi}}{d\psi} \nu^{-1}(\cdot) \right\| \left\| \hat{\psi} - \psi \right\|_{\infty, v} dl_t \left| f_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right| \\
&\quad + \left| \beta_m \int m_t \hat{f}_{m_t|k_t, k_{t-1}, m_{t-1, i}, l_{t-1, i}} dm_t \right\| \left\| \nu^{-1}(k_t, k_{t-1}, \mathbf{z}_t^*) \right\| \left\| \hat{f}_2^{\frac{1}{2}} - f_2^{\frac{1}{2}} \right\|_{\infty, v} \\
&\quad + |\beta_m| \int \left| m_t \frac{df_{m_t|k_t, k_{t-1}, m_{t-1, i}, l_{t-1, i}; \psi}}{d\psi} \nu^{-1}(\cdot) \right\| \left\| \hat{\psi} - \psi \right\|_{\infty, v} dm_t \left| f_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right| \\
&\quad + |\beta_k k_t + q(\mathbf{c}(m_{t-1, i}, k_{t-1})' \beta_\omega)| \left\| \nu^{-1}(k_t, k_{t-1}, \mathbf{z}_t^*) \right\| \left\| \hat{f}_2^{\frac{1}{2}} - f_2^{\frac{1}{2}} \right\|_{\infty, v} \Bigg)^2 \Bigg] \\
&\leq B_{1n} E \left\| \hat{\alpha} - \alpha \right\|_{s, \alpha}^2 = o(1),
\end{aligned}$$

because

$$\begin{aligned}
&B_{1n} \\
&\equiv \left[\frac{1}{n} \sum_{i=1}^n E \left\{ \left(\int \left| y_t \hat{f}_{y_t|k_t, k_{t-1}, \mathbf{s}_{ti}} \right| dy_t \right) \nu^{-1}(k_t, k_{t-1}, \mathbf{z}_t^*) \right\} + \int \left| y_t \frac{df_{y_t|k_t, k_{t-1}, \mathbf{s}_{ti}; \psi}}{d\psi} \nu^{-1}(\cdot) \right| dy_t \left| f_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right| \right. \\
&\quad + \left| \beta_l l_{ti} + \beta_k k_t + \beta_m m_{ti} + \mathbf{c}(m_{ti}, k_t)' \beta_\omega \right| \left\| \nu^{-1}(k_t, k_{t-1}, \mathbf{z}_t^*) \right\| \Bigg)^2 \\
&\quad + \left(\left| \int y_t \hat{f}_{y_t|k_t, k_{t-1}, m_{t-1, i}, l_{t-1, i}} dy_t \right| \left\| \nu^{-1}(k_t, k_{t-1}, \mathbf{z}_t^*) \right\| + \int \left| y_t \frac{df_{y_t|k_t, k_{t-1}, m_{t-1, i}, l_{t-1, i}; \psi}}{d\psi} \nu^{-1}(\cdot) \right| dy_t \left| f_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right| \right. \\
&\quad + \left| \beta_l \int l_t \hat{f}_{l_t|k_t, k_{t-1}, m_{t-1, i}, l_{t-1, i}} dl_t \right\| \left\| \nu^{-1}(k_t, k_{t-1}, \mathbf{z}_t^*) \right\| + |\beta_l| \int \left| l_t \frac{df_{l_t|k_t, k_{t-1}, m_{t-1, i}, l_{t-1, i}; \psi}}{d\psi} \nu^{-1}(\cdot) \right| dl_t \left| f_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right| \\
&\quad + \left| \beta_m \int m_t \hat{f}_{m_t|k_t, k_{t-1}, m_{t-1, i}, l_{t-1, i}} dm_t \right\| \left\| \nu^{-1}(k_t, k_{t-1}, \mathbf{z}_t^*) \right\| \\
&\quad + |\beta_m| \int \left| m_t \frac{df_{m_t|k_t, k_{t-1}, m_{t-1, i}, l_{t-1, i}; \psi}}{d\psi} \nu^{-1}(\cdot) \right| dm_t \left| f_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right| \\
&\quad + \left| \beta_k k_t + q(\mathbf{c}(m_{t-1, i}, k_{t-1})' \beta_\omega) \right\| \left\| \nu^{-1}(k_t, k_{t-1}, \mathbf{z}_t^*) \right\| \Bigg)^2 \Bigg] = O(1)
\end{aligned}$$

and because $\|\hat{\alpha} - \alpha\|_{s,\alpha}^2 = o_p(1)$.

We therefore obtain $\sum_{i=1}^n \left\| \hat{g}(\mathbf{x}_{ti}; \theta) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} - g(\mathbf{x}_{ti}; \theta) f_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right\|_E^2 / n = o_p(1)$ by the Markov inequality. Note that $Q_n(\theta) = Q(\theta) + o_p(1)$ by the weak law of large numbers. Therefore, we have verified Lemma A.2 Condition (ii) of Newey and Powell (2003) by the triangle inequality.

Next we verify Condition (iii) of their Lemma A.2. Since $\|\cdot\|_{s,\theta}^\nu$ is bounded on $\Theta \times \Theta$ by the compactness of the parameter space, there are constants $C, \tilde{C}$ such that

$$\begin{aligned}
& \left| \hat{Q}_n(\tilde{\theta}) - \hat{Q}_n(\theta) \right| \\
& \leq \frac{C}{n} \cdot \left| \sum_{i=1}^n \int \left(\left\| \hat{g}(\mathbf{x}_{ti}; \tilde{\theta}) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right\|_E^2 - \left\| \hat{g}(\mathbf{x}_{ti}; \theta) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right\|_E^2 \right) dk_t dk_{t-1} \right| \\
& \leq \frac{C}{n} \cdot \left(\sum_{i=1}^n \int \left(\left\| \hat{g}(\mathbf{x}_{ti}; \tilde{\theta}) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} - \hat{g}(\mathbf{x}_{ti}; \theta) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right\|_E^2 \right) dk_t dk_{t-1} \right. \\
& \quad + 2 \sum_{i=1}^n \int \left\| \hat{g}(\mathbf{x}_{ti}; \theta) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right\|_E dk_t dk_{t-1} \\
& \quad \times \sum_{i=1}^n \int \left(\left\| \hat{g}(\mathbf{x}_{ti}; \tilde{\theta}) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} - \hat{g}(\mathbf{x}_{ti}; \theta) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right\|_E \right) dk_t dk_{t-1} \Big) \\
& = \frac{C}{n} \cdot \left(\sum_{i=1}^n \int \left(\left\| \hat{g}(\mathbf{x}_{ti}; \tilde{\theta}) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} - \hat{g}(\mathbf{x}_{ti}; \theta) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right\|_E^2 \right) dk_t dk_{t-1} \right. \\
& \quad + 2 \sum_{i=1}^n \int \left(\left\| \hat{g}(\mathbf{x}_{ti}; \theta) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} - \hat{g}(\mathbf{x}_{ti}; \theta_0) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right\|_E \right) dk_t dk_{t-1} \\
& \quad \times \sum_{i=1}^n \int \left(\left\| \hat{g}(\mathbf{x}_{ti}; \tilde{\theta}) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} - \hat{g}(\mathbf{x}_{ti}; \theta) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right\|_E \right) dk_t dk_{t-1} \\
& \quad + 2 \sum_{i=1}^n \int \left\| \hat{g}(\mathbf{x}_{ti}; \theta_0) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right\|_E dk_t dk_{t-1} \\
& \quad \times \sum_{i=1}^n \int \left(\left\| \hat{g}(\mathbf{x}_{ti}; \tilde{\theta}) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} - \hat{g}(\mathbf{x}_{ti}; \theta) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right\|_E \right) dk_t dk_{t-1} \Big)
\end{aligned}$$

