Solving the Step-Tolled Bottleneck Model with General User Heterogeneity

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August 30, 2015

Abstract

Two new numerical methods, a semi-analytical method and an exact method, are proposed in this paper for solving the step-tolled user equilibrium problem for the bottleneck model with general user heterogeneity. The semi-analytical method transforms the step-tolled user equilibrium problem into a static, asymmetric traffic assignment problem with side constraints, which is formulated and solved as a variational inequality problem (VIP). Existence and uniqueness results are obtained by examining the properties of the VIP. The exact method, on the other hand, locates the equilibrium solution by enumerating all possible combinations of user departure orders. To improve the computational efficiency of the exact method, a simple heuristic is developed to reduce the number of solutions to be enumerated. Examples of up to 1000 classes are tested. The results confirm that ignoring user heterogeneity may lead to sub-optimal design of step tolls. In all experiments, the semi-analytical method is found to be more efficient than the heuristic version of the exact method by an order of magnitude, albeit they consistently produce nearly identical equilibrium solutions.

Keywords: step toll; general user heterogeneity; bottleneck model; semi-analytical method; variational inequality problem

1 Introduction

The classic morning commute analyses based on the bottleneck model (Vickrey, 1969; Hendrickson and Kocur, 1981; Smith, 1984; Daganzo, 1985; Cohen, 1987; Newell, 1987; Arnott et al., 1990,

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1993; Yang and Huang, 1997; Liu and Nie, 2011) views the rush-hour traffic pattern as a *user equilibrium* of competing commuters, who seek to minimize their own cost associated with both queuing and schedule delays by making departure time (and in some cases also route) choices. An important policy implication from these analyses is that the rush-hour congestion may be significantly reduced through congestion pricing. Vickrey (1969), for example, had shown that a time-varying toll could completely eliminate queuing delay, and thereby maximizing system efficiency. Despite its theoretical appeal, however, a continuously varying toll is rarely implemented in practice. Instead, a so-called *step toll* is often adopted in practice, which keeps toll rates constant in predefined discrete time windows (see e.g. Lindsey et al., 2012, for a recent review). Researchers have reported that travelers dislike time-varying tolls more than flat tolls (Higgins, 1994; Chu, 1999).

User heterogeneity has attracted much attention in the morning commute analysis because of its potential impacts on the user equilibrium solutions and welfare effects of demand management policies (Small, 1982; Cohen, 1987; Arnott et al., 1992, 1994; Lindsey, 2004; Small et al., 2005; van den Berg and Verhoef, 2011b; Liu and Nie, 2011; Hall, 2013; Liu et al., 2015). In our context, user heterogeneity typically consists of the desired arrival time (t^*), the value of time (α) and the values of schedule punctuality (β for early arrival and γ for late arrival). Previous studies often make assumptions about the relationship between α , β and γ to simplify the analysis. These assumptions range from the most restrictive “proportional heterogeneity” (Vickrey, 1973; Xiao et al., 2011; Arnott and Kraus, 1995; van den Berg and Verhoef, 2011a; van den Berg, 2014) or a less restrictive structure of heterogeneity that allows α and β to vary freely while assuming γ is proportional to β (Cohen, 1987; Newell, 1987; Arnott et al., 1988, 1994; van den Berg and Verhoef, 2011b; Liu and Nie, 2011; Hall, 2013). Yet, since welfare effects of congestion pricing could vary substantially with the imposed heterogeneity structure (van den Berg and Verhoef, 2011b), it is unclear how well these restricted systems would approximate the real world with general user heterogeneity. This remains an open question, for restricting the heterogeneity structure one way or another has so far been considered necessary to maintain tractability. *Numerical methods proposed in this paper address exactly this question.*¹

Recently, Liu et al. (2015) proposed a *semi-analytical method* for solving the bottleneck model with general user heterogeneity. Their approach transforms the dynamic user equilibrium (DUE) problem of the bottleneck model into an equivalent static asymmetric traffic assignment problem, which is then formulated and solved as a variational inequality problem (VIP). The first part of the paper further develops their semi-analytical approach to tackle the step tolled problem with general user heterogeneity. Unlike Liu et al. (2015), we formulate the commute cost function in the unit of travel time instead of travel cost in order to handle the capacity constraint, which ensures the number of commuters served in each step toll window does not exceed the limit imposed by the bottleneck capacity. We prove that an equivalent asymmetric traffic assignment problem (TAP) also exists for the step-tolled DUE problem and give the corresponding VIP formulation. A specialized algorithm that includes two loops is developed to solve the VIP. The outer loop iteratively determines the penalty (interpreted as extra delay) associated with the capacity constraint, and the inner loop solves an unconstrained VIP with the extra delay

¹This paper still assumes the same desired arrival time t^* for all travelers. At the end of the paper, we briefly explain how this might be relaxed.

determined from the outer loop.

While the semi-analytical approach is convenient for finding step-tolled user equilibrium solutions, it is not particularly amenable to designing optimal step toll scheme because of the difficulty of relating design parameters to the corresponding equilibrium solutions in closed form. In light of this limitation, a new numerical scheme, referred to as the *exact method* here, is developed in this paper to locate an exact solution by enumerating all possible combinations of the commuter classes' presence and departure orders within each arrival window. The underlying idea is that, once a combination is given, an equilibrium solution can be easily solved from a linear equation system, thanks to the affine property of travel cost function, and subsequently verified against the equilibrium conditions. Our focus here is twofold. The first has to do with conceptualizing the method and establishing its correctness. The second focus is concerned with the operationalization of the exact method, noting that it is impractical except for extremely small problems since the number of possible departure orders grow exponentially with the problem size (i.e., the number of users and the number of toll windows). In particular, a *heuristic method* is developed to guide the search direction for the commuter classes' presence and departure orders. Our preliminary tests suggest that the heuristic method consistently finds the same solution identified by either the exact method (when the exact method can be used), or the semi-analytical method. The proposed numerical methods are then used in numerical experiments to demonstrate their utility in analyzing step toll designs.

In what follows, Section 2 first introduces notations and presents the bottleneck model with general user heterogeneity and step toll. Sections 3 and 4 present, respectively, the semi-analytical method and the exact method. Results of numerical examples are reported in Section 5 and Section 6 concludes the paper with a summary of main findings and directions for future research.

2 Step-tolled bottleneck model with general user heterogeneity

Consider a fixed number of individuals (N) who commute from home to work through a corridor during morning rush hour. If the incoming flow rate exceeds its capacity s , a bottleneck will form and a first-in-first-out (FIFO) queue develops. Accordingly, the en-route travel time of any individual who departs from the bottleneck at time t (corresponding to the *arrival time* at the workplace) contains two parts: a waiting time $T(t)$ in the queue and a free flow travel time T_0 . Because the free flow travel time T_0 will not affect the following analysis, we assume $T_0 = 0$. It is also assumed that all commuters prefer to pass the bottleneck and arrive at work at t^* . However, due to the restriction of the bottleneck, some must depart earlier and others later. Whenever an early or late arrival occurs, a schedule cost is incurred. At user equilibrium, each commuter chooses an arrival time t at the workplace to minimize a general cost which includes travel delay $T(t)$ and schedule cost.

To model user heterogeneity, commuters are divided into n classes, each including N_i ($i = 1, \dots, n$) commuters who have identical unit cost of travel time (α_i) and unit cost of schedule delay (β_i for early arrival and γ_i for late arrival). To guarantee the existence of equilibrium, it is assumed that $\beta_i < \alpha_i, \forall i$ (Lindsey, 2004). To avoid potential degenerative solutions, we also assume $\beta_i/\alpha_i \neq \beta_j/\alpha_j$ and $\gamma_i/\alpha_i \neq \gamma_j/\alpha_j$ for any $i \neq j$. Let m^E denote the number of toll steps for commuters arriving at workplace before t^* and let m^L denote the number of toll steps for

Figure 1: Illustration of isocurves at UE (Thick lines depict the upper envelop of the isocost curves)

commuters arriving at workplace after t^* . And let $\pi_j^A (j = 1, \dots, m^A, A = E, L)$ denote the uniform toll imposed within time windows $[t^* - p_j^E, t^* - p_{j+1}^E]$ or $[t^* + p_{j+1}^L, t^* + p_j^L]$, where p_j^A is the time distance between t^* and the start (end) time of j^{th} step toll for early (late) arrival windows, and $p_{m^A+1}^A = 0 (A = E, L)$. A detailed description of the parameters and variables may be found in Appendix A.

A user equilibrium (UE) is attained if no commuter could reduce his/her commute cost by changing the departure time. The equilibrium solution may be illustrated and analyzed graphically using an *isocost curve* (e.g., Hendrickson and Kocur, 1981; Newell, 1987; Cohen, 1987; Arnott et al., 1994; Lindsey, 2004). Figure 1 shows the isocost curves in a three-class UE problem under a single-step toll scheme. Commuters may be seen as bidding a desired arrival time slot using their experienced queuing delay as the “currency”. The isocost curve of each class actually represents the class’s willingness to pay for each arrival time slot. At UE, an arrival time slot is always assigned to the class with highest willingness to pay. Hence, the isocost curve for each class of commuters changes in height so that each class obtains just enough time slots to cover their demand. Graphically, this means that commuters should always stay on the upper envelope of all the isocost curves at UE, whereas the corresponding arrival time slots in x -axis are “assigned” to them and the total length of these time slots equals N_i/s .

