Probabilistic speed-density relationship for pedestrian traffic

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Abstract

We propose a probabilistic modeling approach to represent the speed-density relationship of pedestrian traffic. The approach is data-driven, and it is motivated by the presence of high scatter in the raw data that we have analyzed. We show the validity of the proposed approach, and its superiority compared to deterministic approaches from the literature using a dataset collected from a real scene and another from a controlled experiment.

Keywords: speed-density relationship, probabilistic model, individual trajectories, Voronoi tessellations, statistical validation

1. Introduction

Understanding, reproducing and forecasting phenomena that characterize pedestrian traffic is necessary in order to provide services related to pedestrian safety and convenience. This becomes of utmost importance in areas of high congestion, which is a growing problem of many public spaces (transportation hubs, shopping malls, large sports and cultural events, etc.). Congestion in pedestrian facilities represents a phenomenon with a negative impact on pedestrian dynamics. It prevents pedestrians from achieving efficient movements and may lead to an increase in travel time, delays and potential collisions among pedestrians.

Data collection for pedestrian flow and behavior analysis used to be particularly cumbersome. Typically, manual counting methods (on-site or on videos) and surveys distributed to randomly selected individuals were the main sources of data. Nowadays, automatic pedestrian detection and tracking methods have evolved tremendously, allowing for more comprehensive analyses (Bauer et al., 2009).

Using a direct analogy with vehicular traffic, the main stream of the literature characterizes pedestrian traffic with three fundamental quantities, that is density (k), speed (v) and flow (q), as well as deterministic relationships among them. Density (in ped/m²) is

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the number of pedestrians present in an area at a given moment in time; speed (in m/s) may be averaged over space or over time; and flow (in ped/ms) refers to the number of pedestrians passing a cross section of an area per unit of time (Daamen, 2004). Assuming that the traffic system is at equilibrium (stationary and homogenous), the relationships between density and flow, density and speed, and flow and speed is referred to as the *fundamental diagram* (Weidmann, 1993; Daamen, 2004).

In this paper, we propose a methodology to represent the speed-density relationship of pedestrian traffic in a probabilistic way. This is motivated by the analysis of data collected from the train station in Lausanne, Switzerland, as well as data collected from a controlled experiment by the Technical University of Delft (Daamen and Hoogendoorn, 2003). The empirical analysis of these pieces of data rules out the use of a unique deterministic fundamental diagram, due to a high scatter in the data. A possible approach to capture this complex phenomenon consists in modeling explicitly the exact underlying walking process and the explicit interactions at the disaggregate level. It would allow to test various hypotheses, at the expense of the collection of a great deal of disaggregate data. We propose in this paper an alternative approach, based on an aggregate representation of the pedestrian traffic. The scatter is explicitly represented by relaxing the homogeneity assumption of the equilibrium speed-density relationship.

The structure of the paper is as follows. A review of the literature is provided in Section 2. In Section 3, we first define the variables involved in the model. We then derive the macroscopic relationships between these indicators, starting from first principles. Based on this theoretical model, we introduce a probabilistic speed-density relationship in Section 4. Section 5 presents the two case studies from Lausanne and Delft mentioned above. It reports the empirical analysis of the two case studies and emphasizes the limitations of the state of the art approaches on these concrete examples. Section 6 and Section 7 illustrate the proposed model on the two case studies. Parameter estimation, model validation and comparisons with the existing models are discussed in details. Section 8 summarizes the outcomes of the proposed methodology and suggests future research directions.

2. Literature review

This section is organized into two parts. The first focuses on models from the vehicular traffic theory, that are relevant for pedestrian as well. The second deals with models for the pedestrian traffic.

2.1. Vehicular traffic

The fundamental diagram that corresponds to stationary homogenous traffic was first introduced empirically by Greenshields et al. (1935). This study established the relationship between spacing (the inverse of density) and speed in a form of a simple linear equation. Since then there have been many empirical studies that were aimed at improving this relationship. A comprehensive review of the models proposed in this field is given in Wang et al. (2009). Some of the established deterministic empirical

relationships are listed in Table 1, where v_f is the free-flow speed, v_0 is the average travel speed in stop-and-go conditions, k_j is the jam density, k_c is the critical density, and λ , θ , θ_1 and θ_2 are parameters.

Source	Specification	Parameters
Greenshields et al. (1935)	$v(k) = v_f \left(1 - \frac{k}{k_j}\right)$	v_f, k_j
Underwood (1961)	$v(k) = v_f \exp\left(-\frac{k}{k_c}\right)$	v_f, k_c
Newell (1961)	$v(k) = v_f \left(1 - \exp\left(-\frac{\lambda}{v_f} \left(\frac{1}{k} - \frac{1}{k_j}\right)\right)\right)$	v_f, k_j, λ
Drake et al. (1967)	$v(k) = v_f \exp\left(-\theta k^2\right)$	v_f, θ
Wang et al. (2009)	$v(k) = v_0 + \frac{v_f - v_0}{(1 + \exp(\frac{k - k_c}{\theta_1}))^{\theta_2}}$	$v_0, v_f, k_c, \theta_1, \theta_2$
Units: $k[\text{veh/km}]$, $v[\text{km/h}]$		

Table 1: Deterministic fundamental relationships - vehicular traffic

Another stream of the literature establishes the fundamental relations by analyzing the behavior of microscopic car-following models under the equilibrium conditions at the aggregate level. Different car-following models result in different specifications of fundamental relations (Jabari et al., 2014)..

In several recent studies, it has been recognized that plots of speed-density or flow-density data are usually widely scattered. Wang et al. (2009) addressed this phenomenon through the probabilistic extension of the existing macroscopic relations. Treiber and Helbing (2003) explained the observed scattering by non-stationary dynamical aspects of the traffic system. In Kim and Zhang (2008) the impact of driver heterogeneity on the scatter in the fundamental relations is demonstrated. Jabari et al. (2014) also addressed this scatter through the modeling of macroscopic flow characteristics dependent on drivers' characteristics. The modeling approaches that take this observation into account are shown to explain the data better at the macroscopic level (Jabari et al., 2014).

2.2. Pedestrian traffic

In the context of pedestrian traffic, fundamental relations are usually established by fitting deterministic curves to empirical data. Both linear and nonlinear speed-density models have been proposed, as reported in Table 2, where v_f is the free flow speed, k_j the jam density, and θ and γ are parameters. The linearity of the speed-density relationship has long been questioned for both vehicular and pedestrian flows (Daamen, 2004). An alternative specification has been proposed by Tregenza (1976) where speed decreases exponentially with the increase in density, whereas Weidmann (1993) proposed the so-called Kladek-formula, with a double S-form. The exponential specifications of the relationship appeared to be better for describing the behavior of pedestrian walking speed (Cheah and Smith, 1994). In comparison to fundamental relationships from vehicular traffic, the relationship proposed by Weidmann (1993) corresponds to the model proposed by Newell (1961), while the relationship proposed by Tregenza (1976) can be regarded as the generalization of the model proposed by Underwood (1961). Rastogi et al. (2013) have

shown that the speed-density relationship of pedestrian flow on sidewalks also follows the model presented in Underwood (1961).

