Concise presentation of the Coulomb electrostatic potential of a uniformly charged cube

Orion Ciftja^{1,*}

¹*Department of Physics, Prairie View A&M University, Prairie View, TX 77446, USA*

(Dated: March 31, 2015)

We use a novel method to calculate in closed form the Coulomb electrostatic potential created by a uniformly charged cube at an arbitrary point in space. We apply a suitable transformation of variables that allows us to obtain a simple presentation of the electrostatic potential in one-dimensional integral form. The final concise closed form expression of the Coulomb electrostatic potential of the uniformly charged cube is obtained after completing the calculation of the resulting one-dimensional integrals. Such integrals consist of combinations of products of error functions and power functions that can be solved exactly despite their intimidating appearance. The exact analytic formula for the Coulomb electrostatic potential that we derive reflects the symmetry of the cube and is easy to implement. We illustrate its use by calculating the exact values of the electrostatic potential at some points of symmetry such as the center of cube, center of face of cube, center of edge of cube and corner of cube.

PACS numbers: 73.43.Cd.

I. INTRODUCTION

Calculation of the electrostatic potential created by charged bodies is very important to many fields [1–3]. Obviously, if the charged body under consideration has arbitrary shape and if the distribution of charge is arbitrary, then no exact analytical results are available. However, if the body has regular features/symmetries and is uniformly charged, then one may have a chance to calculate its electrostatic potential in analytic form. That's why regular bodies have always drawn a huge interest over long time periods for their importance to various models in physics and mathematics. In particular, uniformly charged bodies with spherical symmetry (spheres, spherical shells, etc) or cylindrical symmetry (cylinders, disks, etc) allow one to obtain exact expressions [4, 5] for the electrostatic potential in some cases quite trivially (for instance, the case of spheres). On the other hand, uniformly charged bodies with nonspherical shape are notoriously more challenging. Interest on them stems from the fact that geometric figures with nonspherical shape (for instance, polyhedra where the cube is the simplest of them) can be used as a model for irregular bodies. Therefore, it is not surprising to find out that some of the earlier work found in the literature dealt with cubes. In particular, astrophysical studies of gravitational forces dealt with the calculation of the gravitational potential created by a uniformly filled (homogeneous) polyhedra (with cube as a special case) [6, 7]. To our knowledge, the gravitational potential of a uniformly filled parallelepiped (cube) was apparently first calculated explicitly by MacMillan [8]. Calculations are difficult and the work represents a masterful use of direct integration techniques. The final analytic expression ob-

tained in closed form contains a very large number of terms and is quite long [8].

The key objective of this work is to calculate the Coulomb electrostatic potential created by a uniformly charged cube at some arbitrary point in space by adopting a different approach that simplifies the calculations and leads to a much more concise presentation of the final result. The method used differs from the direct integration approach implemented by other authors [8–11]. Another benefit of this method is that it allows us to obtain a very simple presentation of the electrostatic potential as a one-dimensional integral function. Explicit calculation of the resulting one-dimensional integrals leads us to the final closed form exact result, a compact sum of only eight terms.

The model under consideration consists of a cube with length, L uniformly filled with some positive charge, Q . We want to calculate exactly the Coulomb electrostatic potential created by such a uniformly charged cube at some arbitrary point, $\vec{r} = (x, y, z)$ in three dimensions (3D). We write the expression for the electrostatic potential at any arbitrary point in 3D space as:

$$V(x, y, z) = k_e \rho_0 \int_D d^3 r' \frac{1}{|\vec{r} - \vec{r}'|}, \quad (1)$$

where k_e is Coulomb's electric constant, $\rho_0 = Q/L^3$ is the uniform 3D charge density of the cube, $\vec{r}'$ and $\vec{r}$ are 3D position vectors and D is the cubic domain of integration. We choose the origin of a Cartesian system of coordinates at the center of cube and the axes parallel to its edges, thus:

$$D : -\frac{L}{2} \leq x', y', z' \leq +\frac{L}{2}. \quad (2)$$

Our approach hinges upon a suitable transformation of $1/|\vec{r} - \vec{r}'|$ as a function of auxiliary variables, a transformation that eventually leads us to a simpler expression for the electrostatic potential. In the present case,

---

*Electronic address: ogciftja@pvamu.edu

the method leads to a very concise presentation of the potential, firstly, in one-dimensional integral form and, secondly, in closed analytic form, once the integrals are calculated. The integral presentation of the Coulomb electrostatic potential is convenient for numerical calculations since the integrand function is easy to handle numerically. Even though the calculation of the resulting one-dimensional integrals is challenging, we were able to calculate them exactly in analytic form. Differently from the direct integration approach sketched in Appendix A, our method leads to certain type of integrals that are not commonly found in textbooks. For this reason, we deem it important to provide a detailed description of several integration formulae that we derive. These mathematical results can be found in Appendix B, Appendix C, Appendix D, Appendix E and Appendix F. These formulae allow us to complete the calculations rather easily, thus, they are essential to this work. Only after having obtained all the required analytical formulae for various non-standard integrals appearing during the solution process, we proceed to obtain an explicit analytical expression. Since the calculation of the electrostatic potential at points of symmetry is important, we also provide the exact values of the electrostatic potential at the center of cube, center of face of cube, center of edge of cube and corner of cube.

In Section II we explain the general solution method. In Section III, Section IV, Section V and Section VI we list the results for the value of the Coulomb electrostatic potential created by a uniformly charged cube at given special points of symmetry. The general closed form result for the Coulomb electrostatic potential created by

an arbitrarily charged cube in the surrounding space is given in Section VII. A brief summary of the work is given in Section VIII.

II. METHOD

As outlined in Appendix A, calculation of $V(x, y, z)$ in Eq.(1) by direct integration does not appear to be extremely challenging at the beginning. However, as shown by many authors [8–11], this approach grows in complexity very quickly and, at the end, one has to deal with long cumbersome mathematical expressions. As a result, one is always tempted to find alternative routes to solve the problem in a quicker and more transparent way. In this work we solve the problem by adopting a different approach that does not start with direct integration. To this effect, we write the quantity $1/|\vec{r} - \vec{r}'|$ in Eq.(1) as an integral function of a new auxiliary variable:

$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{2}{\sqrt{\pi}} \int_0^\infty du e^{-u^2 (\vec{r} - \vec{r}')^2} . \quad (3)$$

The anticipation is that, this way, the integration over the variables x' , y' and z' that appear in the expression in Eq.(1) will be carried out without much difficulty. Note that:

$$(\vec{r} - \vec{r}')^2 = (x - x')^2 + (y - y')^2 + (z - z')^2 . \quad (4)$$

By substituting the result from Eq.(3) into Eq.(1), one has:

$$V(x, y, z) = k_e \rho_0 \frac{2}{\sqrt{\pi}} \int_0^\infty du \int_{-\frac{L}{2}}^{+\frac{L}{2}} dx' e^{-u^2 (x-x')^2} \int_{-\frac{L}{2}}^{+\frac{L}{2}} dy' e^{-u^2 (y-y')^2} \int_{-\frac{L}{2}}^{+\frac{L}{2}} dz' e^{-u^2 (z-z')^2} . \quad (5)$$

