On flux terms in volume averaging

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Abstract

This note examines the modelling of non-convective fluxes (e.g. stress, heat flux and others) as they appear in the general, unclosed form of the volume-averaged equations of multiphase flows. By appealing to the difference between slowly and rapidly varying quantities, it is shown that the natural closure of these terms leads to the use of a single, slowly-varying combined average flux, common to both phases, plus rapidly-varying local contributions for each phase. **The result is general and only rests on the hypothesis that the spatial variation of the combined average flux is adequately described by a linear function of position within the averaging volume. No further hypotheses on the nature of the flow (e.g., about specific flow regimes) prove necessary.** The result agrees with earlier ones obtained by ensemble averaging, is illustrated with the example of disperse flows and discussed in the light of some earlier and current literature. A very concise derivation of the general averaged balance equation is also given.

Keywords: volume averaging, separation of scales

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1. Introduction

Volume averaging is a standard method for the derivation of averaged equations of balance for the modelling of multiphase flows (see e.g. Nigmatulin et al., 1996; Whitaker, 1999; Prosperetti and Tryggvason, 2009; Ishii and Ibiki, 2011). While the formal application of the method is relatively straightforward, the interpretation of the resulting equations is subtle. This

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7 is a crucial point as, in the absence of physical transparency, it becomes
8 harder to develop physically relevant closures.

9 The specific aspect on which we focus in this note is the modelling of non-
10 convective fluxes (e.g, stress, heat flux, diffusive flux) in the averaged balance
11 equations. The basis of the approach is the recognition that, in a spatially
12 non-homogeneous flow, simultaneously with the slow spatial dependence of
13 the macroscopic averaged quantities, there is a faster, local spatial depen-
14 dence of the microscopic fields. A procedure which takes into consideration
15 this multi-scale nature of the actual situation results in a physically trans-
16 parent form of the averaged equations (section 3) which, in turn, helps to
17 close the equations as we show with the example of a disperse flow in section
18 4. Section 5 provides a discussion of the results in the context of the existing
19 literature. Section 2 contains a very synthetic (“efficient”) derivation of the
20 volume-averaged equations in the standard form in which they are usually
21 presented.

22 **2. The general averaged balance law**

23 We consider for simplicity a general system consisting of two phases,
24 denoted by indices 1 and 2, although extension of the procedure to the more
25 general case of three or more phases is straightforward.

26 We attach to each point $\mathbf{x}$ in space an averaging volume $V(\mathbf{x})$ of fixed
27 shape and orientation; $V_j(\mathbf{x}, t)$, with $j = 1$ or 2 , denotes the (generally time-
28 dependent) part of V occupied by the j -th phase so that $V = V_1 + V_2$. The
29 surface S of V is also decomposed in the same way, $S = S_1 + S_2$, with S_j
30 the portion of S occupied by the j -th phase (see figure 1). Inside V the
31 two phases are separated by an interface S_i , possibly consisting of disjoint
32 parts as, e.g., in the case of droplets suspended in a continuous phase. It is
33 important to keep in mind that the volume occupied by the j -th phase inside
34 the averaging volume is bounded by $S_j + S_i$ which, therefore, is a closed
35 surface.

36 The volume average of a generic quantity q_j of arbitrary tensorial order
37 belonging to the j -th phase is defined as

$$\langle q_j \rangle(\mathbf{x}, t) = \frac{1}{V_j(\mathbf{x}, t)} \int_{V_j(\mathbf{x}, t)} q_j(\boldsymbol{\xi}, t) d^3\xi, \quad (1)$$

38 where $\mathbf{x}$ is the position of the averaging volume. Upon using the (generalized)

Figure 1: Averaging volume in a two-phase system. The surface of the averaging volume, $S = S_1 + S_2$, consists of a portion S_1 in contact with phase 1 (continuous thick line) and a portion S_2 in contact with phase 2 (dashed thick line). Inside the averaging volume the two phases are separated by an interface S_i (thin lines) which may consist of disjoint parts as in this figure.

Figure 2: The unit normals are defined to be directed out of the corresponding phase.