The proposed relationships clearly differ in terms of functional form, but also in terms of the values of their parameters and supports. For instance, jam density (the maximum density achieved under congestion) goes from 3.8 ped/m² to 10 ped/m², the reported critical density (the maximum density achievable under free flow) ranges from 1.7 ped/m² to 7 ped/m² (Seyfried et al., 2010) and the mean of the free-flow speed estimated in different studies is 1.34 m/s while its standard deviation is 0.37 m/s (Daamen, 2004). The researchers have suggested several explanations for these deviations: the cultural differences, the differences between pedestrian facilities and the effects of the environment, flow composition, measurement methods, etc. (Seyfried et al., 2010).

Source	Specification	Parameters
Older (1968) Navin and Wheeler (1969) Fruin (1971) Tanaboriboon et al. (1986) Lam et al. (1995)	$v(k) = v_f - \theta k$	v_f, θ
DiNenno (2002)	$v(k) = v_f - v_f \theta k$	v_f, θ
Tregenza (1976)	$v(k) = v_f \exp\left(-\left(\frac{k}{\theta}\right)^\gamma\right)$	v_f, γ, θ
Weidmann (1993)	$v(k) = v_f \left\{1 - \exp\left(-\gamma \left(\frac{1}{k} - \frac{1}{k_f}\right)\right)\right\}$	v_f, k_f, γ
Rastogi et al. (2013)	$v(k) = v_f \exp\left(-\frac{k}{\theta}\right)$	v_f, θ
Units: $k[\text{ped/m}^2]$, $v[\text{m/s}]$		

Table 2: Deterministic fundamental relationships - pedestrian traffic

Simulation-based fundamental relationships are predominantly obtained via cellular automaton models. For instance, Blue and Adler (1998) specified an unidirectional cellular automaton model that produces a speed-density relationship similar to the one proposed by Weidmann (1993). Few studies propose the specifications of fundamental relations derived from first principles (Flötteröd and Lämmel, 2015; Hoogendoorn et al., 2014).

The findings from several studies (Cheung and Lam, 1998; Daamen et al., 2005; Steffen and Seyfried, 2010), question the deterministic approach of the listed studies. They indeed report a significant scatter in the empirical speed-density relationship. The observed scatter is not possible to predict by the proposed deterministic models. Cheung and Lam (1998) have reported different distributions of the speed data observed for various ranges of density. In this study, speeds are less evenly distributed for lighter traffic conditions, which is explained by higher freedom that pedestrians have in controlling their movements. This indicates that in addition to density, other factors are likely to influence the speed of pedestrians. Weidmann (1993) has empirically shown that the trip purpose of pedestrians represents one of the relevant factors. According to this study free-flow speed of shopping pedestrians is 1.04 m/s, it is 1.45 m/s for commuters and 0.99 m/s for tourists. The speed of pedestrians appears to be affected by the age and the gender as well. According to Bowman and Vecellio (1994), the walking speed of

pedestrians who are 60 years old and older is significantly lower than for the rest of the adult population. Weidmann (1993) has reported that children (under 12 years) are not capable of attaining the same speed as adults. According to the same study, walking speed of men is found to be 1.41 m/s, whereas for women it is lower (1.27 m/s).

Existing models are not designed to capture these complex aspects. This is where our study makes a contribution. We propose a probabilistic speed-density relationship that is able to implicitly account for the heterogeneity of pedestrian flows.

3. Foundation

This section first presents the assumptions related to the quantities involved in our analysis, that is speed and density indicators at the microscopic level. We then present the derivation of the macroscopic relationships between the quantities. It serves as a basis for our modeling framework in Section 4.

3.1. Density and speed indicators

The *trajectory* of pedestrian i is a curve in space and time, that is

$$p_i(t) = (x_i(t), y_i(t), t), \quad (1)$$

where time t spans the horizon of the analysis $[t_0, t_f]$, and $x_i(t)$ and $y_i(t)$ are the coordinates of the position of pedestrian i at time t in a given system of coordinates (typically, we express time in seconds, and use an orthonormal basis for the spatial dimensions).

In practice, the pedestrian trajectory data are collected through an appropriate tracking technology (e.g. Daamen and Hoogendoorn, 2003; Alahi et al., 2011). In this case, the time is discretized and the trajectory is described as a finite collection of triples

$$p_{is} = (x_{is}, y_{is}, t_s), \quad (2)$$

where $t_s = (t_0, t_1, \dots, t_f)$ corresponds to the available sample. We assume that the position of each pedestrian is known at each time t_s of the discretization.

Different measurement methods have been proposed in the literature in order to obtain density and speed indicators from pedestrian trajectories. For a comprehensive analysis of several measurement methods and their influence on the fundamental diagram we refer to Zhang (2012). The measurement methods usually rely on a discretization scheme chosen arbitrarily, in both space and time (Seyfried et al., 2010; Daamen and Hoogendoorn, 2003). The results may be highly sensitive to minor changes of discretization (Openshaw, 1983; Liddle et al., 2011). Our aim is to be as much independent from the aggregation level as possible and to preserve the heterogeneity of pedestrian population. We rely on a data-driven measurement method inspired by the one proposed by Steffen and Seyfried (2010). This method is based on the spatial discretization that is adjusted to the data through the use of Voronoi diagrams (Okabe et al., 2000).

The Voronoi space decomposition assigns a personal region to each pedestrian i , in such a way that each point in the personal region is closer to i than to any other

pedestrian, with respect of the Euclidean distance. In the presence of sampled data defined by (2), for each $s = 0, \dots, f$ and each pedestrian i , the personal region V_{is} is defined as

$$V_{is} = \left\{ \left(\begin{array}{c} x \\ y \end{array} \right) \left\| \left(\begin{array}{c} x \\ y \end{array} \right) - \left(\begin{array}{c} x_{is} \\ y_{is} \end{array} \right) \right\|_2 \leq \left\| \left(\begin{array}{c} x \\ y \end{array} \right) - \left(\begin{array}{c} x_{js} \\ y_{js} \end{array} \right) \right\|_2, \forall j \right\}. \quad (3)$$

We assume that each point (x, y) in space is associated with a unique Voronoi region at time t_s , corresponding to the region associated with pedestrian i , that is $V(x, y, t_s) = V_{is}$. Note that if (x, y) is exactly on the border between two or more regions, the unique region associated to it has to be arbitrarily defined.

Given the space discretization specified above, the density of pedestrians at position (x, y) at time t_s is

$$k_{is} = k(x, y, t_s) = \frac{1}{|V_{is}|}, \quad (4)$$

where V_{is} is the unique Voronoi region that contains (x, y) at time t_s , and $|V_{is}|$ is the area of V_{is} . The unit is the number of pedestrians per surface unit (typically, square meter). Figure 1 illustrates the map of density values obtained through the utilization of (4) for a given time instant. We refer to Steffen and Seyfried (2010) and Nikolić et al. (2014) for detailed discussions of this approach.

