Unilateral problem for the Stokes equations: the well-posedness and finite element approximation

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Abstract

We consider the stationary Stokes equations under a unilateral boundary condition of Signorini's type, which is one of artificial boundary conditions in flow problems. Well-posedness is discussed through its variational inequality formulation. We also consider the finite element approximation for a regularized penalty problem. The well-posedness, stability and error estimates of optimal order are established. The lack of a coupled Babuška and Brezzi's condition makes analysis difficult. We offer a new method of analysis. Particularly, our device to treat the pressure is novel and of some interest. Numerical examples are presented to validate our theoretical results.

Keywords: Stokes equations, finite element approximation, unilateral boundary condition

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1. Introduction

We suppose that Ω is a bounded domain in $\mathbb{R}^d$ with $d = 2, 3$ and that the boundary $\partial\Omega$ is comprises of three parts S_1 , S_2 and Γ . Those S_1 , S_2 and Γ are assumed to be smooth but the whole boundary $\partial\Omega$ is not necessarily smooth. One might imagine a branched pipe resembling that depicted in Fig. 1. The first purpose of this paper is to study the well-posedness of the following unilateral boundary value problem for the Stokes equations

$$-\nu\Delta u + \nabla p = f, \quad \nabla \cdot u = 0 \quad \text{in } \Omega, \quad (1a)$$

$$u = 0 \quad \text{on } S_1 \cup S_2, \quad (1b)$$

$$u_n + g_n \geq 0, \quad \text{on } \Gamma, \quad (1c)$$

$$\tau_n(u, p) + \alpha_n \geq 0 \quad \text{on } \Gamma, \quad (1d)$$

$$(u_n + g_n)(\tau_n(u, p) + \alpha_n) = 0 \quad \text{on } \Gamma, \quad (1e)$$

$$\tau_T(u) + \alpha_T = 0 \quad \text{on } \Gamma \quad (1f)$$

for velocity $u = (u_1, \dots, u_d)$ and pressure p with density $\rho = 1$ and kinematic viscosity ν of the viscous incompressible fluid under consideration. Therein,

$$\tau(u, p) = \sigma(u, p)n$$

denotes the traction vector on $\partial\Omega$, where n is the outward normal vector to $\partial\Omega$, $\sigma(u, p) = (\sigma_{ij}(u, p))_{1 \leq i, j \leq d} = -pI + 2\nu D(u)$ the stress tensor, $D(u) = (D_{ij}(u))_{1 \leq i, j \leq d} = \frac{1}{2}(\nabla u + \nabla u^T)$ the deformation-rate tensor and I the identity matrix. For a vector-valued function v on $\partial\Omega$, its normal and tangential components are denoted, respectively, as

$$v_n = v \cdot n, \quad v_T = v - v_n n.$$

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Figure 1: Example of Ω (branched pipe)

Particularly, $\tau_n(u, p) = \tau(u, p) \cdot n$ and $\tau_T(u) = \tau(u, p) - \tau_n(u, p)n$ respectively denote normal and tangential traction vectors. Moreover, f , g and α are prescribed functions. We also consider the finite element approximation for a regularized penalty problem to (1) which is given as

$$-\nu \Delta u + \nabla p = f, \quad \nabla \cdot u = 0 \quad \text{in } \Omega, \quad (2a)$$

$$u = 0 \quad \text{on } S_1 \cup S_2, \quad (2b)$$

$$\tau_n(u, p) + \alpha_n = \frac{1}{\varepsilon} \phi_\delta(u_n + g_n) \quad \text{on } \Gamma, \quad (2c)$$

$$\tau_T(u) + \alpha_T = 0 \quad \text{on } \Gamma \quad (2d)$$

with $0 < \varepsilon \ll 1$ and $0 < \delta \ll 1$. Therein, $\phi_\delta(s)$ is a C^1 regularization of $[s]_- = \max\{0, -s\}$. We can take, for example,

$$\phi_\delta(s) = \begin{cases} 0 & (s \geq 0) \\ (\sqrt{s^2 + \delta^2} - \delta) & (s < 0). \end{cases} \quad (3)$$