39 Reynolds transport theorem we have

$$\frac{\partial}{\partial t} (V_j \langle q_j \rangle) = \int_{V_j} \frac{\partial q_j}{\partial t} d^3\xi + \oint_{S_j+S_i} q_j \mathbf{v} \cdot \mathbf{n}_j dS, \quad (2)$$

40 where $\mathbf{v}$ is the local velocity of the integration surface and $\mathbf{n}_j$ is the unit
41 normal directed out of the j -th phase (figure 2). Since V is fixed, $\mathbf{v} = 0$ on
42 S_j and we are left with

$$\frac{\partial}{\partial t} (V_j \langle q_j \rangle) = \int_{V_j} \frac{\partial q_j}{\partial t} d^3\xi + \int_{S_i} q_j \mathbf{v}_i \cdot \mathbf{n}_j dS_i, \quad (3)$$

43 where $\mathbf{v}_i$ is the velocity of the interface contained within V . We assume that
44 the quantity q_j satisfies a general balance equation of the form

$$\frac{\partial q_j}{\partial t} = -\nabla \cdot (\mathbf{u}_j q_j) + \nabla \cdot \boldsymbol{\phi}_j + \theta_j, \quad (4)$$

45 where $\mathbf{u}_j$ is the j -th phase velocity, $\boldsymbol{\phi}_j$ the non-convective flux of q_j and θ_j the
46 volume source of q_j . Upon substituting into the first term in the right-hand
47 side of (3) we find

$$\frac{\partial}{\partial t} (V_j \langle q_j \rangle) = \int_{V_j} [-\nabla \cdot (\mathbf{u}_j q_j - \boldsymbol{\phi}_j) + \theta_j] d^3\xi + \oint_{S_i} q_j \mathbf{v}_i \cdot \mathbf{n}_j dS_i, \quad (5)$$

48 or, upon using the divergence theorem,

$$\frac{\partial}{\partial t} (V_j \langle q_j \rangle) + \int_{S_j} q_j \mathbf{u}_j \cdot \mathbf{n}_j dS_j = \int_{S_j} \boldsymbol{\phi}_j \cdot \mathbf{n}_j dS_j + \int_{S_i} [-q_j (\mathbf{u}_j - \mathbf{v}_i) + \boldsymbol{\phi}_j] \cdot \mathbf{n}_j dS_i + \int_{V_j} \theta_j d^3\xi. \quad (6)$$

49 Here we have separated the surface integrals over the interface (in the right-
50 hand side) from those over the surface of the averaging volume (in the left-
51 hand side).

52 Now we use the exact, purely geometric theorem (see e.g. Gray and Lee,
53 1977; Prosperetti and Tryggvason, 2009)

$$\int_{S_j} \boldsymbol{\phi}_j \cdot \mathbf{n}_j dS_j = \nabla \cdot \int_{V_j} \boldsymbol{\phi}_j d^3\xi = \nabla \cdot (V_j \langle \boldsymbol{\phi}_j \rangle), \quad (7)$$

54 (actually valid for any vector or higher-order tensor) to rewrite this result as

$$\frac{\partial}{\partial t} (V_j \langle q_j \rangle) + \nabla \cdot (V_j \langle q_j \mathbf{u}_j \rangle) = \nabla \cdot (V_j \langle \boldsymbol{\phi}_j \rangle) + \int_{S_i} [-q_j (\mathbf{u}_j - \mathbf{v}_i) + \boldsymbol{\phi}_j] \cdot \mathbf{n}_j dS_i + V_j \langle \theta_j \rangle. \quad (8)$$

55 Upon division by V , assumed to be independent of $\mathbf{x}$ as already stated, and
 56 upon introduction of the volume fraction α_j of the j -phase defined by

$$\alpha_j = \frac{V_j}{V}, \quad (9)$$

57 we find the general averaged balance law

$$\begin{aligned} \frac{\partial}{\partial t} (\alpha_j \langle q_j \rangle) + \nabla \cdot (\alpha_j \langle q_j \mathbf{u}_j \rangle) &= \nabla \cdot (\alpha_j \langle \phi_j \rangle) \\ &+ \frac{1}{V} \int_{S_i} [-q_j (\mathbf{u}_j - \mathbf{v}_i) + \phi_j] \cdot \mathbf{n}_j dS_i + \alpha_j \langle \theta_j \rangle. \end{aligned} \quad (10)$$

58 3. Separation of scales

59 While (10) is formally exact, its physical transparency can be enhanced
 60 as we now show. We focus on the terms containing the non-convective flux
 61 ϕ_j in (8), namely

$$\Phi_j \equiv \nabla \cdot (V_j \langle \phi_j \rangle) + \int_{S_i} \phi_j \cdot \mathbf{n}_j dS_i. \quad (11)$$

62 We show the development explicitly for $j = 1$; the results for $j = 2$ can be
 63 recovered by simply interchanging the indices 1 and 2.

