Supplementary Appendix to: Stock Return and Cash Flow Predictability: The Role of Volatility Risk

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A Model Solution

Our basic solution method for the model is adopted from Bansal and Yaron (2004), Bansal, Kiku, and Yaron (2007), and Drechsler and Yaron (2011). To begin, we follow Campbell and Shiller (1988) and solve for the return on consumption by log-linearizing $r_{c,t+1}$ around the unconditional mean of the wealth-consumption ratio v_t ,

$$r_{c,t+1} \approx \kappa_0 + \kappa_1 v_{t+1} - v_t + \Delta c_{t+1}, \quad (\text{A.1})$$

where $\kappa_1 = \frac{\exp(E(v))}{1 + \exp(E(v))}$, and $\kappa_0 = \log[1 + \exp(E(v))] - \kappa_1 E(v)$. We then conjecture a solution for v_t as a linear function of the state vector Y_t ,

$$v_t = A_0 + A' Y_t, \quad (\text{A.2})$$

where A_0 is a scalar, and $A = (0, A_x, A_\sigma, A_q, 0)$ refer to the pricing coefficients. Next, by substituting v_t and v_{t+1} into equation (A.1), both $r_{c,t+1}$ and the stochastic discount factor m_{t+1} defined in the main text may be expressed as linear functions of the state vector,

$$m_{t+1} = \mu_m - (\gamma e'_1 + (1 - \theta)\kappa_1 A') Y_{t+1} - (\theta - 1) A' Y_t, \quad (\text{A.3})$$

$$r_{c,t+1} = \mu_{r_c} + (e'_1 + \kappa_1 A') Y_{t+1} - A' Y_t. \quad (\text{A.4})$$

Going one step further, it follows that the innovations to the pricing kernel and the return on the wealth claim may be expressed as,

$$m_{t+1} - E_t(m_{t+1}) = -\Lambda' H G_t z_{t+1}, \quad (\text{A.5})$$

$$r_{c,t+1} - E_t(r_{c,t+1}) = \Lambda'_c H G_t z_{t+1}, \quad (\text{A.6})$$

where Λ denotes the price of risk for the factor shocks,

$$\Lambda = \gamma e_1 + \kappa_1 (1 - \theta) A,$$

for $e_1 \equiv [1, 0, 0, 0, 0]$, and $\Lambda_c = e_1 + \kappa_1 A$. The magnitude and sign of Λ are determined by the preference parameter θ and the pricing coefficient vector A . If investors prefer early resolution of uncertainty, i.e., $\gamma > \phi^{-1}$, A reveals the sensitivity of the market prices for the different shocks to higher order consumption dynamics. When $\gamma = \phi^{-1}$ (CRRA case), Λ collapses to γe_1 , and only transient shocks to consumption growth level $z_{g,t+1}$ are priced.

Since the no-arbitrage condition must hold regardless of the realization of the state vector Y_t , it is possible to solve for A by imposing the Euler equation,

$$0 = \mu_m + \mu_{r_c} + [(-\Lambda + \Lambda_c)' F - \theta A'] Y_t + \frac{1}{2} (-\Lambda + \Lambda_c)' H G_t G_t' H' (-\Lambda + \Lambda_c). \quad (\text{A.7})$$

This in turn implies that

$$\theta A_{\{i\}} + (\tilde{\Lambda}'_c F)_{\{i\}} = \frac{1}{2} [1_{i=3} \sum_{j=1,5} (\tilde{\Lambda}'_c h_j)^2 + 1_{i=4} \sum_{j=2,3,4} (\tilde{\Lambda}'_c h_j)^2], \quad (\text{A.8})$$

$$0 = \mu_m + \mu_{r_c}, \quad (\text{A.9})$$

where $\tilde{\Lambda}_c = -\Lambda_c + \Lambda = (\gamma - 1)e_1 - \kappa_1 \theta A$, i refers to the i^{th} element of vector, and $1_{i=n}$ is an indicator function. The solutions are,

$$A_x = -\frac{\gamma - 1}{\theta(1 - \kappa_1 \rho_x)}, \quad (\text{A.10})$$

$$A_\sigma = \frac{(\gamma - 1)^2}{2\theta(1 - \kappa_1 \rho_\sigma)}, \quad (\text{A.11})$$

while A_q solves the equation $\frac{1}{2}a_q \theta^2 A_q^2 + (b_q + (1 - \kappa_1 \rho_q))(-\theta A_q) + \frac{1}{2}c_q = 0$, where

$$\begin{aligned} a_q &= \kappa_1^2 (\varphi_x^2 s_{q,x}^2 + s_{q,\sigma}^2 + \varphi_q^2) > 0, \\ b_q &= \kappa_1^2 (\varphi_x^2 (-A_x \theta - A_\sigma \theta s_{\sigma,x}) s_{q,x} - A_\sigma \theta s_{q,\sigma}), \\ c_q &= \kappa_1^2 (\varphi_x^2 (-A_x \theta - A_\sigma \theta s_{\sigma,x})^2 + A_\sigma^2 \theta^2) > 0. \end{aligned}$$

Since $a_q > 0$ and $c_q > 0$, the two roots are either negative or positive. We choose the larger root for $-\theta A_q$ if $b_q + (1 - \kappa_1 \rho_q) > 0$, or the smaller root if $b_q + (1 - \kappa_1 \rho_q) < 0$. In both cases A_q reduces to zero when $s_{q,x}$, $s_{q,\sigma}$ and φ_q are zero.

Even though no closed-form expressions for A are available when we consider κ_0 and κ_1 as endogenous, the system of equations are still solvable. As shown in equation (A.8), A depends on κ_1 , μ , F , H , as well as the preference parameters. Considering the definitions of κ_1 and κ_0 , κ_1 and A are the only unknowns in the constant term in the Euler equation, so that κ_1 may be solved endogenously together with A . Finally, κ_0 and A_0 can be expressed as functions of A and κ_1 . For detailed numerical solutions, see Drechsler and Yaron (2011) Appendix A.1 and A.2.

Applying a similar conjecture-evaluation type method, it is possible to solve for the aggregate market return $r_{t,t+1}$. Denote the price-dividend ratio by w_t , and consider the conjecture solution $w_t = A_{d,0} + A'_d Y_t$. Log-linearize $r_{t,t+1}$ around the unconditional mean of the price-dividend ratio yields,

$$r_{t,t+1} \approx \kappa_{d,0} + \kappa_{d,1} w_{t+1} - w_t + \Delta d_{t+1}. \quad (\text{A.12})$$

Substituting out w_t and w_{t+1} in the above equation, the return on the market may be rewritten as,

$$r_{t,t+1} = \mu_{r_d} + (e'_5 + \kappa_{d,1} A'_d) Y_{t+1} - A'_d Y_t, \quad (\text{A.13})$$

where $\Lambda_d = e_5 + \kappa_{d,1} A_d$ and $A_d = [0, A_{d,x}, A_{d,\sigma}, A_{d,q}, A_{d,d}]'$ is a vector of pricing coefficients.