The integrals with respect to x' , y' and z' are easy to calculate [12, 13]. In addition, we can simplify the expressions if we introduce dimensionless variables: $X = x/L$, $Y = y/L$, $Z = z/L$ and $X' = x'/L$, $Y' = y'/L$, $Z' = z'/L$. With the change of variables, we have: $-1/2 \leq X', Y', Z' \leq +1/2$. In terms of the new variables, $du dx' dy' dz' = L^2 dt dX' dY' dZ'$, where $t = uL$ is another dummy variable. Straightforward calculations lead us to the following expression for the Coulomb electrostatic potential created by a uniformly charged cube:

$$V(X, Y, Z) = \frac{k_e Q}{L} \frac{2}{\sqrt{\pi}} \int_0^\infty dt f(t, X) f(t, Y) f(t, Z) , \quad (6)$$

where $f(t, X)$, $f(t, Y)$ and $f(t, Z)$ are auxiliary functions. Specifically:

$$f(t, X) = \frac{\sqrt{\pi}}{2t} \left\{ \operatorname{erf} \left[t \left(\frac{1}{2} - X \right) \right] + \operatorname{erf} \left[t \left(\frac{1}{2} + X \right) \right] \right\} , \quad (7)$$

where

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt e^{-t^2} , \quad (8)$$

is an error function (See Pg. 297 of Ref. [14]). Note that $f(t, Y)$ and $f(t, Z)$ are obtained from Eq.(7) by, respectively, replacing X with Y and X with Z . Note that the form of the auxiliary functions reflects the symmetry of the problem under consideration. For instance, one can see that:

$$f(t, -X) = f(t, X) . \quad (9)$$

Obviously, the result in Eq.(9) mirrors the fact that $V(-X, Y, Z) = V(X, Y, Z)$ which is easy to deduce from the cubic symmetry. The integral presentation in Eq.(6) is not only simple, but also very convenient for numerical calculations since the auxiliary functions in the integrand are smooth and do not have singularities. For instance, it is easy to verify that, for a given finite value of X :

$$\lim_{t \rightarrow 0} f(t, X) = 1 \quad ; \quad \lim_{t \rightarrow \infty} f(t, X) = 0 . \quad (10)$$

The usefulness of the result in Eq.(6) is self-evident if, for example, one wants to calculate the Coulomb self-energy (E_S) of a uniformly charged cube:

$$E_S = \frac{k_e \rho_0^2}{2} \int_D d^3 r \int_D d^3 r' \frac{1}{|\vec{r} - \vec{r}'|} = \frac{\rho_0}{2} \int_D d^3 r V(\vec{r}) . \quad (11)$$

It is straightforward to prove that:

$$E_S = \frac{Q}{2} \int_{-\frac{1}{2}}^{+\frac{1}{2}} dX \int_{-\frac{1}{2}}^{+\frac{1}{2}} dY \int_{-\frac{1}{2}}^{+\frac{1}{2}} dZ V(X, Y, Z) . \quad (12)$$

If one substitutes the expression of $V(X, Y, Z)$ from Eq.(6) into Eq.(12), the result is:

$$E_S = \frac{k_e Q^2}{L} \frac{1}{\sqrt{\pi}} \int_0^\infty dt [g(t)]^3 , \quad (13)$$

where

$$g(t) = \int_{-\frac{1}{2}}^{+\frac{1}{2}} dX f(t, X) = \int_{-\frac{1}{2}}^{+\frac{1}{2}} dY f(t, Y) = \int_{-\frac{1}{2}}^{+\frac{1}{2}} dZ f(t, Z) . \quad (14)$$

The specific form of the function $g(t)$ as well as the final result for E_S can be found in Ref. [15].

At this point, we reiterate that our ultimate goal is to obtain a closed form exact solution of the problem. Thus, we do not want to limit ourselves to deriving one-dimensional integral presentations (that, nevertheless, are quite convenient for numerical calculations). Therefore, we must overcome the last remaining obstacle towards this goal, namely, we must calculate the one-dimensional integrals appearing in Eq.(6). To this effect, we rewrite the expression in Eq.(6) in a more explicit form as:

$$V(X, Y, Z) = \frac{k_e Q}{L} \frac{\pi}{16} \int_0^\infty \frac{du}{u^3} \left\{ \operatorname{erf}[u(1-2X)] + \operatorname{erf}[u(1+2X)] \right\} \left\{ \operatorname{erf}[u(1-2Y)] + \operatorname{erf}[u(1+2Y)] \right\} \left\{ \operatorname{erf}[u(1-2Z)] + \operatorname{erf}[u(1+2Z)] \right\} , \quad (15)$$

where we introduced the dummy variable, $u = t/2$ to simplify the calculations. A look at Eq.(15) indicates that an exact calculation of $V(X, Y, Z)$ hinges upon the calculation of integrals of the form:

$$F(a, b, c) = \int_0^\infty dx \frac{\operatorname{erf}(ax) \operatorname{erf}(bx) \operatorname{erf}(cx)}{x^3} , \quad (16)$$

where a, b, c are arbitrary real constants. There are $2^3 = 8$ integrals of this form in Eq.(15). Clearly, the explicit calculation of $F(a, b, c)$, where a, b and c are real parameters is a key moment. One expects the calculations to be simpler for special values of the parameters, a, b and c which typically would correspond to points of symmetry in the cube (a cube has 6 faces, 12 edges and 8 corners). We identify these important points of symmetry as the center of cube ($X = 0, Y = 0, Z = 0$), center of face of cube ($X = 1/2, Y = 0, Z = 0$) (one of the 6 faces), center of edge of cube ($X = 1/2, Y = 1/2, Z = 0$ (one of the 12 edges) and corner of cube ($X = 1/2, Y = 1/2, Z = 1/2$) (one of the 8 corners). Therefore, we first calculate the

value of the electrostatic potential created by the uniformly charged cube at these points of symmetry, before providing the general closed form expression for the electrostatic potential at any arbitrary point in space.

III. ELECTROSTATIC POTENTIAL AT THE CENTER OF CUBE

The center of cube corresponds to the origin of the coordinative system, ($X = 0, Y = 0, Z = 0$). If we start from Eq.(15), the expression for the electrostatic potential created by the uniformly charged cube at its own center becomes:

$$V(X = 0, Y = 0, Z = 0) = \frac{k_e Q}{L} \frac{\pi}{2} \int_0^\infty du \left[\frac{\operatorname{erf}(u)}{u} \right]^3 . \quad (17)$$

We now use the formula derived in Appendix B. As a result, we can write the value of the electrostatic potential created by a uniformly charged cube at its center in

several equivalent forms with some of them listed below:

$$\begin{aligned}
 V(X=0, Y=0, Z=0) &= \left[3 \ln \frac{\sqrt{3}+1}{\sqrt{3}-1} - \frac{\pi}{2} \right] \frac{k_e Q}{L} = \left[6 \ln \frac{\sqrt{3}+1}{\sqrt{2}} - \frac{\pi}{2} \right] \frac{k_e Q}{L} = \\
 &= \left[3 \ln (2 + \sqrt{3}) - \frac{\pi}{2} \right] \frac{k_e Q}{L} = \left[6 \sinh^{-1} \left(\frac{1}{\sqrt{2}} \right) - \frac{\pi}{2} \right] \frac{k_e Q}{L} \approx \\
 &\approx 2.380077364(0) \frac{k_e Q}{L}.
 \end{aligned} \tag{18}$$

In Eq.(18) we also display the numerically approximated value of the potential with an accuracy of ten digits after the decimal point where the last digit has been rounded. The function $\sinh^{-1}(x)$ appearing in Eq.(18) represents the inverse hyperbolic sine function (See Pg. 87 of Ref. [14]) which reads:

$$\sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1}), \tag{19}$$

where $\ln(x)$ is the usual natural logarithm function. We remind the reader that there are other common nota-

tions used in the literature for the inverse hyperbolic functions, for instance, $\sinh^{-1}(x)$, may be found written as $\operatorname{arcsinh}(x)$, $\operatorname{ArcSinh}(x)$, $\operatorname{arcsinh}$, and so on. As shown in Appendix B, we have $\sinh^{-1}(1/\sqrt{2}) = \ln \left(\frac{\sqrt{3}+1}{\sqrt{2}} \right)$.

