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A novel finite element two-step solution scheme for fully coupled hydro-mechanical processes in poroelastic media

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Abstract

A novel two-step solution scheme (TSSS) is described for fully coupled hydro-mechanical (FCHM) analysis of saturated poroelastic media. The TSSS is based on the *pressure formulation* in two-steps. In step one, the pressure field is obtained directly by solving the sub-problems with a reduced scale of displacement variables. This process is fully decoupled. In step two, the displacement field is calculated by staggered iteration of pressure variables. The finite element method (FEM) is used for discretization of the FCHM differential equations in the space domain. The precise time step integration method is performed for the time derivatives. The stability and convergence of the TSSS is proved using a matrix-based spectral analysis in the time domain. It is demonstrated that the TSSS is unconditionally stable, fully explicit and highly precise. The algorithmic error estimation results indicate that the numerical performance in the time domain can match the computer precision. Theoretically, the algorithmic error is caused only by mesh discretization. The stability and accuracy of the TSSS are verified and calibrated by numerical examples. By comparing with the analytical or reference solutions, it is shown that the TSSS result is highly precise, and it is remarkably better than the standard FEM in terms of precision. In addition, the numerical results are stable and independent of the time step size. The numerical experiments also demonstrate the stability, convergence and precision for the *pressure formulation* of TSSS. The proposed scheme has great potential in engineering applications for long timescale problems, especially the problems focusing on the evolution of pressure field.

Keywords: Coupled hydro-mechanical analysis; Fully explicit; Precise integration method; Pressure formulation; Poroelastic

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1. Introduction

The interaction between the pore fluid and solid skeleton of porous geomaterials is a fully coupled fluid-flow and solid-deformation problem (such as rocks, soils, etc.), which has been studied in many engineering fields, such as reservoir engineering, hydraulic and hydropower engineering, geotechnical and environmental engineering [1,2]. The theoretical basis can be dated back to the mid-20th century, and the mathematical framework can accurately describe the fully coupled hydro-mechanical (FCHM) process of saturated poroelastic media [2-4]. However, due to the complex physical interactions and complicated boundary conditions, the analytical solutions of FCHM problems are available for only a few situations. The extensive use of computers and the concomitant development of numerical techniques have made precise analysis possible for practical problems and shifted attention from problem solving to modeling observed phenomena. The eventual aim is to develop the capability for making high-efficiency and precise simulators [1,5,6]. In order to achieve this goal, numerical techniques have developed and extended towards different directions, such as the numerical approach pertinent to differential equations [1,7-13], the numerical stabilized formulation [14-19] and solution strategies, and etc..

After spatial discretization by a numerical approach, the ordinary differential equations are transformed into monolithic equations. The monolithic equation is commonly discretized in time. Then, three solution strategies can be adopted in the discretization system, as direct evaluation, partitioned schemes and two single-field analysis. Direct evaluation [1,20], i.e. the fully coupled $\mathbf{u}$ - p equation system directly solved simultaneously for the solid displacements $\mathbf{u}$ and fluid pressures p at every time step. In such case, numerical efficiency becomes the primary concern. The overall efficiency depends to a great extent on the algorithm used for solution of the algebraic equation system resulted from the discretization process. However, with increasing scale of the fully coupled $\mathbf{u}$ - p equation system, the computational cost and memory requirements for the scheme has increased dramatically, especially for long timescale problems. Hence, the sequential solution methods have been proposed later. Partitioned schemes, including simultaneous and staggered solutions [21-25]. In the partitioned schemes, an appropriate partitioning of the matrices is performed on the algebraic system allowing the equations to be solved sequentially. Either the flow or mechanical problem is solved first, with the other field variables frozen. Then the other problem is solved using the intermediate solution information. One may iterate this sequential procedure at each time step until convergence [21-25]. During this process, the coupled problem is partitioned and the sub-problems are solved sequentially. The partitioning allows for the use of existing robust

simulators for the sub-problems, producing smaller systems of equations to be solved than the fully coupled methods. However, the stability and convergence of the sequential schemes can be quite different when addressing various problems [24, 25], which leads to difficulties in practical applications. Therefore the correctness and precision of these sequential schemes should be carefully assessed each time [1]. The fully coupled “true” solution is always the target of these sequential schemes. The fully coupled method is internally consistent and rigorous because the fluid flow and the solid deformation are solved simultaneously on the same discretized grid [26].

In many practical problems, only the $\mathbf{u}/p$ field or the physical field for a particular moment is concerned. For instance, the long-term surface subsidence connected with the change of hydraulic equilibrium in systems comprising aquitards and/or hydrocarbon reservoirs may be the focus, when there is extensive groundwater withdrawal and/or oil and gas pumpage for a long time; on the contrary, the distribution and evolution of the p field is more concerned during hydraulic fracturing in oilfield, rather than the distribution of the $\mathbf{u}$ field. In this case, either the $\mathbf{u}$ or p field variables can be eliminated from the time-discretized equation. Smith [27] proposed the so-called *displacement formulation*, which yields directly the $\mathbf{u}$ field by eliminating p . Alternatively, if $\mathbf{u}$ is eliminated from the monolithic equation and a *pressure formulation* is obtained after discretization in the time domain, which was done by Krause [28]. Either the *displacement* or *pressure formulation* can be used, i.e. the coupled two-step procedure (or the two single-field analysis), if both fields are concerned. It should be emphasized that the two-step procedure gives the fully coupled solutions, which is consistent with the results of direct evaluation. The elimination process can be considered as the decoupling process for the $\mathbf{u}$ or p field variables. The sub-problems with reduced scale can then be solved.

