Models for space-time random functions

M. Grigoriu

Cornell University, Ithaca NY 14853–3501, USA

Abstract

Models are developed for random functions $Q(x, t)$ of space $x \in D$ and time $t \in [0, \tau]$ from samples of these functions and any other information when available. Most of the models in the paper can be viewed as extensions of Karhunen-Loève (KL) representations for random fields. Their samples are linear forms of basis functions with random coefficients which are extracted from samples of $Q(x, t)$ by singular value decomposition. The coefficients of these forms are stochastic processes rather than random variables. The proposed models can be used to generate large sets of samples whose statistics are similar to those of target random functions. Theoretical arguments and numerical examples are presented to establish properties of the proposed models, assess their accuracy, and illustrate their implementation.

Keywords: Karhunen-Loève (KL) representation, random field, singular value decomposition, space-time random function, stochastic process

1 Introduction

Random functions $Q(x, t)$ of space $x \in D$ and time $t \in [0, \tau]$ abound in applications, e.g., temperatures and ozone levels in cities during a day or season, pressure fields on aircrafts, buildings, and other physical systems, wave forces on offshore platforms, ships, and dikes, and pollution concentration in rivers. These functions can also be used to describe the spatial variation of material properties at small scale and their evolution in time caused by large loads and/or excessive temperatures.

Generally, the number of available records of space-time random functions $Q(x, t)$ is insufficient to estimate their probability laws and find statistics of system responses subjected to actions described by these types of random functions. For example, available records are insufficient to estimate ozone levels and wind speeds of relatively large return periods. Statistics of stress fields in aircrafts during takeoff and landing, which are essential for developing economical and safe designs, cannot be derived from a few records of random actions on these systems. Models $Q_m(x, t)$, which are consistent with the available information on target random functions $Q(x, t)$, need to be used to characterize these functions beyond their records and solve practical problems.

There is a vast literature on space-time random functions in the statistical literature, where they are referred to as spatial random processes, functional time series, and space-time random functions [13, 14, 15]. The objective of most of these studies is to construct

models that match the first two moments of target space-time random functions. In contrast, our objective is to construct models whose samples are similar to those of target space-time random functions $Q(x, t)$ from the available information, which consists of samples of $Q(x, t)$ or samples of $Q(x, t)$ and some information on the probability law of this random function.

Most of the models in our discussion can be viewed as extensions of the Karhunen-Loève (KL) representations for random fields $Q(x)$, $x \in D$, i.e., random functions of space coordinates. These representations are superpositions of eigenfunctions of the correlation function of $Q(x)$ with random coefficients $\{Z_k\}$. The proposed models have the same functional form except that stochastic processes $\{Z_k(t)\}$ are used in place of the random coefficient $\{Z_k\}$. These types of models have been used to solve stochastic partial differential equations (SPDEs) in both applied [21] and theoretical [2] (Sect. 3.2) studies. In these context, the model components are solutions of differential equations. In our context, the model components are inferred from the available information on $Q(x, t)$.

The following section restates the objective of this study in formal terms. Sections 3 and 4 develop models for time-stationary and arbitrary random functions $Q(x, t)$, present properties of these models, and develop bounds on the discrepancy between target functions $Q(x, t)$ and models of these functions. Numerical examples are in Sect. 5. They illustrate the construction of the proposed models for various space-time random functions $Q(x, t)$ and levels of information on these functions. The model performance can be assessed precisely since the probability laws of the target functions are known. It is found that the proposed model are satisfactory in all cases examined in this study.

2 Problem definition

Let $Q(x, t)$, $x \in D$, $t \in [0, \tau]$, be an $\mathbb{R}^q$ -valued random function of space x and time t , where D is a bounded subset of $\mathbb{R}^p$, $p = 1, 2, 3$, and $[0, \tau]$ denotes a bounded time interval. Our objective is to construct a family of models $\{Q_m(x, t)\}$, $m = 1, 2, \dots$, for $Q(x, t)$ from (1) a set $\{q_i(x, t)\}$, $i = 1, \dots, N$, of $N \geq 1$ independent samples of $Q(x, t)$ or (2) a set $\{q_i(x, t)\}$, $i = 1, \dots, N$, of $N \geq 1$ independent samples of $Q(x, t)$ and some properties of this random function, e.g., the knowledge that $Q(x, t)$ is a time-stationary random function.

Consider the family of random fields $\{Q(x, t), x \in D\}$ indexed by $t \in [0, \tau]$. The members of this family have the same properties if $Q(x, t)$ is time-stationary but their properties change in time if $Q(x, t)$ is not time-stationary. This observation is used to construct models for this random function. If $Q(x, t)$ is time-stationary, the properties of the random fields $\{Q(x, t), x \in D\}$ are time-invariant and can be estimated from the values of the samples $\{q_i(x, t)\}$, $i = 1, \dots, N$, at a single time $t \in [0, \tau]$. This estimation approach uses partially the available information. It can be improved by using values of $\{q_i(x, t)\}$, $i = 1, \dots, N$, at several times $t = t_r$, $r = 1, \dots, n_r$, which are spaced adequately such that the dependence between the random fields $Q(x, t_r)$ and $Q(x, t_{r'})$, $t_r \neq t_{r'}$ is weak. If $Q(x, t)$ is not time-stationary, the law of the random fields $\{Q(x, t), x \in D\}$ changes in time so that the spatial variation of $Q(x, t)$ cannot be inferred from values of the samples $\{q_i(x, t)\}$, $i = 1, \dots, N$, at a single time.

The following two sections develop models for time-stationary and arbitrary random functions $Q(x, t)$. Theoretical arguments and numerical examples are presented in these

sections to clarify features of the models of $Q(x, t)$ and illustrate their implementation. It is shown that space-time random functions can be represented by linear forms of basis functions of spatial argument $x \in D$ with random coefficients, which are stochastic processes rather than random variables. Both the basis functions and the random coefficients can be extracted from samples of $Q(x, t)$. For simplicity, it is assumed that the random function $Q(x, t)$ is real-valued with mean zero.

3 Models for time-stationary random functions

Denote by $Q(x)$, $x \in D$, an arbitrary member of the family of random fields $\{Q(x, t), x \in D\}$ indexed by $t \in [0, \tau]$. If the random field $Q(x)$, $x \in D$, has continuous square integrable correlation function $r(x, y) = E[Q(x)Q(y)]$, it admits the Karhunen-Loève (KL) representation $Q(x) = \sum_{k=1}^{\infty} Z_k \varphi_k(x)$, $x \in D$, where $\{Z_k\}$ are uncorrelated random variables with means 0 and variances $\{\lambda_k\}$ and $\{\lambda_k, \varphi_k(x)\}$ are the eigenvalues and eigenfunctions of the integral equation $\int_D r(x, y) \varphi(y) dy = \lambda \varphi(x)$, $x \in D$ [11](Sect. 3.6.5). It is common in applications to approximate $Q(x)$ by truncated KL expansions, i.e., models of the type

$$Q(x) \simeq Q_m(x) = \sum_{k=1}^m Z_k \varphi_k(x), \quad x \in D, \quad m = 1, 2, \dots, \quad (1)$$

where m denotes the truncation level.

This fact suggests to describe the space-time random function $Q(x, t)$ considered in this section by versions of KL representations which allow the random variables $\{Z_k\}$ to fluctuate in time. This extension of the model in Eq. 1 has the form

$$Q(x, t) \simeq Q_m(x, t) = \sum_{k=1}^m Z_k(t) \varphi_k(x), \quad x \in D, \quad t \in [0, \tau], \quad m = 1, 2, \dots, \quad (2)$$

where $\{\varphi_k : D \rightarrow \mathbb{R}\}$ are deterministic functions, referred to as basis functions, and $\{Z_k(t)\}$ are real-valued, stationary stochastic processes with mean zero and finite variance, rather than random variables as in Eq. 1. The model $Q_m(x, t)$ of $Q(x, t)$ has been used in [2] (Sect. 3.2) to solve partial differential equations with space-time white noise input.

We note that the models $Q_m(x, t)$ in Eq. 2 and the target random function $Q(x, t)$ have similar properties in the sense clarified by the following property.

Property 1. The random function $Q_m(x, t)$ in Eq. 2 has mean zero and is time-stationary.

Proof: The first two moments of $Q_m(x, t)$ are $E[Q_m(x, t)] = 0$ and $E[Q_m(x, t)Q_m(y, s)] = \sum_{k,l=1}^m E[Z_k(t)Z_l(s)] \varphi_k(x) \varphi_l(y)$. The latter moment simplifies to $E[Q_m(x, t)Q_m(y, s)] = \sum_{k=1}^m E[Z_k(t)Z_k(s)] \varphi_k(x) \varphi_k(y)$ if the processes $\{Z_k(t)\}$ are uncorrelated. Since the expectations $E[Z_k(t)Z_l(s)]$ depend only on the time lag $(t-s)$, the random functions $Q_m(x, t)$ are weakly time-stationary for both correlated and uncorrelated $\{Z_k(t)\}$. Since the processes

$\{Z_k(t)\}$ are stationary by assumption, we have

$$\begin{aligned}
P(Q_m(x, t_1) \leq \xi_1, \dots, Q_m(x, t_d) \leq \xi_d) &= P\left(\sum_{k=1}^m Z_k(t_1) \varphi_k(x) \leq \xi_1, \dots, \sum_{k=1}^m Z_k(t_1) \varphi_k(x) \leq \xi_d\right) \\
&= P\left(\sum_{k=1}^m Z_k(t_1 + s) \varphi_k(x) \leq \xi_1, \dots, \sum_{k=1}^m Z_k(t_1 + s) \varphi_k(x) \leq \xi_d\right) \\
&= P(Q_m(x, t_1 + s) \leq \xi_1, \dots, Q_m(x, t_d + s) \leq \xi_d)
\end{aligned}$$

for arbitrary $d \geq 1$ and times $t_1, \dots, t_d, s$, so that the random functions $Q_m(x, t)$ are time-stationary. $\blacktriangle$

The implementation of the model in Eq. 2 requires to select the basis functions $\{\varphi_k\}$ and specify the properties of the stochastic processes $\{Z_k(t)\}$. We accomplish this task in two steps. First, the basis functions $\{\varphi_k\}$ are constructed from the spatial variation of the samples of $Q(x, t)$ at arbitrary times. Second, the orthogonality property of the basis functions is used to extract samples of $\{Z_k(t)\}$ from samples of $Q(x, t)$, which are subsequently used to construct models for these processes.