$$\begin{aligned}
&= \frac{C}{n} \left\{ \int \sum_{i=1}^n \left[\left(\hat{g}_1(\mathbf{x}_{ti}; \tilde{\beta}) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} - \hat{g}_1(\mathbf{x}_{ti}; \beta) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right)^2 \right. \right. \\
&\quad + \left. \left(\hat{g}_2(\tilde{\mathbf{x}}_{ti}; \tilde{\theta}) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} - g_2(\tilde{\mathbf{x}}_{ti}; \theta) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right)^2 \right] dk_t dk_{t-1} \\
&\quad + 2 \int \left\{ \sum_{i=1}^n \left[\left(\hat{g}_1(\mathbf{x}_{ti}; \beta) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} - \hat{g}_1(\mathbf{x}_{ti}; \beta_0) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right)^2 \right. \right. \\
&\quad + \left. \left(\hat{g}_2(\tilde{\mathbf{x}}_{ti}; \theta) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} - g_2(\tilde{\mathbf{x}}_{ti}; \theta_0) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right)^2 \right] \Big\}^{1/2} dk_t dk_{t-1} \\
&\quad \cdot \int \left\{ \sum_{i=1}^n \left[\left(\hat{g}_1(\mathbf{x}_{ti}; \tilde{\beta}) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} - \hat{g}_1(\mathbf{x}_{ti}; \beta) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right)^2 \right. \right. \\
&\quad + \left. \left(\hat{g}_2(\tilde{\mathbf{x}}_{ti}; \tilde{\theta}) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} - g_2(\tilde{\mathbf{x}}_{ti}; \theta) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right)^2 \right] \Big\}^{1/2} dk_t dk_{t-1} \\
&\quad + 2 \int \left\{ \sum_{i=1}^n \left(\hat{g}_1(\mathbf{x}_{ti}; \beta_0)^2 \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}} + \hat{g}_2(\tilde{\mathbf{x}}_{ti}; \theta_0)^2 \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}} \right) \right\}^{1/2} dk_t dk_{t-1} \\
&\quad \cdot \int \left\{ \sum_{i=1}^n \left[\left(\hat{g}_1(\mathbf{x}_{ti}; \tilde{\beta}) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} - \hat{g}_1(\mathbf{x}_{ti}; \beta) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right)^2 \right. \right. \\
&\quad + \left. \left(\hat{g}_2(\tilde{\mathbf{x}}_{ti}; \tilde{\theta}) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} - g_2(\tilde{\mathbf{x}}_{ti}; \theta) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right)^2 \right] \Big\}^{1/2} dk_t dk_{t-1} \Big\} \\
\\
&= \frac{C}{n} \left\{ \int \sum_{i=1}^n \left[\left(\int (\rho_1(\mathbf{y}_t; \tilde{\beta}) - \rho_1(\mathbf{y}_t; \beta)) \hat{f}_{\mathbf{y}_t | \mathbf{x}_{ti}} d\mathbf{y}_t \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right)^2 \right. \right. \\
&\quad + \left. \left(\int (\rho_2(\mathbf{y}_{t,t-1}; \tilde{\theta}) - \rho_2(\mathbf{y}_{t,t-1}; \theta)) \hat{f}_{\mathbf{y}_{t,t-1} | \tilde{\mathbf{x}}_{ti}} d\mathbf{y}_{t,t-1} \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right)^2 \right] dk_t dk_{t-1} \\
&\quad + 2 \int \left\{ \sum_{i=1}^n \left[\left(\int (\rho_1(\mathbf{y}_t; \beta) - \rho_1(\mathbf{y}_t; \beta_0)) \hat{f}_{\mathbf{y}_t | \mathbf{x}_{ti}} d\mathbf{y}_t \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right)^2 \right. \right. \\
&\quad + \left. \left(\int (\rho_2(\mathbf{y}_{t,t-1}; \theta) - \rho_2(\mathbf{y}_{t,t-1}; \theta_0)) \hat{f}_{\mathbf{y}_{t,t-1} | \tilde{\mathbf{x}}_{ti}} d\mathbf{y}_{t,t-1} \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right)^2 \right] \Big\}^{1/2} dk_t dk_{t-1} \\
&\quad \cdot \int \left\{ \sum_{i=1}^n \left[\left(\int (\rho_1(\mathbf{y}_t; \tilde{\beta}) - \rho_1(\mathbf{y}_t; \beta)) \hat{f}_{\mathbf{y}_t | \mathbf{x}_{ti}} d\mathbf{y}_t \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right)^2 \right. \right. \\
&\quad + \left. \left(\int (\rho_2(\mathbf{y}_{t,t-1}; \tilde{\theta}) - \rho_2(\mathbf{y}_{t,t-1}; \theta)) \hat{f}_{\mathbf{y}_{t,t-1} | \tilde{\mathbf{x}}_{ti}} d\mathbf{y}_{t,t-1} \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}}^{\frac{1}{2}} \right)^2 \right] \Big\}^{1/2} dk_t dk_{t-1} \Big\}
\end{aligned}$$

$$\begin{aligned}
& +2 \int \left\{ \sum_{i=1}^n \left[\left(\int (\rho_1(\mathbf{y}_t; \beta_0)) \hat{f}_{\mathbf{y}_t|\mathbf{x}_{ti}} d\mathbf{y}_t \right)^2 \hat{f}_{k_t, k_{t-1}|\mathbf{s}_{ti}} \right. \right. \\
& + \left. \left(\int \rho_2(\mathbf{y}_{t,t-1}; \theta_0) \hat{f}_{\mathbf{y}_{t,t-1}|\tilde{\mathbf{x}}_{ti}} d\mathbf{y}_{t,t-1} \right)^2 \hat{f}_{k_t, k_{t-1}|\mathbf{s}_{ti}} \right] \Big\}^{1/2} dk_t dk_{t-1} \\
& \cdot \int \left\{ \sum_{i=1}^n \left[\left(\int (\rho_1(\mathbf{y}_t; \tilde{\beta}) - \rho_1(\mathbf{y}_t; \beta)) \hat{f}_{\mathbf{y}_t|\mathbf{x}_{ti}} d\mathbf{y}_t \hat{f}_{k_t, k_{t-1}|\mathbf{s}_{ti}}^{\frac{1}{2}} \right)^2 \right. \right. \\
& + \left. \left(\int (\rho_2(\mathbf{y}_{t,t-1}; \tilde{\theta}) - \rho_2(\mathbf{y}_{t,t-1}; \theta)) \hat{f}_{\mathbf{y}_{t,t-1}|\tilde{\mathbf{x}}_{ti}} d\mathbf{y}_{t,t-1} \hat{f}_{k_t, k_{t-1}|\mathbf{s}_{ti}}^{\frac{1}{2}} \right)^2 \right] \Big\}^{1/2} dk_t dk_{t-1} \\
& \leq \frac{C}{n} \left\{ \int \sum_{i=1}^n \left(\int b_1(\mathbf{y}_t) \hat{f}_{\mathbf{y}_t|\mathbf{x}_{ti}} d\mathbf{y}_t \hat{f}_{k_t, k_{t-1}|\mathbf{s}_{ti}}^{\frac{1}{2}} \right)^2 \cdot \|\tilde{\theta} - \theta\|_{s, \theta}^{2\nu} dk_t dk_{t-1} \right. \\
& + \int \sum_{i=1}^n \left(\int b_2(\mathbf{y}_{t,t-1}) \hat{f}_{\mathbf{y}_{t,t-1}|\tilde{\mathbf{x}}_{ti}} d\mathbf{y}_{t,t-1} \hat{f}_{k_t, k_{t-1}|\mathbf{s}_{ti}}^{\frac{1}{2}} \right)^2 \cdot \|\tilde{\theta} - \theta\|_{s, \theta}^{2\nu} dk_t dk_{t-1} \\
& + 2 \int \left\{ \sum_{i=1}^n \left[\left(\int b_1(\mathbf{y}_t) \hat{f}_{\mathbf{y}_t|\mathbf{x}_{ti}} d\mathbf{y}_t \hat{f}_{k_t, k_{t-1}|\mathbf{s}_{ti}}^{\frac{1}{2}} \right)^2 + \left(\int b_2(\mathbf{y}_{t,t-1}) \hat{f}_{\mathbf{y}_{t,t-1}|\tilde{\mathbf{x}}_{ti}} d\mathbf{y}_{t,t-1} \hat{f}_{k_t, k_{t-1}|\mathbf{s}_{ti}}^{\frac{1}{2}} \right)^2 \right] \right\}^{1/2} \\
& \cdot \|\theta - \theta_0\|_{s, \theta}^\nu dk_t dk_{t-1} \\
& \cdot \int \left\{ \sum_{i=1}^n \left[\left(\int b_1(\mathbf{y}_t) \hat{f}_{\mathbf{y}_t|\mathbf{x}_{ti}} d\mathbf{y}_t \hat{f}_{k_t, k_{t-1}|\mathbf{s}_{ti}}^{\frac{1}{2}} \right)^2 + \left(\int b_2(\mathbf{y}_{t,t-1}) \hat{f}_{\mathbf{y}_{t,t-1}|\tilde{\mathbf{x}}_{ti}} d\mathbf{y}_{t,t-1} \hat{f}_{k_t, k_{t-1}|\mathbf{s}_{ti}}^{\frac{1}{2}} \right)^2 \right] \right\}^{1/2} \\
& \cdot \|\tilde{\theta} - \theta\|_{s, \theta}^\nu dk_t dk_{t-1} \\
& + 2 \int \left\{ \sum_{i=1}^n \left[\left(\int (\rho_1(\mathbf{y}_t; \beta_0)) \hat{f}_{\mathbf{y}_t|\mathbf{x}_{ti}} d\mathbf{y}_t \right)^2 \right. \right. \\
& + \left. \left(\int \rho_2(\mathbf{y}_{t,t-1}; \theta_0) \hat{f}_{\mathbf{y}_{t,t-1}|\tilde{\mathbf{x}}_{ti}} d\mathbf{y}_{t,t-1} \right)^2 \right] \Big\}^{1/2} \hat{f}_{k_t, k_{t-1}|\mathbf{s}_{ti}}^{\frac{1}{2}} dk_t dk_{t-1} \\
& \cdot \int \left\{ \sum_{i=1}^n \left[\left(\int b_1(\mathbf{y}_t) \hat{f}_{\mathbf{y}_t|\mathbf{x}_{ti}} d\mathbf{y}_t \hat{f}_{k_t, k_{t-1}|\mathbf{s}_{ti}}^{\frac{1}{2}} \right)^2 + \left(\int b_2(\mathbf{y}_{t,t-1}) \hat{f}_{\mathbf{y}_{t,t-1}|\tilde{\mathbf{x}}_{ti}} d\mathbf{y}_{t,t-1} \hat{f}_{k_t, k_{t-1}|\mathbf{s}_{ti}}^{\frac{1}{2}} \right)^2 \right] \right\}^{1/2} \\
& \cdot \|\tilde{\theta} - \theta\|_{s, \theta}^\nu dk_t dk_{t-1} \\
& \leq B_{2n} \|\tilde{\theta} - \theta\|_{s, \theta}^\nu,
\end{aligned}$$