However, it is worth noting that no matter which numerical approach and solution strategy is adopted, the finite difference (FD) operator for the time derivative is always needed. According to the generalized trapezoidal θ -method, the variables and their temporal derivative in the time interval can be a two-point FD approximation [1,20], given that θ is an integration parameter varying from 1.0 (fully implicit) to 0 (fully explicit). The value of θ has an important influence on the stability. The approximation is unconditionally stable when $\theta \geq 0.5$, but for any value of $\theta \neq 1$, the numerical solution can exhibit a spurious rippling effect [29]. Thus the fully implicit ($\theta = 1$) scheme is commonly used. However, the computation precision and computational cost always exist in the FD-based schemes. The accuracy of the results is sensitive to the time step size. A large time step size would lead to large errors, but less time-consuming. In order to optimize this problem, to estimate the local truncation error and to control the time step size, the adaptive time-stepping methods and three-point

discretization scheme have been developed [21,30-34]. Nevertheless, these FD based schemes are either first- or second-order accurate approximations, and therefore their accuracy is relatively low. When $\theta = 0$, even though the fully explicit algorithm is simple, it is stable for only an extremely small time step size, and perhaps the step size is so small that a real time of interest would require an excessive number of steps. Therefore, the fully explicit scheme is not of practical value. However, the explicit FCHM simulators are still attractive to users owing to their advantages in that there is no need to solve the linear equations of the global stiffness matrix (in contrast to the more commonly used implicit methods) and also no need to check the convergence of the solution. Thus, the mathematical operation and structure are simpler. The widely used FLAC software is an explicit FD program for geotechnical mechanics computation [7]. However, its algorithm is still conditionally stable, which requires strict control of the time step size. Regrettably, little attention is devoted to the high precise and unconditionally stable explicit FE modeling by which the numerical schemes are suitable for the long time scale FCHM problems. There is no doubt that the development of a time domain FEM simulator for the FCHM analysis, which is explicit, insensitive to the time step size and can be easily implemented, is appealing and of great practical value.

This paper presents a novel two-step solution scheme (TSSS) based on *pressure formulation* for the FCHM FE modeling of saturated poroelastic media. The proposed TSSS is unconditionally stable, highly precise and fully explicit, in which the precise integration method is employed for time derivatives. The initial numerical integration efforts towards the parabolic partial differential systems were presented in [35]. *Section 2* describes the mathematical formulation of the FCHM model of saturated poroelastic media. In *Section 3*, a FE implementation of the TSSS is presented, and the stability and convergence of the TSSS are proved. The accuracy and stability of the numerical scheme are verified and calibrated in *Section 4* by applications in typical geotechnical problems. The conclusions and remarks are presented in *Section 5*.

2. Coupled hydro-mechanical model of saturated poroelasticity

The physical model is based on poroelasticity theories [1,2], and assumptions are made on the fluid-saturated porous medium, isotropic geomaterials, isothermal single-phase flow of a slightly compressible fluid (water in this study), and a slow and small deformation process.

2.1. Differential equations

The governing differential equations for coupled flow and poroelastic media are based on

the linear momentum balance and mass balance. Under the quasi-static assumption for solid-phase displacement, the equilibrium equation for mechanical deformation of the hydro-mechanical system can be written as:

$$\nabla \cdot \boldsymbol{\sigma} + \rho_b \mathbf{g} = \mathbf{0} \quad (1)$$

According to Terzaghi's effective stress principle that links the solid grain stress to the fluid pore pressure

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}' - \alpha p \mathbf{I} \quad (2)$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ are the total and effective Cauchy stress tensor (tension in the solid phase is positive), respectively; $\alpha = 1 - \frac{K_{dr}}{K_s}$ is the Biot's coefficient, K_s is the bulk modulus of the solid grain, and K_{dr} is the drained bulk modulus, p is the (excess) pore pressure (compression in the fluid phase is positive), $\mathbf{I}$ is the second-order unit tensor, ρ_b is the buoyant density of the saturated porous medium, $\mathbf{g}$ is the vector of gravity accelerations. The constitutive law of the solid phase is

$$\boldsymbol{\sigma}' = \boldsymbol{\Theta} : \boldsymbol{\varepsilon} \quad (3)$$

where $\boldsymbol{\Theta}$ is the fourth-order drained elasticity tensor and $\boldsymbol{\varepsilon}$ is the linearized strain tensor under the assumption of infinitesimal transformation, as

$$\boldsymbol{\varepsilon} = \nabla^s \mathbf{u} = \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^T) \quad (4)$$

where $\mathbf{u}$ is the displacement field for the solid matrix.

The mass balance (continuity) equation for fluid is given by

$$\alpha \dot{\varepsilon}_v + \frac{\dot{p}}{Q^*} + \nabla \cdot \mathbf{v} = Q \quad (5)$$

where $\varepsilon_v = \text{tr} \boldsymbol{\varepsilon}$ is the volumetric strain (the trace of the strain tensor), Q^* is the Biot's modulus and $\frac{1}{Q^*} = \frac{\alpha - n}{K_s} + \frac{n}{K_f}$, K_f is the bulk modulus of the fluid, n is the porosity, Q is a volumetric source term and $\mathbf{v}$ is the Darcy's velocity, defined as

$$\mathbf{v} = -\frac{\mathbf{k}}{\rho_f g} \cdot \nabla p \quad (6)$$

where $\mathbf{k}$ is the permeability tensor, ρ_f is the fluid density and $g = \|\mathbf{g}\|$ is the gravitational constant.

2.2. Initial and boundary conditions

To solve a particular problem of poroelasticity initial conditions must be specified, and also boundary conditions. The solid and fluid mixture occupies a domain Ω with the boundary

Γ . This boundary is suitably decomposed into regions where essential and natural boundary conditions are specified for both solid and fluid. In particular, the following notations are used: Γ_u , the solid displacement boundary; Γ_σ , the solid traction boundary; Γ_p , the fluid pressure boundary; Γ_q , the fluid flux boundary. The boundary decomposition is subjected to the restrictions

$$\Gamma = \Gamma_u \cup \Gamma_\sigma = \Gamma_p \cup \Gamma_q \quad (7)$$

$$\Gamma_u \cap \Gamma_\sigma = \Gamma_p \cap \Gamma_q = \emptyset \quad (8)$$

The boundary conditions are given as

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on } \Gamma_u \quad (\text{specified displacement}) \quad (9)$$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \bar{\boldsymbol{\sigma}} \quad \text{on } \Gamma_\sigma \quad (\text{specified traction}) \quad (10)$$

$$p = \bar{p} \quad \text{on } \Gamma_p \quad (\text{specified pressure}) \quad (11)$$

$$\mathbf{v} \cdot \mathbf{n} = \bar{q} \quad \text{on } \Gamma_q \quad (\text{specified flux}) \quad (12)$$

where $\mathbf{n}$ is the outward unit normal to the boundary Γ . Finally, initial conditions must be set for both the displacement and fluid pressure fields at time $t = 0$, as

$$\mathbf{u}|_{t=0} = \mathbf{u}_0, \quad p|_{t=0} = p_0 \quad \text{in } \Omega \quad \text{and on } \Gamma \quad (13)$$

The initial stress field should satisfy mechanical equilibrium.