64 Let us define

$$\bar{\phi} = \alpha_1 \langle \phi_1 \rangle + \alpha_2 \langle \phi_2 \rangle. \quad (12)$$

65 By its definition, this quantity captures the large-scale structure of the ϕ field
 66 and may be expected to be slowly varying over the scale of the averaging
 67 volume. Upon multiplying by V (assumed independent of position) and
 68 taking the divergence we have

$$V \nabla \cdot \bar{\phi} = \nabla \cdot (V_1 \langle \phi_1 \rangle) + \nabla \cdot (V_2 \langle \phi_2 \rangle). \quad (13)$$

69 This relation permits us to re-write (11) with $j = 1$ as

$$\Phi_1 = V \nabla \cdot \bar{\phi} - \nabla \cdot (V_2 \langle \phi_2 \rangle) + \int_{S_i} \phi_1 \cdot \mathbf{n}_1 dS_i. \quad (14)$$

70 On the phase interface S_i the conservation law of of the generic quantity q_j
 71 imposes a relation between ϕ_1 and ϕ_2 of the general form

$$(\phi_1 - \phi_2) \cdot \mathbf{n}_1 = \gamma, \quad (15)$$

72 where γ is a surface source term. For example, when $\boldsymbol{\phi}$ is the stress, γ would
 73 be a vector accounting for surface tension effects at the interface separating
 74 two fluids. For a liquid-vapor system, when $\boldsymbol{\phi}$ is the heat flux, γ would
 75 account for the latent heat effects associated to phase change, and so on.
 76 Upon using the fact that, on the interface S_i , $\mathbf{n}_1 = -\mathbf{n}_2$ as the normals are
 77 defined to be directed out of the corresponding phase (see figure 2), we can
 78 re-write (14) as

$$\Phi_1 = V \nabla \cdot \bar{\boldsymbol{\phi}} - \nabla \cdot (V_2 \langle \boldsymbol{\phi}_2 \rangle) - \int_{S_i} \boldsymbol{\phi}_2 \cdot \mathbf{n}_2 dS_i + \int_{S_i} \gamma dS_i, \quad (16)$$

79 or, by the geometric theorem (7),

$$\Phi_1 = V \nabla \cdot \bar{\boldsymbol{\phi}} - \oint_{S_2+S_i} \boldsymbol{\phi}_2 \cdot \mathbf{n}_2 dS + \int_{S_i} \gamma dS_i. \quad (17)$$

80 Let us now set in this equation

$$\boldsymbol{\phi}_2 = \bar{\boldsymbol{\phi}} + \boldsymbol{\phi}'_2. \quad (18)$$

81 By the definition of $\bar{\boldsymbol{\phi}}$, $\boldsymbol{\phi}'_2$ captures the local small-scale fluctuations of the
 82 field $\boldsymbol{\phi}_2$ in the averaging volume. The contribution of $\bar{\boldsymbol{\phi}}$ to the first integral
 83 in (17) is

$$\oint_{S_2+S_i} \bar{\boldsymbol{\phi}} \cdot \mathbf{n}_2 dS = \int_{V_2} \nabla \cdot \bar{\boldsymbol{\phi}} dV_2. \quad (19)$$

84 Since $\bar{\boldsymbol{\phi}}$ is slowly varying over the scale of the averaging volume, we approx-
 85 imate the volume integral as

$$\oint_{S_2+S_i} \bar{\boldsymbol{\phi}} \cdot \mathbf{n}_2 dS \simeq V_2 \nabla \cdot \bar{\boldsymbol{\phi}}, \quad (20)$$

86 which is equivalent to assuming that $\nabla \cdot \bar{\boldsymbol{\phi}}$ is essentially constant over the
 87 averaging volume V or that, equivalently, $\bar{\boldsymbol{\phi}}$ is adequately represented by a
 88 linear function of position inside V . Upon substituting into (17) we then find

$$\Phi_1 \simeq V \nabla \cdot \bar{\boldsymbol{\phi}} - V_2 \nabla \cdot \bar{\boldsymbol{\phi}} - \oint_{S_2+S_i} \boldsymbol{\phi}'_2 \cdot \mathbf{n}_2 dS + \int_{S_i} \gamma dS_i, \quad (21)$$