The isocost curve analysis corresponds to the definition of equilibrium, i.e., nobody can reduce his/her commute cost by changing the departure time. To explain it mathematically, let us use the example in Figure 1. The commute cost of class i (expressed in the unit of travel time) can be formulated as a function of commuter’s arrival time t :

$$\mu_i(t) = \begin{cases} T(t) + \frac{\beta_i}{\alpha_i}(t^* - t), & t < t^* - p_1^E & \text{untolled early arrival window} \\ T(t) + \frac{\beta_i}{\alpha_i}(t^* - t) + \frac{\pi_1^E}{\alpha_i}, & t^* - p_1^E \leq t < t^* - p_2^E & \text{first tolled early arrival window} \\ \dots & & \\ T(t) + \frac{\beta_i}{\alpha_i}(t^* - t) + \frac{\pi_{mE}^E}{\alpha_i}, & t^* - p_{mE}^E \leq t < t^* & \text{last tolled early arrival window} \\ T(t) + \frac{\gamma_i}{\alpha_i}(t - t^*) + \frac{\pi_{mL}^L}{\alpha_i}, & t^* \leq t < t^* + p_{mL}^L & \text{first tolled late arrival window} \\ \dots & & \\ T(t) + \frac{\gamma_i}{\alpha_i}(t - t^*) + \frac{\pi_1^L}{\alpha_i}, & t^* + p_2^L \leq t < t^* + p_1^L & \text{last tolled late arrival window} \\ T(t) + \frac{\gamma_i}{\alpha_i}(t - t^*), & t \geq t^* + p_1^L & \text{untolled late arrival window} \end{cases}, \quad (1)$$

The equilibrium definition implies $d\mu_i(t)/dt = 0$. Then, it gives the slope of the isocost curve:

$$\frac{dT}{dt} = \begin{cases} \frac{\beta_i}{\alpha_i}, & t < t^* & \text{early arrival} \\ -\frac{\gamma_i}{\alpha_i}, & t \geq t^* & \text{late arrival} \end{cases}. \quad (2)$$

It is well known that the primary complexity in step toll analysis comes from the discontinuity when the price of toll drops at the end of a toll window (e.g., time $t^* + p_1^L$ in Figure 1). To compensate the sudden drop in the toll charge, the first commuter in the lower-toll window must experience a higher non-toll cost than the last commuter in the higher-toll window would do at UE. This calls for additional assumptions about how commuters behave at the end of the higher-priced tolling window. Arnott et al. (1990) assume that a mass arrival (MA) occurs at the point of toll drop such that the above two critical consumers would have the same “expected cost”. The braking-induced idle (BI) assumption of Lindsey et al. (2012) implies that the bottleneck will be operated below capacity at the point of discontinuity. Both assumptions lead to substantial changes to the isocost curves: the MA assumption changes the cost functions for the commuters arriving in the mass, whereas the BI assumption changes the capacity of the time window before each toll drop (hence the commuters assigned to these time windows have to raise their isocost curves up to obtain more time slots). Consequently, they require a configuration of isocost curves different from that depicted in Figure 1, which complicates the analysis.

In contrast, Lai (1994, 2004) shows that, in the case of homogenous users, a separate-waiting (SW) assumption avoids the complexity of deviating from the simple configuration of isocost curves shown in Figure 1. The SW assumption introduces a separate waiting queue to “hold” some commuters so that those who come after them but are willing to pay a higher toll can jump in front of them. This resolves the violation of first-in-first-out principle because there are two queues. In effect, the assumption allows the aforementioned two critical commuters to have different waiting time (even though they pass the bottleneck at the same time), without changing anyone’s preference for time window or the bottleneck capacity. It can be shown (omitted here for brevity) that this nice property of the SW assumption still holds in the heterogeneous case.

We note that the above three behavioral assumptions give somewhat different UE solutions. Yet the results are expected to be relatively insensitive to them, especially if the behavioral parameters fall into realistic ranges (see e.g. Nie, 2013). Therefore, the SW assumption is adopted in what follows for its simplicity.

3 Semi-analytical method

3.1 Formulation

We proceed in this section to show that the step-tolled user equilibrium problem can be transformed into an equivalent static traffic assignment problem, along the same line as in Liu (2013) and Liu et al. (2015).

We first divide the time, according to the discrete toll rates, into $(m^E + m^L + 2)$ windows, corresponding to m^E early-arrival tolled windows, m^L late-arrival tolled windows and 2 untolled windows. By setting $\pi_0^A = 0$ and $p_0^A = \infty$, the two untolled windows are treated as tolled windows with zero tolls. Therefore, each arrival window represents a time period with a uniform toll rate either before or after t^* .

According to Arnott et al. (1994), the departure order of class i can be determined based on their time inflexibility (β_i/α_i for early arrival or γ_i/α_i for late arrival). Namely, the higher the time inflexibility, the closer the class will travel to their desired arrival time t^* . Define $E(i)$ ($L(i)$) as the class ID of the class whose $\frac{\beta}{\alpha}$ ($\frac{\gamma}{\alpha}$) value ranks at the i^{th} place in all classes in ascending order. Without loss of generality, let:

$$E(i) = i. \quad (3)$$

Hence, we have:

$$\frac{\beta_1}{\alpha_1} < \frac{\beta_2}{\alpha_2} < \dots < \frac{\beta_n}{\alpha_n}; \quad (4a)$$

$$\frac{\gamma_{L(1)}}{\alpha_{L(1)}} < \frac{\gamma_{L(2)}}{\alpha_{L(2)}} < \dots < \frac{\gamma_{L(n)}}{\alpha_{L(n)}}. \quad (4b)$$

In the semi-analytical method, isocost curves are used to impose the user equilibrium condition within each arrival window, i.e., the travel cost of each class remains identical given the constant slope derived in Equation (2). The equilibrium conditions across the arrival windows are completed by treating each arrival window as a route in the equivalent TAP, and thereby solving the equivalent TAP actually achieves the user equilibrium conditions across arrival windows.

We now formally state the equivalent traffic assignment problem as follows (see Liu (2013) for more details).

Equivalent static traffic assignment problem: Consider a network with n origin-destination pairs, each from node i to $i + 1$ corresponding to a user class i with a demand N_i . As shown in Figure 2, there are $(m^E + m^L + 2)$ routes connecting each O-D pair, representing the $(m^E + m^L + 2)$ arrival windows. Each commuter class i is assigned to the $(m^E + m^L + 2)$ arrival windows (i.e., $m^E + m^L + 2$ routes between node i and $i + 1$) based on the Wardrop principle (Wardrop, 1952). Let $N_i^{A_j}$ ($i = 1, 2, \dots, n, j = 0, 1, \dots, m^A, A = E, L$) denote the flows of class i in arrival window A_j .

An extra side constraint is needed to ensure that the number of commuters assigned to each arrival window does not exceed the bottleneck capacity. Note that the capacity of each arrival window A_j is shared by the aggregated flows of all classes assigned to that window, that is, the total flows on the A_j -th route of all O-D pairs in Figure 2 is subject to that constraint. We denote $D^{A_j}(i)$ as the *ratio rank ID* of the class whose inflexibility ranks at the i^{th} place among all classes

Figure 2: Network representation of the equivalent static traffic assignment problem

present in arrival window A_j . Hence, its true class ID should be $A(D^{A_j}(i))(A = E, L)$. For early arrival windows, $E(i) = i$, hence the true class ID can be simplified as $D^{A_j}(i)$.

Figures 3(a) and 3(b) illustrate the cases with unsaturated and saturated capacity constraints, respectively. The maximum number of travelers that can pass the bottleneck within a tolled arrival window is determined by the window length $(p_{j+1}^A - p_j^A)$ and the bottleneck capacity (s). If the capacity constraint of an arrival window A_j is saturated, all commuters in the arrival window will be subject to an identical penalty v_j^A , which is the multiplier associated with the constraint and may be interpreted as an extra delay. Hence, we have the following complementarity conditions associated with the capacity constraint:

$$\left(\sum_{i=1}^n N_i^{A_j} - (p_{j+1}^A - p_j^A)s \right) v_j^A = 0, \quad j = 0, 1, \dots, m^A, A = E, L; \quad (5a)$$

$$\sum_{i=1}^n N_i^{A_j} - (p_{j+1}^A - p_j^A)s \leq 0, \quad j = 0, 1, \dots, m^A, A = E, L; \quad (5b)$$

$$v_j^A \geq 0, \quad j = 0, 1, \dots, m^A, A = E, L; \quad (5c)$$

where $N_i^{A_j}$ is the number of commuters of class i in the arrival window A_j .

In each time window A_j and for the commuter of class i furthest from the center, its travel

(a) With Idle Time (Unconstrained)

(b) Without Idle Time (Constrained)

Figure 3: Illustration of user equilibrium in an arrival window

delay ($T_i^{A_j}$) and schedule delay ($t^* - t_i^{E_j}$ or $t_{L(i)}^{L_j} - t^*$), can be written as follows:

$$T_i^{E_j} = \sum_{k=1}^{i-1} \frac{\beta_k}{\alpha_k} \frac{N_k^{E_j}}{s} + v_j^E, \quad i = 1, 2, \dots, n, j = 0, 1, \dots, m^E; \quad (6a)$$

$$t^* - t_i^{E_j} = \sum_{k=i}^n \frac{N_k^{E_j}}{s} + p_{j+1}^E, \quad i = 1, 2, \dots, n, j = 0, 1, \dots, m^E; \quad (6b)$$

$$T_{L(i)}^{L_j} = \sum_{k=1}^{i-1} \frac{\gamma_{L(k)}}{\alpha_{L(k)}} \frac{N_{L(k)}^{L_j}}{s} + v_j^L, \quad i = 1, 2, \dots, n, j = 0, 1, \dots, m^L; \quad (6c)$$

$$t_{L(i)}^{L_j} - t^* = \sum_{k=i}^n \frac{N_{L(k)}^{L_j}}{s} + p_{j+1}^L, \quad i = 1, 2, \dots, n, j = 0, 1, \dots, m^L. \quad (6d)$$