IV. ELECTROSTATIC POTENTIAL AT THE CENTER OF FACE OF CUBE

One can easily verify from Eq.(15) that:

$$V\left(X = \pm \frac{1}{2}, Y = 0, Z = 0\right) = V\left(X = 0, Y = \pm \frac{1}{2}, Z = 0\right) = V\left(X = 0, Y = 0, Z = \pm \frac{1}{2}\right). \tag{20}$$

The six points above represent the centers of each of the six faces of the cube. Starting from Eq.(15), we obtain:

$$V\left(X = \frac{1}{2}, Y = 0, Z = 0\right) = \frac{k_e Q}{L} \frac{\pi}{4} \int_0^\infty du \frac{\operatorname{erf}(2u) \operatorname{erf}(u)^2}{u^3}. \tag{21}$$

Integrals of this form have been calculated in Appendix C. As a result, we can write the value of the electrostatic potential created by a uniformly charged cube at the center of its own face as:

$$\begin{aligned}
 V\left(X = \frac{1}{2}, Y = 0, Z = 0\right) &= \left[4 \sinh^{-1} \left(\frac{1}{\sqrt{5}} \right) + \sinh^{-1}(\sqrt{2}) - 2 \tan^{-1} \left(\frac{1}{2\sqrt{6}} \right) - \tan^{-1} \left(\frac{2}{\sqrt{6}} \right) \right] \frac{k_e Q}{L} = \\
 &= \left[4 \ln \frac{1 + \sqrt{6}}{\sqrt{5}} + \ln \frac{2 + \sqrt{6}}{\sqrt{2}} - 2 \tan^{-1} \left(\frac{1}{2\sqrt{6}} \right) - \tan^{-1} \left(\frac{2}{\sqrt{6}} \right) \right] \frac{k_e Q}{L} \approx \\
 &\approx 1.792810243(2) \frac{k_e Q}{L},
 \end{aligned} \tag{22}$$

where we numerically approximated the exact value with an accuracy of ten digits after the decimal point rounding the last digit. The function $\tan^{-1}(x)$ in Eq.(22) is the

inverse tangent trigonometric function. We remind the reader that there are other common notations used in the literature for the inverse trigonometric functions, for

instance, $\tan^{-1}(x)$, may be found in the literature written as $\arctan(x)$, $\text{ArcTan}(x)$, $\text{arctg}(x)$, and so on.

V. ELECTROSTATIC POTENTIAL AT THE CENTER OF EDGE OF CUBE

One can easily verify from Eq.(15) that:

$$V\left(X = \pm\frac{1}{2}, Y = \pm\frac{1}{2}, Z = 0\right) = V\left(X = 0, Y = \pm\frac{1}{2}, Z = \pm\frac{1}{2}\right) = V\left(X = \pm\frac{1}{2}, Y = 0, Z = \pm\frac{1}{2}\right). \quad (23)$$

These twelve points represent the centers of each of the twelve edges of the cube. If we start from Eq.(15), we have:

$$V\left(X = \frac{1}{2}, Y = \frac{1}{2}, Z = 0\right) = \frac{k_e Q}{L} \frac{\pi}{8} \int_0^\infty du \frac{\text{erf}(2u)^2 \text{erf}(u)}{u^3}. \quad (24)$$

This type of integral has been calculated in Appendix D. As a result, we write the value of the electrostatic potential created by a uniformly charged cube at the center of edge as:

$$V\left(X = \frac{1}{2}, Y = \frac{1}{2}, Z = 0\right) = \left[2 \sinh^{-1}\left(\frac{1}{\sqrt{8}}\right) + 2 \sinh^{-1}\left(\frac{2}{\sqrt{5}}\right) - 2 \tan^{-1}\left(\frac{1}{3}\right) - \frac{1}{4} \tan^{-1}\left(\frac{4}{3}\right)\right] \frac{k_e Q}{L}. \quad (25)$$

Note that:

$$\sinh^{-1}\left(\frac{1}{\sqrt{8}}\right) = \ln\left(\sqrt{2}\right) \quad ; \quad \sinh^{-1}\left(\frac{2}{\sqrt{5}}\right) = \ln\left(\sqrt{5}\right). \quad (26)$$

With a little bit of algebra, one can write the result in Eq.(25) as:

$$V\left(X = \frac{1}{2}, Y = \frac{1}{2}, Z = 0\right) = \left[\ln(10) - 2 \tan^{-1}\left(\frac{1}{3}\right) - \frac{1}{4} \tan^{-1}\left(\frac{4}{3}\right)\right] \frac{k_e Q}{L} \approx 1.427260179(7) \frac{k_e Q}{L}, \quad (27)$$

where we numerically approximated the exact value to ten digits after the decimal point rounding the last digit.

VI. ELECTROSTATIC POTENTIAL AT THE CORNER OF CUBE

One can easily verify from Eq.(15) that:

$$V\left(X = \pm\frac{1}{2}, Y = \pm\frac{1}{2}, Z = +\frac{1}{2}\right) = V\left(X = \pm\frac{1}{2}, Y = \pm\frac{1}{2}, Z = -\frac{1}{2}\right). \quad (28)$$

The eight points listed above represent each of the eight corners (vertices) of the cube. Starting from Eq.(15), we obtain:

$$V\left(X = \frac{1}{2}, Y = \frac{1}{2}, Z = \frac{1}{2}\right) = \frac{k_e Q}{L} \frac{\pi}{16} \int_0^\infty du \frac{\text{erf}(2u)^3}{u^3}. \quad (29)$$

This type of integral has been calculated in Appendix E. As a result, we write the value of the electrostatic potential created by uniformly charged cube at its corner as:

$$V\left(X = \frac{1}{2}, Y = \frac{1}{2}, Z = \frac{1}{2}\right) = \frac{1}{2} \left[3 \ln \frac{\sqrt{3}+1}{\sqrt{3}-1} - \frac{\pi}{2} \right] \frac{k_e Q}{L} \approx 1.190038682(0) \frac{k_e Q}{L}, \quad (30)$$

where we numerically approximated the exact value to ten digits after the decimal point rounding the last digit. By comparing Eq.(30) to Eq.(18), one notes that the Coulomb electrostatic potential at the corner of cube is half the electrostatic potential at the center of cube. This result can be proven rigorously:

$$V\left(X = \frac{1}{2}, Y = \frac{1}{2}, Z = \frac{1}{2}\right) = \frac{1}{2} V(X = 0, Y = 0, Z = 0). \quad (31)$$

Interestingly, the same result, namely, that the value of the potential at the corner is half the value of the poten-

tial at the center is also true for the case of a uniformly charged square [16].