89 i.e.

$$\Phi_1 \simeq V_1 \nabla \cdot \bar{\boldsymbol{\phi}} - \oint_{S_2+S_i} \boldsymbol{\phi}'_2 \cdot \mathbf{n}_2 dS + \int_{S_i} \gamma dS_i. \quad (22)$$

90 With this development the general averaged balance law (10) becomes

$$\begin{aligned} \frac{\partial}{\partial t} (\alpha_1 \langle q_1 \rangle) + \nabla \cdot (\alpha_1 \langle q_1 \mathbf{u}_1 \rangle) = & \alpha_1 \nabla \cdot \bar{\boldsymbol{\phi}} - \frac{1}{V} \oint_{S_2+S_i} \boldsymbol{\phi}'_2 \cdot \mathbf{n}_2 dS \\ & + \frac{1}{V} \int_{S_i} [-q_1 (\mathbf{u}_1 - \mathbf{v}_i) \cdot \mathbf{n}_1 + \gamma] dS_i + \alpha_1 \langle \theta_1 \rangle, \end{aligned} \quad (23)$$

91 with a similar equation for the other phase. This is the main result of this
 92 work. It shows that *both phases* are acted upon the by *same* non-convective
 93 flux $\bar{\boldsymbol{\phi}}$ apportioned according to the respective volume fractions. The first
 94 surface integral in the right-hand side accounts for the local interaction be-
 95 tween the phases and can be calculated as if the *situation* was locally uniform
 96 since the large-scale structure *of the flow* is captured by $\bar{\boldsymbol{\phi}}$.

97 4. An example: Disperse flow

98 Let us consider the application of the result (23) to the case of disperse
 99 flow in which *arbitrarily shaped* drops or particles (phase 2) are suspended
 100 in a continuous phase (phase 1). The same result would apply to bubbles
 101 provided they maintain a constant volume as, in the derivation that follows
 102 (*but not in the general argument of the previous section*), we assume the two
 103 phases to be incompressible. For simplicity of exposition we use the word
 104 “drop” to designate any type of disperse entity.

105 In this application the interface S_i consists of the collection of the surfaces
 106 of the drops entirely contained in the averaging volume, plus the surfaces of
 107 the incomplete drops only part of which is in the averaging volume (see
 108 figure 1).

109 For momentum balance, for which $q_j = \rho_j \mathbf{u}_j$, $\boldsymbol{\phi}_j = \boldsymbol{\sigma}_j$ is the stress tensor
 110 for phase j . The total force exerted by the continuous phase on each complete
 111 drop (i.e., on each drop entirely contained within the averaging volume V)
 112 is given by

$$\mathbf{f}_t = \oint_{s_d} \boldsymbol{\sigma}_1 \cdot \mathbf{n}_d ds_d = \oint_{s_d} \boldsymbol{\sigma}_2 \cdot \mathbf{n}_d ds_d, \quad (24)$$

113 in which s_d is the drop surface and $\mathbf{n}_d$ is the outwardly directed unit normal.
 114 Actually $\boldsymbol{\sigma}_2 \cdot \mathbf{n}_d$ differs from $\boldsymbol{\sigma}_1 \cdot \mathbf{n}_d$ because of the effect of surface tension
 115 but, after integration over the entire drop surface, this contribution vanishes
 116 as surface tension does not exert any net force on the whole drop (see e.g.

117 Prosperetti and Jones, 1984; Hesla et al., 1993). We write

$$\mathbf{f}_t = \oint_{s_d} \boldsymbol{\sigma}_2 \cdot \mathbf{n}_d ds_d = \oint_{s_d} (\bar{\boldsymbol{\sigma}} + \boldsymbol{\sigma}'_2) \cdot \mathbf{n}_d ds_d \simeq v_d \nabla \cdot \bar{\boldsymbol{\sigma}} + \oint_{s_d} \boldsymbol{\sigma}'_2 \cdot \mathbf{n}_d ds_d, \quad (25)$$