Remark 1. It is sometimes desirable or convenient to calculate the immediate response of a poroelastic medium to a sudden load, there is hardly any movement of the fluid, i.e. ignoring all drainage. The time-dependent FCHM process of the medium may follow this initial behaviour. The initial response at time $t = 0$, the moment of loading, can be determined as the solution of a problem of elasticity, for poroelastic medium with shear modulus G and undrained compression modulus $K_u = K_{dr} + \alpha^2 / S$, where S is storativity. Then,

$p_0 = \alpha \sigma_0 / K_u S$, where σ_0 denotes the average stress immediately after application of the load. In the special case that the solid particles and the pore fluid are both incompressible the storativity $S = 0$. Verruijt [2] introduced a good approximation of the undrained response as $\alpha^2 / S = 100 K_{dr}$. Another, and perhaps simpler approach is to let the original computer program find the undrained solution, by setting all the boundary conditions for the pore pressure as impermeable, and assuming a small time step in which the undrained response may be calculated. The value of time after that initial time step can be reset to zero, before the progress of the coupled process is calculated, using the actual boundary conditions for the pore pressure.

3. Finite element two-step solution implementation

3.1. Discretization of spatial domain

For the spatial discretization of the FCHM problem, it is necessary to construct a variational formulation of the governing partial differential system with boundary and initial conditions based on the weighted residual FEM for the primary variables of the problem to transform the weak formulation into a matrix form. First, the two spaces of the trial functions used to solve $\mathbf{u}$ and p can be defined as follows:

$$S_u = \{\mathbf{u} : \Omega \rightarrow \mathbb{R}^{\dim} \mid \mathbf{u} \in \mathbf{H}^1, \mathbf{u} = \bar{\mathbf{u}} \text{ on } \Gamma_u\} \quad (14)$$

$$S_p = \{p : \Omega \rightarrow \mathbb{R}^1 \mid p \in H^1, p = \bar{p} \text{ on } \Gamma_p\} \quad (15)$$

where $\dim$ (=1 or 2 or 3) denotes the space dimension; and H^1 denotes a Sobolev space of degree one. The corresponding spaces of the weighting functions $\mathbf{w}$ and φ can also be defined for mechanics and flow, respectively, with homogeneous conditions on the essential boundaries as follows:

$$V_u = \{\mathbf{w} : \Omega \rightarrow \mathbb{R}^{\dim} \mid \mathbf{w} \in \mathbf{H}^1, \mathbf{w} = \mathbf{0} \text{ on } \Gamma_u\} \quad (16)$$

$$V_p = \{\varphi : \Omega \rightarrow \mathbb{R}^1 \mid \varphi \in H^1, \varphi = 0 \text{ on } \Gamma_p\} \quad (17)$$

Let the space domain be partitioned into non-overlapping elements, $\Omega = \bigcup_{i=1}^{n_{ele}} \Omega_i^h$, where n_{ele} represents the number of elements; and the superscript h indicates the spatially discrete quantities. S_u^h , V_u^h , S_p^h and V_p^h are the corresponding finite dimensional subspaces. Then, a discrete approximation of the weak form of the governing equations (Eqs. (1) and (5)) can be used to find $\{\mathbf{u}^h, p^h\} \in S_u^h \otimes S_p^h$ such that for all $\{\mathbf{w}^h, \varphi^h\} \in V_u^h \otimes V_p^h$

$$\int_{\Omega} (\nabla^s \mathbf{w}^h : \boldsymbol{\sigma}^h - \alpha p \nabla \cdot \mathbf{w}^h - \mathbf{w}^h \cdot \rho_b \mathbf{g}) d\Omega - \int_{\Gamma_{\sigma}} \mathbf{w}^h \cdot \bar{\boldsymbol{\sigma}} d\Gamma = 0 \quad \forall \mathbf{w}^h \in V_u^h \quad (18)$$

$$\int_{\Omega} (\alpha \varphi^h \dot{\varepsilon}_v + \frac{1}{Q^*} \varphi^h \dot{p} - \nabla \varphi^h \cdot \mathbf{v} - \varphi^h Q) d\Omega + \int_{\Gamma_q} \varphi^h \bar{q} d\Gamma = 0 \quad \forall \varphi^h \in V_p^h \quad (19)$$

The displacement and pore pressure fields can be approximated as follows:

$$\mathbf{u}^h = \sum_{i=1}^{n_{nod}} N_i^u \mathbf{u}_i \quad (20)$$

$$p^h = \sum_{i=1}^{n_{nod}} N_i^p p_i \quad (21)$$

where n_{nod} is the number of nodes; $\mathbf{u}_i$ and p_i are the displacement vectors and pore pressures at the element nodes (vertices), respectively; and N_i^u and N_i^p are the displacement and pore pressure shape functions of the i -th node, respectively. The weighting functions can be represented as follows:

$$\mathbf{w}^h = \sum_{i=1}^{n_{nod}} N_i^u \delta \mathbf{u}_i \quad (22)$$

$$\phi^h = \sum_{i=1}^{n_{nod}} N_i^p \delta p_i \quad (23)$$

By substituting Eqs. (2)-(4) and (20)-(23) into Eq. (18), the discrete functions are introduced into the residual equations, and the arbitrary nature of the weights $\delta \mathbf{u}$ and δp is recognized. The equilibrium system matrix form:

