Synthetic Turbulence: A Wavelet Based Simulation

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Abstract: The treatment of wind-induced vibrations is an important consideration in the design of civil structures with increasing span-lengths and heights. The turbulence in the atmospheric boundary layer has been treated as Gaussian in conventional stochastic simulation schemes, wherein higher-order statistics have been disregarded. However, experimental evidence points at turbulence being a typical multifractal process, which suggests that the statistics at different scales of atmospheric turbulence are not strictly self-similar but exhibit stronger non-Gaussianity as the length scale decreases. Intermittency characterized by the occasional bursts in the wind velocity leads to non-Gaussianity. Intermittency and its potential impact on wind-induced response have been neglected though. Recent studies have addressed the multifractal property of turbulence in wind with wavelet-basis representations of the log-Normal or log-Poisson models. These schemes offer new insight into the simulation of turbulent wind, but suffer from several significant drawbacks, such as the inappropriate sampling of the wavelet coefficients. To overcome these shortcomings, a new simulation scheme, based on the Haar wavelet representation,

is proposed in this study where the exact relation between the wind velocities and the wavelet coefficients is introduced. In addition, the effects of intermittency on the wind-induced response of structures are evaluated for a number of cases. A quasi-steady theory-based assessment indicates that the intermittency results in amplifying extremes of the wind-induced response and exhibits higher impact on relatively more rigid structures. It is likely that intermittency may invoke flow-structure interaction with possible enhancement in load effects at related scales, which may further amplify the extremes.

Keywords: Stochastic simulation; turbulent flow; wind process; intermittency; log-Poisson model; wavelet.

1 INTRODUCTION

Civil structures, such as bridges, buildings, transmission towers and residential structures, are vulnerable to the winds and the associated turbulence. Turbulence has attracted a wide range of studies including the chaotic features, which imply high level of nonlinearity and unpredictability. In general, there are two distinct approaches to describe a turbulent flow: one developed from first principles using the equations of motion, e.g. Navier-Stokes equations, or the probabilistic framework utilizing statistics of turbulent flows based on observations.

Numerical simulation of turbulent wind field using the Navier-Stokes equations is computationally very intensive, especially at high Reynold numbers, whereas statistical approaches can significantly reduce the computational effort by utilizing turbulence statistics. Conventionally, turbulence in the wind field is characterized by the second-order statistics. For example, both the spectral representation and the time series models are

46 based on the second-order statistics with implied assumption of Gaussianity (Shinozuka
47 and Deodatis 1991; Li and Kareem 1990a; 1990b). However, turbulence has significant
48 higher-order information, which is usually described in terms of velocity increment
49 between two time instants. Experiments have shown that the statistics of the velocity
50 increment depart gradually away from Gaussianity as the time interval decreases (Atta and
51 Chen 1970; Castaing et al. 1990; Frisch 1995). The K41 theory of turbulence divides the
52 velocity increment into energy-containing, inertial and dissipation sub-ranges
53 (Kolmogorov 1941). The probability density functions (PDFs) of velocity increments in
54 the inertial sub-range present longer tails than the Gaussian distribution. The non-
55 Gaussianity is highly correlated with the intermittency in turbulent flows. Theoretically,
56 non-Gaussianity features of the PDF reflect the presence of intermittency in the velocity
57 increments and gradients (e.g., Sreenivasan and Antonia 1997). The intermittency has
58 potential effects on wind-induced response because it occurs suddenly with attendant high
59 energy, which may be underestimated by a tacit Gaussian assumption. The higher-order
60 statistics of the wind velocity process can be approximately derived according to their
61 relation to the second-order statistics. For example, both the log-Normal model
62 (Kolmogorov 1962) and the log-Poisson model (She and Leveque 1994) show high-fidelity
63 for the derivation of higher-order statistics. The log-Poisson model is shown to perform
64 better in She and Waymire (1995), and thus it has been used in the present simulation
65 scheme. Indeed, both the log-Normal model and the log-Poisson model introduce a
66 multifractal description of wind fluctuations, rendering the simulation process non-trivial.
67 In this context, a wavelet expansion, which is a multi-scale modeling tool, is a preferred
68 basis (e.g., Arneodo et al. 1998; Gurley and Kareem 1999; Kitagawa and Nomura 2003).

In this approach, the sampling of wavelet coefficients from the multifractal statistics is crucial to a high-fidelity simulation of the wind field. In these references initial work in this context was introduced which is further explored and refined here.

This study proposes a new scheme for simulating wind velocity based on the Haar wavelet. This entails several assumptions, i.e., (1) the wind velocity increment in the energy-containing sub-range is only affected by the boundary conditions; (2) statistics of the wind velocity increment in the inertial sub-range follow a log-Poisson model for both stationary and non-stationary winds; (3) the wind velocity increment in the energy-dissipation sub-range has negligible contribution to wind fluctuations with practical implications, and is therefore disregarded. The second assumption relates to the Kolmogorov's conjecture that within a small time interval the statistics of velocity increment are approximately steady even when the wind velocity is not (Kolmogorov 1941).

The higher-order effects in velocity increments related to intermittency are highlighted here. Typically, the wind-induced response is based on the gust loading factor which involves a peak factor derived under Gaussian assumption. Three major issues, which have been usually ignored, are taken into account in the study to examine their contribution: the quadratic velocity term; the inclusion of relative motion in the formulation of aerodynamic force; the higher-order statistics of velocity fluctuations.