Excluding the extra delay associated with the capacity constraints, the “actual” travel cost $c_i^{A_j}$ can be written as:

$$\begin{aligned} c_i^{E_j} &= \alpha_i T_i^{E_j} + \beta_i (t^* - t_i^{E_j}) + \pi_j^E \\ &= \alpha_i \sum_{k=1}^{i-1} \frac{\beta_k}{\alpha_k} \frac{N_k^{E_j}}{s} + \beta_i \sum_{k=i}^n \frac{N_k^{E_j}}{s} + \beta_i p_{j+1}^E + \pi_j^E, \quad i = 1, 2, \dots, n, j = 0, 1, \dots, m^E; \end{aligned} \quad (7a)$$

$$\begin{aligned} c_{L(i)}^{L_j} &= \alpha_{L(i)} T_{L(i)}^{L_j} + \gamma_{L(i)} (t_{L(i)}^{L_j} - t^*) + \pi_j^L \\ &= \alpha_{L(i)} \sum_{k=1}^{i-1} \frac{\gamma_{L(k)}}{\alpha_{L(k)}} \frac{N_{L(k)}^{L_j}}{s} + \gamma_{L(i)} \sum_{k=i}^n \frac{N_{L(k)}^{L_j}}{s} + \gamma_{L(i)} p_{j+1}^L + \pi_j^L, \quad i = 1, 2, \dots, n, j = 0, 1, \dots, m^L, \end{aligned} \quad (7b)$$

Define the actual travel cost measured in the unit of travel time as

$$C_i^{A_j} \equiv \frac{c_i^{A_j}}{\alpha_i}, \quad i = 1, 2, \dots, n, j = 0, 1, \dots, m^A, A = E, L. \quad (8)$$

Then, UE conditions across arrival windows can be represented as follows:

$$(C_i^{A_j} + v_j^A - \mu_i) N_i^{A_j} = 0, \quad (9a)$$

$$C_i^{A_j} + v_j^A - \mu_i \geq 0, \quad (9b)$$

$$N_i^{A_j} \geq 0, \quad i = 1, 2, \dots, n, j = 0, 1, \dots, m^A, A = E, L. \quad (9c)$$

That is, if class i has positive flows in arrival window A_j , its corresponding travel cost ($C_i^{A_j} + v_j^A$) must be the minimum among all windows. Moreover, the demand conservation and nonnegativity conditions are given by

$$\sum_{A \in \{E, L\}} \sum_{j=0}^{m^A} N_i^{A_j} = N_i, \quad i = 1, 2, \dots, n; \quad (10a)$$

$$N_i^{A_j} \geq 0, \quad i = 1, 2, \dots, n; j = 0, 1, \dots, m^A; A = E, L; \quad (10b)$$

Let

$$\Omega = \{\mathbf{N} | \mathbf{N} \text{ satisfies Constraints (10a) and (10b)}\}.$$

and

$$\Sigma = \{\mathbf{N} | \mathbf{N} \text{ satisfies Constraints (5b)}\}.$$

Then, the traffic assignment problem depicted in Figure 2 can be formulated as a variational inequality problem. The result below follows directly from Liu et al. (2015) (the proof is omitted here for brevity).

Proposition 1. *A route flow pattern $\mathbf{N}^* \in \Omega \cap \Sigma$ is a user equilibrium solution of the equivalent static traffic assignment problem if and only if it solves the variational inequalities problem $VIP(\Omega \cap \Sigma, \mathbf{C})$:*

$$\langle \mathbf{C}(\mathbf{N}^*)^T, \mathbf{N} - \mathbf{N}^* \rangle \geq 0 \quad (11)$$

where $\mathbf{C}$ are defined in Equation (8).

Remark: Unlike Liu et al. (2015) (which does not consider step toll), the above VIP formulation requires that the general cost be measured in the unit of travel time. This requirement is derived from the fact that the multiplier associated with the capacity constraint must be identical for all classes. Clearly, only if the multiplier itself is measured in the unit of time can this requirement be fulfilled. Accordingly, all the cost items in the complementarity conditions (9) must also be in the unit of time.

Proposition 2 (Solution existence). *There exists a solution to $VIP(\Omega \cap \Sigma, \mathbf{C})$ as defined in (11), if (1) $N_i \geq 0, \forall i$ and (2) $\beta_i < \alpha_i, \forall i$.*

Proof: Note that $\Omega \cap \Sigma$ is a nonempty polyhedral. It is nonempty because of condition (1) and the fact that not all routes are constrained; and it is a polyhedral because all constraints are affine. Thus, the solution existence easily follows from Proposition 1 of Liu et al. (2015). Note that Condition (2) is required to ensure the validity of the UE conditions within each window (e.g. Lindsey, 2004). $\square$

Proposition 3 (Solution uniqueness). *The problem $VIP(\Omega \cap \Sigma, \mathbf{C})$ defined in (11) has a unique solution if (1) there exists at least one solution and (2) $\beta_i/\alpha_i \neq \beta_j/\alpha_j, \gamma_i/\alpha_i \neq \gamma_j/\alpha_j, \forall i \neq j$.*

Proof: Note that the Jacobian matrix $\mathbf{J}$ of $\mathbf{C}$ is in the form of

$$\mathbf{J} = \begin{pmatrix} J_E & & & \\ & J_E & & \\ & & \ddots & \\ & & & J_L \\ & & & & J_L \end{pmatrix}. \quad (12)$$

There are $(m^E + 1)$ matrices J_E and $(m^L + 1)$ matrices J_L in the Jacobian matrix $\mathbf{J}$, and

$$J_E = \frac{1}{s} \begin{pmatrix} \frac{\beta_{E(1)}}{\alpha_{E(1)}} & \frac{\beta_{E(1)}}{\alpha_{E(1)}} & \dots & \frac{\beta_{E(1)}}{\alpha_{E(1)}} \\ \frac{\beta_{E(1)}}{\alpha_{E(1)}} & \frac{\beta_{E(2)}}{\alpha_{E(2)}} & \dots & \frac{\beta_{E(2)}}{\alpha_{E(2)}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\beta_{E(1)}}{\alpha_{E(1)}} & \frac{\beta_{E(2)}}{\alpha_{E(2)}} & \dots & \frac{\beta_{E(n)}}{\alpha_{E(n)}} \end{pmatrix}, J_L = \frac{1}{s} \begin{pmatrix} \frac{\gamma_{L(1)}}{\alpha_{L(1)}} & \frac{\gamma_{L(1)}}{\alpha_{L(1)}} & \dots & \frac{\gamma_{L(1)}}{\alpha_{L(1)}} \\ \frac{\gamma_{L(1)}}{\alpha_{L(1)}} & \frac{\gamma_{L(2)}}{\alpha_{L(2)}} & \dots & \frac{\gamma_{L(2)}}{\alpha_{L(2)}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\gamma_{L(1)}}{\alpha_{L(1)}} & \frac{\gamma_{L(2)}}{\alpha_{L(2)}} & \dots & \frac{\gamma_{L(n)}}{\alpha_{L(n)}} \end{pmatrix}. \quad (13)$$

The structure of $\mathbf{J}$ and the $\mathbf{C}$ is identical to that from the multi-route model in Liu et al. (2015). Hence solution uniqueness follows from Proposition 4 (Liu et al., 2015). $\square$

Remark. It is worth noting that Proposition 3 only establishes the uniqueness of the UE flow pattern. Lindsey (2004) pointed out that the UE cost μ may not be unique with the step tolls, since some classes may be able to “shift (their isocost curve) up or down slightly” without affecting other classes. This is expected since μ includes the multipliers for the capacity constraints, which may not be unique. Appendix B demonstrates this issue using a small example.

3.2 Solution algorithm

In this section, an algorithm is developed to solve the VIP formulation. To handle the capacity constraints, the algorithm consists of two loops. The inner loop solves an unconstrained UE problem with fixed multipliers associated with capacity constraints. The outer loop adjusts the multipliers according to the saturation status of the capacity constraints. We note that Liu et al. (2015) solve their problem using an algorithmic idea similar to what is adopted in our inner loop, but little details of their algorithm were reported.

In the inner loop, the unconstrained UE problem is solved based on Gauss-Seidel decomposition. The idea is to equilibrate the flows of only one class (or O-D pair as in Figure 2) at a time, while fixing the flows of all other classes. This equilibration is performed sequentially for each class and repeated until a desired convergence measure is achieved. According to Equation (8), if the multiplier and the flows of all other classes are known, the cost functions for a given class can be written as a one-dimensional linear function of flows $N_i^{A_j}$

$$\begin{aligned} C_i^{E_j} &= \frac{\beta_i}{\alpha_i} \frac{N_i^{E_j}}{s} + \sum_{k=1}^{i-1} \frac{\beta_k}{\alpha_k} \frac{N_k^{E_j}}{s} + \frac{\beta_i}{\alpha_i} \sum_{k=i+1}^n \frac{N_k^{E_j}}{s} + \frac{\beta_i}{\alpha_i} p_{j+1}^E + \frac{\pi_j^E}{\alpha_i} + v_j^E \\ &=^{def} e_i^{E_j} N_i^{E_j} + f_i^{E_j}, i = 1, 2, \dots, n, j = 0, 1, \dots, m^E; \end{aligned} \quad (14a)$$

$$\begin{aligned} C_{L(i)}^{L_j} &= \frac{\gamma_{L(i)}}{\alpha_{L(i)}} \frac{N_{L(i)}^{L_j}}{s} + \sum_{k=1}^{i-1} \frac{\gamma_{L(k)}}{\alpha_{L(k)}} \frac{N_{L(k)}^{L_j}}{s} + \frac{\gamma_{L(i)}}{\alpha_{L(i)}} \sum_{k=i+1}^n \frac{N_{L(k)}^{L_j}}{s} + \frac{\gamma_{L(i)}}{\alpha_{L(i)}} p_{j+1}^L + \frac{\pi_j^L}{\alpha_{L(i)}} + v_j^L \\ &=^{def} e_{L(i)}^{L_j} N_{L(i)}^{L_j} + f_{L(i)}^{L_j}, i = 1, 2, \dots, n, j = 0, 1, \dots, m^L; \end{aligned} \quad (14b)$$

where $e_i^{A_j}$ and $f_i^{A_j}$ are the slopes and intercepts, respectively. Such a basic traffic assignment problem with linear cost functions is easy to solve. Algorithm 1 details a procedure, which first ranks the intercepts of the cost functions, and then solves a series of linear equation systems until the UE conditions are satisfied.