Using the same solution method as the one previously used for A , it follows by the no-arbitrage condition,

$$0 = \mu_m + \mu_{r_d} + [(-\Lambda + \Lambda_d)'F - (\theta - 1)A' - A_d']Y_t + 0.5(-\Lambda + \Lambda_d)'HG_tG_t'H'(-\Lambda + \Lambda_d), \quad (\text{A.14})$$

which implies that

$$(\theta - 1)A_{\{i\}} + A_{d,\{i\}} + (\tilde{\Lambda}_d'F)_{\{i\}} = 0.5[1_{i=3} \sum_{j=1,5} (\tilde{\Lambda}_d'h_j)^2 + 1_{i=4} \sum_{j=2,3,4} (\tilde{\Lambda}_d'h_j)^2], \quad (\text{A.15})$$

$$0 = \mu_m + \mu_{r_d}, \quad (\text{A.16})$$

where $\tilde{\Lambda}_d = -\Lambda_d + \Lambda = \gamma e_1 - e_5 + \kappa_1(1 - \theta)A - \kappa_{d,1}A_d$.

The solution for A_d may therefore be expressed as,

$$A_{d,d} = \frac{\rho_d}{1 - \kappa_{d,1}\rho_d} \quad (\text{A.17})$$

$$\frac{1 - \kappa_1\rho_x}{1 - \kappa_{d,1}\rho_x}(1 - \theta)A_x - A_{d,x} = -\frac{-\gamma + \phi_{dx}(1 + \kappa_{d,1}A_{d,d})}{1 - \kappa_{d,1}\rho_x} \quad (\text{A.18})$$

$$\frac{1 - \kappa_1\rho_\sigma}{1 - \kappa_{d,1}\rho_\sigma}(1 - \theta)A_\sigma - A_{d,\sigma} = -\frac{1}{2} \frac{\gamma^2 + \varphi_d^2(1 + \kappa_{d,1}A_{d,d})^2}{1 - \kappa_{d,1}\rho_\sigma} < 0 \quad (\text{A.19})$$

$$\frac{1 - \kappa_1\rho_q}{1 - \kappa_{d,1}\rho_q}(1 - \theta)A_q - A_{d,q} = -\frac{1}{2} \frac{a_{d,q} \left(\kappa_1(1 - \theta)A_q - \kappa_{d,1}A_{d,q} \right)^2 + 2b_{d,q} \left(\kappa_1(1 - \theta)A_q - \kappa_{d,1}A_{d,q} \right) + c_{d,q}}{1 - \kappa_{d,1}\rho_q} \quad (\text{A.20})$$

where

$$\begin{aligned} a_{d,q} &= a_q = (\varphi_x^2 s_{q,x}^2 + s_{q,\sigma}^2 + \varphi_q^2) > 0 \\ b_{d,q} &= \left(\varphi_x^2 \left(\kappa_1(1 - \theta)A_x - \kappa_{d,1}A_{d,x} + (\kappa_1(1 - \theta)A_\sigma - \kappa_{d,1}A_{d,\sigma}) s_{\sigma,x} + \frac{-1}{1 - \kappa_{d,1}\rho_d} s_{d,x} \right) s_{q,x} \right) \\ &\quad + \left(\left(\kappa_1(1 - \theta)A_\sigma - \kappa_{d,1}A_{d,\sigma} + \frac{-1}{1 - \kappa_{d,1}\rho_d} s_{d,\sigma} \right) s_{q,\sigma} + \frac{-1}{1 - \kappa_{d,1}\rho_d} s_{d,q} \varphi_q^2 \right) \\ c_{d,q} &= \left(\varphi_x^2 \left(\kappa_1(1 - \theta)A_x - \kappa_{d,1}A_{d,x} + (\kappa_1(1 - \theta)A_\sigma - \kappa_{d,1}A_{d,\sigma}) s_{\sigma,x} + \frac{-1}{1 - \kappa_{d,1}\rho_d} s_{d,x} \right)^2 \right) \\ &\quad + \left(\left(\kappa_1(1 - \theta)A_\sigma - \kappa_{d,1}A_{d,\sigma} + \frac{-1}{1 - \kappa_{d,1}\rho_d} s_{d,\sigma} \right)^2 + \left(\frac{-1}{1 - \kappa_{d,1}\rho_d} s_{d,q} \right)^2 \varphi_q^2 \right) > 0 \end{aligned}$$

In other words, $A_{d,q}$ solves the equation (A.20) and we choose the root with smaller absolute value.

Both A_q and $A_{d,q}$ reduce to zero when $s_{q,x}$, $s_{q,\sigma}$ and φ_q are all zero. We will discuss the sign of $A_{d,q}$ later on in the parameter implication section. We can explicitly express $A_{d,x}$ and $A_{d,\sigma}$ as

$$A_{d,x} = \frac{(1 - \gamma)/\theta - 1 + \phi_{dx}/(1 - \kappa_{d,1}\rho_d)}{1 - \kappa_{d,1}\rho_x} = \frac{-\psi^{-1} + \phi_{dx}/(1 - \kappa_{d,1}\rho_d)}{1 - \kappa_{d,1}\rho_x} \quad (\text{A.21})$$

$$A_{d,\sigma} = \frac{(\gamma - 1)^2 + 2\theta\gamma + \theta(\varphi_d^2/(1 - \kappa_{d,1}\rho_d)^2 - 1)}{2\theta(1 - \kappa_{d,1}\rho_\sigma)}. \quad (\text{A.22})$$

B Variance Risk Premium

In order to determine the factor structure for the variance risk premium, we first need to solve for the second order moment of the return $r_{t,t+1}$. It follows from above that $r_{t,t+1} - E_t(r_{t,t+1}) = \Lambda'_d H G_t z_{t+1}$, so that the conditional variance of the return is affine in σ_t^2 and q_t ,

$$\begin{aligned} \text{Var}_t(r_{t,t+1}) &= \sum_{j=1,5} \Lambda'_d h_j h'_j \Lambda_d \sigma_t^2 + \sum_{j=2,3,4} \Lambda'_d h_j h'_j \Lambda_d q_t \\ &= (1 + \kappa_{d,1} A_{d,d})^2 \varphi_d^2 \sigma_t^2 + \sum_{j=2,3,4} \Lambda'_d h_j h'_j \Lambda_d q_t. \end{aligned} \quad (\text{B.23})$$

The first term is associated with the volatility of cash flow shocks, and the second term represents the consumption uncertainty. Accordingly, the equity risk premium may be expressed as,

$$\begin{aligned} \log(E_t R_{t,t+1}) - r_{f,t} &= E_t(r_{t,t+1}) + \frac{1}{2} \text{Var}_t(r_{t,t+1}) - r_{f,t} \\ &= -\text{Cov}_t(m_{t+1}, r_{t,t+1}) = \sum_{j=2,3,4} \Lambda'_d h_j h'_j \Lambda_d q_t \end{aligned} \quad (\text{B.24})$$