VII. ELECTROSTATIC POTENTIAL AT AN ARBITRARY POINT

One can write the expression for the Coulomb electrostatic potential created by a uniformly charged cube at an arbitrary point in space as:

$$\begin{aligned} V(X, Y, Z) = & \frac{k_e Q}{L} \frac{\pi}{16} \int_0^\infty \frac{du}{u^3} \{ \text{erf}[u(1-2X)] \text{erf}[u(1-2Y)] \text{erf}[u(1-2Z)] + \\ & \text{erf}[u(1-2X)] \text{erf}[u(1+2Y)] \text{erf}[u(1-2Z)] + \\ & \text{erf}[u(1+2X)] \text{erf}[u(1-2Y)] \text{erf}[u(1-2Z)] + \\ & \text{erf}[u(1+2X)] \text{erf}[u(1+2Y)] \text{erf}[u(1-2Z)] + \\ & \text{erf}[u(1-2X)] \text{erf}[u(1-2Y)] \text{erf}[u(1+2Z)] + \\ & \text{erf}[u(1-2X)] \text{erf}[u(1+2Y)] \text{erf}[u(1+2Z)] + \\ & \text{erf}[u(1+2X)] \text{erf}[u(1-2Y)] \text{erf}[u(1+2Z)] \\ & \text{erf}[u(1+2X)] \text{erf}[u(1+2Y)] \text{erf}[u(1+2Z)] \} . \end{aligned} \quad (32)$$

Each of the $2^3 = 8$ integrals in the expression above is a function of the coordinates (X, Y, Z) . Such a function was denoted as $F(a, b, c)$ in Eq.(16). The parameters a , b and c depend, respectively, on the values of X , Y and Z , thus, are considered real. One can immediately identify them by looking at the specific form of the integrals in Eq.(32). For instance, the first integral in Eq.(32), would correspond to: $a = 1 - 2X$, $b = 1 - 2Y$ and

$c = 1 - 2Z$ and so on. The explicit analytic expression of $F(a, b, c) = \int_0^\infty dx \frac{\text{erf}(ax) \text{erf}(bx) \text{erf}(cx)}{x^3}$ is given in Appendix F. Based on this information, one can write the expression of the Coulomb electrostatic potential created by a uniformly charged cube at an arbitrary point in 3D space as:

$$V(X, Y, Z) = \frac{k_e Q}{L} \frac{\pi}{16} \{ F(1-2X, 1-2Y, 1-2Z) + F(1-2X, 1+2Y, 1-2Z) + F(1+2X, 1-2Y, 1-2Z) + F(1+2X, 1+2Y, 1-2Z) + F(1-2X, 1-2Y, 1+2Z) + F(1-2X, 1+2Y, 1+2Z) + F(1+2X, 1-2Y, 1+2Z) + F(1+2X, 1+2Y, 1+2Z) \}. \quad (33)$$

We can simplify the presentation of the result further by setting up Table. I that lists all the possible values of the parameters, (a_i, b_i, c_i) for $i = 1, \dots, 8$. As a result, one

TABLE I: All values of (a_i, b_i, c_i) where $i = 1, \dots, 8$.

i	a_i	b_i	c_i
1	$1-2X$	$1-2Y$	$1-2Z$
2	$1-2X$	$1+2Y$	$1-2Z$
3	$1+2X$	$1-2Y$	$1-2Z$
4	$1+2X$	$1+2Y$	$1-2Z$
5	$1-2X$	$1-2Y$	$1+2Z$
6	$1-2X$	$1+2Y$	$1+2Z$
7	$1+2X$	$1-2Y$	$1+2Z$
8	$1+2X$	$1+2Y$	$1+2Z$

may write the expression in Eq.(33) in a very concise and compact way as:

$$V(X, Y, Z) = \frac{k_e Q}{L} \frac{\pi}{16} \sum_{i=1}^8 F(a_i, b_i, c_i), \quad (34)$$

where the closed form analytic expression of $F(a, b, c)$ is given in Appendix F and the parameters, (a, b, c) are listed in Table. I.

One expects $V(X, Y, Z)$ to be proportional to $1/\sqrt{X^2 + Y^2 + Z^2}$ at large distances. Therefore, it is quite instructive to compare the exact potential $V(X, Y, Z)$ given from Eq.(33) or Eq.(34) to the Coulomb potential created by a point charge, Q localized at the center of the cube. Such a potential, in terms of dimensionless coordinates, $X = x/L$, $Y = y/L$ and $Z = z/L$ is written as:

$$V_C(X, Y, Z) = \frac{k_e Q}{L} \frac{1}{\sqrt{X^2 + Y^2 + Z^2}}. \quad (35)$$

We select two directions, $(X, Y = 0, Z = 0)$ and $(X, Y = X, Z = X)$ for such a comparison. The Coulomb potential of the point charge along these two directions is:

$$V_C(X, 0, 0) = \frac{k_e Q}{L} \frac{1}{|X|} ; \quad V_C(X, X, X) = \frac{1}{\sqrt{3}} \frac{k_e Q}{L} \frac{1}{|X|}. \quad (36)$$

It is clearly shown in Fig. 1 that the electrostatic potential due to the uniformly charged cube, $V(X, Y, Z)$ behaves like a Coulomb potential, $V_C(X, Y, Z)$ at large distances.

FIG. 1: Electrostatic potential, $V(X, Y, Z)$ created by a uniformly charged cube along the $(X, Y = 0, Z = 0)$ direction (dashed line) and the $(X, Y = X, Z = X)$ direction (dotted line). The Coulomb's electrostatic potential for the two corresponding directions is denoted, respectively, as $V_C(X, 0, 0)$ and $V_C(X, X, X)$ (solid lines). Note that $V_C(X, X, X) = V_C(X, 0, 0)/\sqrt{3}$. The electrostatic potential is given in units of $k_e Q/L$ where Q is the total charge and L is the length of the cube (k_e is Coulomb's electric constant). The dimensionless coordinates are defined as $X = x/L$, $Y = y/L$ and $Z = z/L$. The origin of the 3D Cartesian system of coordinates is chosen at the center of the cube with the axes parallel to its edges.

VIII. CONCLUSIONS

To conclude, we obtained an exact analytical expression for the Coulomb electrostatic potential created by a uniformly charged cube at an arbitrary point in 3D space. We used a suitable transformation of variables that allowed us to obtain a simple presentation of the electrostatic potential in one-dimensional integral form. We obtain the closed form exact solution of the problem after completing the calculation of the resulting one-dimensional integrals. Such integrals consist of combinations of products of error functions and power functions and are not standard. Even though they look challenging to the point of defying an analytic solution, we show in this work that they can be calculated exactly. We illus-

trate the results obtained by calculating the electrostatic potential created by a uniformly charged cube at some points of symmetry such as the center of cube, center of face of cube, center of edge of cube and corner of cube.

We believe that the exact analytical result shown in Eq.(34) represents the most compact, symmetric and easy to implement expression for the Coulomb electrostatic potential of a uniformly charged cube. We verified that the formula obtained reproduces exactly the known values for the electrostatic potential at all points of symmetry of the cube in agreement with other results [8–10]. We also compared the values of the exact analytical Coulomb electrostatic potential to corresponding numerical solutions of the same problem. We found that the results are in agreement to the desired level of numerical accuracy (for instance, ten digits after the decimal point) in all cases considered. Thus, we are confident that our final expression is equivalent to previous results. At this juncture, we remark that all previously available closed form expressions of the electrostatic potential of a uniformly cube were derived by mainly using direct integration methods [8–10]. This means that our approach to the problem is quite different. We do not start with direct integration. We first obtain a convenient one-dimensional integral presentation of the potential and, only afterwards, we solve the resulting one-dimensional integrals exactly. After researching the literature, we became aware of only one previously published work that adopts a similar strategy. This work [17] deals with the energy per particle in an infinite Wigner lattice of electrons in a uniform compensating background. In the process, the authors obtain an expression for the electrostatic potential of the uniformly charged cube as a one-dimensional integral and then proceed to calculate the resulting one-dimensional integral numerically. At first sight, the result appearing in Eq.(2.6) of Ref. [17] (a one-dimensional integral presentation of the electrostatic potential) looks quite different from our result in Eq.(6). However, a more in-depth analysis of the results shows

that the two one-dimensional integral expressions can be proven to be equivalent. The proof of such an equivalence is provided in Appendix. G.