After solving the unconstrained UE solution, the total flow assigned to each arrival window A_j , i.e., N^{A_j} , is compared against the capacity $s(p_A^j - p_A^{j+1})$. If N^{A_j} is larger than $s(p_A^j - p_A^{j+1})$, the associated multiplier will be adjusted upward using the procedure listed from line 12 - 19 in Algorithm 2; otherwise, the multiplier will be adjusted downward using the procedure listed from line 20 - 25 in Algorithm 2.

Algorithm 1 Inner loop algorithm: Gauss-Seidel decomposition

```

1: Output: Unconstrained UE flow solution  $\hat{\mathbf{N}}^*$ .
2: Input:
3: Multipliers associated with capacity constraints:  $v_j^A$ .
4: Initial flow solution  $\mathbf{N}^0$ 
5: Convergence criterion  $\epsilon_0$ .
6: Initialize:
7: Set inner loop counter  $k = 0$ .
8: repeat
9:   Set  $k = k + 1$ .
10:  for class ID  $i = 1$  to  $n$  do
11:    Calculate  $(e_i^{A_{j,k}}, f_i^{A_{j,k}})$  by (14) for all arrival windows using  $N_i^{A_{j,k-1}}$  and  $v_j^A$ .
12:    Sort  $(e_i^{A_{j,k}}, f_i^{A_{j,k}})$  by  $f_i^{A_{j,k}}$  in the ascending order.
13:    Initialize  $l = 0$ .
14:    repeat
15:      Set  $l = l + 1$ .
16:      Create an  $l$ -dimensional linear equation system (a restricted master problem):
17:      (1) Set the costs of the first  $l$  windows to be equal to each other ( $l - 1$  equations).
18:      (2) Set the sum of associated flows to  $N_i$  (one more equation).
19:      Solve the equation system to get the solution to the restricted problem  $N_i^{A_{j,k,l}}$ .
20:      Calculate the actual cost  $\mu_i^{k,l}$ .
21:      until  $\mu_i^{k,l}$  is smaller than the  $(l + 1)^{th}$  intercept
22:      Set  $N_i^{A_{j,k}} = N_i^{A_{j,k,l}}$ 
23:    end for
24:  until  $\|\mathbf{N}^k - \mathbf{N}^{k-1}\| < \epsilon_0$ 
25: Set  $\hat{\mathbf{N}}^* = \mathbf{N}^k$ .

```

Algorithm 2 Outer-loop algorithm

```
1: Output: UE flow solution  $\mathbf{N}^*$ .
2: Input: Convergence criterion  $\epsilon_1$ .
3: Initialize:
4: Set all multipliers  $v_j^A = 0$ .
5: Set outer loop counter  $l = 0$ .
6: Set initial flow solution  $\mathbf{N}^0 = \mathbf{0}$ .
7: repeat
8:   Set  $l = l + 1$ .
9:   Call Algorithm 1 to get unconstrained UE solution  $\hat{\mathbf{N}}^l$ .
10:  for each arrival window do
11:    Calculate the total traffic flows  $N^{A_j, l} = \sum_{i=1}^n N_i^{A_j, l}$ .
12:    if  $N^{A_j, l} > s(p_A^j - p_A^{j+1})$  then
13:      Set  $\Delta = N^{A_j, l} - s(p_A^j - p_A^{j+1})$ .
14:      Initialize  $k = 0$ .
15:      repeat
16:        Set  $k = k + 1$ , and  $\delta = \min(\Delta, N_{A(k)}^{A_j, l})$ .
17:        Update  $v_j^A = v_j^A + \frac{\beta_{E(k)}}{\alpha_{E(k)}} \frac{\delta}{s}$ , or  $v_j^A = v_j^A + \frac{\gamma_{L(k)}}{\alpha_{L(k)}} \frac{\delta}{s}$ .
18:        Update  $\Delta = \Delta - \delta$ .
19:      until  $\Delta \leq 0$ 
20:    else
21:      Set  $\Delta = s(p_A^j - p_A^{j+1}) - N^{A_j, l}$ .
22:      Find the class farthest from  $t^*$  in the window, whose ID is referred to as  $k$ .
23:      Set  $v_j^A = v_j^A - \frac{\beta_{E(k)}}{\alpha_{E(k)}} \frac{\Delta}{s}$ , or  $v_j^A = v_j^A - \frac{\gamma_{L(k)}}{\alpha_{L(k)}} \frac{\Delta}{s}$ .
24:      Set  $v_j^A = \max\{v_j^A, 0\}$ .
25:    end if
26:  end for
27: until  $\|v^l - v^{l-1}\| < \epsilon_1$ 
28: Set  $\mathbf{N}^* = \hat{\mathbf{N}}^l$ 
```

4 Exact method

In this section we describe a new method that analyzes the step-tolled bottleneck problem from a quite different perspective. Unlike the semi-analytical method discussed in the previous section, this method promises to locate the “exact solution”. Perhaps more importantly, the new method makes it possible to express the relationship between step toll parameters and the corresponding UE flows in an explicit form, which opens the door to more effective design of such toll schemes.

Let us first make a formal definition of a variable that describes the appearance of classes in the $(m^E + m^L + 2)$ arrival windows defined by the step toll scheme. For each arrival window A_j , the set of all classes that pass the bottleneck during that window is called the *Class Membership Set (CMS)* of the window, denoted as ρ_j^A . Note that $\rho_j^A \subseteq \{0, 1, 2, \dots, n\}$, where 1 to n are the indexes of real classes and 0 is a dummy user class corresponding to idle arrival time (in other words a time slot occupied by the dummy class is actually an idle slot). Note that for two untolled arrival windows, the dummy class is always present. Accordingly, we call a vector $\rho = \{\rho_0^E, \rho_1^E, \dots, \rho_{m^E}^E, \rho_{m^L}^L, \dots, \rho_1^L, \rho_0^L\}$ a *Class Membership Set Realization (CMSR)*.

The basic idea behind the exact method arises from the concept of CMSR. For any given CMSR, the class-specific flows in each arrival window can be written analytically as functions of (unknown) class-specific UE costs (see Section 4.1 for details). The vector of UE costs can then be solved from a linear equation system that simply links the class-specific flows in each arrival time to the total class flows. Finally, the UE cost vector solved from the above process must be checked to ensure that it satisfies the UE conditions implied by the CMSR (see Section 4.2 for details). If it does, then the solution obtained is confirmed to be an UE solution. Now, Proposition 2 and Proposition 3 state that there is one and only one UE solution. Since this unique UE solution must correspond to a unique CMSR, a brute-force solution method would be to enumerate all CMSRs and identify the “correct” one, i.e. satisfying all UE conditions. Because enumeration is clearly impractical, a heuristic method will be developed (see Section 4.3 for details).

4.1 Find UE costs for a given CMSR

The goal is to establish a linear relationship between the user demands (N_i) and the UE costs (μ_i) for a given CMSR. We first define $n_D^{A_j}$ as the number of classes that present in arrival window A_j . Recall that $D^{A_j}(i)$ is the ratio rank ID of the class whose time inflexibility ranks at the i^{th} place among all presenting classes in arrival window A_j .

By rearranging Equation (1) and setting $t^* = 0$, the isocost curve functions (including the dummy class 0) can be written as

$$T(t) = a_i^{A_j} t + b_i^{A_j} \quad i = 0, 1, 2, \dots, n, j = 0, 1, \dots, m^A, A = E, L, \quad (15)$$

where

$$a_i^{A_j} = \begin{cases} \frac{\beta_i}{\alpha_i} & i = 1, 2, \dots, n, j = 0, 1, \dots, m^E \\ -\frac{\gamma_i}{\alpha_i} & i = 1, 2, \dots, n, j = 0, 1, \dots, m^L \\ 0 & i = 0, j = 0, 1, \dots, m^A, A = E, L \end{cases} ; \quad (16a)$$

$$b_i^{A_j} = \begin{cases} \mu_i - \frac{\pi_j^A}{\alpha_i} & i = 1, 2, \dots, n, j = 0, 1, \dots, m^A, A = E, L \\ 0 & i = 0, j = 0, 1, \dots, m^A, A = E, L \end{cases} . \quad (16b)$$

Figure 4: Illustration of isocost curves in the arrival window E_j

where μ_i is the UE cost of class i . Note that, since the dummy class 0 will occupy any time slot for which no real user has a positive willingness to pay, their isocost curves can be simply set as $T(t) = 0$. Evidently, the time inflexibility ratio for class 0 will be ranked lowest among all classes. Hence, the ranked ID for the dummy class $A(0) = 0 (A = E, L)$.