The first equality comes from the normality distribution of $r_{t,t+1} \equiv \log(R_{t,t+1})$, the second equality comes from the no arbitrage condition and $r_{f,t} = \log(E_t m_{t+1} + \frac{1}{2} \text{Var}_t(m_{t+1}))$. The expectations of $\text{Var}_t(r_{t,t+1})$ under the physical and risk-neutral probability measures are,

$$E_t(\text{Var}_{t+1}(r_{d,t+2})) = \sum_{j=1,5} \Lambda'_d h_j h'_j \Lambda_d (\mu_\sigma + \rho_\sigma \sigma_t^2) + \sum_{j=2,3,4} \Lambda'_d h_j h'_j \Lambda_d (\mu_q + \rho_q q_t), \quad (\text{B.25})$$

$$E_t^Q(\text{Var}_{t+1}(r_{d,t+2})) = \sum_{j=1,5} \Lambda'_d h_j h'_j \Lambda_d (\mu_\sigma + \rho_\sigma \sigma_t^2 + s_{q,1} q_t) + \sum_{j=2,3,4} \Lambda'_d h_j h'_j \Lambda_d (\mu_q + \rho_q q_t + s_{q,2} q_t). \quad (\text{B.26})$$

Under the risk neutral measure, we reweight probabilities according to the pricing kernel $\frac{e^{m_{t,t+1}}}{E_t e^{m_{t,t+1}}}$. If investor prefers early resolution of uncertainty, the shocks z_{t+1} 's conditional mean shifts away from zero. And this shift can be expressed as the conditional covariance between the state vector and SDF $m_{t,t+1}$,

$$\begin{aligned} s_{q,1} q_t &= \text{Cov}_t(e_3 H G_t z_{t+1}, -\Lambda' H G_t z_{t+1}) \\ &= -(\varphi_x s_{\sigma,x} h'_2 + h'_3) \Lambda_d q_t, \end{aligned} \quad (\text{B.27})$$

$$\begin{aligned} s_{q,2} q_t &= \text{Cov}_t(e_4 H G_t z_{t+1}, -\Lambda' H G_t z_{t+1}) \\ &= -(\varphi_x s_{q,x} h'_2 + s_{q,\sigma} h'_3 + \varphi_q h'_4) \Lambda_d q_t, \end{aligned} \quad (\text{B.28})$$

where,

$$\begin{aligned}
s_{q,1} &= -\kappa_1(1-\theta) \left(A_x \varphi_x^2 s_{\sigma,x} + A_\sigma (\varphi_x^2 s_{\sigma,x}^2 + 1) + A_q (\varphi_x^2 s_{\sigma,x} s_{q,x} + s_{q,\sigma}) \right) \\
&= -\kappa_1(1-\theta) (\varphi_x^2 s_{\sigma,x} s_{q,x} + s_{q,\sigma}) \left(\frac{A_x \varphi_x^2 s_{\sigma,x} + A_\sigma (\varphi_x^2 s_{\sigma,x}^2 + 1)}{\varphi_x^2 s_{\sigma,x} s_{q,x} + s_{q,\sigma}} + A_q \right) \\
s_{q,2} &= -\kappa_1(1-\theta) \left(A_x \varphi_x^2 s_{q,x} + A_\sigma (\varphi_x^2 s_{\sigma,x} s_{q,x} + s_{q,\sigma}) + A_q (\varphi_x^2 s_{q,x}^2 + s_{q,\sigma}^2 + \varphi_q^2) \right) \\
&= -1/\kappa_1(1-\theta) a_q \left(\frac{b_q}{-a_q} + A_q \right)
\end{aligned}$$

By definition, $s_{q,1}$ and $s_{q,2}$ represent the market prices of shocks to σ_t^2 and q_t , respectively. Thus, the variance risk premium is naturally defined by,

$$\begin{aligned}
VRP_t &\equiv E_t^Q(Var_{t+1}(r_{d,t+2})) - E_t(Var_{t+1}(r_{d,t+2})) \\
&= (\sum_{j=1,5} \Lambda'_d h_j h'_j \Lambda_d s_{q,1} + \sum_{j=2,3,4} \Lambda'_d h_j h'_j \Lambda_d s_{q,2}) q_t.
\end{aligned} \tag{B.29}$$

In the main text, we will refer to the expected return variation and the variance risk premium as,

$$\begin{aligned}
ERV_t &= \frac{Q_{1,1}}{\rho_\sigma} (\mu_\sigma + \rho_\sigma \sigma_t^2) + \frac{Q_{1,2}}{\rho_q} (\mu_q + \rho_q q_t), \\
VRP_t &= Q_{2,2} q_t
\end{aligned}$$

for short, where

$$Q_{1,1} = \sum_{j=1,5} \Lambda'_d h_j h'_j \Lambda_d \rho_\sigma > 0, \tag{B.30}$$

$$Q_{1,2} = \sum_{j=2,3,4} \Lambda'_d h_j h'_j \Lambda_d \rho_q > 0, \tag{B.31}$$

$$Q_{2,2} = \frac{Q_{1,1}}{\rho_\sigma} s_{q,1} + \frac{Q_{1,2}}{\rho_q} s_{q,2}. \tag{B.32}$$

In order to determine the signs of $A_{d,x}$, $A_{d,\sigma}$ and $A_{d,q}$, it is informative to write out the formula in terms of the estimated B and $\tilde{\rho}$ matrices,

$$\frac{\phi_{d,x}}{-A_{d,x}} = \tilde{\rho}_{3,4}, \quad \frac{A_{d,\sigma}}{Q_{1,1}} = B_{4,1}, \quad \frac{Q_{1,2}}{Q_{2,2}} = -B_{1,2}, \quad \frac{A_{d,q}}{Q_{2,2}} = B_{4,2} - B_{1,2} B_{4,1}. \tag{B.33}$$

Since $\tilde{\rho}_{3,4} < 0$, $\phi_{d,x}$ and $A_{d,x}$ must have the same signs. Thus, by definition $Q_{1,1} > 0$ and $Q_{1,2} > 0$, which together with the estimates for $B_{4,1} = -0.60 < 0$ and $B_{1,2} = -0.02 < 0$, imply that $A_{d,\sigma} < 0$ and $Q_{2,2} > 0$. Consequently $A_{d,q} = Q_{2,2}(B_{4,2} - B_{1,2} B_{4,1}) = -1.45 Q_{2,2} < 0$.

C Alternative Setups

C.1 Separate Volatility Processes

We will consider the following alternative setup for G_t and H , with F unchanged,

$$G_t = \begin{pmatrix} \sigma_t & 0 & 0 & 0 & 0 \\ 0 & \sqrt{q_t} & 0 & 0 & 0 \\ 0 & 0 & \sigma_t & 0 & 0 \\ 0 & 0 & 0 & \sqrt{q_t} & 0 \\ 0 & 0 & 0 & 0 & \sigma_t \end{pmatrix} \quad H = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & s_{\sigma,x} & \varphi_\sigma & 0 & 0 \\ 0 & s_{q,x} & 0 & \varphi_q & 0 \\ 0 & s_{d,x} & \varphi_\sigma s_{d,\sigma} & \varphi_q s_{d,q} & \varphi_d \end{pmatrix} \quad (C.34)$$

This setup is related to Bansal and Shaliastovich (2013), where the volatilities of x_t and σ_t^2 are modeled as two separate processes.