2 DESCRIPTION OF THE WIND VELOCITY PROCESS

A wind velocity as a random process can be expressed in terms of the Karhunen-Loeve (K-L) expansion as

$$V(\boldsymbol{\theta}, t) = \sum_{j=0}^N \xi_j(\boldsymbol{\theta}) \lambda_j \varphi_j(\boldsymbol{\theta}, t) \quad (1)$$

where $V(\boldsymbol{\theta}, t)$ represents a wind velocity indexed by time t ; $\boldsymbol{\theta}$ is the vector of the underlying random parameters of the wind velocity process; $\{\xi_j(\boldsymbol{\theta})\}_{j=0}^N$ denotes a normalized decomposed random vector with a scaling vector $\{\lambda_j\}_{j=0}^N$, which also represents the eigenvalue vector of K-L expansion; $\{\varphi_j(\boldsymbol{\theta}, t)\}_{j=0}^N$ is the corresponding basis.

The Fourier basis is often used in the simulation of a stationary process, whereas a wavelet basis is equally applicable to both stationary and non-stationary processes. Both the spectral representation and the time series (e.g., ARMA) models imply stationarity with the assumption that $\{\xi_j(\boldsymbol{\theta})\}_{j=0}^N$ is a Gaussian random vector (Shinozuka and Deodatis 1991; Li and Kareem 1990a; 1990b). The basis $\{\varphi_j(\boldsymbol{\theta}, t)\}_{j=0}^N$ can be further expanded in terms of a wavelet basis as (e.g., Ghanem and Spanos 2003; Phoon et al. 2004)

$$\varphi_j(\boldsymbol{\theta}, t) = \sum_{k=1}^M \zeta_{j,k}(\boldsymbol{\theta}) \psi_{j,k}(t) \text{ for } j=0, \dots, N \quad (2)$$

where $\{\psi_{j,k}(t)\}_{j=0,k=1}^{N,M}$ is the wavelet basis; $\{\zeta_{j,k}(\boldsymbol{\theta})\}_{k=1}^M$ is the Gaussian random vector.

As a result, the wind velocity can be further expressed as

$$\begin{aligned} V(\boldsymbol{\theta}, t) &= \sum_{j=0}^N \left\{ \xi_j(\boldsymbol{\theta}) \lambda_j \sum_{k=1}^M \tilde{\zeta}_{j,k}(\boldsymbol{\theta}) \psi_{j,k}(t) \right\} \\ &= \sum_{j=0}^N \sum_{k=1}^M \zeta_{j,k}(\boldsymbol{\theta}) m_{j,k} \psi_{j,k}(t) \end{aligned} \quad (3)$$

where $\{\zeta_{j,k}(\boldsymbol{\theta})\}_{k=1}^M$ denotes a normalized Gaussian random vector with a scaling vector

$\{m_{j,k}\}_{k=1}^M$, which also represents the vector of wavelet coefficients.

Turbulent wind may be treated as a $1/f$ noise whose power spectrum obeys the power law (Bak et al. 1988). For example, the power spectrum of a typical turbulent wind has the relation $S(f) \sim f^{5/3}$ in the inertial range (Kolmogorov 1941). The power law implies the existence of self-similarity or scale-invariance in turbulence (Bak et al. 1988), i.e., the PDFs of velocity increments at different length or time scales are similar. Here the velocity increment is defined as $\delta V = V(t) - V(t-\tau)$, in which τ is a time interval. However, further investigation of turbulence structure has shown that the PDFs at different scales are not strictly similar but gradually depart from Gaussian density functions as the length scale decreases. This scale-dependent or multifractal property implies that the turbulent wind is not a Gaussian process (e.g., Basu et al. 2004). Figure 1 presents the PDFs of velocity increments in various time intervals for the comparison of a measured turbulent wind velocity and the corresponding spectral representation. It shows that, the spectral representation naturally leads to the PDFs at different time intervals to be Gaussian, whereas for the measured turbulent wind, the PDFs present tails longer than the Gaussian. The longer tails in the PDFs are a reflection of the intermittency phenomenon of turbulence. The statistical explanation refers to the theory of large-deviation (Frisch 1995). Due to the multifractal nature of turbulence, a multi-scale simulation framework like the wavelet representation would be a more appropriate representation of wind velocity fluctuations, which could be utilized for numerical simulation.

2.1 Multifractal Property of Turbulence

The inertial range of turbulent wind is characterized by two length scales, l_c and l_d , which represent the smallest scales of the energy-containing range and inertial range, respectively. The multifractal property of turbulence can be quantified following two basic parameters,

the energy dissipation rate ε_l and the velocity increment δV_l . Here δV_l denotes the velocity increment with time interval l . For the velocity increment within the inertial range, the p th order moments of δV_l and ε_l are both dependent on l (She and Waymire 1995)

$$\langle \delta V_l^p \rangle \sim l^{\alpha_p} \quad (4)$$

$$\langle \varepsilon_l^p \rangle \sim l^{\tau_p} \quad (5)$$

and

$$\alpha_p = \frac{p}{3} + \frac{\tau_p}{3} \quad (6)$$

where $\langle \cdot \rangle$ denotes the expectation operator; τ_p and α_p are both the exponents related to the p th order moment.