Figure 1: Voronoi-based density map

The velocity of pedestrian i at time t is given by

$$\vec{v}_i(t) = v_i(t) \vec{d}_i(t), \quad (5)$$

where $\vec{d}_i(t)$ is the (normalized) direction of pedestrian i at time t and $v_i(t)$ is the magnitude of the velocity vector, or speed. If the functions $x_i(t)$ and $y_i(t)$ in (1) are differentiable in t , it is defined as

$$v_i(t) = \sqrt{\left(\frac{dx_i(t)}{dt} \right)^2 + \left(\frac{dy_i(t)}{dt} \right)^2}. \quad (6)$$

In the presence of discretized data, the speed is approximated using finite differences, for instance

$$v_{is} = \sqrt{\left(\frac{\Delta x_{is}}{\Delta t} \right)^2 + \left(\frac{\Delta y_{is}}{\Delta t} \right)^2}, \quad (7)$$

where $\Delta x_{is} = x_{i,s+1} - x_{i,s-1}$, $\Delta y_{is} = y_{i,s+1} - y_{i,s-1}$, and $\Delta t = t_{s+1} - t_{s-1}$.

This definition assumes that the direction of the flow is unique at each point in time and space. It is therefore appropriate for the analysis of speed at an individual level. The definition may not result in the desired outcome if more aggregate characteristics are of interest, in particular if pedestrians in the same area do not walk in the same direction. For instance, if half of pedestrians walk in one direction and the rest in the opposite direction, but both with the same speed, their velocity vectors would cancel out at the aggregate level.

3.2. Equilibrium speed-density relationships

In order to derive the relationship between the indicators defined in Section 3.1 at the macroscopic level, we start from first principles. Similar to the approach presented in Hoogendoorn et al. (2014), the microscopic Social Force Model (SFM) proposed by Helbing and Molnar (1995) represents the basis of the derivation. The model explains the acceleration of pedestrian i through the influence of neighboring pedestrians j

$$\vec{a}_i = \frac{\vec{v}_i^f - \vec{v}_i}{\tau_i} - C_i \sum_j \exp\left(-\frac{R_{ij}}{B_i}\right) \vec{n}_{ij} (\lambda_i + (1 - \lambda_i) \frac{1 + \cos(\phi_{ij})}{2}), \quad (8)$$

where $\vec{v}_i^f$ is the desired velocity, $\vec{v}_i$ is the current velocity, τ_i is the relaxation time (the time needed to accelerate/decelerate to the desired velocity), C_i is the interaction strength, R_{ij} is the distance between pedestrians i and j , B_i is a scaling parameter, $\vec{n}_{ij}$ is the unit vector pointing from pedestrian i to j , λ_i denotes the anisotropy parameter and it ranges from 0 to 1 ($\lambda_i = 1$ implies isotropy), and ϕ_{ij} denotes the angle between the direction of i and the position of j .

Consistently with many studies reported in the literature that are focused on the fundamental diagram (Section 2), we assume isotropic traffic conditions (identical properties in all directions). This leads to the model given as

$$a_i = \frac{v_i^f - v_i}{\tau_i} - F_i^r, \quad (9)$$

where F_i^r indicates the influence of the interaction between pedestrian i with other pedestrians j for isotropic conditions (the isotropic repulsive force). It is given as

$$F_i^r = C_i \sum_j \exp\left(-\frac{R_{ij}}{B_i}\right). \quad (10)$$

A pedestrian i experiences a greater repulsive force F_i^r as other pedestrians are closer to i . This can be also reflected through a concept of a personal space associated with a pedestrian. Voronoi diagrams provide a way to derive a personal space "belonging" to a pedestrian, based on the position of i and the positions of neighbors j (see (3)). Given that the inverse of a personal space corresponds to density (as defined in (4)), the equation (9) can be rewritten as

$$a_i = \frac{v_i^f - v_i}{\tau_i} - C_i k_i. \quad (11)$$

Under the assumption that traffic conditions are stationary ($a_i = 0$), equation (11) results in the relationship between microscopic speed and density indicators

$$v_i = v_i^f - \gamma_i k_i, \quad (12)$$

where $\gamma_i = \tau_i C_i$. To establish one-to-one speed-density relationship we further assume that pedestrian population is homogenous. This means that all the pedestrians have the same movement parameters (free-flow speed v_f and sensitivity to congestion γ). In homogenous traffic under stationarity the distances between the pedestrians are also the same, so the densities are. Under these equilibrium conditions the fundamental speed-density relationship is

$$v_e = v_e(k) = v_f - \gamma k, \quad (13)$$

where subscript 'e' denotes equilibrium. The resulting relationship correspond to the ones proposed by Older (1968), Navin and Wheeler (1969), Fruin (1971), Tanaboriboon et al. (1986) and Lam et al. (1995) (Table 2). It is also in agreement with the model of Hoogendoorn et al. (2014) for isotropic, homogenous and stationary traffic conditions.

Different SFM assumptions can lead to different fundamental relations under the equilibrium assumptions. If the closeness between pedestrians is assumed to affect the free-flow speed (e.g. the free-flow speed is an exponentially decreasing function of density), the resulting isotropic SFM can be specified as

$$a_i = \frac{v_i^f \exp(-(\frac{k_i}{\theta_i})^\gamma) - v_i}{\tau_i}, \quad (14)$$

where θ_i and γ_i are the pedestrian-specific parameters. For pedestrian traffic at equilibrium (acceleration is zero and all pedestrians possess the same movement parameters) the relationship between speed and density resulting from (14) is given as

$$v_e = v_e(k) = v_f \exp(-(\frac{k}{\theta})^\gamma). \quad (15)$$

This specification coincides with the speed-density relationship proposed by Tregenza (1976) (Table 2).

4. Methodology

We propose a probabilistic model to characterize speed-density relations. The model is derived by preserving the stationarity assumption and relaxing the homogeneity assumption of equilibrium relations derived in Section 3.2. In addition to the general modeling framework, we also present concrete suggestions for the operationalization of the derived model.

4.1. Model derivation

We assume that the speed of pedestrians is a random variable (V), such that for each density level there is a distribution of speed values rather than one deterministic value. We assume that the following properties characterize this distribution:

P_1 : The distribution of speed is continuous with positive support;

P_2 : The distribution of speed is unimodal.

The first property is in accordance with the physical characteristic of the speed, being that the speed is a continuous variable whose values cannot be negative. The second property is introduced with a purpose of maintaining the model parsimonious. Furthermore, we denote by $f_{\text{slow}}(\xi|k)$ the probability density function of the speed values lower than the equilibrium speed $v_e(k)$, and by $f_{\text{fast}}(\xi|k)$ the probability density function of the speed values greater than the equilibrium speed $v_e(k)$, both of which are conditional on density. We define the probability density function of the speed as

$$f_V(\xi|v_e(k), \theta_{\text{slow}}(k), \theta_{\text{fast}}(k)) = \begin{cases} f_{\text{slow}}(\xi|v_e(k), \theta_{\text{slow}}(k)), & \xi \leq v_e(k) \\ f_{\text{fast}}(\xi|v_e(k), \theta_{\text{fast}}(k)), & \xi \geq v_e(k), \end{cases} \quad (16)$$

where $\theta_{\text{slow}}(k)$ and $\theta_{\text{fast}}(k)$ are the density-dependent parameters that characterize slow, respectively fast, regime.