For simplicity, we use the same general notation as in the main setup for A , Λ , Λ_c and Λ_d . However, the solutions for the pricing coefficients are obviously different from the main setup, except for $A_{d,d} = \frac{\rho_d}{1 - \kappa_{d,1}\rho_d}$,

$$\theta A_{\{i\}} + (\tilde{\Lambda}'_c F)_{\{i\}} = 0.5[1_{i=3} \sum_{j=1,3,5} (\tilde{\Lambda}'_c h_j)^2 + 1_{i=4} \sum_{j=2,4} (\tilde{\Lambda}'_c h_j)^2], \quad (C.35)$$

$$(\theta - 1)A_{\{i\}} + A_{d,\{i\}} + (\tilde{\Lambda}'_d F)_{\{i\}} = 0.5[1_{i=3} \sum_{j=1,3,5} (\tilde{\Lambda}'_d h_j)^2 + 1_{i=4} \sum_{j=2,4} (\tilde{\Lambda}'_d h_j)^2]. \quad (C.36)$$

Since $r_{t,t+1} - E_t(r_{t,t+1}) = \Lambda'_d H G_t z_{t+1}$, the conditional variance of the return is again affine,

$$Var_t(r_{t,t+1}) = \sum_{j=1,3,5} \Lambda'_d h_j h'_j \Lambda_d \sigma_t^2 + \sum_{j=2,4} \Lambda'_d h_j h'_j \Lambda_d q_t \quad (C.37)$$

The expectations of $Var_t(r_{t,t+1})$ under the physical and risk-neutral probability measures may further be expressed as,

$$E_t(Var_{t+1}(r_{d,t+2})) = \sum_{j=1,3,5} \Lambda'_d h_j h'_j \Lambda_d (\mu_\sigma + \rho_\sigma \sigma_t^2) + \sum_{j=2,4} \Lambda'_d h_j h'_j \Lambda_d (\mu_q + \rho_q q_t) \quad (C.38)$$

$$E_t^Q(Var_{t+1}(r_{d,t+2})) = \sum_{j=1,3,5} \Lambda'_d h_j h'_j \Lambda_d (\mu_\sigma + \rho_\sigma \sigma_t^2 + s_{q,1} q_t + s_{\sigma,1} \sigma_t^2) + \sum_{j=2,4} \Lambda'_d h_j h'_j \Lambda_d (\mu_q + \rho_q q_t + s_{q,2} q_t) \quad (C.39)$$

If investors prefer early resolution of uncertainty, the conditional means of the z_{t+1} shocks shift away from zero under the risk-neutral measure,

$$\begin{aligned} s_{\sigma,1} \sigma_t^2 + s_{q,1} q_t &= Cov_t(e'_3 H G_t z_{t+1}, -\Lambda' H G_t z_{t+1}) \\ &= -\varphi_\sigma h'_3 \Lambda \sigma_t^2 - (\varphi_x s_{\sigma,x} h'_2) \Lambda q_t, \end{aligned} \quad (C.40)$$

$$\begin{aligned} s_{q,2} q_t &= Cov_t(e'_4 H G_t z_{t+1}, -\Lambda' H G_t z_{t+1}) \\ &= -(\varphi_x s_{q,x} h'_2 + \varphi_q h'_4) \Lambda q_t. \end{aligned} \quad (C.41)$$

Defining the variance risk premium as before,

$$\begin{aligned} VRP_t &\equiv E_t^Q(Var_{t+1}(r_{d,t+2})) - E_t(Var_{t+1}(r_{d,t+2})) \\ &= \sum_{j=1,3,5} \Lambda'_d h_j h'_j \Lambda_d (s_{\sigma,1} \sigma_t^2 + s_{q,1} q_t) + \sum_{j=2,4} \Lambda'_d h_j h'_j \Lambda_d s_{q,2} q_t, \end{aligned} \quad (C.42)$$

we may express the expected return variation and premium in short-hand form as,

$$\begin{aligned} ERV_t &= \frac{Q_{1,1}}{\rho_\sigma} (\mu_\sigma + \rho_\sigma \sigma_t^2) + \frac{Q_{1,2}}{\rho_q} (\mu_q + \rho_q q_t), \\ VRP_t &= Q_{2,1} \sigma_t^2 + Q_{2,2} q_t, \end{aligned}$$

where

$$\begin{aligned} Q_{1,1} &= \sum_{j=1,3,5} \Lambda'_d h_j h'_j \Lambda_d \rho_\sigma > 0 & Q_{1,2} &= \sum_{j=2,4} \Lambda'_d h_j h'_j \Lambda_d \rho_q > 0 \\ Q_{2,1} &= \frac{Q_{1,1}}{\rho_q} s_{\sigma,1} & Q_{2,2} &= \frac{Q_{1,1}}{\rho_\sigma} s_{q,1} + \frac{Q_{1,2}}{\rho_q} s_{q,2}. \end{aligned}$$

C.2 Long-Run Stochastic Volatility

We will consider the following alternative setup for G_t , H , and F ,

$$F = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & \rho_x & 0 & 0 & 0 \\ 0 & 0 & \rho_\sigma & 1 & 0 \\ 0 & 0 & 0 & \rho_q & 0 \\ 0 & \phi_{dx} & 0 & 0 & \rho_d \end{pmatrix} \quad G_t = \begin{pmatrix} \sigma_t & 0 & 0 & 0 & 0 \\ 0 & \sigma_t & 0 & 0 & 0 \\ 0 & 0 & \sigma_t & 0 & 0 \\ 0 & 0 & 0 & \sqrt{q_t} & 0 \\ 0 & 0 & 0 & 0 & \sigma_t \end{pmatrix} \quad H = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \varphi_x & 0 & 0 & 0 \\ 0 & \varphi_x s_{\sigma,x} & \varphi_\sigma & 0 & 0 \\ 0 & \varphi_x s_{q,x} & 0 & \varphi_q & 0 \\ 0 & \varphi_x s_{d,x} & \varphi_\sigma s_{d,\sigma} & \varphi_q s_{d,q} & \varphi_d \end{pmatrix} \quad (C.43)$$

This setup is motivated by the model analyzed by Branger and Vòlkert (2012), among others, allowing for a time-varying mean of the consumption variance σ_t^2 .