3.1 Spatial variation

Let $Q(x)$, $x \in D$, continue to denote $Q(x, t)$ at an arbitrary time $t \in [0, \tau]$. Let $\mathcal{H}$ be a Hilbert space with norm induced by an inner product $\langle \cdot, \cdot \rangle$ that contains the samples of $Q(x)$ and let $\{e_i\}$, $i \in I$, be an orthonormal basis of $\mathcal{H}$, where I is an index set. An arbitrary element e of $\mathcal{H}$ admits the representation $e = \sum_{i \in I} \langle e, e_i \rangle e_i$, where $\{\langle e, e_i \rangle\}$ are Fourier coefficients. It can be shown [12] (Theorem 3.4.10) that the set of non-zero Fourier coefficients is countable, so that $e = \sum_{k=1}^{\infty} \langle e, e_k \rangle e_k$. Accordingly, the samples $Q(x, \omega)$ of $Q(x)$ admit the representation

$$Q(x, \omega) = \sum_{k=1}^{\infty} Z_k(\omega) \varphi_k(x), \quad x \in D, \quad (3)$$

where the functions $\{\varphi_k(x)\}$ define an orthonormal basis, i.e., $\int_D \varphi_k(x) \varphi_l(x) dx = \delta_{kl}$, and $Z_k(\omega) = \langle Q(\cdot, \omega), \varphi_k(\cdot) \rangle = \int_Q Q(x, \omega) \varphi_k(x) dx$ are Fourier coefficients.

Truncated versions,

$$Q_m(x, \omega) = \sum_{k=1}^m Z_k(\omega) \varphi_k(x), \quad x \in D, \quad (4)$$

of the representation in Eq. 3 have the form of the model of $Q(x, t)$ in Eq. 2 for an arbitrary time. The representation of $Q(x)$ in Eq. 4 is an element of the subspace $\mathcal{H}_m$ of $\mathcal{H}$ spanned by the basis functions in this equation, i.e., $\mathcal{H}_m = \text{span}(\varphi_1, \dots, \varphi_m)$, and represents the projection of $Q(x)$ on $\mathcal{H}_m$.

The error field $\varepsilon_m(x) = Q(x) - Q_m(x)$ with samples

$$\varepsilon_m(x, \omega) = Q(x, \omega) - Q_m(x, \omega) = \sum_{k=m+1}^{\infty} Z_k(\omega) \varphi_k(x), \quad x \in D. \quad (5)$$

has the covariance

$$\Sigma(x, y) = E[\varepsilon_m(x) \varepsilon_m(y)] = \sum_{k, l=m+1}^{\infty} E[Z_k Z_l] \varphi_k(x) \varphi_l(y), \quad (6)$$

and the norm

$$\varepsilon_m(x, \omega)^2 = \|Q(\cdot, \omega) - Q_m(\cdot, \omega)\|^2 = \|Q(\cdot, \omega)\|^2 - \sum_{k=1}^m |Z_k(\omega)|^2 \quad (7)$$

by bounded convergence and the Bessel equation. The magnitude of the error $\varepsilon_m(x)$ depends on the model size m and the set of basis functions $\{\varphi_k\}$. For given $\{\varphi_k\}$, the error of the model $Q_m(x)$ decreases with m . The selection of the model size m is largely dictated by computational demand. For a given size m , the model accuracy depends on the basis functions used in the definition of $Q_m(x, t)$. The following property is a classical result of the approximation theory.

Property 2. An element $\tilde{e}_0$ of $\mathcal{H}_m$ is the best approximation of $e \in \mathcal{H}$ if and only if its error $e - \tilde{e}_0$ is orthogonal to $\mathcal{H}_m$, i.e., $\langle e - \tilde{e}_0, \tilde{e} \rangle = 0$ for all $\tilde{e} \in \mathcal{H}_m$.

Proof: Suppose there is $\tilde{e} \in \mathcal{H}_m$ such that $\langle e - \tilde{e}_0, \tilde{e} \rangle \neq 0$ and note that $\tilde{e}_0 + \lambda \tilde{e} \in \mathcal{H}_m$, $\lambda \in \mathbb{R}$. Then $\|e - \tilde{e}_0 - \lambda \tilde{e}\|^2 = \|(e - \tilde{e}_0) - \lambda \tilde{e}\|^2 = \|\tilde{e}\|^2 \lambda^2 - 2 \langle e - \tilde{e}_0, \tilde{e} \rangle \lambda + \|e - \tilde{e}_0\|^2$ attains its minimum at $\lambda = \langle e - \tilde{e}_0, \tilde{e} \rangle / \|\tilde{e}\|^2 \neq 0$ by assumption, which implies that $\tilde{e}_0 + \lambda \tilde{e} \in \mathcal{H}_m$ with $\lambda = \langle e - \tilde{e}_0, \tilde{s} \rangle / \|\tilde{e}\|^2$ is a better approximation for $e \in \mathcal{H}$ than $\tilde{e}_0$.

Suppose now $\langle e - \tilde{e}_0, \tilde{e} \rangle = 0$, $\forall \tilde{e} \in \mathcal{H}_m$. For $e^* \in \mathcal{H}_m$ arbitrary we have

$$\begin{aligned} \|e - e^*\|^2 - \|e - \tilde{e}_0\|^2 &= \|e^*\|^2 - \|\tilde{e}_0\|^2 - 2 \langle e, e^* \rangle + 2 \langle e, \tilde{e}_0 \rangle \\ &= \|e^* - \tilde{e}_0\|^2 - 2 \langle e - \tilde{e}_0, \tilde{e}_0 - e^* \rangle = \|e^* - \tilde{e}_0\|^2, \end{aligned}$$

where the first two equalities hold by properties of the inner product and the latter equality holds since $\tilde{e}_0 - e^* \in \mathcal{H}_m$ and $e - \tilde{e}_0$ is orthogonal to $\mathcal{H}_m$. We have $\|e - e^*\|^2 = \|e - \tilde{e}_0\|^2 + \|e^* - \tilde{e}_0\|^2 \geq \|e - \tilde{e}_0\|^2$ which means that $\tilde{e}_0$ is the best approximation. $\blacktriangle$

The model $Q_m(x)$ in Eq. 4 is completely defined by its size m , basis functions $\{\varphi_k(x)\}$, and properties of the stochastic processes $\{Z_k(t)\}$. The following two sections construct basis functions for $Q_m(x)$ and calculate samples of the processes $\{Z_k(t)\}$ from samples of $Q(x, t)$ for a specified model size m .

3.1.1 Basis functions

The samples of $Q(x, t)$ are recorded at discrete space and time arguments so that the corresponding samples of $Q(x)$ are available at the nodes of a mesh in D . If the mesh is too coarse, essential information on properties of the samples of $Q(x)$ may be missed. If the mesh is too fine, the resulting representation, although accurate, can be computational prohibitive. The following property provides a mesh size that is optimal in the sense that representations of $Q(x)$ based on it are accurate and require a minimum computational effort.

Property 3. If $Q(x)$ is a homogeneous random field and the support of its spectral density is included in the rectangle $R = \times_{u=1}^d [-\nu_u^*, \nu_u^*]$, $\nu_u^* > 0$, then

$$Q^{(n)}(x) = \sum_{k_1=n_{x_1}-n_1}^{n_{x_1}+n_1+1} \cdots \sum_{k_d=n_{x_d}-n_d}^{n_{x_d}+n_d+1} Q(k_1 x_1^*, \dots, k_d x_d^*) \prod_{u=1}^d \beta_{k_u}(x_u; x_u^*) \quad (8)$$

becomes a version of $Q(x)$ as $n_u \rightarrow \infty$, $u = 1, \dots, d$, where $x_u^* = \pi/\nu_u^*$, n_{x_u} is the integer part of x_u/x_u^* , $n_u \geq 1$ are integers, $n = (n_1, \dots, n_d)$, and $\beta_{k_u}(x_u; x_u^*) = \sin[\pi(x_u/x_u^* - k_u)]/[\pi(x_u/x_u^* - k_u)]$.

Proof: We only note that meshes with nodes spaced at $\{\pi/\nu_u^*\}$, $u = 1, \dots, d$, or less characterize the samples of $Q(x)$ accurately provided the components of $n = (n_1, \dots, n_d)$ are sufficiently large. For proof see, e.g., [9] (Sects. 5.3.1.3 and 5.3.1.4). Similar results are available for nonstationary random functions [6]. $\blacktriangle$

Let A be an $\mathbb{R}^d$ -valued column random vector whose entries are values of $Q(x) := Q(x, t)$ at d distinct arguments x at an arbitrary time $t \in [0, \tau]$. Let $\{a_i\}$ be d -dimensional column vectors collecting recorded values of $\{q_i(x, t)\}$, $i = 1, \dots, N$, at a particular time, so that they are samples of $Q(x)$. The rows of the (N, d) -matrix $\mathcal{A}$ are samples of A , i.e., row i of $\mathcal{A}$ is a_i' . The singular value decomposition (SVD) of $\mathcal{A}$ has the form

$$\mathcal{A} = U S V' = \sum_{k=1}^r \sigma_k u_k v_k' \quad (9)$$

where U , S , and V are (N, N) -, (N, d) -, and (d, d) -matrices, $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r$ denote the non-zero entries of the diagonal matrix S and are the eigenvalues of the symmetric positive definite matrices $\mathcal{A}'\mathcal{A}$ and $\mathcal{A}\mathcal{A}'$ with rank $r \leq \min(N, d)$, the columns $\{u_k\}$ of U are the eigenvectors of the (N, N) -matrix $\mathcal{A}\mathcal{A}'$, and the columns $\{v_k\}$ of V are the eigenvectors of the (d, d) -matrix $\mathcal{A}'\mathcal{A}$. The eigenvectors $\{u_k\}$ and $\{v_k\}$ are orthonormal, i.e., $u_k' u_l = \delta_{kl}$ and $v_k' v_l = \delta_{kl}$, and have some notable properties [17, 18]. For example, let v_k be an eigenvector of $\mathcal{A}'\mathcal{A}$ corresponding to eigenvalue σ_k , i.e., $(\mathcal{A}'\mathcal{A}) v_k = \sigma_k v_k$. We have (1) $v_k' (\mathcal{A}'\mathcal{A}) v_k = \sigma_k v_k' v_k$ which implies $\|\mathcal{A} v_k\|^2 = \sigma_k \|v_k\|^2 = \sigma_k$ and (2) $\mathcal{A} (\mathcal{A}'\mathcal{A}) v_k = \sigma_k \mathcal{A} v_k$ which implies $(\mathcal{A}\mathcal{A}') (\mathcal{A} v_k) = \sigma_k (\mathcal{A} v_k)$ and $(\mathcal{A}\mathcal{A}') u_k = \sigma_k u_k$, where $u_k = \mathcal{A} v_k / \|\mathcal{A} v_k\|$.