We do not give a specific formulation to individual random effects that may exist. Instead, we assume that individual random effects (i) are independent from other sources of randomness and do not correlate with any of the observed explanatory variable (e.g. density); (ii) do not affect the parameters' estimates and (iii) enter in an additive way the location parameters of the speed distribution.

For the illustration of the proposed methodology we need to specify the exact form of $v_e(k)$, $f_{\text{slow}}(\xi|v_e(k), \theta_{\text{slow}}(k))$ and $f_{\text{fast}}(\xi|v_e(k), \theta_{\text{fast}}(k))$. In principle any specification may be chosen, as long as the required properties P_1 and P_2 are satisfied. We give some operational comments and propose concrete examples below.

4.2. Operational comments

The proposed specification of the speed-density relationship (16) does not satisfy the properties of the C^1 class functions. The maximum likelihood estimation of its parameters may therefore involve complex non-smooth optimization algorithms. Also, the value of $v_e(k)$ is not observed and has to be modeled. To address these issues we adopt a mixing technique, where $v_e(k)$ is assumed to be distributed according to a parametric distribution $f_{v_e(k)}(\zeta; \theta_{v_e(k)})$ with density-dependent parameters $\theta_{v_e(k)}$.

The resulting likelihood function is

$$f_{\text{PedProb-vk}}(\xi|k; \theta_{\text{slow}}(k), \theta_{\text{fast}}(k), \theta_{v_e(k)}) = \int_{\zeta=0}^{\infty} f_V(\xi|\zeta; \theta_{\text{slow}}(k), \theta_{\text{fast}}(k)) f_{v_e(k)}(\zeta; \theta_{v_e(k)}) d\zeta, \quad (17)$$

where $f_V(\xi|\zeta; \theta_{\text{slow}}(k), \theta_{\text{fast}}(k))$ is defined by (16). The model is called *PedProb-vk*, which stands for **P**edestrian **P**robabilistic speed (**v**) - density (**k**) relationship.

4.3. Exemplary specification

We suggest a linear model for the distribution of the speed of the slow component, since it is well adjusted to impose the lower bound of the distribution at zero

$$f_{\text{slow}}(\xi|v_e(k), \alpha_k, \beta_k) = \frac{\beta_k - \alpha_k}{v_e(k)} \xi + \alpha_k, \quad (18)$$

where $0 \leq \xi \leq v_e(k)$, and $\alpha_k \geq 0$ and $\beta_k \geq 0$ are parameters dependent on k . They are such that $f_{\text{slow}}(0|k) = \alpha_k$ and $f_{\text{slow}}(v_e(k)|k) = \beta_k$. We adopt a simple specification for the parameters α_k, β_k . We assume that they depend on the density k in the following way

$$\alpha_k(a_\alpha, b_\alpha) = a_\alpha k + b_\alpha, \quad (19)$$

and

$$\beta_k(a_\beta, b_\beta) = a_\beta k + b_\beta \quad (20)$$

where $a_\alpha \geq 0$, $b_\alpha \geq 0$, $a_\beta \geq 0$ and $b_\beta \geq 0$. They are such that $\alpha_0 = b_\alpha$ and $\beta_0 = b_\beta$.

We propose an exponential model for the distribution of the speed of the fast component

$$f_{\text{fast}}(\xi|v_e(k), \beta_k, \lambda) = \exp(-\lambda\xi + \log(\beta_k) + \lambda v_e(k)), \quad (21)$$

where $\xi \geq v_e(k)$, $v_e(k)$ and β_k are defined as above, and $\lambda \geq 0$ is an additional parameter, defining the rate of the exponential distribution. The choice of the exponential specification is well suited to avoid any arbitrariness in imposing the upper bound, while preventing arbitrary high values at the same time. The normalizing constant is in this example equal to $\frac{\alpha_k + \beta_k}{2} v_e(k) + \frac{\beta_k}{\lambda}$.

Note that, the value determined by $v_e(k)$ represents the mode of the distribution ($\frac{df_V(\xi|v_e(k), \alpha_k, \beta_k, \lambda)}{d\xi} = 0$ when $\xi(k) = v_e(k)$). If $\xi = v_e(k)$, the values of $f_{\text{slow}}(\xi|k)$ and $f_{\text{fast}}(\xi|k)$ coincide, and are both equal to β_k . When $k = 0$, then $f_{\text{slow}}(0|k) = b_\alpha$, $f_{\text{slow}}(v_e(k)|k) = f_{\text{fast}}(v_e(k)|k) = b_\beta$ and $f_{\text{fast}}(\xi|k) < b_\beta$ if $\xi > v_e(k)$. When $k \rightarrow \infty$, then $f_{\text{slow}}(0|k) = f_{\text{fast}}(0|k) \rightarrow \infty$ and $f_{\text{fast}}(\xi|k) \rightarrow 0$. The parameters characterizing the distribution of each component are illustrated in Figure 2.

We use a symmetric triangular distribution for mixing defined on the interval $[\bar{v}_e(k) - \sigma(k), \bar{v}_e(k) + \sigma(k)]$, where $\bar{v}_e(k)$ is the mean of the distribution and $\sigma(k) = \frac{1}{1 + \exp(\eta)} \bar{v}_e(k)$

$$f_{v_e(k)}(\zeta; \bar{v}_e(k), \sigma(k)) = \begin{cases} \frac{\zeta - \bar{v}_e(k) + \sigma(k)}{\sigma(k)^2}, & \bar{v}_e(k) - \sigma(k) \leq \zeta \leq \bar{v}_e(k) \\ \frac{\bar{v}_e(k) + \sigma(k) - \zeta}{\sigma(k)^2}, & \bar{v}_e(k) < \zeta \leq \bar{v}_e(k) + \sigma(k) \\ 0, & \zeta < \bar{v}_e(k) - \sigma(k) \text{ or } \zeta > \bar{v}_e(k) + \sigma(k). \end{cases} \quad (22)$$

The choice of this specific distribution is motivated by the simple closed form of its probability density function and cumulative density function. The proposition for the specification of $\sigma(k)$ insures that the property P_1 is satisfied. The specification of $\bar{v}_e(k)$ is typically a deterministic speed-density relationship such as those presented in Section 3.2.

Putting everything together, the distribution of the speed in our example is given by

$$f_{\text{PedProb-vk}}(\xi|k; \alpha_k, \beta_k, \lambda, \bar{v}_e(k), \sigma(k)) = \int_{\zeta = \bar{v}_e(k) - \sigma(k)}^{\bar{v}_e(k) + \sigma(k)} f_V(\xi|\zeta; \alpha_k, \beta_k, \lambda) f_{v_e(k)}(\zeta; \bar{v}_e(k), \sigma(k)) d\zeta, \quad (23)$$

where the distributions of the slow and the fast component are defined by (18), respectively (21), and $f_{v_e(k)}(\zeta; \bar{v}_e(k), \sigma(k))$ is defined by (22). For this specification, the integral (23) does not have a closed-form, and numerical integration or Monte-Carlo simulation is required. However, other specifications of $f_{v_e(k)}(\zeta; \theta_{v_e(k)})$, $f_{\text{slow}}(\xi|v_e(k), \theta_{\text{slow}(k)})$ and $f_{\text{fast}}(\xi|v_e(k), \theta_{\text{fast}(k)})$ may lead to a closed-form formulation, as illustrated in Appendix.