2.2 A Hierarchical Relation of Velocity Increments

In K41 theory, $\tau_p = 0$, implying that the statistics of ε is scale-invariant. However, for a measured turbulence, the statistics of ε are actually scale-dependent ($\tau_p \neq 0$). Various models have been proposed to model the statistics of δv and ε through the modification of τ_p (Kolmogorov 1962; She and Leveque 1994). Among them one of the most promising models may be the log-Poisson model (She and Leveque 1994; She and Waymire 1995). Let l_j denote the length of the j th scale. As j decreases, the scale becomes larger. The log-Poisson model suggests that, within the inertial range, the velocity increments δV_{l_1} and δV_{l_2} have the following hierarchical relation (She and Waymire 1995)

$$\delta V_{l_2} = W_{l_1 l_2} \delta V_{l_1} = \left[\left(\frac{l_1}{l_2} \right)^{\gamma-1} \beta^{K_{H/2}} \right]^{\frac{1}{3}} \delta V_{l_1} \quad (7)$$

150 where $W_{l_1 l_2}$ is a multiplicative factor from δV_{l_1} to δV_{l_2} ; γ and β are scale-related
 151 parameters estimated from the real turbulent wind. In this study, the values are set to $\gamma =$
 152 $2/3$ and $\beta = 2/3$ as suggested by She and Waymire (1995). $\kappa_{l_1 l_2}$ is a Poisson random
 153 variable with mean $\lambda_{l_1 l_2}$, which satisfies the following relation

$$154 \quad \lambda_{l_1 l_2} = -\frac{\gamma}{\beta-1} \ln \frac{l_1}{l_2} \quad (8)$$

155 It should be noted that the hierarchical relation from scale l_1 to l_3 is identical to the relation
 156 from scale l_1 to l_3 via an immediate scale l_2 , i.e.,

$$157 \quad \delta V_{l_3} = W_{l_2 l_3} W_{l_1 l_2} \delta V_{l_1} = W_{l_1 l_3} \delta V_{l_1} \quad (9)$$

158 **3 SYNTHETIC TURBULENCE USING A WAVELET**

159 **REPRESENTATION**

160 **3.1 Wavelet Representation**

161 Equation (3) suggests that a general random process can be represented in the wavelet
 162 domain. Here an orthogonal wavelet basis is employed for efficiency. Thus the wind
 163 velocity process can be further expressed as

$$164 \quad V(\theta, t) = \zeta_\phi(\theta) m_\phi \phi(t) + \sum_{j=1}^n \sum_{k=1}^{2^{j-1}} \zeta_{j,k}(\theta) m_{j,k} \psi_{j,k}(t) \quad (10)$$

165 where $\zeta_\phi(\theta)$ is the constant 1; m_ϕ represents the wavelet coefficient corresponding to the
 166 father wavelet $\phi(t)$; the definition of $\{\zeta_{j,k}(\theta)\}_{j=1, k=1}^{n, 2^{j-1}}$ is in Section 3.3; $m_{j,k}$ represents the

167 wavelet coefficient at the j th scale and k th time position corresponding to the wavelet
 168 $\psi_{j,k}(t)$, which is defined as:

$$169 \quad \psi_{j,k}(t) = 2^{-(n-j+1)/2} \psi\left(\frac{t - 2^{n-j+1}(k-1)}{2^{n-j+1}}\right) \quad \text{for} \quad 1 \leq j \leq n, 1 \leq k \leq 2^{j-1} \quad (11)$$

170 in which $\psi(t)$ is the mother wavelet. It should be noted that, for the purpose of relating it
 171 to the scale order in turbulence, the scale order in the above equation is opposite to the
 172 common definition used in the wavelet analysis.

173 **3.2 Selection of Wavelet Basis**

174 An appropriate wavelet basis should establish a connection between the velocity
 175 increments and the wavelet coefficients. For example, the Haar wavelet basis has the
 176 following father and mother wavelets, respectively

$$177 \quad \phi(x) = \begin{cases} 1 & 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad (12)$$

$$178 \quad \psi(x) = \begin{cases} 1 & 0 < x \leq \frac{1}{2} \\ -1 & \frac{1}{2} < x \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad (13)$$

179 as shown in Fig. 2(a). The wavelet coefficient at the smallest scale ($j=n$) can be calculated
 180 as

$$181 \quad \begin{aligned} m_{n,k} &= \int_{-\infty}^{\infty} v(t) \psi_{n,k}(t) dt \\ &= \frac{1}{\sqrt{2}} [v(2k-1) - v(2k)] \quad \text{for} \quad k = 1, \dots, 2^{n-1} \end{aligned} \quad (14)$$

182 which coincides with the velocity increment in the time interval 1 scaled by $\frac{1}{\sqrt{2}}$. In general,

$$\begin{aligned}
 m_{j,k} &= \int_{-\infty}^{\infty} v(t) \psi_{j,k}(t) dt \\
 &= \frac{1}{2^{(n-j+1)/2}} \sum_{i=2^{n-j} \times (k-1)+1}^{2^{n-j} \times k} [v(i) - v(i+2^{n-j})] \quad \text{for } k=1, \dots, 2^{j-1}
 \end{aligned}
 \tag{15}$$

184 which represents the sum of several velocity increments in the time interval 2^{n-j} . Here $\delta v_{l_j,i}$
 185 is used to denote the velocity increment at time instant i with time interval $l_j = 2^{n-j}$. Thus,
 186 according to Eq. (15), there is

$$m_{j,k} = \frac{1}{2^{(n-j+1)/2}} \sum_{i=2^{n-j} \times (k-1)+1}^{2^{n-j} \times k} \delta v_{l_j,i}
 \tag{16}$$

188 The above equations demonstrate that the Haar wavelet coefficients can be exactly derived
 189 from the velocity increments at the same scale. A simple example is provided here as an
 190 illustration. Suppose there is a wind velocity process $v(t)$ indexed by 8 time instants as
 191 follows

192 1. The smallest scale:

$$\begin{aligned}
 m_{3,1} &= \frac{1}{\sqrt{2}} \delta v_{l_3,1} = \frac{1}{\sqrt{2}} (v_1 - v_2) \\
 m_{3,2} &= \frac{1}{\sqrt{2}} \delta v_{l_3,3} = \frac{1}{\sqrt{2}} (v_3 - v_4) \\
 m_{3,3} &= \frac{1}{\sqrt{2}} \delta v_{l_3,5} = \frac{1}{\sqrt{2}} (v_5 - v_6) \\
 m_{3,4} &= \frac{1}{\sqrt{2}} \delta v_{l_3,7} = \frac{1}{\sqrt{2}} (v_7 - v_8)
 \end{aligned}
 \tag{17}$$