The low rank approximation of $\mathcal{A}$ is obtained by truncating the representation in Eq. 9 and has the form

$$\mathcal{A}_m = \sum_{k=1}^m \sigma_k u_k v_k', \quad m < r. \quad (10)$$

The eigenvalues, $\{\sigma_k\}$, and the eigenvectors, $\{u_k\}$ and $\{v_k\}$, in Eqs. 9 and 10 depend on the particular set of samples used in the definition of $\mathcal{A}$. They are estimates whose properties are examined in a subsequent section, where they will be denoted by $\{\hat{\sigma}_k\}$, $\{\hat{u}_k\}$, and $\{\hat{v}_k\}$. Yet, we continue to call them $\{\sigma_k\}$, $\{u_k\}$, and $\{v_k\}$ in this section for simplicity.

Property 4. The accuracy of the low rank approximation $\mathcal{A}_m$ of $\mathcal{A}$ can be quantified by

$$\begin{aligned} \|\mathcal{A} - \mathcal{A}_m\|_F &= \left(\sum_{k=m+1}^r \sigma_k^2 \right)^{1/2} \\ \|\mathcal{A} - \mathcal{A}_m\|_2 &= \sup_{\{x \in \mathbb{R}^d, \|x\|=1\}} \|(\mathcal{A} - \mathcal{A}_m) x\| = \sigma_{m+1} \end{aligned} \quad (11)$$

where $\|\cdot\|_F$ and $\|\cdot\|_2$ denote the Frobenius and the induced 2-norms.

Proof: We first note that $\mathcal{A}_m$ is optimal in the sense $\|\mathcal{A} - \mathcal{A}_m\|_F \leq \|\mathcal{A} - \mathcal{B}\|_F$, where $\mathcal{B}$ is an arbitrary (N, d) -deterministic matrix with rank m [4]. The latter equality in Eq. 11 follows from

$$\begin{aligned} \|(\mathcal{A} - \mathcal{A}_m) z\|_2^2 &= \sum_{s=1}^N \left(\sum_{k=m+1}^r \alpha_k u_{k,s} \right) \left(\sum_{l=m+1}^r \alpha_l u_{l,s} \right) = \sum_{k,l=m+1}^r \alpha_k \alpha_l \sum_{s=1}^N u_{k,s} u_{l,s} = \sum_{k=m+1}^r \alpha_k^2 \\ &= \sum_{k=m+1}^r \sigma_k^2 (v'_k z)^2 \leq \sigma_{m+1}^2 \sum_{k=m+1}^r (v'_k z)^2 \leq \sigma_{m+1}^2 \|z\|_2^2, \end{aligned}$$

where $\alpha_k = \sigma_k (v'_k z) \in \mathbb{R}$ and $z \in \mathbb{R}^d$, which implies that the minimum of $\|(\mathcal{A} - \mathcal{A}_m) z\|_2^2$ is reached for $z = v_{m+1}$ since $v'_k v_{m+1} = \delta_{k,m+1}$ so that $\sum_{k=m+1}^r \sigma_k^2 (v'_k z)^2 = \sigma_{m+1}^2$. $\blacktriangle$

The SVD of $\mathcal{A}$ in Eq. 9 implies that an arbitrary row a'_i of this matrix, i.e., an arbitrary sample of A or $Q(x)$, has the representation

$$a'_i = \sum_{k=1}^r (\sigma_k u_{k,i}) v'_k, \quad (12)$$

where $u_{k,i}$ is component i of u_k . This shows that the samples of A are elements of the r -dimensional space spanned by the vectors $\{v_k\}$ and that the samples $A(\omega)$ of A admit the representation

$$A(\omega) = \sum_{k=1}^r W_k(\omega) v_k, \quad (13)$$

where $W_k(\omega) = \sigma_k u_{k,i}$ is the sample of $\{W_k\}$ corresponding to a_i . The representations in Eqs. 12 and 13 can be simplified for calculations by retaining the top $m \leq r$ modes, i.e.,

$$A_m(\omega) = \sum_{k=1}^m (\sigma_k u_{k,i}) v_k = \sum_{k=1}^m W_k(\omega) v_k, \quad m < r, \quad (14)$$

and corresponds to the low-rank approximation $\mathcal{A}_m$ of $\mathcal{A}$. This model has the functional form in Eq. 2 with basis functions $\{\varphi_k(x)\}$ given by the eigenvectors $\{v_k\}$ and random coefficients $\{Z_k\}$ characterized by the samples $\{W_k(\omega)\}$ of $\{W_k\}$.

This completes the building of parametric models for the spatial variation of $Q(x, t)$. In summary, samples of $Q(x)$ recorded at discrete spatial locations are used to ensemble the matrix $\mathcal{A}$, develop its SVD in Eq. 9, and approximate it by $\mathcal{A}_m$ in Eq. 10. Property 4 provides information on one source of modeling error, the truncation error. It shows that the discrepancy between $\mathcal{A}_m$ and $\mathcal{A}$ depends on the rate at which the eigenvalues $\{\sigma_k\}$ decrease with k . Other sources of errors are examined in a subsequent section. We also note that the samples of the random coefficients $\{W_k\}$ in Eq. 14 can be used to estimate the joint distribution of A and generate samples of this random vector from the resulting estimate of this distribution.

3.2 Temporal variation

The samples of $Q(x, t)$ include information on both the spatial and temporal variation. We use the orthogonality of the basis functions in the definition of $Q_m(x, t)$ to extract samples of $\{Z_k(t)\}$ from samples $Q(x, t)$. The resulting samples are employed to model the temporal variation of $Q(x, t)$.

Let $q_{v,i}(t)$ denote a d -dimensional column vector whose entries are values of $q_i(x, t)$ at the nodes of a mesh used to discretize the x -space, i.e., the entries of the vector A in Eq. 13 coincide with $q_{v,i}(t)$ at a time. According to Eq. 14, the vector-valued process $q_{v,i}(t)$ admits the representation

$$q_{v,i}(t) \simeq \sum_{k=1}^m z_{k,i}(t) v_k \quad (15)$$

where $\{z_{k,i}(t)\}$ are the samples of $\{Z_k(t)\}$ corresponding to the sample $q_i(x, t)$ of $Q(x, t)$. Note that $\{Z_k(t)\}$ at a time are the random variables $\{W_k\}$ in Eq. 14.

The latter equation and the orthogonality of the eigenvectors $\{v_k\}$ give

$$v'_l q_{v,i}(t) \simeq \sum_{k=1}^m z_{k,i}(t) v'_l v_k = z_{l,i}(t), \quad l = 1, \dots, m, \quad i = 1, \dots, N, \quad (16)$$

by pre-multiplying Eq. 15 with v'_l , which constitutes a simple algorithm for calculating samples of $Z_k(t)$ from samples of $Q(x, t)$. The resulting samples of $\{Z_k(t)\}$ can be used to infer the law of these processes.

The modes $\{v_k\}$ in Eqs. 15 and 16 and the samples $\{z_{l,i}(t)\}$ in Eq. 16 depend on the particular set of N samples used to described $Q(x, t)$. Like in the previous section, we refer to the estimates $\{\hat{v}_k\}$ and $\{\hat{z}_{l,i}(t)\}$ as $\{v_k\}$ and $\{z_{l,i}(t)\}$ for simplicity. The statistical uncertainty of $\{\hat{z}_{l,i}(t)\}$ and $\{\hat{v}_k\}$ is examined in the following section.

3.2.1 Statistical uncertainty

The error of the model $Q_m(x, t)$ of $Q(x, t)$ in Eq. 2 has three sources. The first and second have already been discussed. They relate to the truncation of the representation in Eq. 3 and the discretization of the spatial domain D . The third relates to the construction of $Q_m(x, t)$ from a finite set of samples of $Q(x, t)$. The latter source of error of $Q_m(x, t)$ is examined in this section. We use hats on the symbols denoting properties of $Q_m(x, t)$ if based on finite sets of samples of $Q(x, t)$ to distinguish them from their asymptotic values for $N \rightarrow \infty$.

Let $\{q_i(x) := q_i(x, t)\}$, $i = 1, \dots, N$, be values of N independent samples of $Q(x) := Q(x, t)$ at an arbitrary time $t \in [0, \tau]$. Since $N < \infty$, the models of $Q(x)$ will depend on the particular set of samples used for their construction. Consider the estimator

$$\hat{F}(\xi) = \frac{1}{N} \sum_{i=1}^N 1(Q^{(i)}(x_1) \leq \xi_1, \dots, Q^{(i)}(x_p) \leq \xi_p), \quad \xi = (\xi_1, \dots, \xi_p) \in \mathbb{R}^p, \quad (17)$$

of the finite dimensional distributions $F(\xi)$ of $Q(x)$, where $p \geq 1$ is an integer, $\xi \in \mathbb{R}^p$ is arbitrary, and $\{Q^{(i)}(x)\}$ are independent copies of $Q(x)$. The following two properties assess the quality of the estimator $\hat{F}$.

Property 5. The estimator $\hat{F}$ converges in probability and almost surely (a.s.) to the finite dimensional distributions of $Q(x)$ pointwise, i.e., $\hat{F}(\xi) \rightarrow F(\xi)$ at each continuity point $\xi = (\xi_1, \dots, \xi_p) \in \mathbb{R}^p$ in probability and a.s.

Proof: The Bernoulli random variable $Y_i = 1(Q^{(i)}(x_1) \leq \xi_1, \dots, Q^{(i)}(x_p) \leq \xi_p)$ is 0 and 1 with probabilities $1 - F(\xi)$ and $F(\xi)$, so that $\hat{F}(\xi) = \sum_{i=1}^N Y_i/N$ has mean $E[\hat{F}(\xi)] = F(\xi)$ and variance $\text{Var}[\hat{F}(\xi)] = F(\xi)(1 - F(\xi))/N$. We have $P(|\hat{F}(\xi) - F(\xi)| > \varepsilon) \leq \text{Var}[\hat{F}(\xi)]/\varepsilon^2$ by Chebyshev's inequality so that $\hat{F}(\xi) \rightarrow F(\xi)$ in probability as $N \rightarrow \infty$. The strong law of large numbers implies $\sum_{i=1}^N Y_i/N \rightarrow E[Y_1] = F(x)$ a.s. and $E[\sum_{i=1}^N Y_i/N - F(x)] \rightarrow 0$ [7] (Theorem 14, p. 192) since $E[|Y_1|] = F(\xi)$ is finite. $\blacktriangle$

Property 6. The Glivenko-Cantelli theorem strengthen above pointwise convergence of $\hat{F}$ to F to uniform convergence. We have [3, 20]

$$P\left(\sup_{x \in \mathbb{R}^d} |\hat{F}(\xi) - F(\xi)| > \varepsilon\right) \leq c(1 + N)^d \exp(-N\varepsilon^2/32) \quad \text{and}$$

$$\|\hat{F} - F\|_\infty = \sup_{x \in \mathbb{R}^d} |\hat{F}_n(\xi) - F(\xi)| \rightarrow 0, \quad N \rightarrow \infty, \text{ a.s.}, \quad (18)$$

where $c > 0$ is a constant.