118 where v_d is the drop volume. Note that the normal $\mathbf{n}_d$ has the same direction
 119 as the normal $\mathbf{n}_2$ appearing in (23). The first term is the force due to the
 120 large-scale flow in which the drop is immersed and is akin to a pseudo-
 121 buoyancy force (see e.g. Jackson, 2000; Prosperetti and Tryggvason, 2009),
 122 while the second one is responsible for the force due to the local conditions
 123 in the neighborhood of the drop such as drag, lift, added mass and others;
 124 we set

$$\mathbf{f} = \oint_{s_d} \boldsymbol{\sigma}'_2 \cdot \mathbf{n}_d ds_d. \quad (26)$$

125 The situation of the drops only partially contained in the averaging vol-
 126 ume is different as their contribution to S_i is not via integration over a closed
 127 surface. Thus, for example, the result of the integral analogous to (24) for
 128 these drops would depend on the absolute pressure level, which would intro-
 129 duce an unphysical feature in the modelling of incompressible flows. Further-
 130 more, these drops contribute to the integration over S_2 in (23). The simplest
 131 way to deal with this situation probably is to count the contribution of these
 132 drops proportionally to the fraction of their volume (or surface area) con-
 133 tained within the averaging volume. With this understanding we find for the
 134 second term in the right-hand side of (23)

$$\frac{1}{V} \oint_{S_2+S_i} \boldsymbol{\sigma}'_2 \cdot \mathbf{n}_2 dS = \frac{1}{V} \sum_d \mathbf{f}_d = n \bar{\mathbf{f}}, \quad (27)$$

135 where the summation is over all the drops in the averaging volume, possibly
 136 weighted proportionally to their inclusion in the averaging volume as ex-
 137 plained before. In the last step we have introduced the drop number density
 138 n and the mean hydrodynamic force acting on the drops $\bar{\mathbf{f}}$.

139 Assuming the density of the continuous phase to be constant and ne-
 140 glecting mass transfer phenomena across the drop interfaces, (23) becomes
 141 then

$$\begin{aligned} \rho_1 \left(\frac{\partial}{\partial t} (\alpha_1 \langle \mathbf{u}_1 \rangle) + \nabla \cdot (\alpha_1 \langle \mathbf{u}_1 \mathbf{u}_1 \rangle) \right) &= \alpha_1 \nabla \cdot \bar{\boldsymbol{\sigma}} - \frac{1}{V} \sum_d \mathbf{f}_d \\ &+ \frac{1}{V} \int_{S_i} \boldsymbol{\gamma} dS_i + \alpha_1 \rho_1 \mathbf{g}, \end{aligned} \quad (28)$$

142 where $\boldsymbol{\gamma}$ is the (vector) contribution of the surface forces and the volume
 143 source of momentum per unit mass has been written as $\mathbf{g}$. The mean stress
 144 $\bar{\boldsymbol{\sigma}}$ may be decomposed into pressure and viscous components and closed, e.g.,
 145 as

$$\bar{\sigma}_{ij} = -\bar{p}\delta_{ij} + \bar{\tau}_{ij} = -\bar{p}\delta_{ij} + \mu_{eff} \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right), \quad (29)$$

146 with μ_{eff} an effective shear viscosity and $\bar{\mathbf{u}} = \alpha_1 \langle \mathbf{u}_1 \rangle + \alpha_2 \langle \mathbf{u}_2 \rangle$ as in (12).
 147 A diagonal term proportional to $\nabla \cdot \bar{\mathbf{u}}$ is absent since the mixture is incom-
 148 pressible. With these choices (28) becomes

$$\begin{aligned} \rho_1 \left(\frac{\partial}{\partial t} (\alpha_1 \langle \mathbf{u}_1 \rangle) + \nabla \cdot (\alpha_1 \langle \mathbf{u}_1 \mathbf{u}_1 \rangle) \right) &= -\alpha_1 \nabla \bar{p} + \alpha_1 \nabla \cdot \bar{\boldsymbol{\tau}} - \frac{1}{V} \sum_d \mathbf{f}_d \\ &+ \frac{1}{V} \int_{S_i} \boldsymbol{\gamma} dS_i + \alpha_1 \rho_1 \mathbf{g}. \end{aligned} \quad (30)$$