Again, for simplicity we will use the same general notation as in the main setup for A , Λ , Λ_c and Λ_d . The solution for $A_{d,d} = \frac{\rho_d}{1-\kappa_{d,1}\rho_d}$ remains the same, but the other the pricing coefficients now take the form,

$$\theta A_{\{i\}} + (\tilde{\Lambda}'_c F)_{\{i\}} = \frac{1}{2} [1_{i=3} \sum_{j=1,2,3,5} (\tilde{\Lambda}'_c h_j)^2 + 1_{i=4} \sum_{j=4} (\tilde{\Lambda}'_c h_j)^2], \quad (C.44)$$

$$(\theta - 1) A_{\{i\}} + A_{d,\{i\}} + (\tilde{\Lambda}'_d F)_{\{i\}} = \frac{1}{2} [1_{i=3} \sum_{j=1,2,3,5} (\tilde{\Lambda}'_d h_j)^2 + 1_{i=4} \sum_{j=4} (\tilde{\Lambda}'_d h_j)^2]. \quad (C.45)$$

As before, $r_{t,t+1} - E_t(r_{t,t+1}) = \Lambda'_d H G_t z_{t+1}$, so that the conditional variance of the return may be expressed as,

$$Var_t(r_{t,t+1}) = \sum_{j=1,2,3,5} \Lambda'_d h_j h'_j \Lambda_d \sigma_t^2 + \sum_{j=4} \Lambda'_d h_j h'_j \Lambda_d q_t. \quad (C.46)$$

The expectation of $Var_t(r_{t,t+1})$ under the physical and risk-neutral probability measures are,

$$E_t(Var_{t+1}(r_{d,t+2})) = \sum_{j=1,2,3,5} \Lambda'_d h_j h'_j \Lambda_d (\mu_\sigma + \rho_\sigma \sigma_t^2 + q_t) + \sum_{j=4} \Lambda'_d h_j h'_j \Lambda_d (\mu_q + \rho_q q_t),$$

$$E_t^Q(Var_{t+1}(r_{d,t+2})) = \sum_{j=1,2,3,5} \Lambda'_d h_j h'_j \Lambda_d (\mu_\sigma + \rho_\sigma \sigma_t^2 + q_t + s_{\sigma,1} \sigma_t^2) + \sum_{j=4} \Lambda'_d h_j h'_j \Lambda_d (\mu_q + \rho_q q_t + s_{\sigma,2} \sigma_t^2 + s_{q,2} q_t).$$

The shifts in the conditional means of the z_{t+1} shocks under the risk-neutral measure become,

$$\begin{aligned} s_{\sigma,1} \sigma_t^2 &= Cov_t(e'_3 H G_t z_{t+1}, -\Lambda' H G_t z_{t+1}) \\ &= -\varphi_\sigma h'_3 \Lambda \sigma_t^2 - (\varphi_x s_{\sigma,x} h'_2) \Lambda \sigma_t^2, \end{aligned} \quad (C.47)$$

$$\begin{aligned} s_{\sigma,2} \sigma_t^2 + s_{q,2} q_t &= Cov_t(e_4 H G_t z_{t+1}, -\Lambda' H G_t z_{t+1}) \\ &= -(\varphi_x s_{q,x} h'_2) \Lambda \sigma_t^2 - (\varphi_q h'_4) \Lambda q_t. \end{aligned} \quad (C.48)$$

As before, the expected return variation and variance risk premium, may be conveniently expressed as,

$$\begin{aligned} ERV_t &= \frac{Q_{1,1}}{\rho_\sigma} (\mu_\sigma + \rho_\sigma \sigma_t^2) + \frac{Q_{1,2}}{\rho_q} (\mu_q + \rho_q q_t), \\ VRP_t &= Q_{2,1} \sigma_t^2 + Q_{2,2} q_t, \end{aligned}$$

where

$$\begin{aligned} Q_{1,1} &= \sum_{j=1,2,3,5} \Lambda'_d h_j h'_j \Lambda_d \rho_\sigma > 0 & Q_{1,2} &= \sum_{j=4} \Lambda'_d h_j h'_j \Lambda_d \rho_q > 0 \\ Q_{2,1} &= \frac{Q_{1,1}}{\rho_\sigma} s_{\sigma,1} + \frac{Q_{1,2}}{\rho_q} s_{\sigma,2} & Q_{2,2} &= \frac{Q_{1,2}}{\rho_q} s_{q,2}. \end{aligned}$$

D Detailed Derivations for Section 2.2

Substituting f_t by $Q^{-1}(X_t - \mu_X)$ in the basic relation $f_{t+1} = \mu + \rho f_t + S \epsilon_{t+1}$, it follows that

$$Q^{-1}X_{t+1} = \mu + Q^{-1}\mu_X - \rho Q^{-1}\mu_X + \rho Q^{-1}X_t + S \epsilon_{t+1}. \quad (\text{D.49})$$

Normalizing each element of $Q^{-1}X_{t+1}$ by the corresponding diagonal element of Q^{-1} , the model may be rewritten as,

$$BX_{t+1} = \tilde{\mu} + \tilde{\rho}BX_t + \tilde{S}\tilde{\epsilon}_{t+1},$$

where

$$B \equiv \left(\frac{1}{\text{diag}(Q^{-1})} \otimes I_{1 \times 4} \right) \odot Q^{-1}.$$

To match with equation (D.49),

$$\tilde{\mu} = \left(\frac{1}{\text{diag}(Q^{-1})} \otimes I_{1 \times 4} \right) \odot (\mu - \rho Q^{-1}\mu_X),$$

and

$$\begin{aligned} \tilde{\rho} &= \left[\left(\frac{1}{\text{diag}(Q^{-1})} \otimes I_{1 \times 4} \right) \odot (\rho Q^{-1}) \right] B^{-1} \\ &= \left[\left(\frac{1}{\text{diag}(Q^{-1})} \otimes I_{1 \times 4} \right) \odot (Q^{-1} \text{diag}(\rho) + (\rho - \text{diag}(\rho))Q^{-1}) \right] B^{-1} \\ &= \left[\left(\frac{1}{\text{diag}(Q^{-1})} \otimes I_{1 \times 4} \right) \odot (Q^{-1} \odot (\text{vec}(\text{diag}(\rho)) \otimes I_{1 \times 4}) + (\rho - \text{diag}(\rho))Q^{-1}) \right] B^{-1} \\ &= \left[B \odot (\text{vec}(\text{diag}(\rho)) \otimes I_{1 \times 4}) + \frac{\rho - \text{diag}(\rho)}{-A_{d,x}} B \right] B^{-1} \\ &= \rho + \frac{\rho - \text{diag}(\rho)}{-A_{d,x}} \end{aligned} \quad (\text{D.50})$$

or

Defining $\tilde{\epsilon}_{t+1}$ as

$$\tilde{\epsilon}_{t+1} \equiv \frac{1}{\text{diag}(Q^{-1})} \odot \epsilon_{t+1},$$

it follows again from equation (D.49) that

$$\tilde{S} = \left(\frac{1}{\text{diag}(Q^{-1})} \otimes I_{1 \times 4} \right) \odot S \odot \frac{1}{\text{diag}(Q^{-1})' \otimes I_{1 \times 4}}.$$

Based on the formula for Q in the main text, the inverse Q^{-1} and $\frac{1}{\text{diag}(Q^{-1})}$ may be expressed as,

$$Q^{-1} = \begin{pmatrix} \frac{1}{Q_{1,1}} & \frac{-Q_{1,2}}{Q_{1,1}Q_{2,2}} & 0 & 0 \\ 0 & \frac{1}{Q_{2,2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-A_{d,\sigma}}{Q_{1,1}A_{d,x}} & \frac{-Q_{1,1}A_{d,q} + Q_{1,2}A_{d,\sigma}}{Q_{1,1}Q_{2,2}A_{d,x}} & \frac{-A_{d,d}}{A_{d,x}} & -\frac{1}{A_{d,x}} \end{pmatrix} \quad \frac{1}{\text{diag}(Q^{-1})} = \begin{pmatrix} Q_{1,1} \\ Q_{2,2} \\ 1 \\ -A_{d,x} \end{pmatrix}$$