Consider now the (N, d) matrix $\mathcal{A}$ in Eq. 9 assembled from N independent samples of $Q(x)$ recorded at d locations. The eigenvectors $\{v_k\}$ of the (d, d) matrix $\mathcal{A}'\mathcal{A}$ are the basis functions used to construct the representation in Eq. 14 and depend on the particular set of samples of $Q(x)$ used in the definition of $\mathcal{A}$. Our objective is to quantify this dependence as a function of the samples size N .

Note that the entries of $\mathcal{A}'\mathcal{A}$ are $\{\sum_{i=1}^N R_{ik} R_{il}\}$, where $R_{ik} = Q^{(i)}(x_k)$, $k, l = 1, \dots, d$. If $E[|R_{ik} R_{il}|] < \infty$, then

$$\frac{1}{N} \sum_{i=1}^N R_{ik} R_{il} \rightarrow E[Q^{(i)}(x_k) Q^{(i)}(x_l)] = E[Q(x_k) Q(x_l)], \quad N \rightarrow \infty, \quad (19)$$

almost surely (a.s.) by the strong law of large numbers [7] (Theorem 14, p. 192). This convergence shows that $(1/N) \sum_{i=1}^N R_{ik} R_{il}$ is an unbiased estimator of $E[Q^{(i)}(x_k) Q^{(i)}(x_l)]$. Its variance is of order $O(1/N)$ by the central limit theorem [7] (Theorem 3, p. 275). We also note that the scaled version $\hat{\mathcal{B}} = (1/N) \mathcal{A}'\mathcal{A}$ of $\mathcal{A}'\mathcal{A}$ converges to $\mathcal{B} = \{E[Q^{(i)}(x_k) Q^{(i)}(x_l)]\}$ as $N \rightarrow \infty$ a.s. in any norm, i.e., $\|\hat{\mathcal{B}} - \mathcal{B}\| \rightarrow 0$ as $N \rightarrow \infty$ [16] (Proposition 1, p. 361). The eigenvalues λ and $\hat{\lambda}$ of $\mathcal{B}$ and $\hat{\mathcal{B}}$ are related by

$$\hat{\lambda} = \lambda + v' \mathcal{E} v + O(\|\mathcal{E}\|^2), \quad (20)$$

where $v \in \mathbb{R}^d$, $\|v\| = 1$, is the eigenvector of $\mathcal{B}$ associated with λ and $\mathcal{E} = \hat{\mathcal{B}} - \mathcal{B}$ denotes the error matrix [19] (Theorem 2.3, p. 361). Since $\|\mathcal{E}\| \rightarrow 0$ as $N \rightarrow \infty$, the approximations $\hat{\lambda} \simeq \lambda$ and $(1/N) \sum_{i=1}^N R_{ik} R_{il} \simeq E[Q^{(i)}(x_k) Q^{(i)}(x_l)]$ hold for a sufficiently large N .

Property 7. Let $\{\lambda_k, v_k\}$ denote the eigenvalues and eigenvectors of $\mathcal{B}$ and $\{\hat{\lambda}_k, \hat{v}_k\}$ denote the corresponding eigenvalues and eigenvectors of $\hat{\mathcal{B}}$. If $\mathcal{B}$ has distinct eigenvalues, $\varepsilon > 0$ is a

small parameter, and N is sufficiently large such that $\|\mathcal{E}\| \sim O(\varepsilon)$ almost surely (a.s.), then

$$\begin{aligned}\hat{\lambda}_k &= \lambda_k + v'_k \mathcal{E} v_k + O(\varepsilon^2) \quad \text{and} \\ \hat{v}_k &= v_k + \sum_{l=1, l \neq k}^d \frac{v'_l \mathcal{E} v_k}{\lambda_k - \lambda_l} + O(\varepsilon^2) \quad \text{a.s.}\end{aligned}\tag{21}$$

Proof: The eigenvalues and eigenvectors of $\hat{\mathcal{B}}$ admit the representation

$$\hat{\lambda}_k = \lambda_k + \varepsilon \lambda_{k,1} + \varepsilon^2 \lambda_{k,2} + \dots \quad \text{and} \quad \hat{v}_k = v_k + \varepsilon v_{k,1} + \varepsilon^2 v_{k,2} + \dots$$

so that we have $\mathcal{B} v_k = \lambda_k v_k$ and $\mathcal{B} v_{k,1} + \mathcal{R} v_k = \lambda_k v_{k,1} + \lambda_{k,1} v_k$ by equating terms of orders one and ε , where $\mathcal{R} = \mathcal{E}/\varepsilon$. The left multiplication of the equation of order ε with v'_l gives

$$v'_l \mathcal{B} \sum_{s=1}^d b_{ks} v_s + v'_l \mathcal{R} v_k = \lambda_k v'_l \sum_{s=1}^d b_{ks} v_s + \lambda_{k,1} v'_l v_k$$

since $v_{k,1} = \sum_{s=1}^d b_{ks} v_s$ as an element of the linear space spanned by the eigenvectors $\{v_k\}$ so that we have $\lambda_l b_{kl} + v'_l \mathcal{R} v_k = \lambda_k b_{kl} + \lambda_{k,1} \delta_{kl}$ since $v'_l v_k = \delta_{kl}$. This gives $(\lambda_k - \lambda_l) b_{kl} = v'_l \mathcal{R} v_k$ or $b_{kl} = v'_l \mathcal{R} v_k / (\lambda_k - \lambda_l)$ if $k \neq l$ and $\lambda_{k,1} = v'_k \mathcal{R} v_k$ if $k = l$. The projection b_{kk} of $v_{k,1}$ on v_k remains undetermined. Since $v_{k,1} = \alpha v_k + \sum_{s=1, s \neq k}^d v_s v'_s \mathcal{R} v_k / (\lambda_k - \lambda_s)$ satisfies the equation of order ε for any $\alpha \in \mathbb{R}$, we set $\alpha = 0$. $\blacktriangle$

Since the information on the random function $Q(x, t)$ consists of N independent samples, only estimates can be obtained for the basis functions $\{\varphi_k(x)\}$ and the processes $\{Z_k(t)\}$ in the expression of $Q_m(x, t)$ in Eq. 2. The model $Q_m(x, t)$ and its approximation $\hat{Q}_m(x, t)$ based on N samples of $Q(x, t)$ have the expressions

$$\begin{aligned}Q_m(x, t) &= \sum_{k=1}^m Z_k(t) v'_k \\ \hat{Q}_m(x, t) &= \sum_{k=1}^m \hat{Z}_k(t) \hat{v}'_k,\end{aligned}\tag{22}$$

where $\{v'_k = (\varphi_k(x_1), \dots, \varphi_k(x_d))\}$, $(x_1, \dots, x_d)$ are the nodes of a mesh in D , and $\{\hat{Z}_k(t)\}$ and $\{\hat{v}_k\}$ are estimates of $\{Z_k(t)\}$ and $\{v_k\}$ obtained from N independent samples of $Q(x, t)$.

Property 8. If almost all samples of $Q(x, t)$ are bounded, the discrepancy between the samples of $\{Z_k(t)\}$ and $\{\hat{Z}_k(t)\}$ can be made as small as desired by increasing N .

Proof: The samples of $Z_k(t)$ and $\hat{Z}_k(t)$ corresponding to the samples $q_i(x, t)$ of $Q(x, t)$ result from Eq. 16 with the basis $\{v_k\}$ and $\{\hat{v}_k\}$. They are $z_{k,i}(t) = v'_k q_{v,i}(t)$ and $\hat{z}_{k,i}(t) = \hat{v}'_k q_{v,i}(t)$ so that the discrepancy between them can be bounded by

$$|z_{k,i}(t) - \hat{z}_{k,i}(t)| = |(v_k - \hat{v}_k) q_{v,i}(t)| \leq \|v_k - \hat{v}_k\| \|q_{v,i}(t)\|.\tag{23}$$

Since $\|\mathcal{E}\| \rightarrow 0$ as $N \rightarrow \infty$ implies $\|v_k - \hat{v}_k\| \rightarrow 0$ and $Q(x, t)$ has bounded samples, the samples of $\hat{Z}_k(t)$ can match the samples of $Z_k(t)$ to any accuracy for N sufficiently large. $\blacktriangle$

Property 9. If the norm of the error matrix $\mathcal{E} = \hat{\mathcal{B}} - \mathcal{B}$ is small of order $\varepsilon > 0$, i.e., $\|\mathcal{E}\| \sim O(\varepsilon)$ a.s., and almost all samples of $Q(x, t)$ are bounded, then $|Q_m(x, t) - \hat{Q}_m(x, t)| \sim O(\varepsilon)$ for $x = x_u \in D$, $u = 1, \dots, d$.

Proof: Since $Q_m(x, t) = \sum_{k=1}^m Z_k(t) v'_k$ and $\hat{Q}_m(x, t) = \sum_{k=1}^m \hat{Z}_k(t) \hat{v}'_k$ at $x = x_u$, $u = 1, \dots, d$, we have

$$\begin{aligned} |Q_m(x, t) - \hat{Q}_m(x, t)| &= \left| \sum_{k=1}^m Z_k(t) (v_k - \hat{v}_k) + \sum_{k=1}^m (Z_k(t) - \hat{Z}_k(t)) \hat{v}_k \right| \\ &\leq \sum_{k=1}^m \left(|Z_k(t)| \|v_k - \hat{v}_k\| + |Z_k(t) - \hat{Z}_k(t)| \|\hat{v}_k\| \right) \\ &\leq \sum_{k=1}^m \left(|Z_k(t)| \|v_k - \hat{v}_k\| + |Z_k(t) - \hat{Z}_k(t)| \|\hat{v}_k\| \right) \end{aligned}$$

where $\|v_k - \hat{v}_k\| \sim O(\varepsilon)$ by Eq. 21, $Q(x, t)$ is bounded by assumption, and $|Z_k(t) - \hat{Z}_k(t)|$ is bounded by the product of $\|v_k - \hat{v}_k\|$ and the norm of the d -dimensional vector process defined by $Q(x, t)$ at $x = x_u$, $u = 1, \dots, d$, by Eq. 23. $\blacktriangle$

The above considerations demonstrate the importance of the sample size N . The accuracy of the modes $\{\hat{v}_k\}$ in the expression of $\hat{Q}_m(x, t)$ given by Eq. 22 depends essentially on the sample size N . Moreover, their accuracy determines to a great extent the quality of the approximations $\{\hat{Z}_k(t)\}$ of the processes $\{Z_k(t)\}$ whose samples are calculated from Eq. 16. The above approach in which the modes $\{\hat{v}_k\}$ are extracted from values of the samples of $Q(x, t)$ at a single time uses partially the available information. Superior estimates can be obtained by using values of samples of $Q(x, t)$ at n_r times $\{t_r\}$ that are sufficiently far from each other such that $Q(x, t_r)$ and $Q(x, t_{r'})$, $r \neq r'$, are weakly correlated. Under this formulation, the matrix $\mathcal{A}$ in Eq. 10 has $n_r N$ rows and d columns where, e.g., its first and second set of N rows are the N samples of $Q(x, t_1)$ and the N samples of $Q(x, t_2)$, respectively.