where $B_{2n} \equiv \tilde{C}\tilde{B}_n$ and by Assumption 5.7, we have

$$\begin{aligned}
& \tilde{B}_n \\
& \equiv \frac{1}{n} \left\{ \int \sum_{i=1}^n \left[\left(\int b_1(\mathbf{y}_t) \hat{f}_{\mathbf{y}_t|\mathbf{x}_{ti}} d\mathbf{y}_t \hat{f}_{k_t, k_{t-1}|\mathbf{s}_{ti}}^{\frac{1}{2}} \right)^2 + \left(\int b_2(\mathbf{y}_{t,t-1}) \hat{f}_{\mathbf{y}_{t,t-1}|\tilde{\mathbf{x}}_{ti}} d\mathbf{y}_{t,t-1} \hat{f}_{k_t, k_{t-1}|\mathbf{s}_{ti}}^{\frac{1}{2}} \right)^2 \right] dk_t dk_{t-1} \right. \\
& \quad + 2 \int \left\{ \sum_{i=1}^n \left[\left(\int (\rho_1(\mathbf{y}_t; \beta_0)) \hat{f}_{\mathbf{y}_t|\mathbf{x}_{ti}} d\mathbf{y}_t \right)^2 \right. \right. \\
& \quad \left. \left. + \left(\int \rho_2(\mathbf{y}_{t,t-1}; \theta_0) \hat{f}_{\mathbf{y}_{t,t-1}|\tilde{\mathbf{x}}_{ti}} d\mathbf{y}_{t,t-1} \right)^2 \right] \right\}^{\frac{1}{2}} \hat{f}_{k_t, k_{t-1}|\mathbf{s}_{ti}}^{\frac{1}{2}} dk_t dk_{t-1} \\
& \quad \cdot \int \left\{ \sum_{i=1}^n \left[\left(\int b_1(\mathbf{y}_t) \hat{f}_{\mathbf{y}_t|\mathbf{x}_{ti}} d\mathbf{y}_t \hat{f}_{k_t, k_{t-1}|\mathbf{s}_{ti}}^{\frac{1}{2}} \right)^2 + \left(\int b_2(\mathbf{y}_{t,t-1}) \hat{f}_{\mathbf{y}_{t,t-1}|\tilde{\mathbf{x}}_{ti}} d\mathbf{y}_{t,t-1} \hat{f}_{k_t, k_{t-1}|\mathbf{s}_{ti}}^{\frac{1}{2}} \right)^2 \right] \right\}^{\frac{1}{2}} dk_t dk_{t-1} \\
& = O_p(1).
\end{aligned}$$

Therefore, we satisfy Condition (iii) of Lemma A.2. We have verified all three conditions in Lemma A.1 of Newey and Powell (2003) and hence the result follows.

C.2 Convergence Rate: Proof of Theorem 5.21

The first result (Theorem 5.21.1) follows from a similar argument to Theorem 2 in Hu and Schennach (2008). The second result (Theorem 5.21.2) follows from a similar argument to Ai and Chen (2003) and Song (2012)'s convergence rate results. First we show two lemmas: Lemma C.1 and Lemma C.2 below, which are useful to prove Theorem 5.21.

Lemma C.1. *Suppose Assumptions for Theorem 5.10 hold. Then (a) under Assumptions 5.4, 5.7, 5.11, 5.12, 5.13, 5.16, 5.17, 5.18, 5.20, $\sum_{i=1}^n \left[\int \|\hat{g}(\mathbf{x}_{ti}; \theta) \hat{f}_{k_t, k_{t-1}|\mathbf{s}_{ti}}^{\frac{1}{2}} - g(\mathbf{x}_{ti}; \theta) f_{k_t, k_{t-1}|\mathbf{s}_{ti}}^{\frac{1}{2}}\|_E dk_t dk_{t-1} \right]^2 / n = o_p(n^{-1/2})$ uniformly over $\theta \in \Theta$; (b) Under Assumptions 5.5 and 5.7, $\sum_{i=1}^n \int \|\hat{g}(\mathbf{x}_{ti}; \theta_0) \hat{f}_{k_t, k_{t-1}|\mathbf{s}_{ti}}^{\frac{1}{2}}\|_E^2 dk_t dk_{t-1} / n = O_p(\kappa_{1n}/n)$.*

Proof. (a) Since $\|\hat{\alpha} - \alpha_0\|_\alpha = o_p(n^{-1/4})$ by a similar argument to Theorem 2 in Hu and Schennach (2008), $\|\hat{f}_{k_t, k_{t-1}|\mathbf{s}_t} - f_{k_t, k_{t-1}|\mathbf{s}_t}\|_\alpha^2 = o_p(n^{-1/2})$. From

$$\begin{aligned}
& \sum_{i=1}^n \left[\int \|\hat{g}(\mathbf{x}_{ti}; \theta) \hat{f}_{k_t, k_{t-1}|\mathbf{s}_{ti}}^{\frac{1}{2}} - g(\mathbf{x}_{ti}; \theta) f_{k_t, k_{t-1}|\mathbf{s}_{ti}}^{\frac{1}{2}}\|_E dk_t dk_{t-1} \right]^2 / n \\
& \leq C \cdot \left(\sum_{i=1}^n \left[\int \|\hat{g}(\mathbf{x}_{ti}; \theta) - g(\mathbf{x}_{ti}; \theta)\|_E dk_t dk_{t-1} \right]^2 + \|\hat{f}_{k_t, k_{t-1}|\mathbf{s}_{ti}} - f_{k_t, k_{t-1}|\mathbf{s}_{ti}}\|_\alpha^2 \right) / n + o_p(n^{-1/2}),
\end{aligned}$$

for a generic constant C , it suffices to show $\sum_{i=1}^n \left[\int \|\hat{g}(\mathbf{x}_{ti}; \theta) - g(\mathbf{x}_{ti}; \theta)\|_E dk_t dk_{t-1} \right]^2 / n = o_p(n^{-1/2})$ uniformly over $\theta \in \Theta$. We note that from Corollary 4.4, equation (12), and Assumption 5.20, $\|\hat{f}_{y_t|k_t, k_{t-1}, m_{t-1}, l_{t-1}} - f_{y_t|k_t, k_{t-1}, m_{t-1}, l_{t-1}}\|_\alpha = C \|\hat{f}_{y_t|k_t, k_{t-1}, \mathbf{s}_t} - f_{y_t|k_t, k_{t-1}, \mathbf{s}_t}\|_\alpha + o_p(n^{-1/4})$ for a generic

constant C . We also note that $\|\hat{f}_{r_t|k_t, k_{t-1}, m_{t-1}, l_{t-1}} - f_{r_t|k_t, k_{t-1}, m_{t-1}, l_{t-1}}\|_\alpha \asymp \|\hat{f}_{y_t|k_t, k_{t-1}, m_{t-1}, l_{t-1}} - f_{y_t|k_t, k_{t-1}, m_{t-1}, l_{t-1}}\|_\alpha$ for $r_t \in \{l_t, m_t\}$. It follows that