194 2. The intermediate scale:

$$\begin{aligned}
m_{2,1} &= \frac{1}{2} \sum_{i=1}^2 \delta v_{l_2,i} = \frac{1}{2} [(v_1 - v_3) + (v_2 - v_4)] \\
m_{2,2} &= \frac{1}{2} \sum_{i=3}^4 \delta v_{l_2,i} = \frac{1}{2} [(v_5 - v_7) + (v_6 - v_8)]
\end{aligned}
\tag{18}$$

3. The largest scale:

$$\begin{aligned}
m_{1,1} &= \frac{1}{2\sqrt{2}} \sum_{i=1}^4 \delta v_{l_1,i} \\
&= \frac{1}{2\sqrt{2}} [(v_1 - v_5) + (v_2 - v_6) + (v_3 - v_7) + (v_4 - v_8)]
\end{aligned}
\tag{19}$$

Relations between the Haar wavelet coefficients and velocity increments are detailed in Fig. 3.

By comparison, the relationship between wavelet coefficients and velocity increments for other wavelet bases are relatively vague. For instance, the Meyer mother wavelet, as shown in Fig. 2(b), requires that the calculation of its corresponding wavelet coefficients should involve the velocity increments in different time intervals. As a consequence, the statistics of the wavelet coefficients are not conveniently derived from the velocity increments.

3.3 Simulation of Turbulent Wind

As mentioned previously, turbulent wind consists of energy-containing, inertial and dissipation sub-ranges. A critical length l_c separates the energy-containing sub-range from the inertial sub-range. In this study, l_c is assumed corresponding to the critical frequency f_c which makes $f \hat{S}_V(f)$ maximum, where $\hat{S}_V(f)$ is the power spectrum of the zero mean wind velocity process $V(t)$ (Kitagawa and Nomura 2003), i.e.,

$$f_c = \arg \max_{f \in (0, \infty)} f \widehat{S}_V(f) \quad (20)$$

The critical scale with the Haar wavelet basis can be determined based on its relation to f_c as follows

$$j_c = n + \text{int} \left\{ \log_2 \left(\frac{f_c}{0.742} \right) \right\} \quad (21)$$

Details are shown in Appendix A.

Let $\delta U_{l,j,i}$ be the normalized version of $\delta V_{l,j,i}$. The simulation procedure of the turbulent wind based on the Haar wavelet is illustrated in Fig. 4, and interpreted in details as follows

1. Determine j_c by Eq. (21);
2. For $l \geq l_c$ or $j \leq j_c$, i.e., within the energy-containing sub-range, $\{\delta U_{l,j,i}\}_{i=1}^{2^{n-1}}$ is an identical independent standard Gaussian sequence ([Kitagawa and Nomura 2003](#)), and $\zeta_{j,k}(\boldsymbol{\theta}) = \delta U_{l,j,2^{n-j}(k-1)+1}$ for $k = 1, \dots, 2^{j-1}$;
3. For $l < l_c$ or $j > j_c$, i.e., within the inertial sub-range, $\{\delta U_{l,j,i}\}_{i=1}^{2^{n-1}}$ is produced from $\{\delta U_{l_c,i}\}_{i=1}^{2^{n-1}}$ with an identical independent multiplicative sequence $\{W_{j_c j}(i)\}_{i=1}^{2^{n-1}}$ as:

$$\delta U_{l,j,i} \propto W_{j_c j}(i) \delta U_{l_c,i} \quad (22)$$

where

$$W_{j_c j}(i) = \left(\frac{l_{j_c}}{l_j} \right)^\gamma \beta^{\kappa_{j_c j}}(i) \quad (23)$$

229 and $\{\kappa_{jcj}(i)\}_{i=1}^{2^{n-1}}$ is an identical independent Poisson random sequence with mean
 230 $\lambda_{l_1 l_2}$.

231 $\{\zeta_{j,k}(\boldsymbol{\theta})\}_{k=1}^{2^{j-1}}$ is obtained by

$$232 \quad \zeta_{j,k}(\boldsymbol{\theta}) \propto \frac{1}{2^{(n-j+1)/2}} \sum_{i=2^{n-j} \times (k-1)+1}^{2^{n-j} \times k} \delta U_{l_j, i} \quad (24)$$

233 4. $\{m_{j,k}\}_{j=1,k=1}^{n,2^{j-1}}$ is determined by power spectrum as (see Appendix B)

$$234 \quad m_{j,k} = \sqrt{2^{n-j+1} \int_{-\infty}^{\infty} |\tilde{\psi}(f)|^2 S_V(f) df} \quad (25)$$

235 where $\tilde{\psi}(f)$ is the Fourier spectrum of $\psi(t)$; $S_V(f)$ is the power spectrum without
 236 removing the mean wind velocity. m_ϕ is determined by

$$237 \quad m_\phi = \sqrt{\int_{-\infty}^{\infty} |\tilde{\phi}(f)|^2 S_V(f) df} \quad (26)$$

238 where $\tilde{\phi}(f)$ is the Fourier spectrum of $\phi(t)$. If the target turbulent wind is non-
 239 stationary, as noted in the introduction, $\{m_{j,k}\}_{j=1,k=1}^{j_c, 2^{j_c-1}}$ should be adjusted by
 240 matching the wavelet spectra.

241 5. The simulated turbulent wind is obtained according to Eq. (10).