Combining the expressions for ρ and S , it therefore follows that

$$B = \begin{pmatrix} 1 & -\frac{Q_{1,2}}{Q_{2,2}} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{A_{d,\sigma}}{Q_{1,1}} & \frac{Q_{1,1}A_{d,q}-A_{d,\sigma}Q_{1,2}}{Q_{1,1}Q_{2,2}} & A_{d,d} & 1 \end{pmatrix} \quad \tilde{\rho} = \begin{pmatrix} \rho_\sigma & 0 & 0 & 0 \\ 0 & \rho_q & 0 & 0 \\ 0 & 0 & \rho_d & \frac{\phi_{dx}}{-A_{d,x}} \\ 0 & 0 & 0 & \rho_x \end{pmatrix}$$

$$\tilde{S} = \begin{pmatrix} 1 & 0 & 0 & \frac{Q_{1,1}}{-A_{d,x}} s_{\sigma,x} \\ \frac{Q_{2,2}}{Q_{1,1}} s_{q,\sigma} & 1 & 0 & \frac{Q_{2,2}}{-A_{d,x}} s_{q,x} \\ \frac{1}{Q_{1,1}} s_{d,\sigma} & \frac{1}{Q_{2,2}} s_{d,q} & 1 & \frac{1}{-A_{d,x}} s_{d,x} \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \tilde{\epsilon}_{t+1} = \begin{pmatrix} Q_{1,1} \\ Q_{2,2} \\ 1 \\ -A_{d,x} \end{pmatrix} \odot \epsilon_{t+1}.$$

D.1 Separate Volatility Dynamics

In the alternative setup with separate volatility dynamic, ρ , ϵ_{t+1} and S may be expressed as,

$$\rho = \begin{pmatrix} \rho_\sigma & 0 & 0 & 0 \\ 0 & \rho_q & 0 & 0 \\ 0 & 0 & \rho_d & \phi_{dx} \\ 0 & 0 & 0 & \rho_x \end{pmatrix} \quad \epsilon_{t+1} = \begin{pmatrix} \varphi_\sigma \sigma_t z_{\sigma,t+1} \\ \varphi_q \sqrt{q_t} z_{q,t+1} \\ \varphi_d \sigma_t z_{d,t+1} \\ \sqrt{q_t} z_{x,t+1} \end{pmatrix} \quad S = \begin{pmatrix} 1 & 0 & 0 & s_{\sigma,x} \\ 0 & 1 & 0 & s_{q,x} \\ s_{d,\sigma} & s_{d,q} & 1 & s_{d,x} \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (D.51)$$

$$X_t = \mu_X + Q f_t \quad Q = \begin{pmatrix} Q_{1,1} & Q_{1,2} & 0 & 0 \\ Q_{2,1} & Q_{2,2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -A_{d,\sigma} & -A_{d,q} & -A_{d,d} & -A_{d,x} \end{pmatrix}$$

Consequently,,

$$Q^{-1} = \begin{pmatrix} \frac{Q_{2,2}}{Q_{1,1}Q_{2,2}-Q_{1,2}Q_{2,1}} & \frac{-Q_{1,2}}{Q_{1,1}Q_{2,2}-Q_{1,2}Q_{2,1}} & 0 & 0 \\ \frac{-Q_{2,1}}{Q_{1,1}Q_{2,2}-Q_{1,2}Q_{2,1}} & \frac{Q_{1,1}}{Q_{1,1}Q_{2,2}-Q_{1,2}Q_{2,1}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{Q_{2,1}A_{d,q}-Q_{2,2}A_{d,\sigma}}{(Q_{1,1}Q_{2,2}-Q_{1,2}Q_{2,1})A_{d,x}} & \frac{-Q_{1,1}A_{d,q}+Q_{1,2}A_{d,\sigma}}{(Q_{1,1}Q_{2,2}-Q_{1,2}Q_{2,1})A_{d,x}} & \frac{-A_{d,d}}{A_{d,x}} & -\frac{1}{A_{d,x}} \end{pmatrix} \quad \frac{1}{\text{diag}(Q^{-1})} = \begin{pmatrix} \frac{Q_{1,1}Q_{2,2}-Q_{1,2}Q_{2,1}}{Q_{2,2}} \\ \frac{Q_{1,1}Q_{2,2}-Q_{1,2}Q_{2,1}}{Q_{1,1}} \\ 1 \\ -A_{d,x} \end{pmatrix}$$

Combining these expressions, it follows that

$$B = \begin{pmatrix} 1 & -\frac{Q_{1,2}}{Q_{2,2}} & 0 & 0 \\ \frac{-Q_{2,1}}{Q_{1,1}} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-Q_{2,1}A_{d,q}+Q_{2,2}A_{d,\sigma}}{Q_{1,1}Q_{2,2}-Q_{1,2}Q_{2,1}} & \frac{+Q_{1,1}A_{d,q}-Q_{1,2}A_{d,\sigma}}{Q_{1,1}Q_{2,2}-Q_{1,2}Q_{2,1}} & A_{d,d} & 1 \end{pmatrix} \quad \tilde{\rho} = \begin{pmatrix} \rho_\sigma & 0 & 0 & 0 \\ 0 & \rho_q & 0 & 0 \\ 0 & 0 & \rho_d & \frac{\phi_{dx}}{-A_{d,x}} \\ 0 & 0 & 0 & \rho_x \end{pmatrix}$$

$$\tilde{S} = \begin{pmatrix} 1 & 0 & 0 & \frac{Q_{1,1}Q_{2,2}-Q_{1,2}Q_{2,1}}{-Q_{2,2}A_{d,x}} s_{\sigma,x} \\ 0 & 1 & 0 & \frac{Q_{1,1}Q_{2,2}-Q_{1,2}Q_{2,1}}{-Q_{1,1}A_{d,x}} s_{q,x} \\ \frac{Q_{2,2}}{Q_{1,1}Q_{2,2}-Q_{1,2}Q_{2,1}} s_{d,\sigma} & \frac{Q_{1,1}}{Q_{1,1}Q_{2,2}-Q_{1,2}Q_{2,1}} s_{d,q} & 1 & \frac{1}{-A_{d,x}} s_{d,x} \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \tilde{\epsilon}_{t+1} = \begin{pmatrix} \frac{Q_{1,1}Q_{2,2}-Q_{1,2}Q_{2,1}}{Q_{2,2}} \\ \frac{Q_{1,1}Q_{2,2}-Q_{1,2}Q_{2,1}}{Q_{1,1}} \\ 1 \\ -A_{d,x} \end{pmatrix} \odot \epsilon_{t+1}.$$