$$\begin{aligned}
& E \sum_{i=1}^n \left[\int \|\hat{g}(\mathbf{x}_{ti}; \theta) - g(\mathbf{x}_{ti}; \theta)\|_E dk_t dk_{t-1} \right]^2 / n \\
&= \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^2 E \left(\int \left\{ \int \rho_j(\mathbf{y}_{t,t-1}; \theta) \hat{f}_{\mathbf{y}_{t,t-1}|\mathbf{x}_{ti}} d\mathbf{y}_{t,t-1} - \int \rho_j(\mathbf{y}_{t,t-1}; \theta) f_{\mathbf{y}_{t,t-1}|\mathbf{x}_{ti}} d\mathbf{y}_{t,t-1} \right\} dk_t dk_{t-1} \right)^2 \\
&= \frac{1}{n} \sum_{i=1}^n E \left[\left(\int \left\{ \left(\int y_t \hat{f}_{y_t|k_t, k_{t-1}, \mathbf{s}_{ti}} dy_t - \beta_l l_{ti} - \beta_k k_t - \beta_m m_{ti} - \mathbf{c}(m_{ti}, k_t)' \beta_\omega \right) \right. \right. \right. \\
&\quad \left. \left. \left. - \left(\int y_t f_{y_t|k_t, k_{t-1}, \mathbf{s}_{ti}} dy_t - \beta_l l_{ti} - \beta_k k_t - \beta_m m_{ti} - \mathbf{c}(m_{ti}, k_t)' \beta_\omega \right) \right\} dk_t dk_{t-1} \right)^2 \right. \\
&\quad \left. + \left(\int \left\{ \left(\int y_t \hat{f}_{y_t|k_t, k_{t-1}, m_{t-1}, i, l_{t-1}, i} dy_t - \beta_l \int l_t \hat{f}_{l_t|k_t, k_{t-1}, m_{t-1}, i, l_{t-1}, i} dl_t - \beta_k k_t \right. \right. \right. \right. \\
&\quad \left. \left. \left. - \beta_m \int m_t \hat{f}_{m_t|k_t, k_{t-1}, m_{t-1}, i, l_{t-1}, i} dm_t - q(\mathbf{c}(m_{t-1}, i, k_{t-1})' \beta_\omega) \right) \right. \right. \\
&\quad \left. \left. - \left(\int y_t f_{y_t|k_t, k_{t-1}, m_{t-1}, i, l_{t-1}, i} dy_t - \beta_l \int l_t f_{l_t|k_t, k_{t-1}, m_{t-1}, i, l_{t-1}, i} dl_t - \beta_k k_t \right. \right. \right. \\
&\quad \left. \left. \left. - \beta_m \int m_t f_{m_t|k_t, k_{t-1}, m_{t-1}, i, l_{t-1}, i} dm_t - q(\mathbf{c}(m_{t-1}, i, k_{t-1})' \beta_\omega) \right) \right\} dk_t dk_{t-1} \right)^2 \Big] \\
&= \frac{1}{n} \sum_{i=1}^n E \left\{ \left(\int y_t \left(\hat{f}_{y_t|k_t, k_{t-1}, \mathbf{s}_{ti}} - f_{y_t|k_t, k_{t-1}, \mathbf{s}_{ti}} \right) dy_t dk_t dk_{t-1} \right)^2 \right. \\
&\quad \left. + \left(\int y_t \left(\hat{f}_{y_t|k_t, k_{t-1}, m_{t-1}, i, l_{t-1}, i} - f_{y_t|k_t, k_{t-1}, m_{t-1}, i, l_{t-1}, i} \right) dy_t dk_t dk_{t-1} \right. \right. \\
&\quad \left. \left. - \beta_l \int l_t \left(\hat{f}_{l_t|k_t, k_{t-1}, m_{t-1}, i, l_{t-1}, i} - f_{l_t|k_t, k_{t-1}, m_{t-1}, i, l_{t-1}, i} \right) dl_t dk_t dk_{t-1} \right. \right. \\
&\quad \left. \left. - \beta_m \int m_t \left(\hat{f}_{m_t|k_t, k_{t-1}, m_{t-1}, i, l_{t-1}, i} - f_{m_t|k_t, k_{t-1}, m_{t-1}, i, l_{t-1}, i} \right) dm_t dk_t dk_{t-1} \right)^2 \right\} \\
&\leq \frac{C}{n} \sum_{i=1}^n E \left\{ \left(\int y_t \left(\hat{f}_{y_t|k_t, k_{t-1}, \mathbf{s}_{ti}} - f_{y_t|k_t, k_{t-1}, \mathbf{s}_{ti}} \right) dy_t dk_t dk_{t-1} \right)^2 \right\} + o_p(n^{-1/2}) \\
&= \frac{C}{n} \sum_{i=1}^n E \left\{ \left(\int y_t \left(f_{k_t^*, k_{t-1}^* | \mathbf{z}_{t,i}} f_{k_t, k_{t-1} | \mathbf{z}_{t,i}^*} \right)^{-1} \right. \right. \\
&\quad \left. \left. \cdot \frac{d}{d\psi} f_{y_t | \mathbf{z}_{t,i}; \psi_0} [\hat{\psi} - \psi_0] f_{k_t^*, k_{t-1}^* | \mathbf{z}_{t,i}} f_{k_t, k_{t-1} | \mathbf{z}_{t,i}^*} dy_t dk_t dk_{t-1} \right)^2 \right\} + o_p(n^{-1/2}) \\
&\leq \frac{C}{n} \sum_{i=1}^n E \left\{ \int y_t \left(f_{k_t^*, k_{t-1}^* | \mathbf{z}_{t,i}} f_{k_t, k_{t-1} | \mathbf{z}_{t,i}^*} \right)^{-1} \right. \\
&\quad \left. \cdot \left(\int \frac{d}{d\psi} f_{y_t | \mathbf{z}_{t,i}; \psi_0} [\hat{\psi} - \psi_0] f_{k_t^*, k_{t-1}^* | \mathbf{z}_{t,i}} f_{k_t, k_{t-1} | \mathbf{z}_{t,i}^*} dk_t dk_{t-1} \right)^2 dy_t \right\} + o_p(n^{-1/2})
\end{aligned}$$

$$\begin{aligned}
&= \frac{C}{n} \sum_{i=1}^n \int \sup_{\{k_t^*, k_{t-1}^*, \mathbf{z}_t^*, \mathbf{z}_{t-1}^*\}} \left\{ y_t \left(f_{k_t^*, k_{t-1}^* | \mathbf{z}_{t,i}} f_{k_t, k_{t-1} | \mathbf{z}_{t,i}^*} \right)^{-1} f_{y_t, k_t^*, k_{t-1}^* | \mathbf{z}_{t,i}^*; \alpha_0} \right\} \\
&\quad \cdot E \left\{ \left(\frac{1}{f_{y_t, k_t^*, k_{t-1}^* | \mathbf{z}_{t,i}^*; \alpha_0}} \int \frac{d}{d\psi} f_{y_t | \mathbf{z}_{t,i}; \psi} [\hat{\psi} - \psi_0] f_{k_t^*, k_{t-1}^* | \mathbf{z}_{t,i}} f_{k_t, k_{t-1} | \mathbf{z}_{t,i}^*} dk_t dk_{t-1} \right)^2 \right\} dy_t + o_p(n^{-1/2}) \\
&= \frac{C}{n} \sum_{i=1}^n \int \sup_{\{k_t^*, k_{t-1}^*, \mathbf{z}_t^*, \mathbf{z}_{t-1}^*\}} \left\{ y_t \left(f_{k_t^*, k_{t-1}^* | \mathbf{z}_{t,i}} f_{k_t, k_{t-1} | \mathbf{z}_{t,i}^*} \right)^{-1} f_{y_t, k_t^*, k_{t-1}^* | \mathbf{z}_{t,i}^*; \alpha_0} \right\} \\
&\quad \cdot E \left\{ \left(\frac{d \ln f_{y_t, k_t^*, k_{t-1}^* | \mathbf{z}_{t,i}^*; \alpha_0}}{d\alpha} [\hat{\alpha} - \alpha_0] \right)^2 \right\} dy_t + o_p(n^{-1/2}) \\
&\leq \frac{C}{n} \sum_{i=1}^n \int \sup_{\{k_t^*, k_{t-1}^*, \mathbf{z}_t^*, \mathbf{z}_{t-1}^*\}} \left\{ y_t \left(f_{k_t^*, k_{t-1}^* | \mathbf{z}_{t,i}} f_{k_t, k_{t-1} | \mathbf{z}_{t,i}^*} \right)^{-1} f_{y_t, k_t^*, k_{t-1}^* | \mathbf{z}_{t,i}^*; \alpha_0} \right\} dy_t \|\hat{\alpha} - \alpha_0\|_\alpha^2 + o_p(n^{-1/2}) \\
&= o_p(n^{-1/2}),
\end{aligned}$$

where the last result holds since $\int \sup_{\{k_t^*, k_{t-1}^*, \mathbf{z}_t^*, \mathbf{z}_{t-1}^*\}} \left\{ y_t (f_{k_t^*, k_{t-1}^* | \mathbf{z}_{t,i}} f_{k_t, k_{t-1} | \mathbf{z}_{t,i}^*})^{-1} f_{y_t, k_t^*, k_{t-1}^* | \mathbf{z}_{t,i}^*; \alpha_0} \right\} dy_t = O(1)$. Thus the result follows by the Markov inequality.