242 It should be noted that the wavelet coefficient sequence at a specific scale is
 243 stationary if the wind velocity process is stationary (see Appendix B). For a non-stationary
 244 turbulent wind, the non-stationarity is assumed only related to the wavelet coefficients in
 245 the energy-containing range, as mentioned in the second assumption.

246 3.4 Simulation Example

The wind velocity data recorded during Hurricane Katrina was fitted to the Karman spectrum model to simulate the turbulence, where the turbulence integral scale was estimated to be 247m and $j_c = 7$. The simulation results based on the proposed and the spectral representation schemes are shown in Fig. 5. The Karman spectrum and the power spectrum of the simulation are presented in Fig. 6. Also, PDFs of wavelet coefficients at several typical scales in the inertial sub-range are given in Fig. 7. The results show that the non-Gaussianity in turbulent wind is captured by the proposed simulation scheme.

3.5 Comparison with Other Simulation Schemes in the Wavelet Domain

It should be emphasized that, in the inertial sub-range, the statistics of the velocity increments at various scales, where energy production and dissipation balance, are obtained based on the stationarity assumption (She and Leveque 1994; She and Waymire 1995). While the multi-scale system may not be in equilibrium at a particular time instant because of the intermittency, the averaged statistics of the velocity increments at respective scales may be stationary. The present simulation scheme is based on this premise and illustrated in Fig. 8, where $\zeta_{j,\cdot}$ represents a random variable that has the identical statistics with $\zeta_{j,1}$, $\zeta_{j,2}$, ..., $\zeta_{j,k}$. The statistics of $\zeta_{j_c+1,\cdot}$ are derived from $\zeta_{j_c,\cdot}$ based on the log-Poisson model. Figure 8 illustrates that the wavelet coefficients at a particular scale in the inertial sub-range are identical. The conditional relations of these wavelet coefficients to those at other scales are also identical. Thus it may not be reasonable to discriminate the identical conditional relations. The next two subsections will revisit two typical simulation schemes based on wavelet representations and make remarks accordingly.

3.5.1 Wavelet dyadic trees model

269 The wavelet dyadic trees (WDT) model (Arneodo et al. 1998), as illustrated in Fig. 9, is
 270 expressed as follows

$$\begin{aligned}
 m_{0,1} &= 1 \\
 m_{j,2k-1} &= W_{j-1,k}^l m_{j-1,k} \\
 m_{j,2k} &= W_{j-1,k}^r m_{j-1,k} \quad \text{for } 1 \leq j \leq n, 1 \leq k \leq 2^{j-1}
 \end{aligned} \tag{27}$$

272 where $m_{j,k}$ is a wavelet coefficient; multiplicative factors $W_{j-1,k}^l$ and $W_{j-1,k}^r$ are identical
 273 random variables and have the same statistics as defined in Eq. (7). Note that $m_{j,k}$ is not a
 274 deterministic but a random variable here. Both $m_{j,2k-1}$ and $m_{j,2k}$ derive from the same father
 275 variable $m_{j-1,k}$ and have identical statistics. However, the sequence $\{m_{j,k}\}_{k=1}^{2^{j-1}}$ is not
 276 stationary because the correlation function between $m_{j,2k-1}$ and $m_{j,2k}$ is not equal to that
 277 between $m_{j,2k}$ and $m_{j,2k+1}$. A brief explanation is that, for example, the correlation function
 278 between $m_{2,1}$ and $m_{2,2}$ depends on the sampling of $W_{1,1}^l$ and $W_{1,1}^r$, while the correlation
 279 function between $m_{2,2}$ and $m_{2,3}$ depends on the sampling of not only $W_{1,1}^r$ and $W_{1,2}^l$ but also
 280 $W_{0,1}^l$ and $W_{0,1}^r$ at the higher level. $W_{0,1}^l$ and $W_{0,1}^r$ introduce additional influence on the
 281 calculation of the correlation function between $m_{2,2}$ and $m_{2,3}$ in comparison with that
 282 between $m_{2,1}$ and $m_{2,2}$, and hence causes nonstationarity of the wavelet coefficients at scale
 283 $j = 2$. A simple example here helps to further explain it. Suppose there are two random
 284 variables, Z_1 and Z_2 , which are respectively defined as

$$\begin{aligned}
 Z_1 &= X + \sum_{i=1}^{20} Y_i = X + \sum_{i=1}^{20} W_i X \\
 Z_2 &= X + \sum_{i=1}^{20} \tilde{Y}_i = X + \sum_{i=1}^{20} W_i \tilde{X}_i
 \end{aligned} \tag{28}$$

286 where X is a standard Gaussian random variable; $\{W_i\}_{i=1}^{20}$ and $\{\tilde{X}_i\}_{i=1}^{20}$ are both identical
 287 independent standard Gaussian random sequences. Although Y_i has the identical statistics
 288 with $\tilde{Y}_i$, Y_i indeed depends on X while $\tilde{Y}_i$ is independent of X . Therefore, to simulate Z_1 , the
 289 sampling of $\{Y_i\}_{i=1}^{20}$ is actually conditionally dependent on the sampling of X . Four
 290 sampling pairs of X and $\{Y_i\}_{i=1}^{20}$ are plotted in Fig. 10(a), which shows a strong correlation
 291 between X and Y_i . It may be imagined that the concatenated samplings of $\{Y_i\}_{i=1}^{20}$ are not
 292 stationary. For comparison Fig. 10(b) plots four sampling pairs of X and $\{\tilde{Y}_i\}_{i=1}^{20}$. Obviously,
 293 the concatenated samplings of $\{\tilde{Y}_i\}_{i=1}^{20}$ approach stationary.