The exact compact formula derived in Eq.(34) can be useful to studies of finite electronic systems in a cubic geometry. Finite systems where particles are confined in a small region of space are known as quantum dots [18, 19]. A situation where few electrons are confined in a depleted positively charged jellium region with a cubic shape ("the donor charge" region) would fit well this description [20–22]. The results obtained can also help to gauge the accuracy of various theoretical approximations and numerical methods used in computational physical sciences.

Acknowledgments

This research was supported in part by US Army Research Office (ARO) grant no. W911NF-13-1-0139 and National Science Foundation (NSF) grants no. DMR-1104795 and DMR-1410350.

Appendix A: Direct Integration

Let the origin of a Cartesian system of coordinates be at the center of the cube and let the respective axes be parallel to its edges. The cube has length, L and is uniformly charged by a total amount of charge, Q . To simplify notation, let's introduce dimensionless variables of the form:

$$\vec{r}' = \frac{\vec{r}'}{L} = (X', Y', Z') \quad ; \quad \vec{R} = \frac{\vec{r}}{L} = (X, Y, Z) . \quad (\text{A1})$$

In terms of the dimensionless variables above, the Coulomb electrostatic potential in Eq.(1) can be expressed as:

$$V(X, Y, Z) = \frac{k_e Q}{L} \int_{-\frac{1}{2}}^{+\frac{1}{2}} dX' \int_{-\frac{1}{2}}^{+\frac{1}{2}} dY' \int_{-\frac{1}{2}}^{+\frac{1}{2}} dZ' \frac{1}{\sqrt{(X - X')^2 + (Y - Y')^2 + (Z - Z')^2}} . \quad (\text{A2})$$

One can rewrite the result in Eq.(A2) in a more symmetric form by introducing new variables:

$$u = X' - X \quad ; \quad v = Y' - Y \quad ; \quad w = Z' - Z . \quad (\text{A3})$$

This transformation of variables allows us to write the electrostatic potential in Eq.(A2) as:

$$V(X, Y, Z) = \frac{k_e Q}{L} \int_{-\frac{1}{2}-X}^{+\frac{1}{2}-X} du \int_{-\frac{1}{2}-Y}^{+\frac{1}{2}-Y} dv \int_{-\frac{1}{2}-Z}^{+\frac{1}{2}-Z} dw \frac{1}{\sqrt{u^2 + v^2 + w^2}} . \quad (\text{A4})$$

The first integral with respect to w in Eq.(A4) is a standard integral that can be written in many equivalent forms with one of them being:

$$\int \frac{dx}{\sqrt{x^2 + y^2 + z^2}} = \ln \left(x + \sqrt{x^2 + y^2 + z^2} \right) . \quad (\text{A5})$$

However, one should not be deceived by this result to expect that integration with respect to the remaining variables would be straightforward. It turns out that the calculation of the remaining integrals is not easy and grows very quickly in complexity with the final expression occupying about two book pages [8].

Appendix B: Calculation of the integral

$$\int_0^\infty dx \left[\frac{\text{erf}(x)}{x} \right]^3$$

We want to calculate the integral:

$$\int_0^\infty dx \left[\frac{\text{erf}(x)}{x} \right]^3 , \quad (\text{B1})$$

where $\text{erf}(x) = 2/\sqrt{\pi} \int_0^x dt e^{-t^2}$ is an error function. We consider the indeterminate integral, $\int dx \left[\frac{\text{erf}(x)}{x} \right]^3$ and use integration by parts:

$$\int u dv = uv - \int v du , \quad (\text{B2})$$

where $u = \text{erf}(x)^3$ and $dv = dx/x^3$. It is easy to see that: $du = 3 \text{erf}(x)^2 \frac{2}{\sqrt{\pi}} e^{-x^2} dx$ and $v = -1/(2x^2)$. We

recall that:

$$\frac{d}{dx} \text{erf}(x) = \frac{2}{\sqrt{\pi}} e^{-x^2} . \quad (\text{B3})$$

The integration by part formula leads to:

$$\int dx \left[\frac{\text{erf}(x)}{x} \right]^3 = -\frac{1}{2} \frac{\text{erf}(x)^3}{x^2} + \frac{3}{\sqrt{\pi}} \int dx \left[\frac{\text{erf}(x)}{x} \right]^2 e^{-x^2} . \quad (\text{B4})$$

Now, we remind the reader about the limits of the integration which are from 0 to ∞ . For error functions:

$$\lim_{x \rightarrow 0} \frac{\text{erf}(x)}{x} = \frac{2}{\sqrt{\pi}} ; \quad \lim_{x \rightarrow \infty} \text{erf}(x) = 1 . \quad (\text{B5})$$

This implies that:

$$\lim_{x \rightarrow 0} \frac{\text{erf}(x)^3}{x^2} = 0 ; \quad \lim_{x \rightarrow \infty} \frac{\text{erf}(x)^3}{x^2} = 0 . \quad (\text{B6})$$

Hence, we have:

$$\int_0^\infty dx \left[\frac{\text{erf}(x)}{x} \right]^3 = \frac{3}{\sqrt{\pi}} \int_0^\infty dx \left[\frac{\text{erf}(x)}{x} \right]^2 e^{-x^2} . \quad (\text{B7})$$

We now carry out another integration by parts on the integral appearing in the right-hand-side of Eq.(B7) where, this time, $u = \text{erf}(x)^2 e^{-x^2}$ and $dv = dx/x^2$. By using the same approach as before we end up with:

$$\int_0^\infty dx \left[\frac{\text{erf}(x)}{x} \right]^3 = \frac{3}{\sqrt{\pi}} \left[\frac{4}{\sqrt{\pi}} \int_0^\infty dx \frac{\text{erf}(x)}{x} e^{-2x^2} - 2 \int_0^\infty dx \text{erf}(x)^2 e^{-x^2} \right] . \quad (\text{B8})$$

Note that the first integral in the right-hand-side of Eq.(B8) is a table integral (see formula 6.293, page 650 of Ref. [23]):

$$\int_0^\infty \frac{dx}{x} \text{erf}(x) e^{-\mu x^2} = \frac{1}{2} \ln \left(\frac{\sqrt{\mu+1}+1}{\sqrt{\mu+1}-1} \right) ; \quad \text{Re}(\mu) > 0 . \quad (\text{B9})$$

The second integral in the right-hand-side of Eq.(B8) is elementary:

$$\int_0^\infty dx \text{erf}(x)^2 e^{-x^2} = \frac{\sqrt{\pi}}{6} . \quad (\text{B10})$$

One obtains the final result by putting together everything. We note that the final result can be written in many equivalent ways as illustrated below:

$$\int_0^\infty dx \left[\frac{\operatorname{erf}(x)}{x} \right]^3 = \frac{6}{\pi} \ln \left(\frac{\sqrt{3}+1}{\sqrt{3}-1} \right) - 1 = \frac{12}{\pi} \ln \left(\frac{\sqrt{3}+1}{\sqrt{2}} \right) - 1 = \frac{6}{\pi} \ln (2 + \sqrt{3}) - 1 = \frac{12}{\pi} \sinh^{-1} \left(\frac{1}{\sqrt{2}} \right) - 1 . \quad (\text{B11})$$