4 Models for arbitrary random function

Suppose $Q(x, t)$, $x \in D$, $t \in [0, \tau]$, is an arbitrary space-time random function, rather than a time-stationary random function, as assumed in the previous section. The model $Q_m(x, t)$ in Eq. 2 can be too restrictive unless the basis functions $\{\varphi_k\}$ are sufficiently general to characterize the members of the family of random fields $\{Q(x, t), x \in D\}$ at all times. We present three models of $Q(x, t)$. The first is a version of $Q_m(x, t)$, the second is a low rank approximation of the SVD of the matrix containing the samples of $Q(x, t)$, and the third represents $Q(x, t)$ by a vector-valued process obtained by space discretization.

Time-stationary model. Since $Q(x, t)$ is not time-stationary, the basis functions extracted from samples of $Q(x, t)$ at a single time are inadequate since the properties of the random fields $\{Q(x, t), x \in D\}$ vary in time. To capture the dominant modes of the samples of the members of the family of random fields $\{Q(x, t), x \in D\}$, $t \in [0, \tau]$, we apply the SVD in Eq. 9 to a matrix $\mathcal{A}$ whose rows are samples of $Q(x, t)$ recorded at many times $\{t_r\}$ in $[0, \tau]$ rather than a single time. The modes of the resulting matrix $\mathcal{A}$ will capture the essential

features of the family of random fields $\{Q(x, t_r), x \in D\}$, $r = 1, 2, \dots$, and are expected to be adequate for the family of random fields $\{Q(x, t), x \in D\}$, $t \in [0, \tau]$, provided the times $\{t_r\}$ define a sufficiently fine mesh in $[0, \tau]$.

Space-time SVD. The model in Eq. 4 used to characterize the spatial variation of $Q(x, t)$ can be extended to capture simultaneously the dependence of $Q(x, t)$ of space and time. It has the functional form

$$Q_m(x, t) = \sum_{k=1}^m Z_k \varphi_k(x, t), \quad x \in D, \quad t \in [0, \tau], \quad (24)$$

where $\{Z_k\}$ are random variables and $\{\varphi_k(x, t)\}$ are basis functions in both space and time. The construction of this model is similar to that of $Q_m(x)$ in Sect. 3.1.

Let Q_{matrix} be an (d, d_t) -matrix whose entries are values of $Q(x, t)$ at the nodes of a mesh in the product space $D \times [0, \tau]$, where d and d_t denote the number of nodes in the x and t coordinates. A formulation as in Eq. 9 can be applied to find the modes $\{v_k\}$ of Q_{matrix} from N independent samples $\{q_i(x, t)\}$, $i = 1, \dots, N$, of $Q(x, t)$, which can be organized in (d, d_t) -matrices, so that Q_{matrix} can be modeled by $\sum_{k=1}^m W_k v_k$, where $\{W_k\}$ are real-valued random variables.

Previous considerations for time-stationary random functions extend directly to this general case. The model can be inefficient if, e.g., the frequency band of spacial and temporal fluctuations of $Q(x, t)$ differ significantly since the numbers of modes needed to describe fluctuations in space and time will also be very different. The method of separation of variables can be applied to construct more efficient representation of $Q(x, t)$.

An extended version of the model in Eq. 24 was used in [21] to solve stochastic partial differential equations. The model has the form in Eq. 24 in which the random variables $\{Z_k\}$ are replaced by stochastic processes $\{Z_k(t)\}$. Constraints have been imposed on the model to deal with its redundancy caused by the dependence on time of both the processes $\{Z_k(t)\}$ and the basis functions $\{\varphi_k(x, t)\}$. The estimation of the properties of the processes $\{Z_k(t)\}$ and the basis functions $\{\varphi_k(x, t)\}$ from samples of $Q(x, t)$, an essential step in the setting of this study, is also complicated by model redundancy. Constraints of the type used in [21] may be needed to calibrate the model to data.

Vector-valued representation. Let $Y(t)$, $t \in [0, \tau]$, be an $\mathbb{R}^d$ -valued stochastic process with components $Y_u(t) = Q(x_u, t)$, $u = 1, \dots, d$. If the mesh in D with nodes $\{x_u\}$ is selected according to Property 3, then $Y(t)$ provides an accurate representation of $Q(x, t)$. The information on $Y(t)$ consists of N independent samples, i.e., the samples $\{q_i(x, t)\}$ of $Q(x, t)$ recorded at $x = x_u$, $u = 1, \dots, d$. Generally, $Y(t)$ is a nonstationary, non-Gaussian process that can be modeled by a translation process $Y_T(t)$.

The components $Y_{T,u}(t)$, $u = 1, \dots, d$, of the translation process $Y_T(t)$ are defined by

$$Y_{T,u}(t) = F_{u,t}^{-1} \circ \Phi(G_u(t)), \quad t \in [0, \tau], \quad (25)$$

where $F_{u,t}$ denotes the distribution of the real-valued random variable $Y_u(t)$ at time $t \in [0, \tau]$, Φ is the distribution of the standard Gaussian variable $N(0, 1)$, and $\{G_r(t)\}$ are nonstationary Gaussian processes with means 0, variances 1, and correlation functions $\rho_{uv}(s, t) =$

$E[G_u(s)G_v(t)]$, see [5] and [8] (Chap. 3). The distributions $\{F_{u,t}\}$ and the correlation functions $\{\rho_{uv}\}$ define the probability law of $Y_T(t)$ and are inferred from data.

Samples of $Y(t)$ are used to estimate the distributions $\{F_{u,t}\}$ in Eq. 25 and the correlation functions $\{\xi_{uv}(s,t) = E[Y_u(s)Y_v(t)]\}$. The estimates of $\{F_{u,t}\}$ and Eq. 25 are used to map samples of $Y(t)$ into samples of $G(t)$. Estimates of the correlation function $\{\rho_{uv}\}$ of $G(t)$ result from samples of this process. The above estimates define both the law of $G(t)$ and the mapping in Eq. 25. Samples of $Y_T(t)$ can now be obtained from samples of the nonstationary vector-valued Gaussian process $G(t)$, which can be generated by, e.g., an algorithm in [10].

The following section shows that the first model, which was proposed originally for time-stationary random functions, also works for functions that are not stationary in time. The implementation of the model for arbitrary space-time random functions is similar to that for time-stationary functions.

5 Applications

Models are developed for a broad range of real-valued space-time random functions $Q(x,t)$. In Sect. 5.1, $Q(x,t)$ is a time-stationary. In Sect. 5.2, the space-time random function $Q(x,t)$ is not time-stationary. It is obtained from the random function $Q(x,t)$ in Sect. 5.1 by modulating the processes $\{Y_k(t)\}$ and/or randomly distorting the spatial coordinate $x \in D$. In Sect. 5.3, the space-time random function is obtained from $Q(x,t)$ in Sect. 5.1 by adding phases to the sine waves in the definition of $Q(x,t)$. The information on $Q(x,t)$ is limited to a single sample of this random function.

5.1 Time-stationary random function

Suppose the target real-valued random function $Q(x,t)$ has the expression

$$Q(x,t) = \sum_{k=1}^{\infty} Y_k(t) \sin(k\pi x/l), \quad x \in D = [0,l], \quad t \in [0,\tau], \quad (26)$$

where $\{Y_k(t)\}$ are independent Ornstein-Uhlenbeck processes defined by the stochastic differential equations

$$dY_k(t) = -\lambda_k Y_k(t) dt + \mu_k^{1/2} dB_k(t), \quad t \in [0,\tau], \quad (27)$$

$\lambda_k = \alpha + \beta k\pi/l$, $\alpha, \beta, \mu_k > 0$ are constants, and $\{B_k(t)\}$ denote independent standard Brownian motions. The initial states $\{Y_k(0)\}$ of the processes in Eq. 27 are Gaussian variables with mean 0 and variance $\sigma_k^2 = \mu_k/(2\lambda_k)$ that are independent of the Brownian motions $\{B_k(t)\}$ and each other. Under these assumptions, $\{Y_k(t)\}$ are independent stationary Gaussian processes with mean 0 and correlation functions $c_k(u) = E[Y_k(t+u)Y_k(t)] = \sigma_k^2 \exp(-\lambda_k|u|)$, $u \in \mathbb{R}$. Note that $Q(x,t)$ is a real-valued stationary Gaussian process with mean 0 and correlation function $c_Q(u|x) = E[Q(x,t+u)Q(x,t)] = \sum_{k=1}^{\infty} c_k(u) \varphi_k(x)^2$ for each $x \in D$ and a real-valued inhomogeneous Gaussian field with mean 0 and correlation function $c_Q(x,y|t) = E[Q(x,t)Q(y,t)] = \sum_{k=1}^{\infty} \sigma_k^2 \varphi_k(x) \varphi_k(y)$ for each $t \in [0,\tau]$, provided the series defining

the correlation functions $c_Q(u | x)$ and $c_Q(x, y | t)$ converge to finite limits. These properties imply that the Gaussian function $Q(x, t)$ is stationary in time and inhomogeneous in space.

For calculations, the target function $Q(x, t)$ is given by the first 50 terms of the series in Eq. 26. Numerical results are for $l = 1$, $\tau = 2$, $\alpha = \beta = 1$, and $\mu_k = 1/k^2$. The samples of $Q(x, t)$ are recorded at a space-time mesh in the product space $D \times [0, \tau]$ with equal steps $\Delta x = 0.01$ and $\Delta t = 0.004$ in space and time, so that $Q(x, t)$ is recorded at $d = 101$ spatial coordinates at each discrete time $t \in [0, \tau]$. The information on $Q(x, t)$ consists of a set of $N = 1000$ independent samples of $Q(x, t)$ and the knowledge that $Q(x, t)$ is time-stationary.