For a given CMSR ρ , the departure order of all classes is known for every arrival window. Consequently, the flows of these classes in that window can be expressed as a function of $\mu_i, \forall i$. Figure 4 provides an illustrative example, where three classes out of five appear in window E_j for a given ρ . Clearly, the smaller is the time inflexibility ratio, the farther away from t^* will the class be located. Define $x_{A(D^{A_j(i)})}^{A_j}$ as the horizontal coordinate for the farthest-from- t^* user of the presenting class that has the i^{th} lowest time inflexibility ratio among all presenting classes, in the arrival window A_j . Then, using the isocost curves in Equation (15), they can be written as

$$x_{D^{E_j(i)}}^{E_j} = \begin{cases} -p_j^E, & i = 1 \\ -\frac{b_{D^{E_j(i-1)}}^{E_j} - b_{D^{E_j(i)}}^{E_j}}{a_{D^{E_j(i-1)}}^{E_j} - a_{D^{E_j(i)}}^{E_j}}, & i = 2, 3, \dots, n_D^{E_j} \\ -p_{j+1}^E, & i = n_D^{E_j} + 1 \end{cases}; \quad (17a)$$

$$x_{L(D^{L_j(i)})}^{L_j} = \begin{cases} p_j^L, & i = 1 \\ -\frac{b_{L(D^{L_j(i-1)})}^{L_j} - b_{L(D^{L_j(i)})}^{L_j}}{a_{L(D^{L_j(i-1)})}^{L_j} - a_{L(D^{L_j(i)})}^{L_j}}, & i = 2, 3, \dots, n_D^{L_j} \\ p_{j+1}^L, & i = n_D^{L_j} + 1 \end{cases}; \quad (17b)$$

where $x_{A(D^{A_j(1)})}^{A_j}$ and $x_{A(D^{A_j(n_D^{A_j}+1)})}^{A_j}$ ($A = E, L$) represent the boundaries of the arrival window A_j .

Once the above coordinates are located, the amount of flows for each class of commuters in A_j can be computed as

$$N_{A(D^{A_j(i)})}^{A_j} = s(x_{A(D^{A_j(i+1))}}^{A_j} - x_{A(D^{A_j(i)})}^{A_j}), i = 1, 2, \dots, n_D^{A_j}. \quad (18)$$

Note that the classes that do not appear in the window for the given ρ would have zero flows. Equations (16-18) together represent class-specific flows in each arrival window as functions of the UE travel cost vector μ . Specifically, since $b_i^{A_j}$ are linear combinations of μ , and $a_i^{A_j}$ are constant, $x_{A(D^{A_j(i)})}^{A_j}$ are in fact linear functions of μ . Finally, the demand conservation conditions of each class dictates that

$$\sum_{A \in \{E, L\}} \sum_{j=0}^{m^A} N_i^{A_j} = N_i, \quad i = 1, 2, \dots, n. \quad (19)$$

(19) is a $n \times n$ linear equation system that has μ as the unknowns. While we expect the system usually has a unique solution, the possibility of having a singular system cannot be ruled out. However, such cases needs not concern us because a CMSR corresponding to a singular (19) cannot be the UE solution and hence can be simply discarded.

4.2 Check the validity of a solution corresponding to a given CMSR

The solution given by the linear equation system (19) is valid only if it produces isocost curves consistent with the same CMSR that leads to (19). This consistency check can be performed by examining several constraints, as explained below.

First, for any class that does appear in A_j , its flow defined in (18) must be nonnegative, which leads to the following constraints:

$$x_{A(D^{A_j(i+1)})}^{A_j} \geq x_{A(D^{A_j(i)})}^{A_j}, i = 1, 2, \dots, n_D^{A_j}. \quad (20)$$

Second, for any class that does not appear in A_j , we must ensure that its isocost curve is “dominated” by those of the classes that do appear. We classify these “missing classes” into three categories.

For those whose $|a_k^A|$ (the absolute value of the slope of the isocost curve) are smaller than $|a_{A(D^{A_j(1)})}^A|$, the vertical intercept of their isocost curves at the far end of the arrival window must be lower than that of $A(D^{A_j(1)})$, i.e.

$$b_{E(D^{E_j(1)})}^{E_j} - a_{E(D^{E_j(1)})}^{E_j} p_j^E \geq b_k^{E_j} - a_k^{E_j} p_j^E, \quad k = 0, E(1), \dots, E(D^{E_j(1)} - 1); \quad (21a)$$

$$b_{L(D^{L_j(1)})}^{L_j} + a_{L(D^{L_j(1)})}^{L_j} p_j^L \geq b_k^{L_j} + a_k^{L_j} p_j^L, \quad k = 0, L(1), \dots, L(D^{L_j(1)} - 1). \quad (21b)$$

For the missing classes whose $|a_k^A|$ satisfies the following condition

$$|a_{A(D^{A_j(i)})}^A| < |a_k^A| < |a_{A(D^{A_j(i+1)})}^A|,$$

the intersection of its isocost curve with that of $A(D^{A_j}(i+1))$ must lie further away from t^* compared to the intersection between isocost curves of $A(D^{A_j}(i))$ and $A(D^{A_j}(i+1))$. Mathematically this requirement can be written as follows:

$$x_{E(D^{E_j}(i+1))}^{E_j} \left(a_{E(D^{E_j}(i))}^{E_j} - a_k^{E_j} \right) \geq b_{E(D^{E_j}(i))}^{E_j} - b_k^{E_j}, \quad i = 1, 2, \dots, n_D^{E_j} - 1, k = E(D^{E_j}(i) + 1), \dots, E(D^{E_j}(i+1) - 1); \quad (22a)$$

$$x_{L(D^{L_j}(i+1))}^{L_j} \left(a_{L(D^{L_j}(i))}^{L_j} - a_k^{L_j} \right) \leq b_{L(D^{L_j}(i))}^{L_j} - b_k^{L_j}, \quad i = 1, 2, \dots, n_D^{L_j} - 1, k = L(D^{L_j}(i) + 1), \dots, L(D^{L_j}(i+1) - 1). \quad (22b)$$

Finally, for all the missing classes whose $|a_k^A| > |a_{A(D^{A_j}(n_D^{A_j}))}^A|$, the vertical intercept of its isocost curve with the boundary of the arrival windows near to t^* must be lower than that of $A(D^{A_j}(n_D^{A_j}))$:

$$b_{E(D^{E_j}(n_D^{E_j}))}^{E_j} - a_{E(D^{E_j}(n_D^{E_j}))}^{E_j} p_{j+1}^E \geq b_k^{E_j} - a_k^{E_j} p_{j+1}^E, \quad k = E(D^{E_j}(n_D^{E_j}) + 1), \dots, n; \quad (23a)$$

$$b_{L(D^{L_j}(n_D^{L_j}))}^{L_j} + a_{L(D^{L_j}(n_D^{L_j}))}^{L_j} p_{j+1}^L \geq b_k^{L_j} + a_k^{L_j} p_{j+1}^L, \quad k = L(D^{L_j}(n_D^{L_j}) + 1), \dots, n. \quad (23b)$$

Note that the total number of constraints defined by (20-23) equals n , the number of classes. These n constraints can be easily validated once the UE cost vector μ is obtained from the linear equation system (18). If they are all satisfied, we conclude that the flow pattern associated with the CMSR is indeed the UE solution.

4.3 Solution algorithm

Having explained how to check whether or not a given CMSR satisfies all UE conditions, we proceed to develop a procedure to identify the “correct” CMSR. The most straightforward approach is to enumerate all possible CMSRs, since their number is finite. Algorithm 3 formally describes the procedure.

The complexity of the above algorithm depends on the maximum number of CMSR K . We now analyze how K is related to the size of the problem, i.e., the number of class n and the number of toll steps $(m^E + m^L)$. For the two untolled windows, there must be an idle arrival time. Hence, the dummy user class 0 will always be present, which means the size of possible CMSR for these period is 2^n . The tolled windows, on the other hand, can have up to $(n+1)$ user classes (including the dummy user). This gives a total of $2^{(n+1)} - 1$ possible combinations since at least one user class will be present. Combining all periods together, we can see that $K = 4^n(2^{(n+1)} - 1)^{(m^E + m^L)}$. Clearly, Algorithm 3 will soon become prohibitively time-consuming as n and/or $(m^E + m^L)$ increases. In fact, a quick calculation indicates that for $n = 10$ and $m^E + m^L = 2$, K will exceed one trillion.

To improve the computational efficiency, a heuristic method is developed below. The method is inspired by the premise that the solution verification step (line 8) in Algorithm 3 may be used to guide the search for the correct CMSR. To see this, first note that there is one and only one class corresponding to each of the CMSR’s flow constraints (20), (21), (22), and (23). Hence, if a constraint is violated, the appearance state of its corresponding class may need to be flipped.

Algorithm 3 Exact method by enumeration

```
1: Output: UE solution  $N^*$ 
2: Initialize:
3: Create a list of CMSRs =  $\{\rho^1, \rho^2, \dots, \rho^K\}$ , where  $K$  is the total number of possible CMSR. Set iteration index  $k = 1$  and  $F = 0$ .
4: repeat
5:   Set the linear equation system using Equations (16), (17), (18) and (19) based on  $\rho^k$ ;
6:   if the linear equation system is solvable then
7:     Solve the linear equation for the actual costs  $\mu^{k*}$ ;
8:     if  $\mu^{k*}$  satisfies all constraints (20), (21), (22), and (23) then
9:       Compute  $N^*$  from Equation (18), set  $F = 1$ .
10:    end if
11:  end if
12:  Set  $k = k + 1$ .
13: until  $k = K$  or  $F = 1$ 
```

This idea gives rise to the following procedure, which starts by choosing a predefined CMSR (e.g. one that places all commuter classes in every arrival period). In each iteration, we identify the constraint subject to the most severe violation in each arrival time, and then flip the appearance status of its corresponding class. The new generated CMSR is then tested in the next iteration until no change in CMSR is needed (which implies that the solution is at UE). It turns out that this seemingly simple idea produces exceedingly satisfactory results. Details of the heuristic method can be found in Algorithm 4.

5 Numerical examples

Our numerical experiments consist of a small example and a large example. The small example is used to demonstrate the procedures in details and explore how user heterogeneity might affect the optimal design of step tolls. The large example, which includes up to 1000 classes, is designed to test the computational efficiency of the semi-analytical and exact methods. All algorithms are coded in C++ and tested on a laptop computer with an Intel(R) Core (TM) i7-2720QM CPU (2.20-3.30 GHz) and 8GB DDR3 1333 RAM.