The last expression in Eq.(B11) is given in terms of the inverse sine hyperbolic function, $\sinh^{-1}(x)$ (oftenly denoted as $\operatorname{arcsinh}(x)$, $\operatorname{ArcSinh}(x)$, $\operatorname{arcsh}(x)$, and so on) which has the following logarithmic representation:

$$\sinh^{-1}(x) = \ln \left(x + \sqrt{x^2 + 1} \right) , \quad (\text{B12})$$

where $\ln(x)$ is the natural logarithm function. One can start from Eq.(B12) to verify that:

$$\sinh^{-1} \left(\frac{1}{\sqrt{2}} \right) = \ln \left(\frac{\sqrt{3}+1}{\sqrt{2}} \right) . \quad (\text{B13})$$

Appendix C: Calculation of the integral

$$\int_0^\infty dx \frac{\operatorname{erf}(ax) \operatorname{erf}(x)^2}{x^3}$$

We want to calculate the following integral:

$$I(a) = \int_0^\infty dx \frac{\operatorname{erf}(ax) \operatorname{erf}(x)^2}{x^3} , \quad (\text{C1})$$

where a is a real constant. Note that:

$$I(-a) = -I(a) . \quad (\text{C2})$$

We use integration by parts ($\int u dv = uv - \int v du$) where $u = \operatorname{erf}(ax) \operatorname{erf}(x)^2$ and $dv = dx/x^3$. After straightforward calculations one obtains:

$$I(a) = \frac{1}{\sqrt{\pi}} \left[a \int_0^\infty dx e^{-a^2 x^2} \frac{\operatorname{erf}(x)^2}{x^2} + 2 \int_0^\infty dx e^{-x^2} \frac{\operatorname{erf}(ax) \operatorname{erf}(x)}{x^2} \right] . \quad (\text{C3})$$

The first integral appearing in the right-hand-side of Eq.(C3) can be calculated by using integration by parts

in combination with standard integration formulas with the result:

$$\int_0^\infty dx e^{-a^2 x^2} \frac{\operatorname{erf}(x)^2}{x^2} = \frac{2}{\sqrt{\pi}} \left[2 \ln \left(\frac{1 + \sqrt{a^2 + 2}}{\sqrt{a^2 + 1}} \right) - a \tan^{-1} \left(\frac{1}{a \sqrt{a^2 + 2}} \right) \right] , \quad (\text{C4})$$

where $\tan^{-1}(x)$ is the inverse tangent trigonometric function (oftenly denoted as $\arctan(x)$, $\operatorname{ArcTan}(x)$, $\operatorname{arctg}(x)$, and so on). The second integral appearing in the right-hand-side of Eq.(C3) reads:

$$\int_0^\infty dx e^{-x^2} \frac{\operatorname{erf}(ax) \operatorname{erf}(x)}{x^2} = \frac{2}{\sqrt{\pi}} \left[\ln \left(\frac{a + \sqrt{a^2 + 2}}{\sqrt{2}} \right) + a \ln \left(\frac{1 + \sqrt{a^2 + 2}}{\sqrt{a^2 + 1}} \right) - \tan^{-1} \left(\frac{a}{\sqrt{a^2 + 2}} \right) \right] . \quad (\text{C5})$$

At this point, one substitutes the results from Eq.(C4) and Eq.(C5) into Eq.(C3) to obtain:

$$I(a) = \frac{4}{\pi} \left[2a \ln \left(\frac{1 + \sqrt{a^2 + 2}}{\sqrt{a^2 + 1}} \right) + \ln \left(\frac{a + \sqrt{a^2 + 2}}{\sqrt{2}} \right) \right] - \frac{2}{\pi} \left[a^2 \tan^{-1} \left(\frac{1}{a \sqrt{a^2 + 2}} \right) + 2 \tan^{-1} \left(\frac{a}{\sqrt{a^2 + 2}} \right) \right] . \quad (\text{C6})$$

The formula in Eq.(C6) can be written in a more compact form after noticing that:

$$\ln \left(\frac{1 + \sqrt{a^2 + 2}}{\sqrt{a^2 + 1}} \right) = \sinh^{-1} \left(\frac{1}{\sqrt{a^2 + 1}} \right) ; \quad \ln \left(\frac{a + \sqrt{a^2 + 2}}{\sqrt{2}} \right) = \sinh^{-1} \left(\frac{a}{\sqrt{2}} \right) , \quad (C7)$$

where $\sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1})$ is the sine inverse hyperbolic function. Hence, one can write the expression

in Eq.(C6) in a rather compact form as:

$$I(a) = \frac{4}{\pi} \left[2a \sinh^{-1} \left(\frac{1}{\sqrt{a^2 + 1}} \right) + \sinh^{-1} \left(\frac{a}{\sqrt{2}} \right) \right] - \frac{2}{\pi} \left[a^2 \tan^{-1} \left(\frac{1}{a\sqrt{a^2 + 2}} \right) + 2 \tan^{-1} \left(\frac{a}{\sqrt{a^2 + 2}} \right) \right] . \quad (C8)$$

Note that for $a = 1$ one has $\tan^{-1} \left(\frac{1}{\sqrt{3}} \right) = \frac{\pi}{6}$. As a result, it is easy to see that:

$$I(a = 1) = \frac{12}{\pi} \sinh^{-1} \left(\frac{1}{\sqrt{2}} \right) - 1 = \frac{12}{\pi} \ln \left(\frac{1 + \sqrt{3}}{\sqrt{2}} \right) - 1 . \quad (C9)$$

in agreement with the results in Eq.(B11).

where $I(a)$ is defined in Eq.(C1) and is calculated in Appendix C. One can prove the result in Eq.(D3) in a straightforward manner. Since $\text{erf}(x)^2$ is an even function of x , one writes:

$$J(a) = \int_0^\infty dx \frac{\text{erf}(|a|x)^2 \text{erf}(x)}{x^3} . \quad (D4)$$

Appendix D: Calculation of the integral

$$\int_0^\infty dx \frac{\text{erf}(ax)^2 \text{erf}(x)}{x^3}$$

Let's consider the following integral:

$$J(a) = \int_0^\infty dx \frac{\text{erf}(ax)^2 \text{erf}(x)}{x^3} , \quad (D1)$$

where a is a real constant. In this case:

$$J(-a) = J(a) . \quad (D2)$$

The key result that we anticipate is:

$$J(a) = |a|^2 I \left(\frac{1}{|a|} \right) , \quad (D3)$$

If we introduce a new variable, $u = |a|x$, then Eq.(D4) reads:

$$J(a) = |a|^2 \int_0^\infty du \frac{\text{erf}(u)^2 \text{erf}(\frac{u}{|a|})}{u^3} . \quad (D5)$$

By rearranging the order of terms in Eq.(D5) one immediately sees that:

$$J(a) = |a|^2 \int_0^\infty du \frac{\text{erf}(\frac{u}{|a|}) \text{erf}(u)^2}{u^3} = |a|^2 I \left(\frac{1}{|a|} \right) , \quad (D6)$$

where $I(a) = \int_0^\infty dx \frac{\text{erf}(ax)^2 \text{erf}(x)}{x^3}$ was first introduced in Eq.(C1). The explicit result for $J(a)$ reads:

$$J(a) = \frac{4}{\pi} \left[|a|^2 \sinh^{-1} \left(\frac{1}{\sqrt{2}|a|} \right) + 2|a| \sinh^{-1} \left(\frac{|a|}{\sqrt{|a|^2 + 1}} \right) \right] - \frac{2}{\pi} \left[2|a|^2 \tan^{-1} \left(\frac{1}{\sqrt{2}|a|^2 + 1}} \right) + \tan^{-1} \left(\frac{|a|^2}{\sqrt{2}|a|^2 + 1}} \right) \right] . \quad (D7)$$