Figure 2: Illustration of the model - one density level

We emphasize that the proposed specification is merely one possibility, and the framework is general and other specifications are possible (e.g. different functional forms of $v_e(k)$, $f_{v_e(k)}(\xi; \theta_{v_e(k)})$, $f_{slow}(\xi|v_e(k), \theta_{slow(k)})$ and $f_{fast}(\xi|v_e(k), \theta_{fast(k)})$, different smoothing strategies, etc). To illustrate and validate our approach, we perform next the analysis on two case studies (Section 6, Section 7).

4.4. Estimation procedure

For the estimation of the model parameters, we use quasi-maximum likelihood estimation.¹ As each pedestrian generates several piece of data, there is serial correlation among these observation. If ignored, the unobserved/ unmodeled covariance among the observations of a same individual in the estimation procedure leads to consistent but inefficient estimates (Gourieroux et al., 1984; Trognon, 1987; Wooldridge, 2010). This holds as long as the first order moment of the distribution of the endogenous variable is not affected by its higher order moments (Gourieroux et al., 1984), i.e. independence between mean and higher moments. In this case, the Rao-Cramer bound is not reached, and a block bootstrap method must be used for the calculation of the standard errors of the estimates (Hall et al., 1995; Davison and Hinkley, 1997; Davidson and MacKinnon,

¹Quasi-maximum likelihood estimation differs from full information maximum likelihood estimation in the sense that it is necessary to explicitly specify the covariance structure for the latter.

2004). It is also convenient to assume that there is no correlation across pedestrians, and that the observations of one and the same pedestrian are serially correlated due to unobserved time invariant specific effects. Bootstrap replications are thus based on the sampling over blocks, where each block contains the series of the observations of one individual. We adopt these assumptions in the case studies presented below.

5. Case studies and empirical investigation

The motivation of this research comes from the analysis of two real datasets, that we use below to illustrate and validate our approach.

5.1. Lausanne train station

The first dataset is collected in a pedestrian underpass of the train station of Lausanne, Switzerland. Figure 3 shows the layout of the studied area. It covers approximately 685 m². The underpass is frequently used especially during the morning and afternoon peak hours since it connects the exterior of the train station to the main platforms. It also acts as a connection between mostly residential south and the center of the city in the north.

Figure 3: Lausanne train station - pedestrian underpass West

To collect the raw data, a large-scale network of smart sensors has been deployed in the station. The underlying technology is based on infrared and depth sensors that detect silhouettes and track each pedestrian in the scene covered by the network. The tracking engine uses a sparsity driven framework (Alahi et al., 2011, 2014) to link detected pedestrians over the network of sensors.

It results in a dataset of 25,603 trajectories, collected during a time period between 07:00 and 08:00 on February 12, 13, 14, 15 and 18, 2013. The temporal resolution of every trajectory ranges from 10 to 25 points per second and it has been processed to obtain the position of every pedestrian in the scene at every second. The average length of the trajectories is 78 meters and the duration of a pedestrians' stay in the underpass ranges from 15 seconds to 2.2 minutes.

Note that we have selected only trajectories collected in the shaded area shown in Figure 3, referring to a corridor. The trajectories from the ramps and stairs (denoted

as P1-P9) are not considered in this study. Indeed, as explained by Daamen (2004) and Weidmann (1993), the walking behavior and, therefore, the speed-density relationship, varies with the type of infrastructure.

In the rest of the paper, we refer to this case study as the Lausanne case study.

5.2. Controlled experiment

The second set of data has been collected during a controlled experiment at the Technical University of Delft in the Netherlands (Daamen and Hoogendoorn, 2003). The individuals participating in the experiment were instructed to walk along a corridor that is 10 meters long and 4 meters wide, at a normal speed, and to pass through a bottleneck of 1 meter in width (see Figure 4, where individuals walk from right to left).

Figure 4: Narrow bottleneck experiment (Daamen and Hoogendoorn, 2003)

The scene was filmed from the top by digital cameras. The individual trajectories were extracted from the digital video sequences.

The experiment lasted about 15 minutes. A total of 1,123 trajectories were collected, where the position of each individual is available every 0.1 second. The average length of the trajectories is similar inside and upstream of the bottleneck and it is approximately 5 meters. The average travel time of the trajectories upstream of the bottleneck is 10 seconds, whereas inside the bottleneck it is lower (approximately 5 seconds).

Note that we have selected trajectories collected in the rectangular area (5 meters long and 4 meters wide) upstream of the bottleneck (Figure 4). As explained by Duives et al. (2014, Figure 4), this is where the variability is observed.

In the rest of the paper, we refer to this case study as the Delft case study.

5.3. Empirical analysis

The speed-density profiles corresponding to the Lausanne and the Delft case studies are obtained from the measurement method presented in Section 3.1. In Figure 5, each circle corresponds to one observation, that is, one pedestrian at one specific time in the horizon. The x coordinate of the circle corresponds to the density, calculated from (4), and its y coordinate corresponds to the speed calculated from (7).

Figure 5a plots 270,291 observations corresponding to the peak hour of February 12, 2013 for the Lausanne case study. The same pattern was observed on any weekday. Figure 5b plots 119,156 observations for the Delft case study.

Figure 5: Speed-density profiles

A high scattering is observed in both cases (Figure 5). The density ranges from 0 to approximately 7 pedestrians per square meter. In the Lausanne case, the speed ranges from 0 to 5.72 meters per second (that is about 21 km/h), and 99% of the observations are between 0 and 2.42 meters per second (that is about 9 km/h). In the Delft case, the speed ranges from 0 to 2.87 meters per second (that is about 10 km/h). The difference in the speed distribution is attributed to the controlled nature of the experiment in Delft, where individuals were instructed to walk at normal speed, resulting to a lower variance compared to Lausanne, where no instruction was given. For the same reason, low speeds were not observed at low density in Delft, contrarily to Lausanne.

To investigate this data in more details, the speed distributions at various density levels are presented in Figures 6 and 7 for the Lausanne and Delft case study, respectively. In both cases, a higher level of variability is noticeable at lower densities, compared to higher densities where the distribution of speed is less spread and shifted towards lower values.

The deterministic models for the speed-density relationship proposed in the literature (Section 2.2) appear to be inadequate for representing the observed patterns. Clearly, density is not the only factor influencing pedestrians' speed. We emphasize that there are different sources of uncertainty that can contribute to the observed pattern: (i) the

variability due to the nature of the system (e.g. heterogeneity of behavior, interaction of behaviors such as people walking in groups, the existence of multi-directional flows that are more chaotic) and (ii) measurement errors (e.g. discretization and measurement noise, technological issues). The Voronoi based approach is completely data-driven and designed to minimize the spatial discretization errors. Given the type of the available data, these sources are not separable from each other. Instead, the proposed probabilistic model for the speed-density relationship accounts for the whole uncertainty at once.