The total system cost used in this section is defined as the total commuter cost excluding the toll revenue, in monetary values:

$$SC = \sum_{i=1}^n [\alpha_i \mu_i N_i - \sum_{A \in \{E, L\}} \sum_{j=0}^{m^A} \pi_j^A N_i^{A_j}], \quad (24)$$

where μ_i and $N_i^{A_j}$ are class-specific UE costs and flows.

5.1 Two-class with a single-step-toll

In this example, the bottleneck has a capacity $s = 1000$ vehicle per hour (vph). There are two classes of commuters. The first class not only have a higher value of time, but also like to be on

Algorithm 4 Exact method by heuristic

```
1: Input: Maximum number of iterations  $M$ .
2: Output: UE solution  $\mathbf{N}^*$ .
3: Initialize:
4: Set iteration index  $k = 0$  and choose the initial CMSR  $\rho^k$ . Set  $F = 0$ .
5: while  $k < M$  and  $F = 0$  do
6:   Set  $k = k + 1$ .
7:   Set the linear equation system using Equations (16), (17), (18) and (19) based on  $\rho^k$ ;
8:   if the linear equation system is solvable then
9:     Solve the linear equation for the actual costs  $\mu^{k*}$ ;
10:    if  $\mu^{k*}$  satisfies all constraints (20), (21), (22), and (23) then
11:      Set  $\rho^* = \rho^k$  and  $F = 1$ 
12:    else
13:      for every arrival window do
14:        Find the constraint subject to the largest violation.
15:        Flip the appearance status of the class associated with the constraint.
16:      end for
17:      Set  $\rho^{k+1}$  as the new CMSR after all flipping operations.
18:    end if
19:  else
20:    Set  $\rho^{k+1}$  as a randomly selected CMSR (i.e., restart from a new initial solution).
21:  end if
22: end while
```

time, hence they have a low early arrival time inflexibility and high late arrival time inflexibility. The second class, on the other hand, have a lower value of time but would rather come to work late (i.e. a relatively low late arrival time inflexibility), than to be early at the workplace (i.e. a high early arrival time inflexibility). The parameters of two user classes are listed in Table 1. The last row in the table reports the average values of these parameters, which mimics the situation where the users cannot be differentiated, and hence must be treated as if there is only one aggregated class with its behavioral parameters being the average of the population.

Table 1: Travel preference and demands in the two-classes example

Class No. i	Demand N_i	Values of time			Time inflexibility	
		α_i (\$/hour)	β_i (\$/hour)	γ_i (\$/hour)	$\frac{\beta_i}{\alpha_i}$	$\frac{\gamma_i}{\alpha_i}$
1	5000	12	4	30	1/3	5/2
2	5000	8	6	10	3/4	5/4
Aggregated	10000	10	5	20	1/2	2

We first find the system optimal single step toll for the “hypothetical” aggregated case, which can be solved analytically as $(p_1^E, \pi_1^E, p_1^L, \pi_1^L) = (4, 20, 1, 20)$ (Laih, 1994). In words, the optimal toll is \$ 20, the start time of the toll would be 4 hours prior to t^* , and the end time would be 1 hour after t^* . In theory, this system optimal toll can reduce the total system cost by 25% if the users are indeed homogeneous with the aggregated parameters. In the following, this toll will then be used as the inputs for setting the two-class model. The objective is to examine its actual effect on heterogeneous users.

Appendix C explains in great details how the UE solutions are obtained using the two methods. A summary of results are reported in Table 2 and Figure 5 plots the isocost curves at UE for both the heterogeneous and homogeneous cases. First, we note that the homogeneous assump-

Table 2: Result information for the two-class with a single-step-toll example

Class Information	No-toll UE				Tolled UE				Total Cost Reduced By Toll
	Total Cost (\$)	Semi-Ana. Inner	Semi-Ana. Outer	Heu. Iter.	Total Cost (\$)	Semi-Ana. Inner	Semi-Ana. Outer	Heu. Iter.	
Two Classes	281250	4	1	2	245909	177	17	3	12.6%
Aggregated	400000	2	1	1	300000	2	1	1	25.0%

tion leads to severe overestimation in the overall system cost for both no-toll and tolled cases (by more than 30%). Second, the system optimal toll of the homogeneous case only reduces the system cost by about 12.6% when user heterogeneity is recognized, instead of 25% as in the homogeneous case. Third and perhaps most important, the welfare effect of the toll is significantly altered. In the homogenous case, it is well known that the toll will reduce congestion at nobody’s cost, i.e., all users will break even after the toll is implemented. Yet, Figure 5 (c^* reported in the plots) reveals that, with this toll both classes will be worse off (before the toll revenue is reimbursed). In fact, the second class’s cost jumps more than 20% after the toll. This toll design has such surprisingly miserable welfare effects exactly because it is derived from the aggregated data and hence is sub-optimal for the actual problem.

Figure 5: Isocost curves for the two-class example with a single-step toll

The above finding highlights the importance of recognizing user heterogeneity in the analysis and design of step tolls. We expect that a toll scheme designed with user heterogeneity in mind would outperform the system optimal toll tested here. However, how to design such a toll is beyond the scope of the present paper and shall be addressed in subsequent studies.

5.2 Experiment with larger examples

The purpose of this experiment is to explore the computational aspects of the two methods proposed in this paper. In total 16 examples are tested, each with a different number of classes and/or a toll scheme.

The bottleneck capacity s is 1600 vph; the numbers of classes vary from 10, 50, 200, to 1000. We assume that: (1) the value of time (α) is uniformly distributed from 5 to 35 dollars per hour; (2) the early arrival time inflexibility ($\frac{\beta}{\alpha}$) is uniformly distributed from 0.05 to 0.95, independent of α ; (3) the late arrival time inflexibility ($\frac{\gamma}{\alpha}$) is uniformly distributed from 1.05 to 3.95, independent of α and $\frac{\beta}{\alpha}$; and (4) the total number of travelers are about 12000, so that the length of rush hour

is about 7.5 hours. To avoid generating classes with very similar time inflexibility (which may affect the results), the sample is generated using the following procedure: (1) create α , $\frac{\beta}{\alpha}$ and $\frac{\gamma}{\alpha}$ as arithmetic sequences from the lowest value to the highest; (2) rearrange the order of β/α and γ/α as random permutations to ensure independence; (3) generate the number of travelers for each class as an independent random variable following a Normal Distribution with mean $\frac{12000}{n}$ and standard deviation $\frac{1200}{n}$.

The four step toll schemes tested in this example are: (I) no-toll scheme; (II) a single-step-toll scheme from time -3 to 1 (hour), with a \$ 20 toll (written as $(-3,1, \$20)$ hereafter for simplicity); (III) a three-step-toll scheme with step tolls $(-4,-3,\$3)$, $(-3,0,\$8)$ and $(0,1,\$5)$; (IV) a four-step-toll scheme of tolls $(-5,-2,\$10)$, $(-2,0,\$20)$, $(0,1,\$10)$, and $(1,2,\$5)$.

Table 3: Test results for the experiments with larger examples

Toll scheme I						
n	Total Cost	Semi-Analytical			Heuristic	
		Inner	Outer	Time	Iter.	Time
10	358734	202	1	0 m.sec.	7	281 m.sec.
50	442119	57	1	0 m.sec.	40	1.42 sec.
200	445029	1413	1	593 m.sec.	158	7.21 sec.
1000	440152	573	1	5.79 sec.	794	14.9 min.
Toll scheme II						
Number of Class	Total Cost	Semi-Analytical			Heuristic	
		Inner	Outer	Time	Iter.	Time
10	270261	198	1	0 m.sec.	11	452 m.sec.
50	327128	236	1	16.0 m.sec.	55	1.97 sec.
200	327727	1098	1	982 m.sec.	232	10.5 sec.
1000	321655	295	1	6.01 sec.	1117	21.2 min.
Toll scheme III						
Number of Class	Total Cost	Semi-Analytical			Heuristic	
		Inner	Outer	Time	Iter.	Time
10	280262	5746	52	47.0 m.sec.	13	530 m.sec.
50	376199	1477	48	156 m.sec.	70	2.52 sec.
200	373631	2810	54	3.06 sec.	320	15.1 sec.
1000	376136	14353	78	5.85 min.	1949	36.8 min.
Toll scheme IV						
Number of Class	Total Cost	Semi-Analytical			Heuristic	
		Inner	Outer	Time	Iter.	Time
10	271647	3093	28	31.0 m.sec.	13	530 m.sec.
50	350633	6247	33	780 m.sec.	78	2.84 sec.
200	348700	2921	35	3.90 sec.	338	15.7 sec.
1000	343034	10730	39	5.32 min.	1834	33.0 min.

Table 3 summarizes the computational performance of the two algorithms in solving the 16 examples. First, because we have used very tight convergence criterion for the semi-analytical method, we observed that both algorithms have consistently produced almost exactly same so-

lution. Such a cross examination offer a strong proof that both algorithms are correctly designed and implemented.

As expected, the results show that the computation efforts required by both methods increase super-linearly with the number of classes. The semi-analytical method consistently outperforms the heuristic method by roughly an order of magnitude. Yet, considering the sheer space that the heuristic method has to search and the primitive nature of its design, the robust performance of the heuristic method seems a success. Moreover, as mentioned before, while it may not be the fastest in finding the step-tolled UE solution, it could provide a competitive alternative to the toll design problem.

Another noteworthy finding from the above table is that the performance of the semi-analytical method seems to be more sensitive to the toll schemes. For the same 1000-class example, the semi-analytical method spent 60 times as much time on solving toll scheme IV as it spent on solving toll scheme I. For the heuristic method, this difference is a little more than 2 times. The reason for the large discrepancy is that for both toll schemes I and II, the solution does not have binding capacity constraints, which effectively reduce the number of outer iterations for the semi-analytical method to 1. Evidently, the heuristic method cannot take this advantage. In fact, when some capacity constraints are binding, the difference in computational performance between the two methods seems relatively stable (toll schemes III and IV). We note that, however, the semi-analytical method can find a coarser solution at much less computational efforts than reported here.