149 The integral over S_i is obviously difficult to model. If the area fraction
 150 occupied by the drops on the surface of the averaging volume is α_s we have,
 151 in order of magnitude,

$$\left| \frac{1}{V} \int_{S_i} \boldsymbol{\gamma} dS_i \right| \sim \frac{\alpha_s S}{V} \frac{2\gamma}{r_d} \quad (31)$$

152 where S is the surface of the averaging volume, γ the surface tension coeffi-
 153 cient and r_d the drop radius. The hydrodynamic force is very often dominated
 154 by drag so that, again in order of magnitude,

$$n |\bar{\mathbf{f}}| \sim n \mu_1 r_d w_d, \quad (32)$$

155 with w_d the magnitude of the drop-fluid relative velocity and μ_1 the viscosity
 156 of the continuous phase. The ratio of these two terms is therefore

$$\frac{2(\alpha_s S/V)\gamma/r_d}{n \mu_1 r_d w_d} = 2 \frac{\alpha_s S/r_d^2}{nV} \frac{\gamma}{\mu_1 w_d} = \frac{N_S}{N_V} \frac{2}{Ca}, \quad (33)$$

157 in which N_S and N_V are the numbers of drops on the surface of, and inside,
 158 the averaging volume, respectively, and $Ca = \mu_1 w_d / \gamma$ is a representative
 159 capillary number. One would typically expect $N_S < N_V$. Thus, provided the
 160 capillary number is not too small, the surface integral in (28) and (30) may
 161 be neglected.

162 The disperse phase can be treated in the same way except that, in the
 163 final result, one can take a step similar to the second one in (27) thus finding
 164 a momentum equation for the “average drop” contained in the averaging
 165 volume, which it is convenient to write in the Lagrangian form:

$$\frac{d}{dt}(\overline{m\mathbf{u}_d}) = \bar{v}_d \nabla \cdot \bar{\boldsymbol{\sigma}} + \bar{\mathbf{f}} + \bar{m}\mathbf{g}, \quad (34)$$

166 where m and $\mathbf{u}_d$ are the drop mass and velocity. Surface tension causes
 167 deformation of the drop but no translational motion, as noted before, and,
 168 accordingly, it has been omitted. An alternative to the use of (34) is to write
 169 the equation for each individual drop, thus adopting a Lagrangian description
 170 as is done, for example, in the point-particle or discrete-element models (see
 171 e.g. Ferrante and Elghobashi, 2003; Eaton, 2009; Deen et al., 2007).

172 The thermal energy balance can be treated in a similar way. For an
 173 incompressible continuous phase with constant properties we would have $q_1 =$
 174 $\rho_1 c_{p1} T_1$, with c_{p1} the constant-pressure specific heat and T_1 the temperature,
 175 while $\phi_1 = -\mathbf{q}_1$ the heat flux. In this case the analog of (25) is

$$\mathbf{Q}_t = \oint_{s_d} (-\bar{\mathbf{q}} - \mathbf{q}'_2) \cdot \mathbf{n}_d ds_d \simeq -v_d \nabla \cdot \bar{\mathbf{q}} - \oint_{s_d} \mathbf{q}'_2 \cdot \mathbf{n}_d ds_d, \quad (35)$$

176 and, upon letting

$$\mathbf{Q} = - \oint_{s_d} \mathbf{q}'_2 \cdot \mathbf{n}_d ds_d, \quad (36)$$

177 we find the energy balance in the form

$$\rho_1 c_{p1} \left(\frac{\partial}{\partial t} (\alpha_1 \langle T_1 \rangle) + \nabla \cdot (\alpha_1 \langle T_1 \mathbf{u}_1 \rangle) \right) = -\alpha_1 \nabla \cdot \bar{\mathbf{q}} - n \bar{\mathbf{Q}}. \quad (37)$$

178 Here we have neglected volume sources of internal energy such as viscous
 179 dissipation. Furthermore, because of the neglect of phase change at the drop
 180 surface, the right-hand side of the interface balance (15) vanishes.