$$\mathbf{K}\mathbf{u} - \mathbf{Q}\mathbf{p} = \mathbf{f}^u \quad (24)$$

An integration in the space domain yields a rate form of the first-order differential matrix system,

$$\mathbf{K}\dot{\mathbf{u}} - \mathbf{Q}\dot{\mathbf{p}} = \dot{\mathbf{f}}^u \quad (25)$$

and rewrite as follows:

$$\dot{\mathbf{u}} = \mathbf{K}^{-1}\mathbf{Q}\dot{\mathbf{p}} + \mathbf{K}^{-1}\dot{\mathbf{f}}^u \quad (26)$$

where $\mathbf{K}$ and $\mathbf{Q}$ are the elastic stiffness and flow-stress coupling matrices, respectively; $\mathbf{u}$, $\mathbf{p}$ and $\dot{\mathbf{u}}$, $\dot{\mathbf{p}}$ are the vectors of unknown variables $\mathbf{u}_i$, p_i and the corresponding time derivatives, respectively; $\mathbf{f}^u$ and $\dot{\mathbf{f}}^u$ are the vectors of the nodal loads and loading rate, respectively. Next, by substituting Eqs. (6), (20)-(23) into Eq. (19), the continuity system matrix form:

$$\mathbf{Q}^T \dot{\mathbf{u}} + \mathbf{S}\dot{\mathbf{p}} + \mathbf{H}\mathbf{p} = \mathbf{f}^p \quad (27)$$

where $\mathbf{S}$ and $\mathbf{H}$ are the flow capacity and flow stiffness matrices, respectively; $\mathbf{f}^p$ are the vectors of the nodal flow sources. These matrices and vectors have the same meaning, as those of the conventional FEM. For the detailed matrices and vectors shown here, the reader is referred to [1,20]. Then, by substituting Eq. (26) into Eq. (27), after such decoupling, we have *pressure formulation*

$$\dot{\mathbf{p}} = \underbrace{-\left(\mathbf{Q}^T \mathbf{K}^{-1} \mathbf{Q} + \mathbf{S}\right)^{-1} \mathbf{H} \mathbf{p}}_{\tilde{\mathbf{H}}} + \underbrace{\left(\mathbf{Q}^T \mathbf{K}^{-1} \mathbf{Q} + \mathbf{S}\right)^{-1} \left(\mathbf{f}^p - \mathbf{Q}^T \mathbf{K}^{-1} \dot{\mathbf{f}}^u\right)}_{\mathbf{F}(t)} \quad (28)$$

Remark 2. It is well known that $\mathbf{K}$, $\mathbf{S}$ and $\mathbf{H}$ are symmetrically positive definite matrices. It is assumed that the dimension of the square matrix $\mathbf{K}$ is n_u , i.e. $\mathbf{K}_{n_u \times n_u}$, and its rank $r(\mathbf{K}) = n_u$. Accordingly, the matrix $\mathbf{Q}$ can be denoted by $\mathbf{Q}_{n_u \times n_p}$, and its rank $r(\mathbf{Q}) = \min\{n_u, n_p\}$. However, for porous media with extremely low permeability, the incompressible poroelastic or incompressible flow, spurious oscillations may appear in the pressure field, i.e. the locking behaviour will arise. Extensive researches have been devoted to this problem [14-16]. A rank verification for the problem was provided in [20] through the patch test. The system matrix requires that $n_u \geq n_p$, in order to avoid the locking behaviour.

Thus $\mathbf{Q}^T \mathbf{K}^{-1} \mathbf{Q}$ has full rank n_p , i.e. $\mathbf{Q}^T \mathbf{K}^{-1} \mathbf{Q}$ is invertible. By substituting Eq. (27) into Eq. (26), $\mathbf{p}$ is eliminated, and $\mathbf{u}$ can be expressed in an explicit form. There is $r(\mathbf{QH}^{-1}\mathbf{Q}^T) = n_p$, which implies that $\mathbf{QH}^{-1}\mathbf{Q}^T$ may not have full rank. In another word, the *displacement formulation* is not applicable for the coupled problem involving omitting of the compressibility of both the pore fluid and solid grains. Instead, it is only applicable for the special case with $n_u = n_p$.

3.2. Precise integration in time domain

In this section, we carry out a novel explicit TSSS to semi-analytically solve the monolithic equation. First, we can rewrite Eq. (28) in concise form as follows:

$$\dot{\mathbf{p}}(t) = \bar{\mathbf{H}}\mathbf{p}(t) + \mathbf{F}(t) \quad (29)$$

The general solution of above equation can be written as follows:

$$\mathbf{p}(t) = \exp(\bar{\mathbf{H}}t)\mathbf{p}(0) + \int_0^t \exp[\bar{\mathbf{H}}(t-\xi)]\mathbf{F}(\xi)d\xi \quad (30)$$

The time-step size is designated as Δt whereas the discrete time t_{k+1} for numerical computation can be given by $t_{k+1} = t_k + \Delta t$ for iteration, then Eq. (30) can be rewritten as follows:

$$\mathbf{p}(t_{k+1}) = \underbrace{\exp(\bar{\mathbf{H}}\Delta t)}_{\mathbf{T}} \mathbf{p}(t_k) + \underbrace{\int_{t_k}^{t_{k+1}} \exp[\bar{\mathbf{H}}(t_{k+1}-t)]\mathbf{F}(t)dt}_{RHS} \quad (31)$$

Let $\tau = \Delta t / m$ be a downscaling extremely small time-step size, where the integer m is suggested as $m = 2^N$, and N is a positive integer. The exponential matrix $\mathbf{T}$ can be obtained accurately using the precise integration method [35] with a downscaling time interval as follows:

$$\mathbf{T} = [\exp(\bar{\mathbf{H}}\tau)]^m \quad (32)$$

Using a Taylor expansion, the finite Taylor series with the remainder term reads as follows:

$$\exp(\bar{\mathbf{H}}\tau) = \sum_{i=0}^L \frac{(\bar{\mathbf{H}}\tau)^i}{i!} + O_L \quad (33)$$