One can simplify the expression above as:

$$J(a) = \frac{4}{\pi} \left[a^2 \sinh^{-1} \left(\frac{1}{\sqrt{2a^2}} \right) + 2a \sinh^{-1} \left(\frac{a}{\sqrt{a^2+1}} \right) \right] - \frac{2}{\pi} \left[2a^2 \tan^{-1} \left(\frac{1}{\sqrt{2a^2+1}} \right) + \tan^{-1} \left(\frac{a^2}{\sqrt{2a^2+1}} \right) \right] . \quad (\text{D8})$$

Note that $a^2 = |a|^2$ since a is real. Also note that $x \sinh^{-1}(x)$ is an even function of x . Based on these simple observations, it is straightforward to verify the equivalence between Eq.(D7) and Eq.(D8).

incorporates this change of sign. We now introduce a new variable, $u = |a|x$. As a result, the expression in Eq.(E3) may be written as:

Appendix E: Calculation of the integral

$$\int_0^\infty dx \frac{\text{erf}(ax)^3}{x^3}$$

Let's consider the following integral:

$$K(a) = \int_0^\infty dx \frac{\text{erf}(ax)^3}{x^3} , \quad (\text{E1})$$

where a is assumed to be a real constant. In this case:

$$K(-a) = -K(a) . \quad (\text{E2})$$

We calculate $K(a)$ by performing a suitable transformation of variables which allows us to reduce it to an integral of the form already calculated. Since $K(a)$ switches sign when the value of a changes from positive ($a = |a| > 0$) to negative ($a = -|a| < 0$), we write:

$$K(a) = \frac{a}{|a|} \int_0^\infty dx \frac{\text{erf}(|a|x)^3}{x^3} , \quad (\text{E3})$$

where

$$\pm 1 = \frac{a}{|a|} , \quad (\text{E4})$$

$$K(a) = a |a| \int_0^\infty du \frac{\text{erf}(u)^3}{u^3} . \quad (\text{E5})$$

The integral in the right-hand-side of Eq.(E5) has already been calculated in Appendix B:

$$\begin{aligned} K(a) &= a |a| \left[\frac{6}{\pi} \ln \frac{\sqrt{3}+1}{\sqrt{3}-1} - 1 \right] = \\ &= a \sqrt{a^2} \left[\frac{6}{\pi} \ln \left(\frac{\sqrt{3}+1}{\sqrt{3}-1} \right) - 1 \right] , \end{aligned} \quad (\text{E6})$$

where we used the first of the several expressions listed in Eq.(B11) of Appendix B.

Appendix F: Calculation of the integral function $F(a, b, c)$

We consider the following integral function:

$$F(a, b, c) = \int_0^\infty dx \frac{\text{erf}(ax) \text{erf}(bx) \text{erf}(cx)}{x^3} , \quad (\text{F1})$$

where a, b, c are arbitrary real constants. We use integration by parts ($\int u dv = uv - \int v du$) where:

$$u = \text{erf}(ax) \text{erf}(bx) \text{erf}(cx) \quad ; \quad dv = \frac{dx}{x^3} . \quad (\text{F2})$$

Based on such a choice, it is straightforward to calculate that:

$$du = \frac{2}{\sqrt{\pi}} \left[a e^{-a^2 x^2} \text{erf}(bx) \text{erf}(cx) + b e^{-b^2 x^2} \text{erf}(cx) \text{erf}(ax) + c e^{-c^2 x^2} \text{erf}(ax) \text{erf}(bx) \right] \quad ; \quad v = -\frac{1}{2x^2} . \quad (\text{F3})$$

It can be easily verified that:

$$\lim_{x \rightarrow 0} \frac{\text{erf}(ax) \text{erf}(bx) \text{erf}(cx)}{x^2} = 0 \quad (\text{F4})$$

and

$$\lim_{x \rightarrow \infty} \frac{\text{erf}(ax) \text{erf}(bx) \text{erf}(cx)}{x^2} = 0 . \quad (\text{F5})$$

Based on these results, the integration by part formula leads to:

$$F(a, b, c) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{dx}{x^2} \left[a e^{-a^2 x^2} \operatorname{erf}(bx) \operatorname{erf}(cx) + b e^{-b^2 x^2} \operatorname{erf}(cx) \operatorname{erf}(ax) + c e^{-c^2 x^2} \operatorname{erf}(ax) \operatorname{erf}(bx) \right] . \quad (\text{F6})$$

At this juncture, we observe that the three integrals appearing on the right-hand-side of Eq.(F6) have the same form. Thus, if we calculate the first one, it is straightforward to obtain the other two. Note the permutations,

(*abc*), (*bca*) and (*cab*). For the sake of brevity, we skip further details. Below, we report the outcome of the calculation of the first integral:

$$\int_0^\infty \frac{dx}{x^2} e^{-a^2 x^2} \operatorname{erf}(bx) \operatorname{erf}(cx) = \frac{2}{\sqrt{\pi}} \left[b \ln \left(\frac{c + \Delta}{\sqrt{a^2 + b^2}} \right) + c \ln \left(\frac{b + \Delta}{\sqrt{a^2 + c^2}} \right) - a \tan^{-1} \left(\frac{bc}{a \Delta} \right) \right] , \quad (\text{F7})$$

---

where

$$\Delta = \sqrt{a^2 + b^2 + c^2} . \quad (\text{F8})$$

By applying the result of Eq.(F7) to each of the three integrals in Eq.(F6), one has:

$$\begin{aligned} F(a, b, c) = & \frac{4}{\pi} \left[a b \ln \left(\frac{c + \Delta}{\sqrt{a^2 + b^2}} \right) + b c \ln \left(\frac{a + \Delta}{\sqrt{b^2 + c^2}} \right) + c a \ln \left(\frac{b + \Delta}{\sqrt{c^2 + a^2}} \right) \right] \\ & - \frac{2}{\pi} \left[a^2 \tan^{-1} \left(\frac{bc}{a \Delta} \right) + b^2 \tan^{-1} \left(\frac{ca}{b \Delta} \right) + c^2 \tan^{-1} \left(\frac{ab}{c \Delta} \right) \right] . \end{aligned} \quad (\text{F9})$$

Though not easy to recognize at first sight, one can verify that:

$$\ln \left(\frac{c + \Delta}{\sqrt{a^2 + b^2}} \right) = \sinh^{-1} \left(\frac{c}{\sqrt{a^2 + b^2}} \right) , \quad (\text{F10})$$

$$\ln \left(\frac{b + \Delta}{\sqrt{c^2 + a^2}} \right) = \sinh^{-1} \left(\frac{b}{\sqrt{c^2 + a^2}} \right) , \quad (\text{F12})$$

$$\ln \left(\frac{a + \Delta}{\sqrt{b^2 + c^2}} \right) = \sinh^{-1} \left(\frac{a}{\sqrt{b^2 + c^2}} \right) , \quad (\text{F11})$$