181 The average heat flux can be closed as before as

$$\bar{\mathbf{q}} = -k_{eff} \nabla \bar{T}, \quad (38)$$

182 with k_{eff} an effective conductivity and $\bar{T} = \alpha_1 \langle T_1 \rangle + \alpha_2 \langle T_2 \rangle$ as in (12). The
 183 internal energy balance for the drops can be readily written down as before:

$$\frac{d}{dt} (c_{p1} \overline{mT_d}) = -\bar{v}_d \nabla \cdot \bar{\mathbf{q}} + \bar{\mathbf{Q}}. \quad (39)$$

184 Similarly to the use of a drag coefficient in the closure of the interphase force,
 185 a heat transfer coefficient h can be used to close $\mathbf{Q}$:

$$\mathbf{Q} = h s_d (\langle T_1 \rangle - T_d) . \quad (40)$$

186 5. Discussion

187 The result (23) shows two significant features concerning the contribution
 188 of the non-convective flux ϕ to the general averaged balance law. The first
 189 one is the appearance of the flux averaged over the entire mixture rather
 190 than over the individual phases. The second one is the presence of the volume
 191 fraction outside, rather than inside, the divergence operator.

192 In an earlier paper (Zhang and Prosperetti, 1997) we have derived the
 193 momentum and energy equations in a dilute fluid-particle flow by means
 194 of ensemble averaging. The results conform precisely with (28) and (37),
 195 respectively, with $\bar{\sigma}$ and $\bar{\mathbf{q}}$ as in (29) and (38). The very different method of
 196 derivation lends considerable support to the validity of these results.

197 It is well appreciated in the literature that, in the pressure term of the
 198 momentum equation, the volume fraction must be outside the gradient op-
 199 erator as in (30) in order to avoid unphysical results, e.g. in the case of a
 200 spatially non-uniform system on which no forces are acting (see e.g. Jackson,
 201 2000; Prosperetti and Tryggvason, 2009). However, the viscous term is often
 202 written as $\nabla \cdot (\alpha_1 \langle \tau_1 \rangle)$. While this form agrees with the result of straight
 203 volume averaging as in the first term in the right-hand side of (10), the prob-
 204 lem remains of closing $\langle \tau_1 \rangle$ as well as the interface integral in (10). The first
 205 objective is often accomplished by simply setting

$$\langle \tau_{1,ij} \rangle = \hat{\mu}_1 \left(\frac{\partial}{\partial x_j} \langle u_{1,i} \rangle + \frac{\partial}{\partial x_i} \langle u_{1,j} \rangle - \frac{2}{3} \nabla \cdot \langle \mathbf{u}_1 \rangle \delta_{ij} \right) , \quad (41)$$

206 with $\hat{\mu}_1$ a possibly modified fluid viscosity. This is the choice made by a
 207 large number of investigators; representative examples of what could be a
 208 very long list are the model employed in the code *MFIX* (Benyahia et al.,
 209 2012) as well as the studies by Deen et al. (2007), Janet et al. (2015) and
 210 many others.

211 In a recent paper, Capecelatro and Desjardins (2013) derive a momen-
 212 tum equation for the continuous phase by a mollified form of volume averag-
 213 ing separating terms slowly and rapidly varying in space. Upon combining

214 their equations (26), (27), (40) and (41) one finds that (upon omitting non-
 215 essential features) the contribution of the stress terms in the right-hand side
 216 of the fluid momentum equation can be written as

$$-\alpha_1 \nabla \bar{p} + \alpha_1 \nabla \cdot \left[\mu_{eff} \left(\nabla \tilde{\mathbf{u}}_1 + \nabla \tilde{\mathbf{u}}_1^T - \frac{2}{3} \nabla \cdot \tilde{\mathbf{u}}_1 \right) \right] - \frac{1}{V_c} \sum_i \mathbf{f}_i, \quad (42)$$

217 with the sum extended to the particles in the computational cell of volume
 218 V_c and $\mathbf{f}$ the hydrodynamic force on each particle. Here $\tilde{\mathbf{u}}_1$ is defined as “the
 219 volume-filtered [i.e., slowly varying] fluid velocity” and is therefore similar,
 220 although not exactly equal, to the slowly varying volumetric flux of the whole
 221 mixture $\bar{\mathbf{u}}$ introduced before in (29); the last term is the same as (27). The
 222 derivation by Capecelatro and Desjardins (2013) confirms the presence of
 223 the volume fraction outside the divergence operator in spite of the (probably
 224 quantitatively) minor difference between $\tilde{\mathbf{u}}_1$ and $\bar{\mathbf{u}}$.

225 The derivation of section 3 shows that the proper position of the volume
 226 fraction is outside the gradient operator not only for the momentum equation,
 227 but for all the balance laws. This is at variance with most of the models in
 228 current use (see e.g. Janet et al., 2015; Patil et al., 2015, for some recent
 229 examples).