Because τ is extremely small, the several term (i.e., the truncation order is L) series expansion should be enough. The exponential matrix $\mathbf{T}$ departs from the unit matrix $\mathbf{I}$ to a small extent $\mathbf{T}_a$ as follows:

$$\exp(\bar{\mathbf{H}}\tau) \approx \mathbf{I} + \sum_{i=1}^L \frac{(\bar{\mathbf{H}}\tau)^i}{i!} = \mathbf{I} + \mathbf{T}_a \quad (34)$$

Thus, Eq. (32) can be updated as follows:

$$\mathbf{T} = (\mathbf{I} + \mathbf{T}_a)^{2^N} = (\mathbf{I} + \mathbf{T}_a)^{2^{(N-1)}} \times (\mathbf{I} + \mathbf{T}_a)^{2^{(N-1)}} = (\mathbf{I} + 2\mathbf{T}_a + \mathbf{T}_a \times \mathbf{T}_a)^{2^{(N-1)}} \quad (35)$$

Furthermore, this factorization can be repeated recursively.

Remark 3. It should be mentioned that $\mathbf{T}_a$ will become an appended part, and its precision will significantly decrease in the round-off operation of the computer arithmetic if $\mathbf{T}_a$ is directly added to $\mathbf{I}$. Therefore, avoiding the loss of significant digits is a key issue. If the following instructions

$$\begin{aligned} &\text{do } i = 1, N \\ &\quad \mathbf{T}_a = 2\mathbf{T}_a + \mathbf{T}_a \times \mathbf{T}_a \\ &\text{end do} \end{aligned} \quad (36)$$

are executed, we have

$$\mathbf{T} = \mathbf{I} + \mathbf{T}_a \quad (37)$$

After being multiplied N times, $\mathbf{T}_a$ is no longer a small matrix, and the addition in Eq. (37) will have no serious numerical round-off error.

Generally, the term RHS in Eq. (31) can be analytically integrated when $\mathbf{f}$ are polynomial, exponential or trigonometric. Substituting $\mathbf{p}(t_{k+1})$ into Eq. (24), the node displacement field can be obtained at time t_{k+1} as,

$$\mathbf{u}(t_{k+1}) = \mathbf{K}^{-1} \mathbf{Q} \mathbf{p}(t_{k+1}) + \mathbf{K}^{-1} \mathbf{f}''(t_{k+1}) \quad (38)$$

The flowchart for the coupled two-step iterative procedure is shown in Fig. 1.

Remark 4. The calculation process for $\mathbf{p}(t_{k+1})$ is fully explicit without solving the linear equations. In addition, for a constant time step size, $\mathbf{T}$ is calculated for only once. Therefore, higher efficiency can be achieved for consecutive computation. Furthermore, $\mathbf{p}$ is decoupled with $\mathbf{u}$, i.e., the *pressure formulation* iterative procedure is decoupled. Hence, acquisition of $\mathbf{u}$ is performed only when necessary, instead of each time step.

Remark 5. For complicate time-dependent load in the interval $[t_k, t_{k+1}]$ can be linearized as

$$\mathbf{f}(t) = \mathbf{r}_0 + \mathbf{r}_1 t \quad (39)$$

where $\mathbf{r}_0$ and $\mathbf{r}_1$ are two constant parts. The RHS can be analytically integrated from t_k to t_{k+1} as follows:

$$RHS = (\mathbf{T} - \mathbf{I}) \mathbf{H}^{-1} \left[(\mathbf{Q}^T \mathbf{K}^{-1} \mathbf{Q} + \mathbf{S}) \mathbf{H}^{-1} \cdot \mathbf{r}_1 - \mathbf{r}_0 \right] + \Delta t \mathbf{H}^{-1} \cdot \mathbf{r}_1 \quad (40)$$

Alternatively, for complicate time-dependent load, RHS in the interval $[t_k, t_{k+1}]$ can be more exactly obtained using the Gauss quadrature [36].

Fig. 1

3.3. Stability analysis

To investigate the stability of a direct time-integration scheme, it is necessary to prove that the spectral radius ρ of the approximation operator matrix $\mathbf{T}$ is less than or equal to unity [24-25]. We perform a modal decomposition of Eq. (31), where RHS has been set to zero, like

$$\mathbf{p}_{k+1} = \mathbf{T}\mathbf{p}_k \quad (41)$$

$\mathbf{Q}^T\mathbf{K}^{-1}\mathbf{Q} + \mathbf{S}$ and $\mathbf{H}$ are positive definite matrices, let $\lambda_1, \lambda_2, \dots, \lambda_{n_p}$ be the distinct generalized eigenvalues of $\mathbf{Q}^T\mathbf{K}^{-1}\mathbf{Q} + \mathbf{S}$ to $\mathbf{H}$. By matrix theory, λ_j ($j = 1, \dots, n_p$) all are positive real number, and the eigenvalues of $\bar{\mathbf{H}}$ are $-\lambda_j$, respectively. Then, the homogeneous form of Eq. (41) decomposes into n_p equations,

$$p_{k+1}^j = T_j p_k^j \quad (42)$$

where p^j denote the scalar values, obtained by premultiplying the vector $\mathbf{p}$ by the eigenvectors $\boldsymbol{\psi}_j^T$ of $\bar{\mathbf{H}}$ as

$$p^j = \boldsymbol{\psi}_j^T \mathbf{p} \quad (43)$$

and

$$T_j = 1 + (-\lambda_j \tau) + \frac{(-\lambda_j \tau)^2}{2} + \frac{(-\lambda_j \tau)^3}{3!} + \dots + \frac{(-\lambda_j \tau)^L}{L!} \rightarrow e^{-\lambda_j \tau} \quad (44)$$

So

$$|T_j| < 1 \quad (45)$$

Hence, for an arbitrary time-step size,

$$\rho(\mathbf{T}) < 1 \quad (46)$$

Remark 6. Meanwhile, $\rho(\mathbf{T})$ can be seen as the amplification factor for whether the numerical errors are bounded or grown. Therefore, the current two-step procedure is unconditionally stable for the calculation of the pore pressure.