294 In addition to nonstationarity, WDT model does not present an exact relation
 295 between the velocity increments and the wavelet coefficients.

296 3.5.2 A modified WDT model

297 A modified WDT model is expressed as follows ([Kitagawa and Nomura 2003](#))

$$\begin{aligned}
 \zeta_{j,2k-1} &= S \times W_{j-1,k} \zeta_{j-1,k} \\
 \zeta_{j,2k} &= S \times W_{j-1,k} \frac{\sqrt{2}(\zeta_{j-1,k} + \zeta_{j-1,k+1})}{2} \quad \text{for } j_c < j \leq n, 1 \leq k \leq 2^{j-1}
 \end{aligned} \tag{29}$$

299 where $\zeta_{j,k}$ is a normalized random variable; $W_{j,k}$ is a multiplicative factor as defined in Eq.
 300 (7); S is a binary random variable having two values: -1 and 1. In particular, $\zeta_{j_c,k}$ is a
 301 standard Gaussian random variable. The simulation scheme is illustrated in Fig. 11. In
 302 order to alleviate the nonstationarity of the WDT model, the modified WDT model
 303 introduces artificial correlation among wavelet coefficients derived from different fathers.
 304 According to Eq. (29), since $\zeta_{j,2k-1}$ derives from only $\zeta_{j-1,k}$ while $\zeta_{j,2k}$ from both $\zeta_{j-1,k}$
 305 and $\zeta_{j-1,k+1}$, $\zeta_{j,2k-1}$ has different statistics from $\zeta_{j,2k}$. The artificial modification of $\zeta_{j,2k}$

introduces additional nonstationarity in $\{\zeta_{j,k}\}_{k=1}^{2^{j-1}}$. Besides, the exact relation between the wavelet coefficients and the velocity increments is still absent in the modified WDT model. In light of the preceding comparison, it is concluded that the Haar wavelet based scheme presented earlier captures required features and reflects necessary correlation structures.

4 ASSESSMENT OF INTERMITTENCY EFFECTS

A simple example of a single degree of freedom (SDOF) structure is utilized for the assessment of the intermittency effects on the alongwind-induced response of structures based on quasi-steady aerodynamics. The wind-induced response is described by

$$m\ddot{X}(t) + c\dot{X}(t) + kX(t) = \frac{1}{2}\rho AC_D [\bar{V} + V(t) - \dot{X}(t)]^2 \quad (30)$$

where m , c and k represent respectively the mass, damping coefficient and stiffness of the structure; $\ddot{X}(t)$, $\dot{X}(t)$ and $X(t)$ represent respectively the acceleration, velocity and displacement of the structure; ρ is the air density; A is the projection area of the wind load; C_D is the drag coefficient; $\bar{V}$ is the mean wind velocity and $V(t)$ is the fluctuating velocity.

4.1 Non-Gaussian Structural Response Peak Factor

Rewrite Eq. (30) as

$$m\ddot{X}(t) + \{c + \rho AC_D [\bar{V} + V(t)]\} \dot{X}(t) + kX(t) = \frac{1}{2}\rho AC_D \left\{ [\bar{V} + V(t)]^2 + [\dot{X}(t)]^2 \right\} \quad (31)$$

As the above equation indicates, the transformation from wind velocity to wind load, $[\bar{V} + V(t)]^2$, involves a quadratic effect that leads to a non-Gaussianity. The interaction between the structural motion and wind forces gives rise to a time-variant damping

coefficient $\rho AC_D [\bar{V} + V(t)]$ and a feedback force $\frac{1}{2} \rho AC_D \dot{X}^2(t)$, which enhance the non-Gaussianity of the response process $X(t)$. As a consequence, the marginal PDF of $X(t)$ may have a longer tail than the corresponding Gaussian density function. The longer tail (or large deviation) has a significant impact on the extreme value $Y(t) = \max(|X(s)| : s \in [0, t])$. The quadratic and interaction effects have been well examined under the assumption of the Gaussian wind velocity fluctuation (e.g., Grigoriu 1986; Kareem et al. 1998). These studies suggest that if the two added effects were disregarded, the extreme value of $X(t)$ would be underestimated. However, since $V(t)$ is strictly not a Gaussian process, it may further impact the extremes. Here the effects of non-Gaussianity caused by the intermittency in $V(t)$ on extremes of a SDOF system is evaluated.

The structural extreme response is examined in terms of the structural response peak factor, which is defined as

$$g(t) = \frac{\langle Y(t) \rangle - \langle X(t) \rangle}{\sigma_X(t)} \quad (32)$$

where $g(t)$ is the structural response peak factor; $\sigma_X(t)$ represents the stand deviation of the structural displacement. The PDF of $Y(t)$ can be expressed as an exponential form (Lutes and Sarkani 1997)

$$p_Y(y, t) = \frac{\partial P_Y(y, t)}{\partial y} = \frac{\partial \left[P_Y(y, 0) \exp \left(- \int_0^t \eta_X(y, s) ds \right) \right]}{\partial y} \quad (33)$$

where $P_Y(y, t)$ denotes the cumulative distribution function of $Y(t)$; $\eta_X(y, s)$ is a hazard function in reliability theory. Based on the assumption of a Poisson process, it is assumed to be equal to the zero crossing rate (Lutes and Sarkani 1997)

$$\begin{aligned} \eta_X(y, s) &\approx \nu_X(y, s) \\ &= \int_0^\infty \dot{x} p_{X\dot{X}}(y, \dot{x} | X(0) \leq y, \text{ no upcrossings in } [0, s]) d\dot{x} \end{aligned} \quad (34)$$