Figure 6: Speed distributions for different density levels - Lausanne case study

6. Case study: Lausanne

We illustrate and validate now the model on the Lausanne dataset, introduced in Section 5.1. The dataset used for the estimation consists of 1,269,393 pairwise speed-density observations corresponding to the peak hour of February 12, 13, 14, 15 and 18, 2013.

The descriptive statistics of the estimation dataset are presented in Table 3. The dataset is categorized according to six levels of service (LOS) proposed by Fruin (1971) for pedestrian facilities, labeled from A to F. Table 3 shows that the largest part of the observations falls below the LOS F. Actually, 99% of the observations are below 2.06 ped/m².

Level Of Service	Number of observations
A ($k \leq 0.31$ ped/m ²)	644546
B ($k \in (0.31 - 0.43$ ped/m ²])	174116
C ($k \in (0.43 - 0.71$ ped/m ²])	229808
D ($k \in (0.71 - 1.11$ ped/m ²])	133812
E ($k \in (1.11 - 2.17$ ped/m ²])	76725
F ($k > 2.17$ ped/m ²)	10386

Table 3: Estimation data classified according to LOS (Fruin, 1971) - Lausanne case study

The estimation results for the model (23) presented in Section 4.3 are shown in Table 4. The number of iterations (200) for the block bootstrap has been determined by computing bootstrap approximation of the mean square error of the estimates (Ross, 2013b, Section 8.3). The parameter $\bar{v}_e(k)$ is specified in the model (13).

All estimates have the expected sign and value. The results also show the low standard errors of all parameters and their statistical significance (t-test) at a usual significance level (0.05).²

The positive sign of the parameter a_α shows that α_k , that is the likelihood of low speeds, increases with density. Similarly, the positive sign of the parameter a_β shows that β_k , that is the likelihood of the mode of the speed distribution, also increases with density.

The signs and the estimated values of the parameters v_f and γ are consistent with the ones reported in the literature (see Section 2) and with the trend observed in the data. The low variance of the parameter η is as expected, since the mixing distribution (22) is introduced for the smoothing purposes.

Various alternative specifications of the model have been investigated in order to represent the speed-density relationship. The first alternative specification (denoted as the specification A), assumes the Kumaraswamy distribution of speed values (e.g. Nikolić et al. (2014)). In the second specification (denoted as the specification B), the speed

²The high significance of the parameters can also be attributed to the large number of observations used for the model estimation.

Parameter	Value	Std err (block bootstrap)	t-test
a_α	0.0393	$4.78e^{-03}$	8.52
b_α	0.00708	$8.85e^{-04}$	7.87
a_β	0.00487	$2.41e^{-03}$	2.10
b_β	0.142	$3.71e^{-03}$	38.4
λ	3.53	$1.70e^{-03}$	$2.07e^{03}$
v_f	1.29	$2.85e^{-03}$	$4.57e^{02}$
γ	0.0512	$3.85e^{-03}$	13.4
η	3.48	$1.25e^{-03}$	$2.77e^{03}$
$\log \mathcal{L}$	-783942.897		
Number of parameters	8		
Number of observations	1269393		

Table 4: Estimation results - Lausanne case study

values are assumed to be generated from the Rayleigh distribution. The third specification (denoted as the specification C), assumes the Weibull distribution of speed values. In all cases the parameters of the distributions are allowed to vary with density. Also, the alternative specification of *PedProb-vk* is considered (denoted as the specification D), where we tried for $\bar{v}_e(k)$ the specification proposed by Weidmann (1993) (Table 2). All alternative specifications resulted in a poorer fit with respect to *PedProb-vk*, when compared by the means of the Bayesian information criterion - BIC (Wasserman, 2000):

$$BIC = -2 \cdot \log \mathcal{L} + r \cdot \log(n) \quad (24)$$

where n is the number of the observations, r is the number of parameters to be estimated and $\mathcal{L}$ is the value of the likelihood function of the model. The model with the lowest BIC is preferred. Indeed, the BIC values shown in Table 5 suggest that the *PedProb-vk* represents the best compromise between the model accuracy and simplicity among all evaluated specifications.

Model specification	A	B	C	D	<i>PedProb-vk</i>
$\log \mathcal{L}$	-891878.142	-897336.414	-895420.690	-845079.532	-783942.897
# parameters	8	3	4	9	8
# observations	1269393	1269393	1269393	1269393	1269393
<i>BIC</i>	1783868.715	1794678.828	1790897.596	1690285.550	1567998.226

Table 5: Goodness of Fit (BIC) - Lausanne case study

6.1. Comparison with deterministic models

The performance of the proposed probabilistic model at the aggregate level is compared with the deterministic models proposed in the literature (Table 2). We have estimated the parameters of these models using linear regression on our dataset.

For *PedProb-vk*, the average speed is given by

$$\bar{v}_{\text{PedProb-vk}}(k) = \int_0^\infty \xi f_{\text{PedProb-vk}}(\xi|k; \alpha_k, \beta_k, \lambda, \bar{v}_e(k), \sigma(k)) d\xi. \quad (25)$$

In (25) $f_{\text{PedProb-vk}}(\xi|k; \alpha_k, \beta_k, \lambda, \bar{v}_e(k), \sigma(k))$ refers to mixture distribution given by (23), with the parameters described in Table 4.

Figure 8 shows a comparison among four deterministic models, the aggregate speed calculated from *PedProb-vk*, and the observed values. The analysis is performed for density levels ranging from 0 to 2.06 ped/m², corresponding to 99% of the observed values. The goodness of fit is reported in Table 6, where

$$MSE = \frac{1}{m} \sum_{\ell=1}^m (\bar{v}_{\text{model}}(k_\ell) - \bar{v}_{\text{data}}(k_\ell))^2, \quad (26)$$

where $m = 22$ and $k_\ell = 2.06(\ell - 1)/(m - 1)$. The value $\bar{v}_{\text{data}}(k_\ell)$ is the average of the observed speeds corresponding to densities ranging from $(k_\ell + k_{\ell-1})/2$ and $(k_\ell + k_{\ell+1})/2$. To account for the number of parameters in the model relative to the sample size we also report the adjusted coefficient of determination ($\bar{R}^2$) (Hair et al., 2009, p. 170) in Table 6. Although the *PedProb-vk* has been calibrated at the disaggregate level, it is interesting to observe that it achieves the best fit at the aggregate level.

Model	Weidmann (1993)	Tregenza (1976)	Rastogi et al. (2013)	Linear	<i>PedProb-vk</i>
<i>MSE</i>	$6.69e^{-03}$	$1.82e^{-03}$	$1.90e^{-03}$	$1.87e^{-03}$	$1.27e^{-03}$
$\bar{R}^2$	$7.67e^{-03}$	$7.31e^{-01}$	$7.18e^{-01}$	$7.22e^{-01}$	$8.12e^{-01}$

Table 6: Goodness of Fit (aggregate level) - Lausanne case study

6.2. Kolmogorov-Smirnov validation

The validation is performed by comparing the distribution functions of the estimated model, and the empirical distributions from the dataset. The analysis is carried out at different density levels. Figure 9 shows the probability density functions of the model (model pdf) and the data (empirical histogram) at the same density level. The corresponding cumulative density functions (model cdf and empirical cdf) are plotted in Figure 10. Qualitatively, the match between the two is pretty satisfactory.