3.4. Error estimation and convergence analysis

To estimate the algorithmic two-step error, it can be defined that

$$\|\mathbf{e}^p\| = \|\mathbf{e}_h^p\| + \|\mathbf{e}_T^p\| + \|\mathbf{e}_R^p\| \quad (47)$$

$$\|\mathbf{e}^u\| = \|\mathbf{e}_h^u\| + \|\mathbf{e}^p\| \quad (48)$$

where superscript p and u denote pore pressure and displacement fields such as $\mathbf{e}^p$ and $\mathbf{e}^u$ are the total error $\mathbf{e}$ between the exact solution and the numerical solution of pore pressure and displacement, respectively; the symbol $\|\bullet\|$ indicates an appropriate norm (e.g., L^2 -norm); $\mathbf{e}_T^p$ and $\mathbf{e}_R^p$ are computational errors of the exponential matrix $\mathbf{T}$ and RHS by precise time step integration, respectively; and $\mathbf{e}_h$ represents the FE discretization error.

If the norm of each part of $\mathbf{e}$ is negligibly small, $\mathbf{e}$ is also negligible. The discretization error $\mathbf{e}_h$ in the space domain is caused by the standard FE process as the derivation of Eqs. (24) and (27) in the finite dimensional approximating space $S_u^h \otimes S_p^h$. If an r -order complete polynomial is used in the FEM for an element with a size of h , the local fitting precision can be of the r -order, i.e. $\mathbf{e}_h = O_h = O(h^{r+1})$. The consistency and convergence of the FE discretization in the space domain can be satisfied by enhancing the spatially discrete quantities and using proper shape functions for $\mathbf{u}$ and p , i.e., proper choice of $\mathbf{u}$ - p elements. Construction of low-order stable elements and study on the stability of isoparametric elements are always very attractive because of its simplicity and numerical efficiency [14-16].

If the RHS can be expressed analytically, $\mathbf{e}_R^p$ can be neglected. For a complicate case, an arbitrary order of accuracy of the RHS can be met by numerical integration (such as Gauss quadrature), theoretically. Except for the round-off error in the conventional computer arithmetic used in the matrix multiplication, $\mathbf{e}_T^p$ can be induced from only the power series expansion truncation. For instance, if $N = 20$, $m = 1,048,576$ and $L = 4$, then $\mathbf{e}_T^p = O_L = O(\Delta t^5) = 10^{-30} O(\tau^5)$, which is on the order of the round-off error for ordinary computers. In theory, the arbitrary numerical accuracy of $\mathbf{T}$ can be met by setting the selected parameters N and L . Note that the double precision of a real number in a 64-bit computer has 16 decimal digits, and the numerical result of the exponential matrix $\mathbf{T}$ reaches the computer precision. Thus the error of the TSSS is entirely caused by the FE mesh discretization ideally.

Remark 7. It is note that $\mathbf{e}_T^p + \mathbf{e}_R^p = \mathbf{x}_{k+1} - \mathbf{x}(t_{k+1})$, $\mathbf{x}_{k+1}$ is the exact solution of Eq. (31) and its numerical solution is $\mathbf{x}(t_{k+1})$ for the time instant t_{k+1} . $\mathbf{e}_T^p + \mathbf{e}_R^p$ can be seen as the iterative stability error. As above analysis, $\|\mathbf{T}\| < 1$ is verified, then the *pressure formulation* system is stable and $\lim_{k \rightarrow \infty} \|\mathbf{e}_T^p + \mathbf{e}_R^p\| = 0$.

Because the initial condition is known, the initial error is zero. Based on the above analysis, the approximate solution of the TSSS converges to the exact solution (consistency), and the round-off errors of the initial values do not increase during the iteration process (stability). Therefore, theoretically speaking, the TSSS can approach the computer precision at an arbitrary timescale and is unconditionally stable.

4. Numerical results

In this section, the proposed TSSS is applied to typical geotechnical problems to investigate its performance. To verify and calibrate the accuracy and stability of the proposed two-step algorithms, the modeling results are compared with the reference solutions and/or the numerical solution of a standard coupled FEM. In the standard FEM [1,20], the FEM mesh consists of bilinear-displacement/bilinear-pressure quadrangular (Q4P4) elements, i.e. equal-order interpolations for both the displacement and pressure fields, using 2×2 Gauss points. The time integration is performed with the default FD implicit scheme ($\theta = 1$) and a constant time step. In the TSSS, let $L = 4$ and $N = 20$. In all modeling, the plane strain condition is assumed.

4.1. Scheme verification

4.1.1. One-dimensional problem

In this problem, the saturated poroelastic medium with a thickness of $H = 10$ m is numerically analyzed, as shown in Fig. 2. The downward direction is positive for the z -axis. The FEM mesh consists of a single column of regular Q4P4 elements (i.e. the element size $h = 1$ m), which are restrained in the horizontal direction but can displace vertically. For a time-step load on the upper boundary, the distributed load of $\sigma_0 = 10^4$ Pa is applied during a time period of $t_0 = 100$ s (see Fig. 2), and the ramp load remains constant thereafter. The lower boundary is rigidly fixed. The upper boundary is fully drained and the other boundaries are impermeable. Assume that both the fluid and solid skeletons are incompressible, the other hydraulic and mechanical properties are provided in Table 1. Both the initial pore pressure and the displacement are zero. The analytical solutions for pore pressure and vertical displacement had been described in [37,38],