Further assumptions are necessary due to the complexity in the computation of the conditional joint PDF $p_{X\dot{X}}$. If $X(t)$ is covariance stationary, $X(t)$ and $\dot{X}(t)$ should be uncorrelated; if $X(t)$ is a Gaussian stationary process, $X(t)$ and $\dot{X}(t)$ should be mutually independent. The structural response peak factor is conventionally calculated based on the assumption that the structural response process is a Gaussian stationary process (Davenport 1964). An improved relationship takes into account the non-Gaussian property of $X(t)$ (e.g., Grigoriu 1986; Kareem et al. 1998). In this study, the Monte Carlo simulation (MCS) is employed in the computation of the intermittency effects. The sampling scheme for $g(t)$ is

$$g(t) \approx \frac{\sum_{i=1}^N [y(\theta_i, t) - x(\theta_i, t)]}{\sqrt{\sum_{i=1}^N x^2(\theta_i, t) - \left[\sum_{i=1}^N x(\theta_i, t) \right]^2}} \quad (35)$$

where N represents the number of samples. Each sample $x(\theta_i, t)$ is calculated by the Newmark- β method.

4.2 Case Study

Consider the SDOF system as a water tower used in Grigoriu (1986). Seven cases, with various combinations of the wind profile, structural mass and damping ratio, are studied. The incident wind velocity for the SDOF system is determined by its height, the reference wind velocity and the power law exponent of the wind profile. The results are listed in Table 1, in which h , M and K represent respectively the height, mass and stiffness of the

water tower. The auto-covariance function of the wind velocity process is given as (Grigoriu 1986)

$$R_v(\tau) = \sigma_v^2 \sum_{i=1}^2 p_i e^{-q_i |\tau|} \quad (36)$$

in which $p_1 = 0.492365$; $p_2 = 1 - p_1$; $q_1 = 0.0646863$; $q_2 = 0.28891$. The time step chosen in the Monte Carlo simulation is 0.1 seconds, which ensures that all structural frequencies are captured. The duration time is $2^{15} \times 0.1 = 3276.8$ seconds.

The following observations are offered based on the results:

1. Under the assumption of Gaussian wind velocity process, the structural response peak factors calculated by the MCS are generally smaller than the closed-form solutions (Grigoriu 1986; Kareem et al. 1998). A possible reason is that, the closed-form solutions are obtained under the Poisson assumption as shown in Eq. (34), which may overestimate the crossing rate. The comparison suggests that the closed-form solutions are conservative in the cases studied here.
2. For the structural response peak factors calculated by the MCS that include the intermittency effects are higher than those that disregard these effects. The likely explanation is that, the PDF of the velocity increment at a small scale has larger deviation than the Gaussian case, and accordingly the velocity increment has a higher-rate of occurrence of the extremes. As the scales of these velocity increments overlap with the structural frequency, the structural response peak factor increases.
3. Generally, rigid structures are more sensitive to the intermittency effects as indicated by the comparison of MCS solutions. Intermittency being related to

384 higher frequencies, i.e., velocity increments at smaller scales, tends to enhance
385 excitation that are typical of relatively rigid structures. For the cases studied here,
386 it is shown that the intermittency effects result in an average increase of 6.3% for
387 the rigid structures (various combinations of the wind speeds and structural
388 damping ratios), while the average increase is 4.7% for the relatively flexible
389 structures.

390 **5 CONCLUDING REMARKS**

391 A wavelet-based turbulent wind simulation scheme is developed in this study. This scheme
392 incorporates higher-order statistics of the turbulent wind fluctuations and is able to simulate
393 intermittency phenomenon. In traditional spectral representation or the time series models
394 based simulation with implied assumption of Gaussianity, the intermittency is masked,
395 therefore is not suitable for synthetic turbulence. Earlier examples of simulations based on
396 wavelet representations did not model necessary statistics of wavelet coefficients, whereas
397 the proposed scheme here incorporates these features. In addition, the effects of
398 intermittency on the wind-induced response based on the quasi-steady theory are assessed
399 for various cases. The results indicate that the intermittency tends to increase the extremes
400 of structural response, especially for relatively rigid structures. Also, structures of small
401 spatial dimensions will be more influenced by intermittency as these scales will engulf the
402 structures leading to enhanced loads. It is likely that intermittency may manifest more
403 energetic interactions with the vortical flow features around structures than those predicted
404 by the quasi-steady theory thus potentially leading to even higher extremes.

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408 **Appendix A. Determination of the Critical Scale**

409 Assume that the maximum point of the wavelet $\tilde{\psi}_j(f)$ portrays the representative
 410 frequency of $\psi_j(t)$ at scale j . The critical scale j_c corresponds to the wavelet whose
 411 representative frequencies is closest to f_c . Figure 12 shows the Fourier spectra of several
 412 normalized Haar wavelets.

413 **Appendix B. Relation between a Wavelet Coefficient and the Power Spectrum**

414 The correlation between two wavelet coefficients, m_{j,k_1} and m_{j,k_2} , at the same scale j is
 415 denoted as $R(\zeta_{j,k_1}(\boldsymbol{\theta})m_{j,k_1}, \zeta_{j,k_2}(\boldsymbol{\theta})m_{j,k_2})$. Let $V(t)$ be a stationary process, and then
 416 express $R(\zeta_{j,k_1}(\boldsymbol{\theta})m_{j,k_1}, \zeta_{j,k_2}(\boldsymbol{\theta})m_{j,k_2})$ further as follows