6 Conclusions

In this paper we developed two numerical methods for solving the step-tolled user equilibrium problem for the bottleneck model with general user heterogeneity. The exact method is based on the idea of enumerating all possible departure orders until it identifies the one that produces the correct user equilibrium solution. While this method is only of theoretical interest, its heuristic version, which navigates through the solution space using the clue provided by “the feasibility cuts”, offers a practically useful algorithm. The semi-analytical method, on the other hand, transforms the step-tolled user equilibrium problem into a static, asymmetric traffic assignment problem with side constraints, which is formulated as a variational inequality problem (VIP). This method allows us to derive existence and uniqueness results for the UE solution. Also, the special structure of the VIP makes it possible to derive very efficient assignment algorithm based on the notion of Gauss-Seidel decomposition.

These methods will enable researchers to solve the step-tolled bottleneck model with very general structure of user heterogeneity, which, to the best of our knowledge, has been an unresolved issue in the literature. Findings from our numerical experiments are summarized below.

- The proposed heuristic method, while primitive, successfully operationalizes the algorithmic idea behind the exact method. In all tested examples, it is always able to find the exact solution with a reasonable amount of computational efforts.
- The semi-analytical method is capable of producing highly precise solutions comparable to that of the exact solutions. It outperforms the heuristic method in terms of computation

time by roughly an order of magnitude.

- Ignoring user heterogeneity may severely mislead the analysis and design of step toll schemes in the bottleneck model.

A logical next step to this work is developing methods for solving the optimal step toll design problem under general heterogeneity, which is an ongoing effort. The heuristic version of the exact method proposed in this paper seems to offer a new and promising approach to the toll design problem. Other possible extensions include, but are not limited to: other dimensions of heterogeneity (such as desired arrival time), simultaneous departure time and route choices. As a final note, accommodating a discrete set of desired arrival times is relatively easy, since each desired arrival time would just divide the arrival window into more sub-windows, each corresponding to a unique arrival time and pricing structure. Consequently, such an extension would not change the analytical derivation of isocost curves as well as the nature of the formulation.

Acknowledgement

The work was partially supported by National Science Foundation under the award number CMMI-1256021, and by Singapore Ministry of Education Academic Research Fund Tier 1. We would also like to thank two anonymous reviewers and the Associate Editor, Professor Robin Lindsey, for their detailed and constructive comments on an earlier draft. The authors are responsible for the remaining errors.

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A Notation

Table 4: Description of notations used in the paper

Variable	Description	Unit	Range
Bottleneck			
s	capacity of the bottleneck	veh./time	0
T_0	free flow travel time	time	
T	waiting time at the bottleneck	time	
Commuter			
n	number of user classes	-	0
t^*	desired arrival time	-	
N_i	demand for class i	vehicle	$i = 1, 2, \dots, n$
$N = \sum_{i=1}^n N_i$	total demand	vehicle	
α_i	unit cost of travel time of class i	\$/unit time	$i = 1, 2, \dots, n$
β_i	unit cost early arrival of class i	\$/unit time	$i = 1, 2, \dots, n$
γ_i	unit cost of late arrival of class i	\$/unit time	$i = 1, 2, \dots, n$
Step toll			
A	Arrival periods	-	E (early), L (late)
m^A	number of toll steps in arrival period A	-	$A = E, L$
A_j	j th arrival window in arrival period A	-	$j = 1, \dots, m^A, A = E, L$
p_j^A	Time distance between the far end of the arrival window A_j and t^*	-	$j = 1, \dots, m^A, A = E, L$
π_j^A	toll in the arrival window A_j	\$	$j = 0, 1, \dots, m^A, A = E, L$
Other			
$E(i)$	class ID of the class whose $\frac{\beta}{\alpha}$ value ranks at the i^{th} place in all classes	-	$i = 1, 2, \dots, n$
$L(i)$	class ID of the class whose $\frac{\gamma}{\alpha}$ value ranks at the i^{th} place in all classes	-	$i = 1, 2, \dots, n$
$A(i)$	Generalization of $E(i)$ and $L(i)$	-	$i = 1, 2, \dots, n, A = E, L$
$n_D^{A_j}$	number of classes present in the arrival window A_j	-	$j = 1, \dots, m^A, A = E, L$
$D^{A_j}(i)$	ratio rank ID of the class whose time inflexibility ranks at the i^{th} place among all classes present in arrival period A_j	-	$i = 1, 2, \dots, n_D^{A_j}, j = 1, \dots, m^A, A = E, L$
$N_i^{A_j}$	number of commuters of class i in the arrival window A_j	vehicle	$i = 1, 2, \dots, n, j = 0, 1, \dots, m^A, A = E, L$
$\mathbf{N}$	vector of all $N_i^{A_j}$	vehicle	$i = 1, 2, \dots, n, j = 0, 1, \dots, m^A, A = E, L$
$C_i^{A_j} (c_i^{A_j})$	actual commute cost of class i in the arrival window A_j , measured in travel time (monetary value)	time (\$)	
$\mathbf{C} (\mathbf{c})$	vector of all $C_i^{A_j}$, measured in travel time (monetary value)	time (\$)	$i = 1, 2, \dots, n$
μ_i	UE cost of class i , in travel time	time	
μ	vector of UE cost	time	$j = 1, \dots, m^A, A = E, L$
v_j^A	multiplier associated with the capacity constraint for the arrival window A_j		
$\rho_i^{A_j}$	class membership set realization (indicating whether class i is present in A_j)		$i = 1, 2, \dots, n, j = 0, 1, \dots, m^A, A = E, L$
ρ	vector of $\rho_i^{A_j}$

Figure 6: Illustration of non-uniqueness in UE cost

B Example of non-unique UE cost

In this example, the bottleneck capacity $s = 1000(\text{veh}/\text{hour})$ and a single step toll of \$20 is applied for an arrival window $[-4, 0.5]$, where $t^* = 0$. The details of the classes and the UE cost solutions are given in Table 5. The reader can verify that for class 1, the UE cost μ_1 range between 2 and 2.5 (hour). However, there is a unique flow pattern (hence a unique CMSR) corresponding to all those different UE costs. Two extreme solutions (corresponding to $\mu_1 = 2, 2.5$) are shown in Figure 6. It is clear from the figure that the difference is derived from the multipliers associated with the arrival window, which can range between 0 to 0.5 (hour).

Table 5: Class information and UE cost

Class No. i	Demand N_i	Values of Time			Time Inflexibility		UE Cost μ_i^* (hour)
		α_i (\$/hour)	β_i (\$/hour)	γ_i (\$/hour)	$\frac{\beta_i}{\alpha_i}$	$\frac{\gamma_i}{\alpha_i}$	
1	4500	20	5	40	1/4	2	[2, 2.5]
2	5500	2	1	6	1/2	3	4.5

C Details of the two-class example

C.1 Semi-analytical method

For this problem, we have eight flows (four for each user class) and two capacity multipliers (one for each tolled window) to balance. From Equation (8), we further write the costs as

$$\begin{aligned} C_1^{E_0} &= \frac{1}{3000}N_1^{E_0} + \frac{1}{3000}N_2^{E_0} + \frac{4}{3}, & C_2^{E_0} &= \frac{1}{3000}N_1^{E_0} + \frac{3}{4000}N_2^{E_0} + 3, \\ C_1^{E_1} + v_1^E &= \frac{1}{3000}N_1^{E_1} + \frac{1}{3000}N_2^{E_1} + \frac{5}{3} + v_1^E, & C_2^{E_1} + v_1^E &= \frac{1}{3000}N_1^{E_1} + \frac{3}{4000}N_2^{E_1} + \frac{5}{2} + v_1^E, \\ C_1^{L_1} + v_1^L &= \frac{5}{2000}N_1^{L_1} + \frac{5}{4000}N_2^{L_1} + \frac{5}{3} + v_1^L, & C_2^{L_1} + v_1^L &= \frac{5}{4000}N_1^{L_1} + \frac{5}{4000}N_2^{L_1} + \frac{5}{2} + v_1^L, \\ C_1^{L_0} &= \frac{5}{2000}N_1^{L_0} + \frac{5}{4000}N_2^{L_0} + \frac{5}{2}, & C_2^{L_0} &= \frac{5}{4000}N_1^{L_0} + \frac{5}{4000}N_2^{L_0} + \frac{5}{4}. \end{aligned}$$

According to Equation (5b), the capacity constraints are

$$\begin{aligned} N_1^{E_1} + N_2^{E_1} &\leq 4000, \\ N_1^{L_1} + N_2^{L_1} &\leq 1000. \end{aligned}$$

For the first outer iteration and its first inner iteration, $\mathbf{v} = \mathbf{0}$ and $\mathbf{N} = \mathbf{0}$. For Class 1:

$$C_1^{E_0} = \frac{1}{3000}N_1^{E_0} + \frac{4}{3}, \quad C_1^{E_1} + v_1^E = \frac{1}{3000}N_1^{E_1} + \frac{5}{3}, \quad C_1^{L_1} + v_1^L = \frac{5}{2000}N_1^{L_1} + \frac{5}{3}, \quad C_1^{L_0} = \frac{5}{2000}N_1^{L_0} + \frac{5}{2};$$

and

$$N_1^{E_0} + N_1^{E_1} + N_1^{L_1} + N_1^{L_0} = 5000.$$

Hence, by sorting the intercepts, we find that the flows should be assigned according to the following order: E_0 , E_1 , L_1 , and L_0 . The equilibrated flows are

$$N_1^{E_0} = 2875, \quad N_1^{E_1} = 1875, \quad N_1^{L_1} = 250, \quad N_1^{L_0} = 0.$$