For $t \leq t_0$,

$$p(z, t) = \sum_{m=0}^{\infty} \frac{2\sigma_0}{ct_0 M^3} \sin \frac{Mz}{H} \left[1 - \exp(-M^2 ct) \right] \quad (49)$$

$$u(z,t) = \frac{\sigma_0}{\lambda + 2G} \left\{ \frac{t}{t_0} (H - z) - \sum_{m=0}^{\infty} \frac{2\alpha H}{ct_0 M^4} \cos \frac{Mz}{H} \left[1 - \exp(-M^2 ct) \right] \right\} \quad (50)$$

and for $t \geq t_0$,

$$p(z,t) = \sum_{m=0}^{\infty} \frac{2\sigma_0}{ct_0 M^3} \sin \frac{Mz}{H} \exp(-M^2 ct) \left[\exp(M^2 ct_0) - 1 \right] \quad (51)$$

$$u(z,t) = \frac{\sigma_0}{\lambda + 2G} \left\{ H - z - \sum_{m=0}^{\infty} \frac{2\alpha H}{ct_0 M^4} \cos \frac{Mz}{H} \exp(-M^2 ct) \left[\exp(M^2 ct_0) - 1 \right] \right\} \quad (52)$$

where $M = (2m+1)\pi/2$, $c = \frac{k}{\rho_f g S H^2}$, $S = \frac{1}{Q^*} + \frac{\alpha^2}{\lambda + 2G}$ and λ is Lamé's coefficient.

As mentioned before, TSSS can provide an analytical solution for a linear load. Let the time-step size $\Delta t = 5$ s. Figs. 3 and 4 compare the numerical results and the analytical solutions at different moments. As seen from the figures, both the pressure profiles and the vertical displacements are consistent with the analytical solutions. Table 2 presents the surface settlement obtained using the numerical modeling and the analytical approach. The table indicates that the results obtained using the TSSS are exactly the same as the analytical solutions and are independent of the time-step size. In other words, the timescale effect is completely eliminated. However, the solutions using the standard FEM are affected significantly by the time-step size, and a larger size leads to a larger error. It should be noted that the results obtained using the TSSS in Table 2 are presented to only 2 digits after the decimal point. In fact, the next 10 digits after the decimal point are also the same for the numerical and analytical solutions. Table 3 presents the computer runtime for TSSS and FEM with various time-step sizes. It can be seen that the runtime for the TSSS is greater than that for the standard FEM when the time-step size is small, or the TSSS is less efficient. However, with an increasing time-step size, the runtime for the TSSS is remarkably reduced and becomes less than that for the standard FEM with a small time-step size. Furthermore, the precision is unaffected. The results demonstrate that the TSSS is suitable for problems considering long time durations and multiple timescales.

Fig. 2

Fig. 3

Fig. 4

Table 1

Table 2

Table 3**4.1.2. Two-dimensional problem**

Mandel's physical problem involves a rectangular poroelastic saturated specimen, which assumed to be infinitely long in the z direction and sandwiched at the top and bottom by two rigid, frictionless impermeable plates, lateral sides are traction-free and permeable, both loaded by a sudden constant vertical force, $2F$, at time $t = 0$, as shown in Fig. 5. The analytical form for the pressure was presented by Mandel [39]; then, Abousleiman et al. [40] extended the results to include the analytical expressions for displacement. Therefore, it is an excellent model for verifying the accuracy of a poroelasticity algorithm. Additionally, Mandel's model is fascinating from the point of view that it clearly illustrates that the FCHM can lead to unexpected behavior, which highlights the need for the poroelasticity theory in practice [18]. Because the problem is symmetrical, only one quarter of the domain is considered in the numerical model. The FE model with a side length of $a = b = 1$ m in the x and y direction is considered, and a regular arrangement of elements, such as $10 \times 10 = 100$, is used. F is set equal to 10^4 N/m. The other boundary conditions are provided in Fig. 5. The properties of the homogeneous and isotropic poroelastic media are provided in Table 1.

The numerical and analytical solutions are compared in Figs. 6-7. Fig. 6a presents the time history of the pore pressure dissipation at $x = 0.7$ m, 0.8 m and 1 m for $\Delta t = 0.05$ s, and Fig. 6b illustrates the time variation of the horizontal displacement at $x = 1$ m. It can be seen that the modeling results obtained using the TSSS are in good agreement with the analytical solutions. Fig. 7 illustrates the pore pressure profile along the x axis at different time steps under $\Delta t = 0.05$ s, which is compared with the simulation results by the standard FEM. It can be seen that the error of the standard FEM results is relatively large at locations with higher pore pressure gradients, and the TSSS results are remarkably better than the standard FEM results, especially in the initial stage of modeling. With time elapsed, the modeling results overlap the analytical solutions, which is in agreement with the previous theoretical analysis. The computation process is convergent and the computation error will not accumulate. In addition, as can be seen from this figure, at initial steps, the pore pressure at $x = 0$ first rises above its initial value and then dissipates gradually as the drainage evolves. This non-monotonic behavior of the pore pressure, known as the Mandel-Cryer effect, has been confirmed experimentally by Verruijt [41]. Finally, it should be emphasized that the TSSS results are independent of time step size and the computation error is purely caused by mesh discretization. Therefore, the TSSS can be considered as an analysis tool for the effect of mesh discretization.

Fig. 5**Fig. 6**

Fig. 7

4.2. Stability investigation

The stability of the numerical solution obtained with the TSSS is investigated in a one-dimensional heterogeneous test problem. This two layered column problem and its reference solution were presented by Ferronato et al. [17]. It consists of a fluid saturated column of height 15m with 10m of poroelastic medium A on top and 5m of poroelastic medium B on bottom as shown in Fig. 8. In this test simulation, let the permeability of poroelastic medium A be 1000 times smaller than poroelastic medium B, $k_B = 10^{-5}$ m/s and $k_A = 10^{-8}$ m/s, i.e., $k_B/k_A = 1000$ and assume that the hydraulic and mechanical properties are the same as listed in Table 1. The FE mesh consists of a single column of regular Q4P4 elements and the element size is of $h = 0.5$ m. There is no displacement at the bottom. Also, pore pressure of 9810 Pa is prescribed on the bottom boundary. The top boundary is traction-free and drainage is allowed on this boundary. The lateral boundaries are restrained in the horizontal direction but can displace vertically. The initial pore pressure and displacement are zero. The upward direction is positive for the z-axis.