5.1.1 Space dependence

Our first objective is to characterize the family of random fields $\{Q(x, t), x \in D\}$ corresponding to an arbitrary time $t \in [0, \tau]$ from N independent samples $\{q_i(x, t), x \in D, t \in [0, \tau]\}$, $i = 1, \dots, N$, of the target random function $Q(x, t)$. Since this function is time-stationary, the model $Q_m(x)$ in Eq. 4 applies. It can be constructed from values of the samples $\{q_i(x, t)\}$ at any $t \in [0, \tau]$ by using developments in Sect. 3.1.1.

Let $\mathcal{A}$ be an (N, d) -matrix whose rows are samples of $Q(x, t)$ recorded at d spatial arguments $\{x_u\}$, $u = 1, \dots, d$, at an arbitrary time $t \in [0, \tau]$, i.e., samples of the d -dimensional random vector $A = (Q(x_1, t), \dots, Q(x_d, t))'$. The SVD of $\mathcal{A}$ in Eq. 9 gives the modes $\{v_k\}$ and the other parameters of the low rank representation A_m of A in Eq. 14. The low rank representation A_m provides an approximate characterization of the spatial variation of $Q(x, t)$. The solid and dash lines in Fig. 1 are the top five eigenvectors $\{v_k\}$ and the top five basis

Figure 1: Top five modes of $Q_m(x, t)$ and the basis functions $\{\sin(k \pi x/l)\}$ in the definition of $Q(x, t)$ based on $N = 100$ (left panel) and $N = 1000$ (right panel)

functions $\{\sin(k \pi x/l)\}$ in the definition of $Q(x, t)$ given by Eq. 26. The eigenvectors $\{v_k\}$ have been obtained from values of N independent samples of $Q(x, t)$ at time $t = 1$. They match closely the basis functions $\{\sin(k \pi x/l)\}$ if based on $N = 1000$ samples of $Q(x, t)$ and satisfactory if based on only $N = 100$ samples of this random function.

The left, middle, and right panels in Fig. 2 show with dash lines five samples of $Q(x, t)$, $t = 1$, and with solid lines the corresponding five samples of $Q_m(x, t)$, $t = 1$, i.e., the samples of A_m in Eq. 14, for $m = 5, 10$, and 20 . As expected, the accuracy of $Q_m(x, t)$ increases with m . The samples of $Q(x, t)$ and $Q_m(x, t)$ nearly coincide for $m = 20$. We conclude that the probability law of the random field $Q(x, t)$ for an arbitrary time $t \in [0, \tau]$ is described satisfactorily by $Q_m(x, t)$ with $m = 10$ or $m = 20$.

Figure 2: Five samples of $Q(x, t)$ in dash lines and corresponding samples of $Q_m(x, t)$ for $m = 5$ (left panel), $m = 10$ (middle panel), and $m = 20$ (right panel) in solid lines

The samples of $\{W_k\}$ given by Eq. 14 provide detailed information on the properties of these random coefficients. Estimates of the means and variances of $\{W_k\}$ are zero and decrease with k , respectively. Estimates of the correlations $E[W_k W_l]$ are approximately zero for $k \neq l$. Estimates of the skewness and kurtosis coefficients of $\{W_k\}$ from these samples are in the ranges $(-0.1460, 0.1067)$ and $(2.7328, 3.1556)$, respectively. These estimates suggest that the random coefficients in $\{W_k\}$ in Eq. 14 can be assumed to be independent Gaussian variables, which indicates that the random fields $Q_m(x, t)$ and $Q(x, t)$, $x \in D$, corresponding to fixed times have similar properties.

5.1.2 Time dependence

The samples of $Q(x, t)$, the modes $\{v_k\}$ of the spatial variation of $Q_m(x, t)$, and the formulation in Eq. 16 are employed to extract samples $\{Z_k(t)\}$ from samples of $Q(x, t)$. The left and right panels in Fig. 3 show five samples of $Z_1(t)$ and $Z_{10}(t)$ corresponding to the

Figure 3: Five samples of $Z_1(t)$ (left panel) and five samples of $Z_{10}(t)$ (right panel) for $m = 20$

samples of $Q(x, t)$ in Fig. 2. The samples of $Z_{10}(t)$ fluctuate much faster and have much smaller magnitudes than those of $Z_1(t)$.

The solid lines in Fig. 4 are estimates of the scaled correlation functions of $Z_1(t)$ (left panel) and $Z_{10}(t)$ (right panel) obtained from samples of these processes. The dash lines are the scaled correlation functions of the processes $Y_1(t)$ (left panel) and $Y_{10}(t)$ (right panel), i.e.,

Figure 4: Estimates of the correlation functions of $Z_1(t)$ (left panel) and $Z_{10}(t)$ (right panel) for $m = 20$ in solid lines. Dash lines are the scaled correlation functions of $Y_1(t)$ and $Y_{10}(t)$

the target correlation functions $E[Y_k(t+s)Y_k(t)] = \exp(-\lambda_k |s|)$. The correlation functions of $\{Z_k(t)\}$ are similar to those of $\{Y_k(t)\}$ an expected result since, in this special case, the target random function $Q(x, t)$ in Eq. 26 and its model in Eq. 2 have the same functional form and similar basis functions. Estimates of the correlation functions $E[Z_k(t)Z_l(s)]$, $k \neq l$, are nearly zero for all $t, s \in [0, \tau]$ indicating that the processes $\{Z_k(t)\}$ can be assumed to be uncorrelated.

5.1.3 Comments

Results of the previous sections show that the proposed model $Q_m(x, t)$ identifies both the basis functions and the stochastic processes in the definition of the target space-time random function $Q(x, t)$ provided the sample size N is sufficiently large. It was shown that (1) the basis functions $\{\varphi_k(x)\}$ of the model $Q_m(x, t)$ match closely the basis functions $\{\sin(k\pi x/l)\}$ in the definition of $Q(x, t)$, (2) the scaled correlation functions of $\{Z_k(t)\}$ and $\{Y_k(t)\}$ are similar, (3) the marginal distributions of $\{Z_k(t)\}$ can be assumed to be Gaussian, and (4) the processes $\{Z_k(t)\}$ are uncorrelated. These observations support the assumption that $\{Z_k(t)\}$ are independent Gaussian processes.

We also have seen that the model $Q_m(x, t)$ is less accurate if the sample size N is not sufficiently large, e.g., the basis functions in the expression of $Q_m(x, t)$ were less accurate when based on $N = 100$ samples of $Q(x, t)$ recorded at a single time. Errors of the basis functions propagate to the samples of the processes $\{Z_k(t)\}$, which are projections of the samples of $Q(x, t)$ on the eigenvectors $\{v_k\}$, see Eq. 16. These errors can be reduced by using values of the samples of $Q(x, t)$ at several times rather than a single time, so that information is not wasted.

Since the target random function is time-stationary, we can construct the modes $\{v_k\}$ in the definition of $Q_m(x, t)$ from samples of $Q(x, t)$ at several times $\{t_r\}$, $r = 1, \dots, n_r$, rather than a single time, provided these times are such that the random fields $\{Q(x, t_r), x \in D\}$ are weakly dependent. The degree of dependence between these fields cannot be determined precisely since the correlation functions of the processes $\{Z_k(t)\}$ are unknown and cannot be estimates till the model $\{v_k\}$ are known. Because of the dependence between the random fields $\{Q(x, t_r), x \in D\}$, the effective number of independent samples available to construct

the modes in the definition of $Q_m(x, t)$ is less than $N n_r$ but larger than N , unless $Q(x, t)$ is time invariant, i.e., is a random field.

The left, middle, and right panels in Fig. 5 show with dash lines five samples of $Q(x, t)$,

Figure 5: Five samples of $Q(x, t)$ in dash lines and corresponding samples of $Q_m(x, t)$ for $m = 5$ (left panel), $m = 10$ (middle panel), and $m = 20$ (right panel) in solid lines

$t = 1$, and with solid lines the corresponding five samples of $Q_m(x, t)$, $t = 1$, i.e., the samples of A_m in Eq. 14, for $m = 5, 10$, and 20 . The plots in Fig. 5 are based on $N = 50$ samples of $Q(x, t)$ recorded at $n_r = 10$ equally spaced times in $[0, \tau]$ rather than $N = 1000$ samples of $Q(x, t)$ recorded at a single time. It is remarkable that accurate models for the spatial variation of $Q_m(x, t)$ can be obtained with only $N = 50$ samples of $Q(x, t)$ if their values at several times are used in analysis. The accuracy of $Q_m(x, t)$ increases with the model size m in agreement with results in Fig. 2.

Although accurate basis functions for $Q_m(x, t)$ can be obtained from a relatively small number of samples of $Q(x, t)$ provided we use their values at several times, satisfactory estimates of the correlation functions of $\{Z_k(t)\}$ may require larger number of samples, unless parametric models are postulated for these correlation functions. The solid and dash lines in Figs. 6 and 7 are estimates of the scaled correlation functions of $Z_1(t)$ and $Z_{10}(t)$

Figure 6: Estimates of the correlation functions of $Z_1(t)$ based on $N = 50$ (left panel), $N = 200$ (middle panel), and $N = 1000$ (right panel) samples of $Q(x, t)$ in solid lines. Dash lines are the scaled correlation functions of $Y_1(t)$

and the correlation functions of $Y_1(t)$ and $Y_{10}(t)$. The samples of the processes $\{Z_k(t)\}$ have been obtained from Eq. 16 with $m = 10$.

That $Q(x, t)$ is a Gaussian function is no restriction since the construction of $Q_m(x, t)$ uses samples of the target random function. The left and middle panels in Fig. 8 show two

Figure 7: Estimates of the correlation functions of $Z_{10}(t)$ based on $N = 50$ (left panel), $N = 200$ (middle panel), and $N = 1000$ (right panel) samples of $Q(x, t)$ in solid lines. Dash lines are the scaled correlation functions of $Y_{10}(t)$

Figure 8: Two samples of $Q(x, t)$ (left panels) and corresponding samples of $Q(x, t)^3$ (middle panels) and $Q_m(x, t)$ for $Q(x, t)^3$ (right panels) for $m = 10$

samples of $Q(x, t)$ in Eq. 26 and the corresponding samples of the non-Gaussian random function $Q(x, t)^3$. The right panels are the corresponding samples of $Q_m(x, t)$ with $m = 10$ based on $N = 50$ independent samples of $Q(x, t)^3$. The spatial modes of $Q_m(x, t)$ have been obtained from the samples of $Q(x, t)^3$ at $n_r = 10$ equally spaced times. Samples of the processes $\{Z_k(t)\}$ have been calculated, as previously, from Eq. 16. The plots show that the samples of $Q_m(x, t)$ match closely the target samples.