(b) The result follows from Corollary A.1 (ii) in Ai and Chen (2003). $\square$

Lemma C.2. (a) Under Assumptions 5.1, 5.3, 5.5, 5.6.2, 5.11, 5.13.1, and 5.19, we obtain $\sum_{i=1}^n \|g(\mathbf{x}_{ti}; \theta)\|_E^2/n - E[\|g(\mathbf{x}_{ti}; \theta)\|_E^2] = o_p(n^{-1/2})$ uniformly over $\theta \in \{\Theta_n : \|\theta - \theta_0\|_\theta = o(1)\}$; (b) Under Assumptions 5.1, 5.3, 5.5, 5.6, 5.7, 5.9, 5.11, 5.13, 5.15, 5.18, 5.19, we obtain $\sum_{i=1}^n \|\hat{g}(\mathbf{x}_{ti}; \theta)\|_E^2/n = o_p(\eta_n^2)$ and $\sum_{i=1}^n \|g(\mathbf{x}_{ti}; \theta)\|_E^2/n = o_p(\eta_n^2)$ uniformly over $\theta \in \{\Theta_n : \|\theta - \theta_0\|_\theta = o(\eta_n)\}$, where $\eta_n = n^{-\tau}$ with $\tau \leq 1/4$.

Proof. (a) The result follows from Corollary A.2 (i) in Ai and Chen (2003).

(b) Note that $E[\|g(\mathbf{x}_{ti}; \theta)\|_E^2] = o(\eta_n^2)$ by Assumptions 5.6.2 and 5.19. Then the result follows from applying Lemma C.1 (a) and Lemma C.2 (a). $\square$

Next we prove Theorem 5.21.

Proof. 1. The result follows from a similar argument to the proof of Theorem 2 in Hu and Schennach (2008).

2. From Lemma C.1 (a) and Assumption 5.6, we obtain $\hat{Q}_n(\theta) - Q_n(\theta) = o_p(n^{-1/4})$ uniformly over $\theta \in \Theta_n$. In addition, from Lemma C.1 and Lemma C.2 (b), we obtain $\hat{Q}_n(\theta) - \hat{Q}_n(\theta_0) - \{Q_n(\theta) - Q_n(\theta_0)\} = o_p(\eta_n n^{-1/4})$ uniformly over $\theta \in \{\Theta_n : \|\theta - \theta_0\|_\theta = o(\eta_n)\}$, where $\eta_n = n^{-\tau}$ with $\tau \leq 1/4$. Then the result follows by a similar argument to the proof of Theorem 3.1 in Ai and Chen (2003). $\square$

C.3 Asymptotic Normality: Proof of Theorem 6.12

The first result (Theorem 6.12.1) follows from a similar argument to Theorem 3 in Hu and Schennach (2008). The second result (Theorem 6.12.2) follows from a similar argument to Ai and Chen (2003) and Song (2012)'s asymptotic normality results. First we show two lemmas: Lemma C.3 and Lemma C.4 below, which are useful to prove Theorem 6.12.

Define

$$\begin{aligned}\frac{dQ_n(\theta)}{d\theta}[v_{2n}^*] &= \frac{1}{n} \sum_{i=1}^n \int \left\{ \frac{dg(\mathbf{x}_{ti}; \theta)}{d\theta}[v_{2n}^*] \right\}' A(\mathbf{x}_{ti}) g(\mathbf{x}_{ti}; \theta) f_{k_t, k_{t-1} | \mathbf{s}_{ti}} dk_t dk_{t-1}, \\ \frac{d\hat{Q}_n(\theta)}{d\theta}[v_{2n}^*] &= \frac{1}{n} \sum_{i=1}^n \int \left\{ \frac{d\hat{g}(\mathbf{x}_{ti}; \theta)}{d\theta}[v_{2n}^*] \right\}' \hat{A}(\mathbf{x}_{ti}) \hat{g}(\mathbf{x}_{ti}; \theta) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}} dk_t dk_{t-1},\end{aligned}$$

and similarly define

$$\begin{aligned}\frac{d^2 Q_{1n}(\theta)}{d\theta d\theta}[v_{2n}^*, v_{2n}^*] &= \frac{1}{n} \sum_{i=1}^n \int \left\{ \frac{dg(\mathbf{x}_{ti}; \theta)}{d\theta}[v_{2n}^*] \right\}' A(\mathbf{x}_{ti}) \left\{ \frac{dg(\mathbf{x}_{ti}; \theta)}{d\theta}[v_{2n}^*] \right\} f_{k_t, k_{t-1} | \mathbf{s}_{ti}} dk_t dk_{t-1}, \\ \frac{d^2 \hat{Q}_{1n}(\theta)}{d\theta d\theta}[v_{2n}^*, v_{2n}^*] &= \frac{1}{n} \sum_{i=1}^n \int \left\{ \frac{d\hat{g}(\mathbf{x}_{ti}; \theta)}{d\theta}[v_{2n}^*] \right\}' \hat{A}(\mathbf{x}_{ti}) \left\{ \frac{d\hat{g}(\mathbf{x}_{ti}; \theta)}{d\theta}[v_{2n}^*] \right\} \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}} dk_t dk_{t-1}, \\ \frac{d^2 Q_{2n}(\theta)}{d\theta d\theta}[v_{2n}^*, v_{2n}^*] &= \frac{1}{n} \sum_{i=1}^n \int \left\{ \frac{d^2 g(\mathbf{x}_{ti}; \theta)}{d\theta d\theta}[v_{2n}^*, v_{2n}^*] \right\}' A(\mathbf{x}_{ti}) g(\mathbf{x}_{ti}; \theta) f_{k_t, k_{t-1} | \mathbf{s}_{ti}} dk_t dk_{t-1}, \\ \frac{d^2 \hat{Q}_{2n}(\theta)}{d\theta d\theta}[v_{2n}^*, v_{2n}^*] &= \frac{1}{n} \sum_{i=1}^n \int \left\{ \frac{d^2 \hat{g}(\mathbf{x}_{ti}; \theta)}{d\theta d\theta}[v_{2n}^*, v_{2n}^*] \right\}' \hat{A}(\mathbf{x}_{ti}) \hat{g}(\mathbf{x}_{ti}; \theta) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}} dk_t dk_{t-1},\end{aligned}$$

where

$$\begin{aligned}\frac{dg(\mathbf{x}_t; \theta)}{d\theta}[v_{2n}^*] &= \int \frac{d\rho(\mathbf{y}_{t,t-1}; \theta)}{d\theta}[v_{2n}^*] f_{\mathbf{y}_{t,t-1} | \mathbf{x}_t} d\mathbf{y}_{t,t-1}, \\ \frac{d\hat{g}(\mathbf{x}_t; \theta)}{d\theta}[v_{2n}^*] &= \int \frac{d\rho(\mathbf{y}_{t,t-1}; \theta)}{d\theta}[v_{2n}^*] \hat{f}_{\mathbf{y}_{t,t-1} | \mathbf{x}_t} d\mathbf{y}_{t,t-1}, \\ \frac{d^2 g(\mathbf{x}_t; \theta)}{d\theta d\theta}[v_{2n}^*, v_{2n}^*] &= \int \frac{d^2 \rho(\mathbf{y}_{t,t-1}; \theta)}{d\theta d\theta}[v_{2n}^*, v_{2n}^*] f_{\mathbf{y}_{t,t-1} | \mathbf{x}_t} d\mathbf{y}_{t,t-1}, \\ \frac{d^2 \hat{g}(\mathbf{x}_t; \theta)}{d\theta d\theta}[v_{2n}^*, v_{2n}^*] &= \int \frac{d^2 \rho(\mathbf{y}_{t,t-1}; \theta)}{d\theta d\theta}[v_{2n}^*, v_{2n}^*] \hat{f}_{\mathbf{y}_{t,t-1} | \mathbf{x}_t} d\mathbf{y}_{t,t-1}.\end{aligned}$$