Fixing the flows of Class 1, we have now for class 2:

$$C_2^{E_0} = \frac{3}{4000}N_2^{E_0} + \frac{95}{24}, \quad C_2^{E_1} + v_1^E = \frac{3}{4000}N_2^{E_1} + \frac{25}{8}, \quad C_2^{L_1} + v_1^L = \frac{5}{4000}N_2^{L_1} + \frac{45}{16}, \quad C_2^{L_0} = \frac{5}{4000}N_2^{L_0} + \frac{5}{4};$$

and

$$N_2^{E_0} + N_2^{E_1} + N_2^{L_1} + N_2^{L_0} = 5000.$$

This time, the flows should be assigned in the order of L_0 , L_1 , E_1 , and E_0 . The equilibrated flows are

$$N_2^{E_0} = \frac{18125}{72}, \quad N_2^{E_1} = \frac{98125}{72}, \quad N_2^{L_1} = \frac{25625}{24}, \quad N_2^{L_0} = \frac{55625}{24}.$$

By now, the first inner iteration is completed. After repeating inner iterations 17 times, the algorithm converges to the unconstrained UE flow solution corresponding to $\mathbf{v} = 0$:

$$\begin{aligned} N_1^{E_0} &= 3758.62, & N_1^{E_1} &= 1241.38, & N_1^{L_1} &= 0, & N_1^{L_0} &= 0; \\ N_2^{E_0} &= 0, & N_2^{E_1} &= 1517.24, & N_2^{L_1} &= 1241.38, & N_2^{L_0} &= 2241.38. \end{aligned}$$

Now, with this unconstrained UE flow solution, the algorithm starts to modify the capacity multipliers in the outer loop. First, by checking the capacity constraints, we have

$$\begin{aligned} N_1^{E_1} + N_2^{E_1} &= 2758.62 \leq 4000, \\ N_1^{L_1} + N_2^{L_1} &= 1241.38 > 1000. \end{aligned}$$

Hence, the multiplier v_1^E should be decreased while v_1^L should be increased. Since v_1^E is already zero, no changes is needed. For the arrival window L_1 , $\Delta = 241.38$, and Class 2 with a late time inflexibility of $\frac{5}{4}$ should be penalized. Hence, the increase amount in v_1^L should be $(\frac{241.38}{1000} \times \frac{5}{4}) = 0.30173$. Therefore, after the first outer iteration, the values of multipliers should be

$$v_1^E = 0, \quad v_1^L = 0.30173.$$

The algorithm then resumes the inner loop with the new $\mathbf{v}$, until both the multipliers and flows converge. The reader can verify that the converged flow solution is

$$\begin{aligned} N_1^{E_0} &= 3840.91, & N_1^{E_1} &= 1159.09, & N_1^{L_1} &= 0, & N_1^{L_0} &= 0; \\ N_2^{E_0} &= 0, & N_2^{E_1} &= 1681.82, & N_2^{L_1} &= 1000.00, & N_2^{L_0} &= 2318.18. \end{aligned}$$

C.2 Exact method

First, it is easy to verify that even in such a simple example, the number of total CMSRs is 784, which makes it impossible to demonstrate the whole process of exact method by enumeration. Since for exact method, the processes of solving linear equation systems are the same for both enumeration algorithm and heuristic algorithm, we will only illustrate how heuristic algorithm solves the step-tolled UE problem with two classes.

The isocost curves for user classes can be written as

$$T_0(t) = 0; T_1(t) = \begin{cases} \frac{1}{3}t + \mu_1, & t < -4 \\ \frac{1}{3}t + \mu_1 - \frac{5}{3}, & -4 \leq t < 0 \\ -\frac{5}{2}t + \mu_1 - \frac{5}{3}, & 0 \leq t < 1 \\ -\frac{5}{2}t + \mu_1, & t \geq 1 \end{cases}; T_2(t) = \begin{cases} \frac{3}{4}t + \mu_2, & t < -4 \\ \frac{3}{4}t + \mu_2 - \frac{5}{2}, & -4 \leq t < 0 \\ -\frac{5}{4}t + \mu_2 - \frac{5}{2}, & 0 \leq t < 1 \\ -\frac{5}{4}t + \mu_2, & t \geq 1 \end{cases}.$$

The starting CMSR assumes all classes are present, which is

$$\rho^0 = \{\rho^{E_0}, \rho^{E_1}, \rho^{L_1}, \rho^{L_0}\} = \{(0, 1, 2), (0, 1, 2), (0, 1, 2), (0, 1, 2)\}.$$

With this ρ^0 given, from Equation (17), we can get the horizontal coordinates for all presenting

classes

$$\begin{array}{llll}
x_0^{E_0} = -\infty, & x_1^{E_0} = -3\mu_1, & x_2^{E_0} = \frac{12}{5}\mu_1 - \frac{12}{5}\mu_2, & x_3^{E_0} = -4, \\
x_0^{E_1} = -4, & x_1^{E_1} = -3\mu_1 + 5, & x_2^{E_1} = \frac{12}{5}\mu_1 - \frac{12}{5}\mu_2 + 2, & x_3^{E_1} = 0, \\
x_0^{L_1} = 1, & x_1^{L_1} = \frac{4}{5}\mu_1 - \frac{4}{5}\mu_2 + \frac{2}{3}, & x_2^{L_1} = \frac{4}{5}\mu_2 - 2, & x_3^{L_1} = 0, \\
x_0^{L_0} = +\infty, & x_1^{L_0} = \frac{4}{5}\mu_1 - \frac{4}{5}\mu_2, & x_2^{L_0} = \frac{4}{5}\mu_2, & x_3^{L_0} = 1.
\end{array}$$

Hence, from Equation (18), the flows of real classes in each window are

$$\begin{array}{ll}
\frac{N_1^{E_0}}{1000} = \frac{27}{5}\mu_1 - \frac{12}{5}\mu_2, & \frac{N_2^{E_0}}{1000} = -\frac{12}{5}\mu_1 + \frac{12}{5}\mu_2 - 4, \\
\frac{N_1^{E_1}}{1000} = \frac{27}{5}\mu_1 - \frac{12}{5}\mu_2 - 3, & \frac{N_2^{E_1}}{1000} = -\frac{12}{5}\mu_1 + \frac{12}{5}\mu_2 - 2, \\
\frac{N_1^{L_1}}{1000} = \frac{4}{5}\mu_1 - \frac{4}{5}\mu_2 + \frac{2}{3}, & \frac{N_2^{L_1}}{1000} = -\frac{4}{5}\mu_1 + \frac{8}{5}\mu_2 - \frac{8}{3}, \\
\frac{N_1^{L_0}}{1000} = \frac{4}{5}\mu_1 - \frac{4}{5}\mu_2 - 1, & \frac{N_2^{L_0}}{1000} = -\frac{4}{5}\mu_1 + \frac{8}{5}\mu_2.
\end{array}$$

By Equation (19), we can set up the 2-dimensional linear equation system of μ as

$$\begin{array}{l}
\frac{62}{5}\mu_1 - \frac{32}{5}\mu_2 - \frac{10}{3} = 5, \\
-\frac{32}{5}\mu_1 + 8\mu_2 - \frac{26}{3} = 5.
\end{array}$$

Hence, for ρ^0 , the general costs in travel delay should be $(\mu_1, \mu_2) = (2.64652, 3.82555)$.

We now examine this solution against constraints (20-23), i.e.,

$$\begin{array}{lll}
\text{For } E_0 : & -3\mu_1 + \infty \geq 0, & \frac{27}{5}\mu_1 - \frac{12}{5}\mu_2 \geq 0, & -\frac{12}{5}\mu_1 + \frac{12}{5}\mu_2 - 4 \geq 0; \\
\text{For } E_1 : & -3\mu_1 + 9 \geq 0, & \frac{27}{5}\mu_1 - \frac{12}{5}\mu_2 - 3 \geq 0, & -\frac{12}{5}\mu_1 + \frac{12}{5}\mu_2 - 2 \geq 0; \\
\text{For } L_1 : & -\frac{4}{5}\mu_2 + 3 \geq 0, & \frac{4}{5}\mu_1 - \frac{4}{5}\mu_2 + \frac{2}{3} \geq 0, & -\frac{4}{5}\mu_1 + \frac{8}{5}\mu_2 - \frac{8}{3} \geq 0; \\
\text{For } L_0 : & -\frac{4}{5}\mu_2 + \infty \geq 0, & \frac{4}{5}\mu_1 - \frac{4}{5}\mu_2 - 1 \geq 0, & -\frac{4}{5}\mu_1 + \frac{8}{5}\mu_2 \geq 0.
\end{array}$$

Only for window E_1 , all three constraints are satisfied. The constraints that are violated are: the third one for E_0 , the first one and second one for L_1 , and the second one for L_0 . The most severely violated constraints for each window are: the third one for E_0 (associated with Class 2), the second one for L_1 (associated with Class 1), and the second one for L_0 (associated with Class 1). Hence, the new CMSR to go in the heuristic algorithm should be

$$\rho^1 = \{\rho^{E_0}, \rho^{E_1}, \rho^{L_1}, \rho^{L_0}\} = \{(0, 1), (0, 1, 2), (0, 2), (0, 2)\}.$$

The new ρ gives rise to a solution of $(\mu_1, \mu_2) = (2.58621, 4.05172)$. For this solution, the constraints are all satisfied for windows E_0 , E_1 and L_0 . For window L_1 , the most severely violated

constraint is associated with Class 0. Hence, the new CMSR becomes

$$\rho^2 = \{\rho^{E_0}, \rho^{E_1}, \rho^{L_1}, \rho^{L_0}\} = \{(0, 1), (0, 1, 2), (2), (0, 2)\}.$$

The solution corresponding to ρ^2 is $(\mu_1, \mu_2) = (2.61364, 4.14773)$. The reader can verify that in this case all constraints are satisfied. Consequently, the heuristic algorithm finds the UE solution after only three iterations (the exact method would need to iterate 784 times in the worst case).