230 6. Conclusions

231 By taking advantage of the separation of scales between macroscopic and
 232 local microscopic fields, we have recast the mathematical form in which non-
 233 convective fluxes ϕ_j (e.g. stress, heat flux and others) appear in the volume-
 234 averaged equation of the j -th phase. We have shown that, in place of fields
 235 $\langle \phi_j \rangle$ averaged over the individual phases, it is preferable to use a single, glob-
 236 ally averaged field $\bar{\phi} = \alpha_1 \langle \phi_1 \rangle + \alpha_2 \langle \phi_2 \rangle$, the action of which is apportioned
 237 according to the volume fraction of each phase as shown in (23). This step
 238 has several advantages.

239 In the first place, it provides a clear guidance as to how to interpret
 240 and use the terms containing ϕ_j in the unclosed form (10) of the general
 241 volume-averaged balance equation usually encountered in the literature.

242 Secondly, the derivation shows the much greater significance of the mix-
 243 ture flux compared to that of the individual phases. When the normal com-
 244 ponent of ϕ_j is continuous across the interface, while $\langle \nabla \cdot \phi_j \rangle \neq \nabla \cdot \langle \phi_j \rangle$
 245 as differentiation and averaging do not commute, it can be shown that it is

246 nonetheless true that $\alpha_1 \langle \nabla \cdot \phi_1 \rangle + \alpha_2 \langle \nabla \cdot \phi_2 \rangle = \nabla \cdot (\alpha_1 \langle \phi_1 \rangle + \alpha_2 \langle \phi_2 \rangle) \equiv \nabla \cdot \bar{\phi}$.
 247 Similarly, if some quantity such as the temperature T_j is continuous across the
 248 interface, it is true that $\alpha_1 \langle \nabla T_1 \rangle + \alpha_2 \langle \nabla T_2 \rangle = \nabla (\alpha_1 \langle T_1 \rangle + \alpha_2 \langle T_2 \rangle) \equiv \nabla \bar{T}$.
 249 In the case of heat conduction, for example, this circumstance justifies the
 250 introduction of an effective thermal conductivity k_{eff} of the mixture defined
 251 so that the mixture heat flux $\bar{\mathbf{q}}$ is closed as $\bar{\mathbf{q}} = -k_{eff} \nabla \bar{T}$ as done be-
 252 fore in (38) (see e.g. Sangani and Acrivos, 1983; Chang and Acrivos, 1986;
 253 Bonnetcaze and Brady, 1990; Marchioro and Prosperetti, 1999). A similar
 254 relation permits the introduction of an effective viscosity in a mean stress-
 255 volumetric flux closure relation (see e.g. Batchelor, 1970, 1974; Nunan and
 256 Keller, 1984). There is therefore a direct connection between the voluminous
 257 literature on effective properties of disperse systems (see e.g. Torquato, 2011)
 258 and the volume averaged equations.

259 Thirdly, the derivation clarifies the proper position of the volume fractions
 260 in the terms involving the divergence of the non-convective fluxes.

261 Our result is general and only rests on the hypothesis that the spatial
 262 variation of the combined average flux $\bar{\phi}$ is adequately represented by a lin-
 263 ear function of position within the averaging volume, which seems a rather
 264 mild assumption whenever the use of volume averaging is useful. No further
 265 hypotheses on the nature of the flow (e.g., about specific flow regimes) are
 266 necessary.

267 Finally, it is to be noted that the present results coincide with earlier ones
 268 obtained by the completely different method of ensemble averaging for the
 269 special case of a dilute suspension of spherical particles in a fluid (Zhang and
 270 Prosperetti, 1997). This coincidence, together with the difference between
 271 the methods of derivation, lends a strong support to their validity.

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337 **Figure captions**

338 **Figure 1:** Averaging volume in a two-phase system. The surface of the
339 averaging volume, $S = S_1 + S_2$, consists of a portion S_1 in contact with phase
340 1 (continuous thick line) and a portion S_2 in contact with phase 2 (dashed
341 thick line). Inside the averaging volume the two phases are separated by an
342 interface S_i (thin lines) which may consist of disjoint parts as in this figure.

343 **Figure 2:** The unit normals are defined to be directed out of the correspond-
344 ing phase.