Lemma C.3. (a) Under Assumptions 5.1, 5.3-5.4, 5.6-5.7, 5.9, 6.6.2-6.9, we have

$$\sup_{\tilde{\theta} \in \mathcal{N}_{02n}} \frac{d^2 \hat{Q}_{1n}(\tilde{\theta})}{d\theta d\theta}[v_{2n}^*, v_{2n}^*] = \frac{d^2 Q_{1n}(\theta_0)}{d\theta d\theta}[v_{2n}^*, v_{2n}^*] + o_p(n^{-1/4}).$$

(b) Under Assumptions 5.1-5.4, 5.6-5.9, 5.11-5.20, 6.11, we have

$$\sup_{\tilde{\theta} \in \mathcal{N}_{02n}} \frac{d^2 \hat{Q}_{2n}(\tilde{\theta})}{d\theta d\theta} [v_{2n}^*, v_{2n}^*] = o_p(n^{-1/4}).$$

Proof. (a) We have that uniformly over $\tilde{\theta} \in \mathcal{N}_{02n}$,

$$\begin{aligned} & \frac{d^2 \hat{Q}_{1n}(\tilde{\theta})}{d\theta d\theta} [v_{2n}^*, v_{2n}^*] - \frac{d^2 Q_{1n}(\theta_0)}{d\theta d\theta} [v_{2n}^*, v_{2n}^*] \\ = & \frac{1}{n} \sum_{i=1}^n \int \left[\left\{ \frac{d\hat{g}(\mathbf{x}_{ti}; \theta)}{d\theta} [v_{2n}^*] - \frac{dg(\mathbf{x}_{ti}; \theta_0)}{d\theta} [v_{2n}^*] \right\}' \hat{A}(\mathbf{x}_{ti}) \left\{ \frac{d\hat{g}(\mathbf{x}_{ti}; \theta)}{d\theta} [v_{2n}^*] \right\} \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}} \right. \\ & + \left\{ \frac{dg(\mathbf{x}_{ti}; \theta_0)}{d\theta} [v_{2n}^*] \right\}' \left\{ \hat{A}(\mathbf{x}_{ti}) - A(\mathbf{x}_{ti}) \right\} \left\{ \frac{d\hat{g}(\mathbf{x}_{ti}; \theta)}{d\theta} [v_{2n}^*] \right\} \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}} \\ & + \left\{ \frac{dg(\mathbf{x}_{ti}; \theta_0)}{d\theta} [v_{2n}^*] \right\}' A(\mathbf{x}_{ti}) \left\{ \frac{d\hat{g}(\mathbf{x}_{ti}; \theta)}{d\theta} [v_{2n}^*] - \frac{dg(\mathbf{x}_{ti}; \theta_0)}{d\theta} [v_{2n}^*] \right\} \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}} \\ & \left. + \left\{ \frac{dg(\mathbf{x}_{ti}; \theta_0)}{d\theta} [v_{2n}^*] \right\}' A(\mathbf{x}_{ti}) \left\{ \frac{dg(\mathbf{x}_{ti}; \theta_0)}{d\theta} [v_{2n}^*] \right\} \left\{ \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}} - f_{k_t, k_{t-1} | \mathbf{s}_{ti}} \right\} \right] dk_t dk_{t-1} \\ \equiv & A_1 + A_2 + A_3 + A_4. \end{aligned}$$

Since

$$\begin{aligned} & E \left[\frac{1}{n} \sum_{i=1}^n \left\| \frac{d\hat{g}(\mathbf{x}_{ti}; \theta)}{d\theta} [v_{2n}^*] - \frac{dg(\mathbf{x}_{ti}; \theta_0)}{d\theta} [v_{2n}^*] \right\|_E^2 \right] \\ = & \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^2 E \left[\left(\int \frac{d\rho_j(\mathbf{y}_{t,t-1}; \theta)}{d\theta} [v_{2n}^*] \left(\hat{f}_{\mathbf{y}_{t,t-1} | \mathbf{x}_{ti}} - f_{\mathbf{y}_{t,t-1} | \mathbf{x}_{ti}} \right) d\mathbf{y}_{t,t-1} \right)^2 \right] \\ = & o(n^{-1/2}), \end{aligned}$$

uniformly over $\tilde{\theta} \in \mathcal{N}_{02n}$ from $\|\hat{\alpha} - \alpha_0\|_\alpha = o_p(n^{-1/4})$, we obtain $A_1 = A_3 = A_4 = o_p(n^{-1/4})$ by the Markov inequality, Assumption 5.6, and Lemma C.2 (b). Also we obtain $A_2 = o_p(n^{-1/4})$ from Assumption 5.15 and Lemma C.2 (b).

(b) By Assumption 5.6 and Lemma C.2 (b), we obtain that for some generic constant C ,

$$\begin{aligned} & \left| \frac{d^2 \hat{Q}_{2n}(\tilde{\theta})}{d\theta d\theta} [v_{2n}^*, v_{2n}^*] \right| \\ \leq & C \sqrt{\frac{1}{n} \sum_{i=1}^n \int \left\| \frac{d^2 \hat{g}(\mathbf{x}_{ti}; \tilde{\theta})}{d\theta d\theta} [v_{2n}^*, v_{2n}^*] \right\|_E^2 dk_t dk_{t-1}} \cdot \sqrt{\frac{1}{n} \sum_{i=1}^n \int \|\hat{g}(\mathbf{x}_{ti}; \tilde{\theta})\|_E^2 dk_t dk_{t-1}} \\ = & o_p(n^{-1/4}), \end{aligned}$$

since Assumption 6.11 implies that uniformly over $\tilde{\theta} \in \mathcal{N}_{02n}$, for a measurable function $c(\mathbf{y}_{t,t-1})$

with bounded second moment,

$$\left\| \frac{d^2 \hat{g}(\mathbf{x}_{ti}; \tilde{\theta})}{d\theta d\theta} [v_{2n}^*, v_{2n}^*] \right\|_E^2 \leq \frac{1}{n} \sum_{i=1}^n c(\mathbf{y}_{t,t-1,i})^2 = O_p(1).$$

□

Lemma C.4. (a) Under Assumptions 5.1-5.9, 6.6.2-6.9, we have uniformly over $\tilde{\theta} \in \mathcal{N}_{02n}$,

$$\begin{aligned} & \frac{d\hat{Q}_n(\tilde{\theta})}{d\theta} [v_{2n}^*] \\ &= \frac{1}{n} \sum_{i=1}^n \int \left\{ \frac{dg(\mathbf{x}_{ti}; \theta_0)}{d\theta} [v_2^*] \right\}' A(\mathbf{x}_{ti}) \hat{g}(\mathbf{x}_{ti}; \tilde{\theta}) f_{k_t, k_{t-1} | \mathbf{s}_{ti}} dk_t dk_{t-1}. \end{aligned}$$

(b) Under Assumptions 5.1, 5.3, 5.12, 6.6.2-6.9, we have uniformly over $\tilde{\theta} \in \mathcal{N}_{02n}$,

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n \int \left\{ \frac{dg(\mathbf{x}_{ti}; \theta_0)}{d\theta} [v_2^*] \right\}' A(\mathbf{x}_{ti}) \left\{ \hat{g}(\mathbf{x}_{ti}; \tilde{\theta}) - \hat{g}(\mathbf{x}_{ti}; \theta_0) \right\} f_{k_t, k_{t-1} | \mathbf{s}_{ti}} dk_t dk_{t-1} \\ &= \langle v_2^*, \tilde{\theta} - \theta_0 \rangle_\theta + o_p(n^{-1/2}). \end{aligned}$$

(c) Under Assumptions 5.1, 5.3, 5.4, 5.6.2, 5.12, 5.16, 6.6.3, we have

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n \int \left\{ \frac{dg(\mathbf{x}_{ti}; \theta_0)}{d\theta} [v_2^*] \right\}' A(\mathbf{x}_{ti}) \hat{g}(\mathbf{x}_{ti}; \theta_0) f_{k_t, k_{t-1} | \mathbf{s}_{ti}} dk_t dk_{t-1} \\ &= \frac{1}{n} \sum_{i=1}^n \int \left\{ \frac{dg(\mathbf{x}_{ti}; \theta_0)}{d\theta} [v_2^*] \right\}' A(\mathbf{x}_{ti}) \rho(\mathbf{y}_{t,t-1}; \theta_0) f_{k_t, k_{t-1} | \mathbf{s}_{ti}} dk_t dk_{t-1} + o_p(n^{-1/2}). \end{aligned}$$