Fig. 8

For comparison, the pore pressure solution is provided based on a standard FE model with the same mesh and equal order pressure and displacement approximations. Fig. 9 compares the simulation results at $t = 60$ s for time-step size of $\Delta t = 3$ s, 30s and 60s, respectively, with the reference solutions. It can be seen that, similar to the standard FE, the TSSS exhibits pronounced initial pressure oscillations at the interface between media A and B. The challenge caused by FE discretization in space domain still exists. However, the pressure profiles obtained by the standard FEM are unstable for various time-step sizes, indicating that it is very sensitive to time-step size. On the other hand, the results by TSSS are stable, consistent and close to the reference solution, and the timescale effect is completely eliminated. Fig. 10 compares the modeling results at $t = 600$ s for a time-step size of $\Delta t = 20$ s with the reference solutions. With the time elapses, the disordered distribution of pore pressure in the initial stage tends to be stable and the oscillations at the interface between media A and B gradually disappear. It should be noted that the initial errors in the pressure field for the Q4P4 element slowly dissipate with progressing time. A detailed study on this effect has been described in [42,43], and some stabilized enhanced elements have been developed to overcome the pressure oscillations [14,44,45]. Clearly, the oscillations in the early stages are unacceptable. Fig. 11 compares the modeling results at $t = 3600$ s for a time-step size of $\Delta t = 40$ s and 60s,

respectively, with the reference solution. As can be seen from this figure, the results by standard FEM are still unstable. It is noted that the Ferronato's reference solution was obtained by progressively refining both the FE mesh and the time-step until convergence [17]. The reference solution here is obtained with element size $h = 0.03125\text{m}$ and $\Delta t = 0.01\text{s}$. The results by TSSS are independent of time-step size. Even one-step calculation, i.e., $\Delta t = 3600\text{s}$ can give solutions consistent with the reference solution. For $\Delta t = 5000\text{s}$, the simulated pore pressure profiles at different time are shown in Fig. 12. The pore pressure distribution can reach the steady state within very short duration. Fig. 13 compares the simulated vertical displacements with the reference solutions at various times. The TSSS results agree reasonably well with the reference solution. It should be pointed out that the results by standard FEM are highly unstable with large errors, hence not plotted in this figure.

Fig. 9 - Fig. 13

5. Conclusions and remarks

In this paper, a novel finite element two-step solution scheme is presented for the fully coupled hydro-mechanical analysis of saturated poroelastic media. The FE configuration is used for a spatial discretization of the differential equations. Then, *pressure formulation* two-step procedure is constructed for the discretization in the time domain based on the precise integration method. Different from the conventional FD scheme, the advantages of the proposed TSSS can be summarized as follows:

- The novel *pressure formulation* is fully explicit, unconditionally stable and decoupled pressure iterative procedure.
- The numerical results of the transfer matrix can reach the computer precision. When simple loading (constant or linear) is applied, the error is entirely caused by the mesh discretization. For a time-dependent load, the loading item is expressed using integration. It should be noted that the approximation of the integration is typically better than that of the FD in numerical computation [35]. Therefore, theoretically speaking, the proposed TSSS can be applied for any timescale.
- The proposed scheme has the advantages of having a simple algorithm structure, in which the mathematical operation is easy to high-efficiency implement.

The stability and convergence of the proposed TSSS is verified by numerical experiments. The precision and stability of the modeling results is remarkably better than the standard FEM. Especially, for a larger time step size, the numerical precision of the standard FEM notably decreases, but high precision of the proposed scheme is maintained. An explicit global

coupled pressure-displacement formulation has been presented in another paper [6]. It is shown that the TSSS has its advantage in terms of relatively low space-time computational cost and decoupled pressure iterative procedure. Particularly, it is suitable for studies on the evolution of pore pressure field, where the coupling displacement field is not the concern or only required at a certain moment. In addition, the TSSS can be used as an analysis tool for studying the effect of mesh discretization on the pressure or displacement field. Accordingly, it can be used to evaluate the effect of FD scheme in the standard FEM on the discretization of time derivatives. Therefore, the TSSS can be applied for decoupled evaluation of the space-time discretization error for the differential equations solved by standard FEM. It should be emphasized that the TSSS also is applicable to 3D problems because the numerical operation is essentially the same. The proposed scheme has a great potential in engineering applications for problems involving long time durations and multiple timescales.

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The results by TSSS are independent of time-step size.

Figures

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Tables

Table 1 Hydraulic and mechanical parameters of the poroelastic medium.

Parameters	Values
Young's modulus, E	100 MPa
Poisson's ratio, ν	0.25
Porosity, n	0.375
Permeability, k	10^{-5} m/s
Biot's coefficient, α	1.0
Fluid compressibility, $1/K_f$	4.4×10^{-4} MPa ⁻¹
Fluid density, ρ_f	1000 kg/m ³

Table 2 Analytical solutions and simulation results of surface settlement (Unit: 10^{-4} m)

Time (s)	Exact solution	TSSS				FEM				
		$\Delta t = 5$	$\Delta t = 10$	$\Delta t = 20$	$\Delta t = 100$	$\Delta t = 2$	$\Delta t = 5$	$\Delta t = 10$	$\Delta t = 20$	$\Delta t = 100$
20	0.19	0.19	0.19	0.19	-	0.20	0.21	0.23	0.26	-
40	0.55	0.55	0.55	0.55	-	0.56	0.57	0.60	0.65	-
60	1.01	1.01	1.01	1.01	-	1.02	1.04	1.07	1.13	-
80	1.55	1.55	1.55	1.55	-	1.57	1.59	1.63	1.69	-
100	2.17	2.17	2.17	2.17	2.17	2.19	2.21	2.25	2.33	2.87
140	3.05	3.05	3.05	3.05	-	3.06	3.07	3.09	3.14	-
200	3.97	3.97	3.97	3.97	3.97	3.97	3.98	4.00	4.03	4.25

Table 3 The CPU run-time of numerical computation at time $t = 1200$ s (Unit: s)

Time step size	TSSS	FEM
$\Delta t = 5$	0.450	0.130
$\Delta t = 10$	0.278	0.060
$\Delta t = 20$	0.093	-
$\Delta t = 40$	0.027	-