D.2 Stochastic Volatility in the Long-Run

In the alternative setup with stochastic volatility in the long-run drift, ρ , ϵ_{t+1} and S may be expressed as,

$$\rho = \begin{pmatrix} \rho_\sigma & 1 & 0 & 0 \\ 0 & \rho_q & 0 & 0 \\ 0 & 0 & \rho_d & \phi_{dx} \\ 0 & 0 & 0 & \rho_x \end{pmatrix} \quad \epsilon_{t+1} = \begin{pmatrix} \varphi_\sigma \sigma_t z_{\sigma,t+1} \\ \varphi_q \sqrt{q_t} z_{q,t+1} \\ \varphi_d \sigma_t z_{d,t+1} \\ \varphi_x \sigma_t z_{x,t+1} \end{pmatrix} \quad S = \begin{pmatrix} 1 & 0 & 0 & s_{\sigma,x} \\ 0 & 1 & 0 & s_{q,x} \\ s_{d,\sigma} & s_{d,q} & 1 & s_{d,x} \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (D.52)$$

$$X_t = \mu_X + Q f_t \quad Q = \begin{pmatrix} Q_{1,1} & Q_{1,2} & 0 & 0 \\ Q_{2,1} & Q_{2,2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -A_{d,\sigma} & -A_{d,q} & -A_{d,d} & -A_{d,x} \end{pmatrix}$$

Consequently,

$$Q^{-1} = \begin{pmatrix} \frac{Q_{2,2}}{Q_{1,1}Q_{2,2}-Q_{1,2}Q_{2,1}} & \frac{-Q_{1,2}}{Q_{1,1}Q_{2,2}-Q_{1,2}Q_{2,1}} & 0 & 0 \\ \frac{-Q_{2,1}}{Q_{1,1}Q_{2,2}-Q_{1,2}Q_{2,1}} & \frac{Q_{1,1}}{Q_{1,1}Q_{2,2}-Q_{1,2}Q_{2,1}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{Q_{2,1}A_{d,q}-Q_{2,2}A_{d,\sigma}}{(Q_{1,1}Q_{2,2}-Q_{1,2}Q_{2,1})A_{d,x}} & \frac{-Q_{1,1}A_{d,q}+Q_{1,2}A_{d,\sigma}}{(Q_{1,1}Q_{2,2}-Q_{1,2}Q_{2,1})A_{d,x}} & \frac{-A_{d,d}}{A_{d,x}} & -\frac{1}{A_{d,x}} \end{pmatrix} \quad \frac{1}{\text{diag}(Q^{-1})} = \begin{pmatrix} \frac{Q_{1,1}Q_{2,2}-Q_{1,2}Q_{2,1}}{Q_{2,2}} \\ \frac{Q_{1,1}Q_{2,2}-Q_{1,2}Q_{2,1}}{Q_{1,1}} \\ 1 \\ -A_{d,x} \end{pmatrix}$$

Combining these expressions, it follows that

$$B = \begin{pmatrix} 1 & \frac{-Q_{1,2}}{Q_{2,2}} & 0 & 0 \\ \frac{-Q_{2,1}}{Q_{1,1}} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-Q_{2,1}A_{d,q}+Q_{2,2}A_{d,\sigma}}{Q_{1,1}Q_{2,2}-Q_{1,2}Q_{2,1}} & \frac{+Q_{1,1}A_{d,q}-Q_{1,2}A_{d,\sigma}}{Q_{1,1}Q_{2,2}-Q_{1,2}Q_{2,1}} & A_{d,d} & 1 \end{pmatrix} \quad \tilde{\rho} = \begin{pmatrix} \rho_\sigma & \frac{Q_{1,1}}{Q_{2,2}} & 0 & 0 \\ 0 & \rho_q & 0 & 0 \\ 0 & 0 & \rho_d & \frac{\phi_{dx}}{-A_{d,x}} \\ 0 & 0 & 0 & \rho_x \end{pmatrix}$$

$$\tilde{S} = \begin{pmatrix} 1 & 0 & 0 & \frac{Q_{1,1}Q_{2,2}-Q_{1,2}Q_{2,1}}{-Q_{2,2}A_{d,x}} s_{\sigma,x} \\ 0 & 1 & 0 & \frac{Q_{1,1}Q_{2,2}-Q_{1,2}Q_{2,1}}{-Q_{1,1}A_{d,x}} s_{q,x} \\ \frac{Q_{2,2}}{Q_{1,1}Q_{2,2}-Q_{1,2}Q_{2,1}} s_{d,\sigma} & \frac{Q_{1,1}}{Q_{1,1}Q_{2,2}-Q_{1,2}Q_{2,1}} s_{d,q} & 1 & \frac{1}{-A_{d,x}} s_{d,x} \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \tilde{\epsilon}_{t+1} = \begin{pmatrix} \frac{Q_{1,1}Q_{2,2}-Q_{1,2}Q_{2,1}}{Q_{2,2}} \\ \frac{Q_{1,1}Q_{2,2}-Q_{1,2}Q_{2,1}}{Q_{1,1}} \\ 1 \\ -A_{d,x} \end{pmatrix} \odot \epsilon_{t+1}.$$

Table D.1 Structural Factor GARCH Estimates—Separate Volatility Dynamics

The table reports the “structural” factor GARCH estimates for the alternative setup with separate volatility dynamics described in Sections C.1 and D.1, with the three restrictions: $A_{dd} = \frac{\rho_d}{1-\kappa_{d,1}\rho_d}$, $\Gamma_{4,4} + \Upsilon_{4,4} = \rho_q$, and $\Gamma_{3,3} = 0$. The resulting J -test with 7 degrees-of-freedom for the GMM-based estimation equals 26.31, corresponding to a p-value 0.0004.

B			ERV_{t+1}		VRP_{t+1}		Δd_{t+1}		d_{t+1}/p_{t+1}
ERV_{t+1}			1		-0.490	(0.117)	0		0
VRP_{t+1}			-0.022	(0.030)	1		0		0
Δd_{t+1}			0		0		1		0
d_{t+1}/p_{t+1}			-0.110	(0.141)	-1.595	(0.063)	-0.158		1
$\tilde{\rho}$	constant		ERV_t		VRP_t		Δd_t		d_t/p_t
ERV_{t+1}	0.009	(0.003)	0.827	(0.089)	0		0		0
VRP_{t+1}	0.008	(0.002)	0		0.312	(0.071)	0		0
Δd_{t+1}	-0.002	(0.016)	0		0		-0.187	(0.035)	-0.001 (0.004)
d_{t+1}/p_{t+1}	-0.080	(0.029)	0		0		0		0.980 (0.008)
$\tilde{S}$			$\tilde{\sigma}_{t+1}^2$		$\tilde{\epsilon}_{q,t+1}$		$\tilde{\epsilon}_{\Delta d,t+1}$		$\tilde{\epsilon}_{x,t+1}$
ERV_{t+1}			1		0		0		0.316 (0.038)
VRP_{t+1}			0		1		0		-0.245 (0.017)
Δd_{t+1}			-0.387	(0.080)	-0.134	(0.160)	1		0.095 (0.034)
d_{t+1}/p_{t+1}			0		0		0		1
$\tilde{\epsilon}$	ϖ_u		Γ		Υ				
ERV_{t+1}	0.001	(0.000)	0.153	(0.360)	0.388	(0.138)			
VRP_{t+1}	0.000	(0.000)	0.537	(0.335)	0.116	(0.079)			
Δd_{t+1}	0.001	(0.000)	0		0.449	(0.115)			
d_{t+1}/p_{t+1}	0.002	(0.000)	0.167		0.144	(0.106)			

Table D.2 Structural Model Implications—Separate Volatility Dynamics

The table reports the contemporaneous matrix Φ_0 , the reduced form matrix Φ , and the return equation, implied by the alternative “structural” factor GARCH model defined in Sections C.1 and D.1).