Proof. (a) Note that uniformly over $\tilde{\theta} \in \mathcal{N}_{02n}$,

$$\begin{aligned} & \frac{d\hat{Q}_n(\tilde{\theta})}{d\theta} [v_{2n}^*] - \frac{1}{n} \sum_{i=1}^n \int \left\{ \frac{dg(\mathbf{x}_{ti}; \theta_0)}{d\theta} [v_2^*] \right\}' A(\mathbf{x}_{ti}) \hat{g}(\mathbf{x}_{ti}; \tilde{\theta}) f_{k_t, k_{t-1} | \mathbf{s}_{ti}} dk_t dk_{t-1} \\ &= \frac{1}{n} \sum_{i=1}^n \int \left[\left\{ \frac{d\hat{g}(\mathbf{x}_{ti}; \tilde{\theta})}{d\theta} [v_{2n}^*] - \frac{dg(\mathbf{x}_{ti}; \theta_0)}{d\theta} [v_{2n}^*] \right\}' \hat{A}(\mathbf{x}_{ti}) \hat{g}(\mathbf{x}_{ti}; \tilde{\theta}) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}} \right. \\ & \quad + \left\{ \frac{dg(\mathbf{x}_{ti}; \theta_0)}{d\theta} [v_{2n}^*] \right\}' \left\{ \hat{A}(\mathbf{x}_{ti}) - A(\mathbf{x}_{ti}) \right\} \hat{g}(\mathbf{x}_{ti}; \tilde{\theta}) \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}} \\ & \quad + \left\{ \frac{dg(\mathbf{x}_{ti}; \theta_0)}{d\theta} [v_{2n}^*] \right\}' A(\mathbf{x}_{ti}) \hat{g}(\mathbf{x}_{ti}; \tilde{\theta}) \left\{ \hat{f}_{k_t, k_{t-1} | \mathbf{s}_{ti}} - f_{k_t, k_{t-1} | \mathbf{s}_{ti}} \right\} \\ & \quad \left. + \left\{ \frac{dg(\mathbf{x}_{ti}; \theta_0)}{d\theta} [v_{2n}^* - v_2^*] \right\}' A(\mathbf{x}_{ti}) \hat{g}(\mathbf{x}_{ti}; \tilde{\theta}) f_{k_t, k_{t-1} | \mathbf{s}_{ti}} \right] dk_t dk_{t-1} \\ &\equiv D_1 + D_2 + D_3 + D_4. \end{aligned}$$

Also note that from the proof of Lemma C.3 (a)

$$\sup_{\tilde{\theta} \in \mathcal{N}_{02n}} \frac{1}{n} \sum_{i=1}^n \left\| \frac{d\hat{g}(\mathbf{x}_{ti}; \theta)}{d\theta} [v_{2n}^*] - \frac{dg(\mathbf{x}_{ti}; \theta)}{d\theta} [v_{2n}^*] \right\|_E^2 = o_p(n^{-1/2})$$

and from Corollary C.1 (ii) in Ai and Chen (2003)

$$\sup_{\tilde{\theta} \in \mathcal{N}_{02n}} \frac{1}{n} \sum_{i=1}^n \left\| \frac{d\hat{g}(\mathbf{x}_{ti}; \theta)}{d\theta} [v_{2n}^*] - \frac{dg(\mathbf{x}_{ti}; \theta)}{d\theta} [v_{2n}^*] \right\|_E^2 = o_p(n^{-1/2}).$$

Thus $D_1 = o_p(n^{-1/2})$ by Lemma C.2 (b) and Assumption 5.6. $D_2 = o_p(n^{-1/2})$ by Assumption 5.15 and Lemma C.2 (b). $D_3 = o_p(n^{-1/2})$ by $\|\hat{\alpha} - \alpha_0\|_\alpha = o_p(n^{-1/4})$ and Lemma C.2 (b). $D_4 = o_p(n^{-1/2})$ by Assumption 6.7 and Lemma C.2 (b). Thus the result follows.

(b) Define $\varphi(\mathbf{x}_t, v_2^*) = \left\{ \frac{dg(\mathbf{x}_t; \theta_0)}{d\theta} [v_2^*] \right\}' A(\mathbf{x}_t)$ and

$$\mathcal{F} = \left\{ \varphi(\mathbf{x}_t, v_2^*) \tilde{g}(\mathbf{x}_t; \theta) : \theta \in \mathcal{N}_{02n}, \tilde{g} \in \Lambda_c^\gamma(\mathcal{X}_t) \text{ s.t. } \sup_{\mathbf{x}_t \in \mathcal{X}_t, \theta \in \mathcal{N}_{02n}} |\tilde{g}(\mathbf{x}_t; \theta) - g(\mathbf{x}_t; \theta)| = o(1) \right\}.$$

By the argument of Corollary C.3 (ii) in Ai and Chen (2003), $\mathcal{F}$ is a Donsker class. We also note that $E[\|\varphi(\mathbf{x}_{ti}, v_2^*)\{\hat{g}(\mathbf{x}_{ti}; \theta) - g(\mathbf{x}_{ti}; \theta)\}\|^2] = o_p(1)$ and $\hat{g}(\mathbf{x}_{ti}; \theta) \in \Lambda_c^\gamma(\mathcal{X}_t)$ uniformly over $\theta \in \mathcal{N}_{02n}$ by Assumptions 5.1, 5.3, 5.4, 5.7, 5.18. Thus we obtain uniformly over $\theta \in \mathcal{N}_{02n}$,

$$\frac{1}{n} \sum_{i=1}^n \varphi(\mathbf{x}_{ti}, v_2^*) \{\hat{g}(\mathbf{x}_{ti}; \theta) - g(\mathbf{x}_{ti}; \theta)\} - E[\varphi(\mathbf{x}_{ti}, v_2^*) \{\hat{g}(\mathbf{x}_{ti}; \theta) - g(\mathbf{x}_{ti}; \theta)\}] = o_p(n^{-1/2}) \quad (15)$$

$$\frac{1}{n} \sum_{i=1}^n \varphi(\mathbf{x}_{ti}, v_2^*) \{\hat{g}(\mathbf{x}_{ti}; \theta_0) - g(\mathbf{x}_{ti}; \theta_0)\} - E[\varphi(\mathbf{x}_{ti}, v_2^*) \{\hat{g}(\mathbf{x}_{ti}; \theta_0) - g(\mathbf{x}_{ti}; \theta_0)\}] = o_p(n^{-1/2}) \quad (16)$$

by Lemma 1 of Chen, Linton, and van Keilegom (2003). Combining (15) and (16) we obtain (because $\hat{\theta} \in \mathcal{N}_{02n}$)

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n \varphi(\mathbf{x}_{ti}, v_2^*) \{\hat{g}(\mathbf{x}_{ti}; \hat{\theta}) - \hat{g}(\mathbf{x}_{ti}; \theta_0)\} - \frac{1}{n} \sum_{i=1}^n \varphi(\mathbf{x}_{ti}, v_2^*) \{g(\mathbf{x}_{ti}; \hat{\theta}) - g(\mathbf{x}_{ti}; \theta_0)\} \\ &= E[\varphi(\mathbf{x}_{ti}, v_2^*) \{\hat{g}(\mathbf{x}_{ti}; \hat{\theta}) - \hat{g}(\mathbf{x}_{ti}; \theta_0)\}] - E[\varphi(\mathbf{x}_{ti}, v_2^*) \{g(\mathbf{x}_{ti}; \hat{\theta}) - g(\mathbf{x}_{ti}; \theta_0)\}] + o_p(n^{-1/2}). \end{aligned}$$