5.2 Arbitrary random function

Suppose the space-time random function $Q(x, t)$ is nonstationary in both arguments. The approach in the previous section cannot be applied directly since the members of the family of random fields $\{Q(x, t), x \in D\}$ indexed by $t \in [0, \tau]$ have different properties at different times. We have briefly discussed three methods for constructing models $Q_m(x, t)$

of $Q(x, t)$ in this general case. We implement only the method that assumes that $Q_m(x, t)$ admits the representation in Eq. 24. The modes of this model are extracted from values of the samples of $Q(x, t)$ at multiple times to capture the temporal variation of the family of random fields $\{Q(x, t), x \in D\}$ indexed by $t \in [0, \tau]$.

Numerical results are presented for two space-time random functions $Q(x, t)$. They constitutes generalizations of the random function $Q(x, t)$ in the previous section. In both examples, the available information consists of N independent samples $\{q_i(x, t)\}$, $(x, t) \in D \times [0, \tau]$, of the target function $Q(x, t)$.

Example 1. Suppose $Q(x, t)$ has the expression

$$Q(x, t) = \sum_{k=1}^{\infty} \zeta_k(t) Y_k(t) \sin(k \pi x/l), \quad x \in D = [0, l], \quad t \in [0, \tau]. \quad (28)$$

where $\zeta_k(t) = 1 + \kappa \sin(k \pi t/\tau)$, $t \in [0, \tau]$, $\kappa > 0$ is a constant, and the processes $\{Y_k(t)\}$ are defined by Eq. 27. It is obtained from Eq. 26 by modulation with the functions $\{\zeta_k(t)\}$. The first two moments of $Q(x, t)$ are $E[Q(x, t)] = 0$ and

$$E[Q(x, t) Q(y, s)] = \sum_{k=1}^{\infty} \zeta_k(t) \zeta_k(s) \sigma_k^2 e^{-\lambda_k |t-s|} \sin(k \pi x/l) \sin(k \pi y/l),$$

provided the above sums are convergent with finite limits. This shows that $Q(x, t)$ in Eq. 28 is neither time-stationary nor space-homogeneous. It is assumed that $Q(x, t)$ is completely defined by the first 50 terms of its infinite series representation. The parameters in the definition of $Q(x, t)$ used in the following numerical illustrations are those of $Q(x, t)$ in the previous section and $\kappa = 0.3$.

Since the random fields $\{Q(x, t), x \in D\}$ are for every $t \in [0, \tau]$ elements of the linear space spanned by $\{\sin(k \pi x/l), k = 1, 2, \dots\}$, it is expected that the modes needed to characterize these fields can be extracted from samples of $Q(x, t)$ at a single time. As previously stated, the accuracy of the calculated modes can be increased by using sample values at multiple times.

Figure 9 shows two samples of $Q(x, t)$ (right panels) and the corresponding samples of $Q_m(x, t)$ for $m = 10$ (left panel) and $m = 20$ (middle panel). The models $Q_m(x, t)$ have been constructed from values of $N = 100$ independent samples of $Q(x, t)$ at $n_r = 10$ equally spaced times in $[0, \tau]$. The model accuracy increases with m , e.g., the error $\max_{x \in D, t \in [0, \tau]} |Q(x, t) - Q_m(x, t)|$ is 0.1032 ($m = 10$) and 0.0551 ($m = 20$) for the first sample of $Q(x, t)$ shown in the top panels of the figure.

As anticipated, the modes of $Q_m(x, t)$ can be estimated accurately from the $N = 100$ samples of $Q(x, t)$. They match closely the functions $\{\sin(k \pi x/l)\}$ that spanned the subspace in which the samples of $Q(x, t)$ reside. The accuracy of the modes of $Q_m(x, t)$ is similar to that of the modes in Fig. 1. The processes $\{Z_k(t)\}$ in the definition of $Q_m(x, t)$ are not stationary and their statistics are consistent with the Gaussian assumption. The samples of the processes $\{Z_k(t)\}$ have been calculated from Eq. 16. They can be processed to develop models for $\{Z_k(t)\}$, which together with the modes of $Q_m(x, t)$, can be used to generate large sets of samples of this function.

Figure 9: Two samples of $Q(x, t)$ (right panels) and corresponding samples of $Q_m(x, t)$ for $m = 10$ (left panels) and $m = 20$ (middle panels)

Example 2. Consider an extension of the random function $Q(x, t)$ in Eq. 28 obtained by randomizing the basis functions $\{\sin(k \pi x/l)\}$, i.e.,

$$Q(x, t) = \sum_{k=1}^{\infty} \zeta_k(t) Y_k(t) \sin(k \pi H_k(x/l) + U_k), \quad x \in D = [0, l], \quad t \in [0, \tau], \quad (29)$$

with the notations in Eq. 3, where $\{H_k(x)\}$ are monotonically increasing random functions such that $H_k(0) = 0$ and $H_k(1) = 1$, and $\{U_k\}$ are independent random variables distributed uniformly in $(-\pi, \pi)$. The random functions $\{H_k(x)\}$ and variables $\{U_k\}$ define random space distortions and phases and are independent of the processes $\{Y_k(t)\}$. The first two moments of $Q(x, t)$ in Eq. 4 are $E[Q(x, t)] = 0$ and

$$E[Q(x, t) Q(y, s)] = \sum_{k=1}^{\infty} \zeta_k(t) \zeta_k(s) \sigma_k^2 e^{-\lambda_k |t-s|} E[\sin(k \pi H_k(x/l) + U_k) \sin(k \pi H_k(y/l) + U_k)],$$

which shows that the target function is neither time-stationary nor space-homogeneous. For the numerical illustrations in this section, the functions $\{H_k\}$ are standard Beta distribution functions with shape parameters whose shape parameters are independent variables that are distributed uniformly in $(1, 5)$ and the phase angles $\{U_k\}$ are independent random variable distributed uniformly in $(-\pi, \pi)$.

We note that the target random function $Q(x, t)$ considered in this example has no particular structure since it exhibits random fluctuations in both space and time. The time random fluctuations are defined by the Ornstein-Uhlenbeck processes in Eq. 27 and the deterministic modulation functions $\{\zeta_k\}$. The spatial variation is described by sine waves with randomly distorted arguments and random phases. Since $Q(x, t)$ has no particular structure, the usefulness of the model $Q_m(x, t)$ in Eq. 2 is questionable. It may seem that a

Figure 10: A sample of $Q_m(x, t)$ for $m = 2$ (top left), $m = 5$ (top middle), $m = 10$ (top right), $m = 15$ (bottom left), $m = 20$ (bottom middle), and the target sample of $Q(x, t)$ (bottom right)

more general model is needed to represent $Q(x, t)$ accurately, e.g., the model with modes in the (x, t) -space weighted by random coefficients.

Yet, we further explore the potential of the model $Q_m(x, t)$ in Eq. 2 for $Q(x, t)$ in this general setting. The rows of the matrix $\mathcal{A}$ in Eq. 9 are the samples of the random fields $\{Q(x, t), x \in D\}$ at all available times, so that the modes $\{v_k\}$ of $\mathcal{A}$ include information on $Q(x, t)$ at all recorded space and time arguments. Accordingly, the modes of the low rank approximation $\mathcal{A}_m$ of $\mathcal{A}$ given by Eq. 10, or its version in Eq. 14, should provide adequate basis functions for $Q_m(x, t)$. These basis functions and the algorithm in Sect. 3.2 deliver samples of the processes $\{Z_k(t)\}$ in the definition of $Q_m(x, t)$.

The bottom right panel in Fig. 10 shows a sample of $Q(x, t)$ defined by Eq. 4. The top left, middle, and right panels and the bottom left and middle panels show the corresponding samples of $Q_m(x, t)$ for $m = 2, 5, 10, 15$, and 20 . The model $Q_m(x, t)$ has been constructed from $N = 1000$ independent samples of $Q(x, t)$. Estimates of the mean $\hat{\mu}_{\text{err}}$ and standard deviation $\hat{\sigma}_{\text{err}}$ of the error $\max_{x \in D, t \in [0, \tau]} |Q(x, t) - Q_m(x, t)|$ based on all $N = 1000$ samples are $(\hat{\mu}_{\text{err}}, \hat{\sigma}_{\text{err}}) = (0.5092, 0.1084), (0.2782, 0.0708), (0.1414, 0.0329), (0.0935, 0.0173),$ and $(0.0688, 0.0115)$ for $m = 2, 5, 10, 15$, and 20 . As expected, the expectation of the error and its variability decrease with the model size. The visual inspection of the plots in Fig. 10 is consistent with the above results on the error of $Q_m(x, t)$. It is remarkable that even $Q_2(x, t)$ capture the essential features of $Q(x, t)$. The model resolution increases steadily with the number of modes m included in the analysis.

Figure 11 shows three samples of $Q(x, t)$ to illustrate the broad range of features of the target function $Q(x, t)$. It is remarkable that these features can be captured by just a few modes. Figure 12 shows scaled versions $v_k / \max\{|v_k|\}$ of the top ten modes used to construct $Q_m(x, t)$. The left and right panels show the top five modes and the subsequent five modes,

Figure 11: Three samples of $Q(x, t)$

Figure 12: Top ten modes of $Q_m(x, t)$; the first and the second five modes in the left and right panels

respectively.

The performance of the model $Q_m(x, t)$ is remarkable given that the structure of the target function is rather different from that of the model. The above results show that the samples of $Q_m(x, t)$ match satisfactorily the samples of $Q(x, t)$ even when based on a relatively small number of modes (see Fig. 10). As previously, the samples of the processes $\{Z_k(t)\}$ result from Eq. 16 and can be analyzed to construct models for these processes. Once a model $Q_m(x, t)$ has been constructed, it can be used to generate large sets of samples, which can be viewed as surrogates for samples of $Q(x, t)$.

5.3 Single sample

Suppose the information on a space-time random function $Q(x, t)$ consists of (1) a single, relatively long sample of $Q(x, t)$, i.e., a real-valued deterministic function $q(x, t)$ defined on the product space $D \times [0, \tau]$, and (2) a property of this function related to its evolution in time. It is known that the members of the family of stochastic processes $\{Q(x, t), t \in [0, \tau]\}$ indexed by $x \in D$ are not only stationary but also ergodic. This setting is encountered in various experiments, e.g., $q(x, t)$ can be a sample of the pressure field $Q(x, t)$ on aircrafts, bridges, and other physical systems recorded in a wind tunnel.