$\Phi_0^{-1} \equiv B^{-1}\tilde{S}$			ERV_{t+1}		VRP_{t+1}		Δd_{t+1}		d_{t+1}/p_{t+1}
ERV_{t+1}			1.011	(0.015)	0.496	(0.118)	0		0.198 (0.049)
VRP_{t+1}			0.022	(0.030)	1		0		-0.241 (0.017)
Δd_{t+1}			-0.387	(0.080)	-0.134	(0.160)	1		0.095 (0.034)
d_{t+1}/p_{t+1}			0.085	(0.134)	1.646	(0.079)	0.158	(0.025)	0.652 (0.047)
$\Phi \equiv B^{-1}\tilde{\rho}B$	constant		ERV_t		VRP_t		Δd_t		d_t/p_t
ERV_{t+1}	0.013	(0.003)	0.833	(0.091)	-0.255	(0.075)	0		0
VRP_{t+1}	0.008	(0.001)	0.012	(0.016)	0.306	(0.070)	0.000	(0.000)	0.000 (0.000)
Δd_{t+1}	-0.002	(0.016)	0.000	(0.000)	0.002	(0.006)	-0.187	(0.035)	-0.001 (0.004)
d_{t+1}/p_{t+1}	-0.066	(0.030)	0.002	(0.045)	-1.103	(0.130)	-0.185	(0.035)	0.980 (0.008)
GMM Implied Return Equation									
$r_{t,t+1}$	constant		ERV_t		VRP_t		Δd_t		d_t/p_t
	0.062	(0.029)	-0.002	(0.044)	1.074	(0.127)	-0.007	(0.002)	0.046 (0.009)
			$\tilde{\sigma}_{t+1}^2$		$\tilde{\epsilon}_{q,t+1}$		$\tilde{\epsilon}_{\Delta d,t+1}$		$\tilde{\epsilon}_{x,t+1}$
stru-shocks			-0.470	(0.213)	-1.734	(0.178)	0.846	(0.041)	-0.539 (0.036)

Table D.3 Structural Factor GARCH Estimates—Long-Run Stochastic Volatility

The table reports the “structural” factor GARCH estimates for the alternative setup with long-run stochastic volatility described in Sections C.2 and D.2, with the two restrictions: $A_{d,d} = \frac{\rho_d}{1-\kappa_{d,1}\rho_d}$ and $\Gamma_{3,3} = 0$. The resulting J -test with 6 degrees-of-freedom for the GMM-based estimation equals 37.02, corresponding to a p-value 0.0000.

B			ERV_{t+1}		VRP_{t+1}		Δd_{t+1}		d_{t+1}/p_{t+1}
ERV_{t+1}			1		0.000	(0.189)	0		0
VRP_{t+1}			0.120	(0.039)	1		0		0
Δd_{t+1}			0		0		1		0
d_{t+1}/p_{t+1}			-0.016	(0.086)	-2.007	(0.156)	-0.249		1
$\tilde{\rho}$	constant		ERV_t		VRP_t		Δd_t		d_t/p_t
ERV_{t+1}	0.003	(0.002)	1.001	(0.077)	-0.070	(0.255)	0		0
VRP_{t+1}	0.006	(0.001)	0		0.609	(0.079)	0		0
Δd_{t+1}	-0.000	(0.014)	0		0		-0.329	(0.040)	-0.001 (0.004)
d_{t+1}/p_{t+1}	-0.075	(0.028)	0		0		0		0.982 (0.007)
$\tilde{S}$			$\tilde{\epsilon}_{\sigma_{t+1}^2}$		$\tilde{\epsilon}_{q_{t+1}}$		$\tilde{\epsilon}_{\Delta d_{t+1}}$		$\tilde{\epsilon}_{x,t+1}$
ERV_{t+1}			1		0		0		0.332 (0.036)
VRP_{t+1}			0		1		0		-0.186 (0.016)
Δd_{t+1}			-0.524	(0.061)	-0.069	(0.147)	1		0.061 (0.024)
d_{t+1}/p_{t+1}			0		0		0		1
$\tilde{\epsilon}$	ϖ_u		Γ		Υ				
ERV_{t+1}	0.001	(0.000)	0.001	(0.051)	0.776	(0.062)			
VRP_{t+1}	0.000	(0.000)	0.000	(0.139)	0.322	(0.147)			
Δd_{t+1}	0.001	(0.000)	0		0.454	(0.095)			
d_{t+1}/p_{t+1}	0.002	(0.000)	0.766	(0.061)	0.160	(0.041)			

Table D.4 Structural Model Implications—Long-Run Stochastic Volatility

The table reports the contemporaneous matrix Φ_0 , the reduced form matrix Φ , and the return equation, implied by the alternative “structural” factor GARCH model in Sections C.2 and D.2.

$\Phi_0^{-1} \equiv B^{-1}\tilde{S}$			ERV_{t+1}		VRP_{t+1}		Δd_{t+1}		d_{t+1}/p_{t+1}
ERV_{t+1}			1.000	(0.023)	-0.000	(0.189)	0		0.332 (0.032)
VRP_{t+1}			-0.120	(0.037)	1		0		-0.226 (0.025)
Δd_{t+1}			-0.524	(0.061)	-0.069	(0.147)	1		0.061 (0.024)
d_{t+1}/p_{t+1}			-0.356	(0.052)	1.990	(0.158)	0.249	(0.023)	0.568 (0.051)
$\Phi \equiv B^{-1}\tilde{\rho}B$	constant		ERV_t		VRP_t		Δd_t		d_t/p_t
ERV_{t+1}	0.003	(0.002)	0.993	(0.089)	-0.070	(0.194)	0		0
VRP_{t+1}	0.005	(0.001)	-0.046	(0.020)	0.618	(0.069)	0		0
Δd_{t+1}	-0.000	(0.014)	0.000	(0.000)	0.003	(0.007)	-0.328	(0.040)	-0.001 (0.004)
d_{t+1}/p_{t+1}	-0.065	(0.029)	-0.093	(0.041)	-0.732	(0.174)	-0.326	(0.041)	0.982 (0.007)
GMM Implied Return Equation									
$r_{t,t+1}$	constant		ERV_t		VRP_t		Δd_t		d_t/p_t
	0.063	(0.027)	0.090	(0.040)	0.714	(0.169)	-0.011	(0.002)	0.045 (0.008)
			$\tilde{\epsilon}_{\sigma_{t+1}^2}$		$\tilde{\epsilon}_{q_{t+1}}$		$\tilde{\epsilon}_{\Delta d_{t+1}}$		$\tilde{\epsilon}_{x,t+1}$
stru-shocks			-0.178	(0.124)	-2.004	(0.181)	0.758	(0.050)	-0.491 (0.035)

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