For the quantitative analysis, we use the Kolmogorov-Smirnov probability distance metric (Massey, 1951)

$$D_k = \max_v |F_{\text{model}}(v|k) - F_{\text{data}}(v|k)|, \quad (27)$$

where $F_{\text{model}}(v|k)$ corresponds to the model cdf and $F_{\text{data}}(v|k)$ to the empirical cdf. This metric represents the maximum value of the absolute vertical difference between the two cumulative distribution functions. It is reported in Figure 11a.

We calculate the p -value of the Kolmogorov-Smirnov statistic using simulation (Ross, 2013a, p. 257), with 100 simulation runs. The model (23) is simulated using the rejection method (Ross, 2013b, Section 5.2) on draws from a Rayleigh distribution. The results

are shown in Figure 11b. They suggest that there is no evidence in the data to reject *PedProb-vk* at significance level 0.05 for all levels of density except maybe for the one corresponding to densities close to zero.

6.3. Specification test

In order to test the robustness of the proposed specification, we have performed the validation that consists in splitting the dataset into two subsets. The model is re-estimated on one subset and the remaining data, unused for estimation, is used for validation purposes. The procedure consisting of the following steps is repeated 100 times:

1. A sample of 80% of pairwise speed-density observations is selected using simple random selection.
2. The parameters of the model are estimated using the generated sample.
3. The Kolmogorov-Smirnov statistic D_k (27) is calculated to compare the estimated model and the data on the remaining 20% of the dataset.

In Figure 12a, we compare the value of D_k calculated on the full dataset (in dashed line) with the values calculated with the above mentioned procedure. The 100 values are summarized using a box plot at each level of density. These results are satisfactory. The specification is robust and no overfitting is detected.

To be more precise, we also calculate the p -value for each value of D_k calculated with the above mentioned procedure. For this purpose we use simulation (Ross, 2013a, p. 252), with 100 simulation runs. The box plot of the estimated p -values are shown in Figure 12b. The results do not allow to reject the hypothesis that the data and the model follow the same distribution, at a usual level of significance.

7. Case study: Delft

In this section we illustrate and validate the model on the Delft dataset, described in Section 5.2.

The dataset used for estimation consist of 119,156 pairwise speed-density values observed upstream of the bottleneck. The data has been classified according to the LOS standard of Fruin (1971), showing that now the majority of the observations corresponds to the LOS C, D, E and F (Table 7). Consequently 99% of the data is below the density value of 3.9 ped/m². This is as expected, given the existence of flow constraint (in the form of a narrow bottleneck) that in this case causes congestion upstream of the bottleneck.

The parameters of the model (23) have been estimated, where $\bar{v}_e(k)$ is specified by the model (15). The estimation results are shown in Table 8. The sign and the magnitude of the parameters are as expected. The standard errors of the parameters were computed via block bootstrap method (see Section 4.4). The results also indicate a high significance of the estimated values.

Level Of Service	Number of observations
A ($k \leq 0.31$ ped/m ²)	9288
B ($k \in (0.31 - 0.43$ ped/m ²])	6967
C ($k \in (0.43 - 0.71$ ped/m ²])	20497
D ($k \in (0.71 - 1.11$ ped/m ²])	21540
E ($k \in (1.11 - 2.17$ ped/m ²])	37114
F ($k > 2.17$ ped/m ²)	23750

Table 7: Estimation data classified according to LOS (Fruin, 1971) - Delft case study

Parameter	Value	Std err (block bootstrap)	t-test
a_α	0.165	$6.62e^{-02}$	2.08
b_α	0.244	$6.97e^{-02}$	3.11
a_β	0.166	$6.85e^{-02}$	2.49
b_β	0.965	$5.97e^{-02}$	16.4
λ	6.90	$1.41e^{-02}$	$4.89e^{02}$
v_f	1.87	$4.92e^{-02}$	38.1
θ	1.13	$6.03e^{-02}$	18.3
γ	0.545	$4.31e^{-02}$	13.1
η	3.29	$3.28e^{-02}$	$1.00e^{02}$
$\log \mathcal{L}$	-3642.944		
Number of parameters	9		
Number of observations	119156		

Table 8: Estimation results - Delft case study

The positive sign of the parameter a_α shows that α_k , that is the likelihood of low speeds, increases with density. Similarly, the positive sign of the parameter a_β shows that β_k , that is the likelihood of the mode of the speed distribution, also increases with density.

The signs and the estimated values of the parameters v_f , θ and γ of the model (15) are in accordance with the trend observed in the data.

The value of the parameter λ is higher than the one for the Lausanne case study, which is consistent with the reduced range of speed values in the data. Finally, the value of the parameter η is slightly lower than that of Lausanne case study.

7.1. Comparison with deterministic models

The performance of the proposed model is compared with the deterministic models proposed in the literature (Table 2), using the same procedure as for the Lausanne case study. The models are compared in Figure 13, and the goodness of fit measures are reported in Table 9. Again, *PedProb-vk* exhibits the best fit.

Model	Weidmann (1993)	Tregenza (1976)	Rastogi et al. (2013)	Linear	<i>PedProb-vk</i>
<i>MSE</i>	$3.79e^{-02}$	$1.93e^{-02}$	$3.37e^{-02}$	$4.56e^{-02}$	$1.69e^{-02}$
$\bar{R}^2$	$3.58e^{-01}$	$6.74e^{-01}$	$4.31e^{-01}$	$2.29e^{-01}$	$7.15e^{-01}$

Table 9: Goodness of Fit (aggregate level) - Delft case study

7.2. Kolmogorov-Smirnov validation

The agreement between the model predictions and the observations from the estimation dataset is illustrated in Figure 14 and Figure 15. The Kolmogorov-Smirnov distances (27) between the model cdf and the empirical cdf are illustrated in the Figure 16a.

The agreement between the model predictions with data appears to be less satisfactory for lower density levels, as we have fewer data with low speed at low density levels. This is an artefact of the experimental nature of the data (see discussion in Section 5.3). The quality of the fit for higher density levels (which is of greater interest for applications anyway) is satisfactory. Figure 16a shows that the smallest Kolmogorov-Smirnov distances correspond to the density levels which are characterized by the largest number of observations (Table 7).

The p -values are estimated using the procedure described in Section 6.2 and shown in Figure 16b. Again, there is no evidence in the data to reject *PedProb-vk* at significance level 0.05 for most of the density levels. The p -values less than 0.05 are observed for lower density levels, up to 0.3 ped/m², and for density levels greater than 3.5 ped/m². In the former case low p -values are caused by the experimental nature of the data (as discussed above), while in the latter a low number of observations (0.02% of the data) is insufficient to reach any conclusion.

7.3. Specification test

We test the robustness of the model specification by performing the validation using 80% of the data for estimation and the remaining 20% for validation (see Section 6.3). The Kolmogorov-Smirnov statistics for different density levels from 100 simulation runs and corresponding p -values are shown using box plot representation in Figure 17a and Figure 17b, respectively. The above results validate the model also for the Delft case study.