The target space-time function $Q(x, t)$ is defined by

$$Q(x, t) = \sum_{k=1}^{\infty} Y_k(t) \sin(k \pi x/l + \psi_k), \quad x \in D, \quad t \in [0, \tau], \quad (30)$$

with the notations in Eq. 26, where $\{\psi_k\}$ are deterministic phases taking values in $(-\pi, \pi)$. This form of $Q(x, t)$ is a slight modification of that in Eq. 26. The sample $q(x, t)$ of the target random function $Q(x, t)$ is generated from Eq. 30 and used to construct models $Q_m(x, t)$ of this function. Following numerical results are for the parameters in Sect. 5.1 except for the number of samples of $Q(x, t)$ which is $N = 1$ and the time τ which is longer. The first $n = 50$ terms in Eq. 30 are used to generate the sample $q(x, t)$ of $Q(x, t)$. It is assumed that this truncated version of Eq. 30 is the target random function.

The mean and correlation functions of $Q(x, t)$ are $E[Q(x, t)] = 0$ and

$$E[Q(x, t) Q(y, s)] = \sum_{k=1}^n \sigma_k^2 e^{-\lambda_k |t-s|} \sin(k \pi x/l + \psi_k) \sin(k \pi y/l + \psi_k).$$

Since $Q(x, t)$ is a Gaussian variable with mean 0 and variance $\sum_{k=1}^n \sigma_k^2 \sin^2(k \pi x/l + \psi_k)$ at each (x, t) , we have

$$P(Q(x, t) \leq \xi) = \Phi\left(\xi \left(\sum_{k=1}^n \sigma_k^2 \sin^2(k \pi x/l + \psi_k)\right)^{-1/2}\right), \quad \xi \in \mathbb{R},$$

where Φ denotes the distribution of the standard normal variable.

We also note that $Q(x, t)$ is a stationary Gaussian process for each $x \in D$ as a sum of Gaussian processes, the processes $\{Y_k(t) \sin(k \pi x/l + \psi_k)\}$. It is also ergodic since its correlation function has the property $E[Q(x, t) Q(y, s)] \rightarrow 0$ as $|t - s| \rightarrow \infty$ [1] (Theorem 6.5.4).

As already stated, our objective is to construct a model $Q_m(x, t)$ of the type in Eq. 2 from a sample $q(x, t)$ of $Q(x, t)$, $(x, t) \in D \times [0, \tau]$, and the knowledge that $Q(x, t)$ is ergodic in time. We largely follow the approach in Sects. 3.1 and 3.2. Basis functions are constructed for the spatial variation of $Q(x, t)$ and used subsequently to characterize the time evolution of this function.

Let $0 \leq t_1 < t_2 < \dots < t_N$ be times that are sufficiently far from each other such that the random fields $\{Q(x, t_r)\}$, $r = 1, \dots, N$, are weakly dependent. In applications, the selection of the times $\{t_r\}$ needs to be based on preliminary estimates of the temporal correlation of $Q(x, t)$. Our preliminary statistical analysis suggested to set $(t_{r+1} - t_r) \simeq 0.5$. We view the spatial variation of the sample of $Q(x, t)$ at this times, i.e., the functions $\{q(x, t_r)\}$, $x \in D$, $r = 1, \dots, N$, as independent samples of the spatial variation of $Q(x, t)$. The rows of the (N, d) -matrix $\mathcal{A}$ in Eq. 9 consist of the records $\{q(x, t_r)\}$ at d values of $x \in D$. The SVD of $\mathcal{A}$ in this equation is used to construct low rank approximations $\mathcal{A}_m$ of $\mathcal{A}$ as in Eqs. 10 and 14. The eigenvectors $\{v_k\}$ of the SVD of $\mathcal{A}$ and the records of $Q(x, t)$ in the time intervals $[t_{r-1}, t_r)$, $r = 1, \dots, N$, and the algorithm in Sect. 3.2 are used to extract samples of the processes $\{Z_k(t)\}$, which can be used subsequently to develop models for these processes.

In the previous section, we use multiple records of $Q(x, t)$ with duration $\tau = 2$. In this section, we use a single record of $Q(x, t)$ with length 10τ and 100τ , $\tau = 2$, to construct

Figure 13: Samples of $Q(x, t)$, $(x, t) \in D \times [t_{r-1}, t_r]$ in two time intervals (top panels) and corresponding samples of $Q_m(x, t)$ (bottom panels)

Figure 14: Estimates of the correlation functions of $Z_1(t)$ and $Z_{10}(t)$ from a single sample with length $\tau = 20$ (left two panels) and $\tau = 200$ (right two panels)

models $Q_m(x, t)$ for $Q(x, t)$. The top panels of Fig. 13 show two sections of length τ extracted from the sample $q(x, t)$ of $Q(x, t)$, $(x, t) \in D \times [0, 100\tau]$, i.e., $q(x, t)$ with $(x, t) \in D \times [0, \tau]$ and $(x, t) \in D \times [\tau, 2\tau]$, and the corresponding samples of $Q_m(x, t)$ for $m = 10$. Estimates of the mean and standard deviations of the error $\max_{x \in D, t \in [0, \tau]} |Q(x, t) - Q_m(x, t)|$, $m = 10$, are 0.0831 and 0.0067 when based on a single record of $Q(x, t)$ with length 100τ , $\tau = 2$.

The solid lines in Fig. 14 are estimates of the correlation functions of $Z_1(t)$ and $Z_{10}(t)$ obtained from a single sample of $Q(x, t)$ with length 10τ (left two panels) and a single sample of $Q(x, t)$ with length 100τ (right two panels). The samples of these processes have been scaled to have unit variance. The dash lines are the corresponding correlation functions of the processes $\{Y_k(t)\}$ in the definition of $Q(x, t)$. The estimates of the correlation functions of $\{Z_k(t)\}$ match the correlation functions of $\{Y_k(t)\}$ and improve with the record length.

Figure 15: The distributions $P(Q(x, t) \leq \xi)$ (dash line) and its estimate $P(Q_m(x, t) \leq \xi)$, $m = 10$, (solid line) based a record of length 100τ of $Q(x, t)$ at $x = 0.5$ and $t = 1$

The dash and solid lines in Fig. 15 are the marginal distribution $P(Q(x, t) \leq \xi)$ of $Q(x, t)$ at $(x, t) = (0.5, 1)$ and the approximation $P(Q_m(x, t) \leq \xi)$, $m = 10$, of this distribution derived from a single sample of $Q(x, t)$ with length 100τ . The approximate and the exact distributions are nearly indistinguishable at the scale of the figure.

5.4 Comments

The above numerical examples demonstrate that a broad range of space-time random functions can be approximated satisfactorily by the model $Q_m(x, t) = \sum_{k=1}^m Z_k(t) \varphi_k(x)$ in Eq. 2, which has been originally designed for time-stationary random functions $Q(x, t)$.

The target function $Q(x, t)$ considered in Sect. 5.1 is time-stationary and its functional form is similar to that of $Q_m(x, t)$, see Eqs. 2 and 26. In this case, it is found that not only the probability law of $Q_m(x, t)$ matches that of $Q(x, t)$ but also that the components of $Q_m(x, t)$ and $Q(x, t)$ have similar properties, e.g., the modes of $Q_m(x, t)$ match closely the basis functions $\{\sin(k\pi x/l)\}$ of $Q(x, t)$ (see Fig. 1), the correlation functions of the processes $\{Z_k(t)\}$ approximate satisfactorily the correlation functions of $\{Y_k(t)\}$, see Fig. 4, and $\{Z_k(t)\}$ can be assumed to be stationary Gaussian processes. Once the probability law of $Q_m(x, t)$ has been constructed, samples of this random function can be generated by simple algorithms.

The target functions in Sect. 5.2 are extensions of $Q(x, t)$ in the previous section that are obtained by modulating the processes $\{Y_k(t)\}$, see Eq. 28, and modulating the processes $\{Y_k(t)\}$ and randomly distorting the space coordinate $x \in D$, see Eq. 29. The modes of the model $Q_m(x, t)$ of $Q(x, t)$ in Eq. 28 match closely the basis functions $\{\sin(k\pi x/l)\}$ while those of the model of $Q(x, t)$ in Eq. 29 are distorted sine waves with various phases. For both cases, models can be constructed for the processes $\{Z_k(t)\}$ from their samples. The models of $\{Z_k(t)\}$ and the modes of $Q_m(x, t)$ can be used to generate large sets of samples of this space-time random function.

The target functions in Sect. 5.3 consider the case in which we are given a single sample of a time-stationary and ergodic random function $Q(x, t)$, see Eq. 30. If the sample is sufficiently long, the time-ergodicity of $Q(x, t)$ allows to construct accurate modes for $Q_m(x, t)$ and models for the processes $\{Z_k(t)\}$. As previously, the resulting model can be

used for sample generation. It is shown that the samples of $Q_m(x, t)$ match closely the samples of $Q(x, t)$ and that statistics of $\{Z_k(t)\}$ can be estimated accurately.

6 Conclusions

Models have been developed for random functions $Q(x, t)$ of space $x \in D$ and time $t \in [0, \tau]$ from samples of these functions and any other information when available. Two classes of random functions have been considered, time-stationary and arbitrary random functions. The members of the family of random fields $\{Q(x, t), x \in D\}$ indexed by time $t \in [0, \tau]$ have the same properties for time-stationary random functions but their properties are time-dependent for arbitrary functions. This fact has been considered in the construction and implementation of the proposed models.

One of the models of $Q(x, t)$ is particularly attractive for applications since it is accurate, efficient, and works for both time-stationary and arbitrary random functions $Q(x, t)$. The model can be viewed as an extension of the Karhunen-Loève (KL) representation for random fields. Its samples are linear form of basis functions with random coefficients which are extracted from samples of $Q(x, t)$ by singular value decomposition algorithms. The coefficients of these forms are stochastic processes rather than random variables.

Theoretical arguments have been used to show that the proposed models can match target space-time random functions to any accuracy. Bounds have also been obtained on the discrepancy between models and target random functions. Numerical examples have been presented to illustrate the implementation of the models of $Q(x, t)$ and assess their accuracy. They include time-stationary and arbitrary random functions described by sets of independent samples of these functions and a time-stationary random function characterized by a single sample. The proposed models provide accurate representations for the space-time random functions considered in this study.

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