Now define $\hat{\varphi}(\mathbf{x}_t, v_2^*) = \int \varphi(\mathbf{x}_t, v_2^*) \hat{f}_{\mathbf{y}_{t,t-1}|\mathbf{x}_t} d\mathbf{y}_{t,t-1}$. We then obtain

$$\begin{aligned} E[\varphi(\mathbf{x}_{ti}, v_2^*) \{\hat{g}(\mathbf{x}_{ti}; \hat{\theta}) - \hat{g}(\mathbf{x}_{ti}; \theta_0)\}] &= E[\varphi(\mathbf{x}_{ti}, v_2^*) \int \{\rho(\mathbf{y}_{t,t-1}; \hat{\theta}) - \rho(\mathbf{y}_{t,t-1}; \theta_0)\} \hat{f}_{\mathbf{y}_{t,t-1}|\mathbf{x}_{ti}} d\mathbf{y}_{t,t-1}] \\ &= E[\int \varphi(\mathbf{x}_{ti}, v_2^*) \hat{f}_{\mathbf{y}_{t,t-1}|\mathbf{x}_{ti}} d\mathbf{y}_{t,t-1} E[\rho(\mathbf{y}_{t,t-1}; \hat{\theta}) - \rho(\mathbf{y}_{t,t-1}; \theta_0) | \mathbf{x}_{ti}]] \\ &= E[\hat{\varphi}(\mathbf{x}_{ti}, v_2^*) \{g(\mathbf{x}_{ti}; \hat{\theta}) - g(\mathbf{x}_{ti}; \theta_0)\}] \end{aligned}$$

and

$$\begin{aligned} & E[\hat{\varphi}(\mathbf{x}_{ti}, v_2^*)\{g(\mathbf{x}_{ti}; \hat{\theta}) - g(\mathbf{x}_{ti}; \theta_0)\}] - E[\varphi(\mathbf{x}_{ti}, v_2^*)\{g(\mathbf{x}_{ti}; \hat{\theta}) - g(\mathbf{x}_{ti}; \theta_0)\}] \\ &= E[\{\hat{\varphi}(\mathbf{x}_{ti}, v_2^*) - \varphi(\mathbf{x}_{ti}, v_2^*)\}\{g(\mathbf{x}_{ti}; \hat{\theta}) - g(\mathbf{x}_{ti}; \theta_0)\}] = o_p(n^{-1/2}). \end{aligned}$$

It follows that

$$\frac{1}{n} \sum_{i=1}^n \varphi(\mathbf{x}_{ti}, v_2^*)\{\hat{g}(\mathbf{x}_{ti}; \hat{\theta}) - \hat{g}(\mathbf{x}_{ti}; \theta_0)\} - \frac{1}{n} \sum_{i=1}^n \varphi(\mathbf{x}_{ti}, v_2^*)\{g(\mathbf{x}_{ti}; \hat{\theta}) - g(\mathbf{x}_{ti}; \theta_0)\} = o_p(n^{-1/2}).$$

Then for some $\bar{\theta} \in \mathcal{N}_{02n}$, which lies between $\hat{\theta}$ and θ_0 , the mean value theorem (applied in the third equality) provides

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n \varphi(\mathbf{x}_{ti}, v_2^*)\{\hat{g}(\mathbf{x}_{ti}; \hat{\theta}) - \hat{g}(\mathbf{x}_{ti}; \theta_0)\} \\ &= \frac{1}{n} \sum_{i=1}^n \varphi(\mathbf{x}_{ti}, v_2^*)\{g(\mathbf{x}_{ti}; \hat{\theta}) - g(\mathbf{x}_{ti}; \theta_0)\} + o_p(n^{-1/2}) \\ &= E[\varphi(\mathbf{x}_{ti}, v_2^*)\{g(\mathbf{x}_{ti}; \hat{\theta}) - g(\mathbf{x}_{ti}; \theta_0)\}] + o_p(n^{-1/2}) \\ &= \langle v_2^*, \hat{\theta} - \theta_0 \rangle_{\theta} + E \left[\varphi(\mathbf{x}_{ti}, v_2^*) \left(\frac{dg(\mathbf{x}_{ti}; \bar{\theta})}{d\theta} [\hat{\theta} - \theta_0] - \frac{dg(\mathbf{x}_{ti}; \theta_0)}{d\theta} [\hat{\theta} - \theta_0] \right) \right] + o_p(n^{-1/2}) \\ &= \langle v_2^*, \hat{\theta} - \theta_0 \rangle_{\theta} + o_p(n^{-1/2}) \end{aligned}$$

where the second equality holds because the class of functions $\{\varphi(\mathbf{x}_{ti}, v_2^*)\{g(\mathbf{x}_{ti}; \theta) - g(\mathbf{x}_{ti}; \theta_0)\} : \theta \in \mathcal{N}_{02n}\}$ is a Donsker class and the last result holds by Assumption 6.10. Thus the result follows.

(c) By the definition of $\hat{\varphi}(\mathbf{x}_t, v_2^*)$, we obtain

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n \int \varphi(\mathbf{x}_{ti}, v_2^*)\{\hat{g}(\mathbf{x}_{ti}; \theta_0) - \rho(\mathbf{y}_{t,t-1}; \theta_0)\} f_{k_t, k_{t-1} | \mathbf{s}_{ti}} dk_t dk_{t-1} \\ &= \frac{1}{n} \sum_{i=1}^n \int \{\hat{\varphi}(\mathbf{x}_{ti}, v_2^*) - \varphi(\mathbf{x}_{ti}, v_2^*)\} \rho(\mathbf{y}_{t,t-1}; \theta_0) f_{k_t, k_{t-1} | \mathbf{s}_{ti}} dk_t dk_{t-1}. \end{aligned}$$

Thus the result follows by the Markov inequality since $\hat{\varphi}(\mathbf{x}_{ti}, v_2^*) - \varphi(\mathbf{x}_{ti}, v_2^*) = o_p(1)$ uniformly over $\mathbf{x}_{ti} \in \mathcal{X}_t$ and $g(\mathbf{x}_{ti}; \theta_0) = 0$, as in Ai and Chen (2003). $\square$

We now prove Theorem 6.12.

Proof. 1. The result follows from a similar argument to the proof of Theorem 3 in Hu and Schennach

(2008). In the proof the bound for the pathwise second derivative is obtained by

$$\begin{aligned} & \left| \sup_{\alpha \in \mathcal{N}_{02n}} \frac{d^2}{d\alpha d\alpha} \ln f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \bar{\alpha}) [v_{1n}, (\alpha - \alpha_0)] \right| \\ & \leq \sup_{\alpha \in \mathcal{N}_{02n}} \left[\left| \frac{f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}^{[1]}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \bar{\alpha}, \nu)}{f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \bar{\alpha})} \right|^2 + \left| \frac{f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}^{[2]}(\cdot | \mathbf{z}_t^*; \bar{\alpha}, \nu)}{f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(\cdot | \mathbf{z}_t^*; \bar{\alpha})} \right| \right] \|\alpha - \alpha_0\|_{s, \alpha} \|v_{1n}\|_{s, \alpha} \end{aligned}$$

where $f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}^{[2]}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \bar{\alpha}, \nu)$ is defined as $\frac{d^2}{d\tau^2} f_{r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*}(r_t, k_t^*, k_{t-1}^* | \mathbf{z}_t^*; \bar{\alpha} + \tau\nu^{-1})|_{\tau=0}$ with absolute values of $\bar{f}_1$, $\bar{f}_2$, $\frac{d}{d\psi} f_{r_t | \mathbf{z}_t}$, and $\frac{d^2}{d\psi^2} f_{r_t | \mathbf{z}_t}$.

2. Let $\varepsilon_n = o(n^{-1/2}) > 0$ and $u_{2n}^* = \pm v_{2n}^*$. By a Taylor expansion around $\hat{\theta}$, we obtain that for $\theta_s \in (\hat{\theta}, \hat{\theta} + \varepsilon_n u_{2n}^*)$,

$$\begin{aligned} & \hat{Q}_n(\hat{\theta}) - \hat{Q}_n(\hat{\theta} + \varepsilon_n u_{2n}^*) \\ & = \frac{d\hat{Q}_n(\hat{\theta})}{d\theta} [\varepsilon_n u_{2n}^*] + \frac{d^2 \hat{Q}_{1n}(\theta_s)}{d\theta d\theta} [\varepsilon_n u_{2n}^*, \varepsilon_n u_{2n}^*] + \frac{d^2 \hat{Q}_{2n}(\theta_s)}{d\theta d\theta} [\varepsilon_n u_{2n}^*, \varepsilon_n u_{2n}^*]. \end{aligned}$$

Then by Lemma C.3 and Lemma C.4, we have

$$\sqrt{n} \langle v_{2n}^*, \hat{\theta} - \theta_0 \rangle_{\theta} = -\frac{1}{\sqrt{n}} \sum_{i=1}^n \int \left\{ \frac{dg(\mathbf{x}_{ti}; \theta_0)}{d\theta} [v_{2n}^*] \right\}' A(\mathbf{x}_{ti}) \rho(\mathbf{y}_{t,t-1}; \theta_0) f_{k_t, k_{t-1} | \mathbf{s}_{ti}} dk_t dk_{t-1} + o_p(1).$$

Thus the result follows by the Lindeberg-Lévy central limit theorem. $\square$

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