8. Conclusion and future work

In this paper a novel speed-density relationship for pedestrian traffic is proposed. Different from the deterministic approaches in the literature, it is a probabilistic model designed to account for the heterogeneity of speed at a given density level, as observed in the data.

Various tests on two different case studies validate the specification of the model. Moreover, the model is shown to outperform traditional deterministic models at the aggregate level as well.

The presented work has both theoretical and practical implications. It can be combined with a conservation principle in dynamic continuum and discrete models (Hughes, 2002; Hänseler et al., 2014), leading to probabilistic conservation laws for the representation of the pedestrian dynamics. This would allow the detailed analysis of the effects of heterogeneity on pedestrian flows. The suggested model may be utilized as such by practitioners for the evaluation and optimization of the level of service of pedestrian facilities. Contrasted with existing approaches, it yields a more realistic representation of the empirically observed phenomena.

Nonetheless, some aspects require further investigation.

First, the described framework as such is insufficient to explain the multi-directional nature of pedestrian flows. As further steps we will explore the possibility of addressing this issue by adapting the existing definitions of pedestrian traffic characteristics through a stream-based approach and a data-driven spatio-temporal discretization framework (Nikolić and Bierlaire, 2014).

Second, the structure of the unobservables could be further investigated (e.g. explicit definition of individual random effects - which layers do they affect and how, existence of autocorrelation patterns in the unobservables).

Third, the speed dynamics (e.g. state dependence) may be considered as well. In this case the data generating process would be different from the one assumed in this paper, and all the above statements and estimation method would need to be reconsidered. In the presence of lagged speeds and random individual effects there is correlation between unobservables and the lagged dependent variable. The current quasi-maximum likelihood estimator would lead to inconsistent estimates, while the full information maximum likelihood estimator would involve more simulation-based inference and working on the initial value problem. Therefore conditional maximum likelihood estimation using a correction as proposed by Wooldridge (2010) would be a good starting methodological point to extent the model.

Finally, we plan to combine this data driven approach with a behavioral approach, where the heterogeneity of speeds is explicitly explained, using variables such as trip purpose, walking in groups, train timetables, etc.

Appendix

The integral (17) can be simplified in some circumstances and can be solved in a closed form. Consider the following probability density function of the speed that is defined on the interval $[v_e(k) - \omega(k), v_e(k) + \omega(k)]$

$$f_V(\xi|v_e(k), \omega(k)) = \begin{cases} \frac{\xi - v_e(k) + \omega(k)}{\omega(k)^2}, & v_e(k) - \omega(k) \leq \xi \leq v_e(k) \\ \frac{v_e(k) + \omega(k) - \xi}{\omega(k)^2}, & v_e(k) \leq \xi \leq v_e(k) + \omega(k) \\ 0, & \xi < v_e(k) - \omega(k) \text{ or } \xi > v_e(k) + \omega(k). \end{cases} \quad (28)$$

If the symmetric triangular distribution (22) is the assumed mixing distribution, the

resulting probabilistic speed-density relationship is given as

$$f_{\text{PedProb-vk}}(\xi|k; \bar{v}_e(k), \omega(k), \sigma(k)) = \begin{cases} \frac{\omega(k) + \xi - \bar{v}_e(k)}{\omega^2(k)}, & \xi \leq \bar{v}_e(k) - \sigma(k) \\ \frac{\omega(k) - \frac{\sigma(k)}{3}}{\omega(k)^2} - \frac{(\xi - \bar{v}_e(k))^2}{\sigma(k)\omega(k)^2} - \frac{(\xi - \bar{v}_e(k))^3}{3\sigma(k)^2\omega(k)^2}, & \bar{v}_e(k) - \sigma(k) < \xi \leq \bar{v}_e(k) \\ \frac{\omega(k) - \frac{\sigma(k)}{3}}{\omega(k)^2} - \frac{(\xi - \bar{v}_e(k))^2}{\sigma(k)\omega(k)^2} + \frac{(\xi - \bar{v}_e(k))^3}{3\sigma(k)^2\omega(k)^2}, & \bar{v}_e(k) < \xi \leq \bar{v}_e(k) + \sigma(k) \\ \frac{\omega(k) - \xi + \bar{v}_e(k)}{\omega^2(k)}, & \xi > \bar{v}_e(k) + \sigma(k). \end{cases} \quad (29)$$

Similarly, if we assume the uniform mixing distribution defined on the interval $[\bar{v}_e(k) - \alpha(k), \bar{v}_e(k) + \alpha(k)]$

$$f_{v_e(k)}(\zeta; \bar{v}_e(k), \alpha(k)) = \begin{cases} \frac{1}{2\alpha(k)}, & \bar{v}_e(k) - \alpha(k) \leq \zeta \leq \bar{v}_e(k) + \alpha(k) \\ 0, & \zeta < \bar{v}_e(k) - \alpha(k) \text{ or } \zeta > \bar{v}_e(k) + \alpha(k), \end{cases} \quad (30)$$

the resulting probabilistic speed-density relationship is given as

$$f_{\text{PedProb-vk}}(\xi|k; \bar{v}_e(k), \omega(k), \alpha(k)) = \begin{cases} \frac{\xi + \omega(k) - \bar{v}_e(k)}{\omega(k)^2}, & \xi \leq \bar{v}_e(k) - \alpha(k) \\ -\frac{\alpha(k) - 2\omega(k)}{2\omega(k)^2} - \frac{(\xi - \bar{v}_e(k))^2}{2\alpha(k)\omega(k)^2}, & \bar{v}_e(k) - \alpha(k) < \xi \leq \bar{v}_e(k) + \alpha(k) \\ \frac{-\xi + \omega(k) + \bar{v}_e(k)}{\omega(k)^2}, & \xi > \bar{v}_e(k) + \alpha(k). \end{cases} \quad (31)$$

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Figure 7: Speed distributions for different density levels - Delft case study

Figure 8: Comparison between deterministic models predictions and the aggregated probabilistic model predictions - Lausanne case study

Figure 9: Comparison between model predictions (probability density) and empirical observations - Lausanne case study

Figure 10: Comparison between model predictions (cumulative density) and empirical observations - Lausanne case study

(a) Kolmogorov-Smirnov distance

(b) p-value of Kolmogorov-Smirnov statistic

Figure 11: Kolmogorov-Smirnov validation - Lausanne case study

(a) Kolmogorov-Smirnov statistic

(b) p-value of Kolmogorov-Smirnov statistic

Figure 12: Specification test - Lausanne case study

Figure 13: Comparison between deterministic models predictions and the aggregated probabilistic model predictions - Delft case study

Figure 14: Comparison between model predictions (probability density) and empirical observations - Delft case study

Figure 15: Comparison between model predictions (cumulative density) and empirical observations - Delft case study

(a) Kolmogorov-Smirnov distance

(b) p-value of Kolmogorov-Smirnov statistic

Figure 16: Kolmogorov-Smirnov validation - Delft case study

(a) Kolmogorov-Smirnov statistic

(b) p-value of Kolmogorov-Smirnov statistic

Figure 17: Specification test - Delft case study