$$\begin{aligned}
 & R(\zeta_{j,k_1}(\boldsymbol{\theta})m_{j,k_1}, \zeta_{j,k_2}(\boldsymbol{\theta})m_{j,k_2}) \\
 &= \langle \zeta_{j,k_1}(\boldsymbol{\theta})m_{j,k_1}, \zeta_{j,k_2}^*(\boldsymbol{\theta})m_{j,k_2}^* \rangle \\
 &= \left\langle \int_{-\infty}^{\infty} \psi_{j,k_1}(t_1)V(t_1)dt_1 \int_{-\infty}^{\infty} \psi_{j,k_2}^*(t_2)V^*(t_2)dt_2 \right\rangle \\
 &= \left\langle \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi_{j,k_1}(t_1)\psi_{j,k_2}^*(t_2)V(t_1)V^*(t_2)dt_1dt_2 \right\rangle \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi_{j,k_1}(t_1)\psi_{j,k_2}^*(t_2) \langle V(t_1)V^*(t_2) \rangle dt_1dt_2 \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi_{j,k_1}(t_1)\psi_{j,k_2}^*(t_2)R(V(t_1), V(t_2))dt_1dt_2 \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi_{j,k_1}(t_1)\psi_{j,k_2}^*(t_2) \int_{-\infty}^{\infty} S_V(f)e^{i(t_2-t_1)2\pi f}dfdt_1dt_2 \\
 &= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \psi_{j,k_1}(t_1)e^{-it_1 2\pi f}dt_1 \right] \psi_{j,k_2}^*(t_2)e^{it_2 2\pi f}dt_2 \right\} S_V(f)df \\
 &= \int_{-\infty}^{\infty} \tilde{\psi}_{j,k_1}(f)\tilde{\psi}_{j,k_2}^*(f)S_V(f)df \\
 &= 2^{n-j+1} \int_{-\infty}^{\infty} |\tilde{\psi}(f)|^2 e^{i2\pi f(k_2-k_1)} S_V(f)df
 \end{aligned} \tag{37}$$

418 where the symbol ‘*’ represents the conjugate counterpart; $R(V(t_1), V(t_2))$ is the
 419 correlation between $V(t_1)$ and $V(t_2)$. Since the final expression of
 420 $R(\zeta_{j,k_1}(\boldsymbol{\theta})m_{j,k_1}, \zeta_{j,k_2}(\boldsymbol{\theta})m_{j,k_2})$ depends only the difference between k_1 and k_2 , the
 421 sequence $\{\zeta_{j,k}(\boldsymbol{\theta})\}_{k=1}^{2^{j-1}}$ is stationary. The scaling factor $m_{j,k}$ can be easily obtained as

$$422 \quad m_{j,k} = \sqrt{2^{n-j+1} \int_{-\infty}^{\infty} |\tilde{\psi}(f)|^2 S_V(f) df} \quad (38)$$

423 by setting $k_1 = k_2$.

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477

Figure 1: PDFs of the normalized velocity increments of a wind velocity process in various time intervals.

Figure 2: Different Wavelet bases. (a) Haar mother wavelet; (b) Meyer mother wavelet.

Figure 3: Relations between the Haar wavelet coefficients and velocity increments.

Figure 4: The simulation procedure of turbulent wind.

Figure 5: Measurement and simulation of wind processes.

Figure 6: Power spectrum.

Figure 7: PDF of wavelet coefficients at various scales in the inertial sub-range.

Figure 8: The present simulation scheme in the inertial sub-range.

Figure 9: The simulation scheme of wavelet dyadic tree.

Figure 10: The distribution configuration of samples. Different symbols denote different realizations. (a) Samples of X and $\{Y_i\}_{i=1}^{20}$ to simulate Z_1 ; (b) Samples of X and $\{\tilde{Y}_i\}_{i=1}^{20}$ to simulate Z_2 .

Figure 11: The simulation scheme of the modification model.

Figure 12: Fourier spectra of several Haar wavelets. The bold dash line represents the representative frequency of the wavelet at the scale $j = 1$.

Figure 1

Figure 2a

Figure 2b

Figure 4

Figure 5

Figure 6

Figure 7

Figure 8

Figure 9

Figure 10b

Figure 11

...

Figure 12

Table 1: Comparison of structural response peak factors

Case	Reference	Power	RMS	Damping	g $G-G$	g_{ng}				
	wind	law				$G-NG$	$NG-NG$			
	(m/s)	exponent					(m/s)	(%)	Davenport	Soize
$h = 20\text{m}, M = 2161\text{kg}, K = 85300\text{N/m}$										
1	28	0.15	4.6	5	4.04	4.64	5.18	5.09	4.81	5.33
2	13	0.35	5.8	3	4.10	4.91	6.43	6.50	6.01	6.18
3	13	0.35	5.8	5	4.04	4.86	6.42	6.44	6.03	6.41
$h = 80\text{m}, M = 13500\text{kg}, K = 85300\text{N/m}$										
4	28	0.15	4.6	3	3.87	4.29	4.59	4.49	4.38	4.62
5	28	0.15	4.6	5	3.85	4.31	4.63	4.56	4.44	4.81
6	13	0.35	5.8	3	3.87	4.43	5.13	5.01	4.83	4.90
7	13	0.35	5.8	5	3.85	4.45	5.12	5.14	4.95	5.13

Notation:

RMS: root mean square wind velocity;

 g : Gaussian structural response peak factor; g_{ng} : non-Gaussian structural response peak factor; $G-G$: Gaussian wind velocity process and Gaussian structural response peak factor; $G-NG$: Gaussian wind velocity process and non-Gaussian structural response peak factor; $NG-NG$: turbulent wind velocity process and non-Gaussian structural peak factor;

Davenport: a closed-form solution (Davenport, 1964);

Soize: a closed-form solution (Soize, 1978);

Grigoriu: a closed-form solution (Grigoriu, 1986);

Kareem: a closed-form solution (Kareem et al., 1998).