---

where $\Delta = \sqrt{a^2 + b^2 + c^2}$. Based on these results, one can write the quantity in Eq.(F9) as:

$$\begin{aligned} F(a, b, c) = & \frac{4}{\pi} \left[a b \sinh^{-1} \left(\frac{c}{\sqrt{a^2 + b^2}} \right) + b c \sinh^{-1} \left(\frac{a}{\sqrt{b^2 + c^2}} \right) + c a \sinh^{-1} \left(\frac{b}{\sqrt{c^2 + a^2}} \right) \right] \\ & - \frac{2}{\pi} \left[a^2 \tan^{-1} \left(\frac{bc}{a \Delta} \right) + b^2 \tan^{-1} \left(\frac{ca}{b \Delta} \right) + c^2 \tan^{-1} \left(\frac{ab}{c \Delta} \right) \right] . \end{aligned} \quad (\text{F13})$$

One may write the expression above in a slightly more compact form as:

$$F(a, b, c) = \frac{2}{\pi} \left[2ab \sinh^{-1} \left(\frac{c}{\sqrt{a^2 + b^2}} \right) + 2bc \sinh^{-1} \left(\frac{a}{\sqrt{b^2 + c^2}} \right) + 2ca \sinh^{-1} \left(\frac{b}{\sqrt{c^2 + a^2}} \right) - a^2 \tan^{-1} \left(\frac{bc}{a\Delta} \right) - b^2 \tan^{-1} \left(\frac{ca}{b\Delta} \right) - c^2 \tan^{-1} \left(\frac{ab}{c\Delta} \right) \right], \quad (\text{F14})$$

where $\Delta = \sqrt{a^2 + b^2 + c^2}$.

Appendix G: Other transformations

From Eq.(6) we have:

$$V(X, Y, Z) = \frac{k_e Q}{L} \frac{2}{\sqrt{\pi}} \int_0^\infty dt f(t, X) f(t, Y) f(t, Z). \quad (\text{G1})$$

We remind the reader that the auxiliary functions, $f(t, X)$, and similarly $f(t, Y)$ and $f(t, Z)$ after, respectively, replacing X with Y and X with Z are defined as:

$$f(t, X) = \frac{\sqrt{\pi}}{2t} \left\{ \operatorname{erf} \left[t \left(\frac{1}{2} - X \right) \right] + \operatorname{erf} \left[t \left(\frac{1}{2} + X \right) \right] \right\}. \quad (\text{G2})$$

Let's now introduce new auxiliary functions: $h(t, X)$, $h(t, Y)$ and $h(t, Z)$ so that:

$$\begin{aligned} f(t, X) &= \frac{\sqrt{\pi}}{2t} h(t, X) \quad ; \quad f(t, Y) = \frac{\sqrt{\pi}}{2t} h(t, Y) \quad ; \\ f(t, Z) &= \frac{\sqrt{\pi}}{2t} h(t, Z). \end{aligned} \quad (\text{G3})$$

As a result one can write Eq.(G1) as:

$$V(X, Y, Z) = \frac{k_e Q}{L} \frac{\pi}{4} \int_0^\infty \frac{dt}{t^3} h(t, X) h(t, Y) h(t, Z), \quad (\text{G4})$$

where the functions $h(t, X)$ and similarly $h(t, Y)$ and $h(t, Z)$ after, respectively, replacing X with Y and X with Z read:

$$h(t, X) = \left\{ \operatorname{erf} \left[t \left(\frac{1}{2} - X \right) \right] + \operatorname{erf} \left[t \left(\frac{1}{2} + X \right) \right] \right\}. \quad (\text{G5})$$

Let's integrate the integral in Eq.(G4) by parts ($\int u dv = uv - \int v du$) where:

$$u = h(t, X) h(t, Y) h(t, Z) \quad ; \quad dv = \frac{dt}{t^3}. \quad (\text{G6})$$

Integration by parts results in:

$$V(X, Y, Z) = \frac{k_e Q}{L} \frac{\pi}{8} \int_0^\infty \frac{dt}{t^2} \frac{\partial}{\partial t} [h(t, X) h(t, Y) h(t, Z)]. \quad (\text{G7})$$

Note that, in the process of deriving the result in Eq.(G7), we used the following formulae for limits (that can be easily verified):

$$\lim_{t \rightarrow 0} \frac{h(t, X) h(t, Y) h(t, Z)}{t^2} = 0 \quad (\text{G8})$$

$$\lim_{t \rightarrow \infty} \frac{h(t, X) h(t, Y) h(t, Z)}{t^2} = 0. \quad (\text{G9})$$

These calculations indicate that the quantity in Eq.(G1) is equivalent to that in Eq.(G7). The expression in Eq.(G7) has been reported in Eq.(2.6) of Ref. [17] (apart the factor $k_e Q/L$ that is deliberately omitted there and another factor that we believe is missing because of a typo). We believe that Eq.(2.6) of Ref. [17] is missing a factor of π on the right-hand-side. This fact has also been noted earlier by at least one other author (for instance, see footnote in page 290 of Ref. [9]). The correct expression should be as in Eq.(G7). At this juncture, we note that the formalism and notation adopted in Appendix G, namely, the introduction of the auxiliary functions $h(t, X)$, $h(t, Y)$, $h(t, Z)$ defined in Eq.(G5) follows the notation of Ref. [17] and has the sole purpose to facilitate the comparison of results. In our view, the expression in Eq.(G7) which involves a partial derivative of the product of three auxiliary functions is not particularly useful for practical numerical applications.

[1] J. A. Stratton, Electromagnetic Theory, McGraw-Hill Book Company, New York (1941).

[2] J. D. Jackson, Classical Electrodynamics, Third Edition, John Wiley and Sons, New York (1999).

- [3] D. J. Griffiths, Introduction to Electrodynamics, Third Edition, Prentice Hall, New Jersey (1999).
- [4] O. Ciftja and I. Hysi, Appl. Math. Lett. **24**, 1919 (2011).
- [5] O. Ciftja, Physica B **407**, 2803 (2012).
- [6] J. Waldvogel, J. Appl. Math. Phys. (ZAMP), **30**, 388 (1979).
- [7] R. A. Werner, Celest. Mechan. Dynam. Astronom. **59**, 253 (1994).
- [8] W. D. MacMillan, The Theory of the Potential, pg. 72, Dover, New York (1958).
- [9] G. Hummer, J. Electrostat. **36**, 285 (1996).
- [10] J. Waldvogel, Zeitschr. Angew. Math. Phys. (ZAMP), **27**, 867 (1976).
- [11] J. M. Chappell, M. J. Chappell, A. Iqbal, and D. Abbott, Physics International **3**, 50 (2012).
- [12] O. Ciftja, Phys. Lett. A **374**, 981 (2010).
- [13] O. Ciftja, J. Electrostat. **71**, 102 (2013).
- [14] Handbook of Mathematical Functions, M. Abramowitz and I. A. Stegun, Ninth Printing, Dover (1970).
- [15] O. Ciftja, Phys. Lett. A **375**, 766 (2011).
- [16] O. Ciftja, Eur. J. Phys. **32** L55 (2011).
- [17] B. R. A. Nijboer and Th. W. Ruijgrok, J. Stat. Phys. **53**, 361 (1988).
- [18] O. Ciftja, J. Phys.: Condens. Matter **19**, 046220 (2007).
- [19] O. Ciftja, Physica B **404** 1629 (2009).
- [20] M. Califano and P. Harrison, J. Appl. Phys. **86**, 5054 (1999).
- [21] F. Aristone, J. Sanchez-Dehesa, and G. E. Marques, J. Kor. Phys. Soc. **39** 493 (2001).
- [22] E. Welander and G. Burkard, Phys. Rev. B. **86**, 165312 (2012).
- [23] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, Fourth Edition, Academic Press, New York (1965).