First, we explain our motivation for studying (1) and (2). In numerical simulation of real-world flow problems, we often encounter some issues related to artificial boundary conditions. A typical and important example is the blood flow problem in the large arteries, where the blood is assumed to be a viscous incompressible fluid (see [16, 32]). The blood vessel is modeled as a branched pipe as illustrated, for example, in Fig. 1. Then, for $T > 0$, we consider the Navier–Stokes equations for velocity $v = (v_1, \dots, v_d)$ and pressure q ,

$$v_t + (v \cdot \nabla)v = \nabla \cdot \sigma(v, q) + f, \quad \nabla \cdot v = 0 \quad \text{in } \Omega \times (0, T), \quad (4a)$$

$$v = b \quad \text{on } S_1 \times (0, T), \quad (4b)$$

$$v = 0 \quad \text{on } S_2 \times (0, T) \quad (4c)$$

with the initial condition $v|_{t=0} = v_0$. We are able to give a velocity profile $b = b(x, t)$ at the inflow boundary S_1 . The flow is presumed to be a perfect non-slip flow on the wall S_2 . Then, the blood flow simulation is highly dependent on the choice of artificial boundary conditions posed on the outflow boundary Γ .

An earlier paper by Zhou and Saito [34] presented discussion of the free-traction condition

$$\tau(v, q) = 0 \quad \text{on } \Gamma, \quad (5)$$

which is one of the common outflow boundary conditions (see [18, 21]), and some nonlinear energy-preserving boundary conditions (see [3, 7, 8, 11, 12]) from the view-point of energy inequality. Moreover, we proposed a new outflow boundary condition as

$$v_n \geq 0, \quad \tau_n(v, q) \geq 0, \quad v_n \tau_n(v, q) = 0, \quad \tau_T(v) = 0 \quad \text{on } \Gamma. \quad (6)$$

This is an analogy to Signorini's condition in the theory of elasticity (see [24]). It is indeed a generalization of the free-traction condition (5), as

$$\begin{aligned} &\text{if } v_n > 0 \text{ on } \omega \subset \Gamma, \text{ then } \tau_n(v, q) = 0 \text{ on } \omega; \\ &\text{if } v_n = 0 \text{ on } \omega \subset \Gamma, \text{ then } \tau_n(v, q) \geq 0 \text{ on } \omega. \end{aligned}$$

A benefit of using (6) is that (4) admits energy inequality, *although* it is not guaranteed under (5). To describe it more specifically, we take a *reference flow* (g, π) , which is the solution of the Stokes system

$$\nabla \cdot \sigma(g, \pi) = 0, \quad \nabla \cdot g = 0 \quad \text{in } \Omega, \quad (7a)$$

$$g = b \text{ on } S_1, \quad g = 0 \text{ on } S_2, \quad g = -g_0(x) \int_{S_1} b \cdot n \, dS_1 \text{ on } \Gamma \quad (7b)$$

for all $t \in [0, T]$, where $g_0 = g_0(x) \in C_0^\infty(\Gamma)^d$ is a prescribed function satisfying

$$\int_{\Gamma} g_0 \cdot n \, d\Gamma = 1, \quad g_0 \cdot n \geq 0 \quad \text{on } \Gamma. \quad (8)$$

(Function g is nothing but a lifting function of b .) Using this, we will find (v, q) of the form

$$v = u + g, \quad q = p + \pi.$$

Assuming that (4) admits a smooth solution $(v, q) = (u + g, p + \pi)$ in $0 \leq t \leq T$ and multiplying both sides of (4a) by u , by the integration by parts, we have

$$\begin{aligned} \frac{d}{dt} \|u\|_{L^2(\Omega)^d}^2 + 2\nu \underbrace{\int_{\Omega} D_{ij}(u) D_{ij}(u) \, dx + \frac{1}{2} \int_{\Gamma} v_n |u|^2 \, d\Gamma - \int_{\Gamma} \tau(v, q) \cdot u \, d\Gamma}_{=I} \\ = \int_{\Omega} [f - g_t - (g \cdot \nabla)g] \cdot u \, dx - \int_{\Omega} (u \cdot \nabla)g \cdot u \, dx. \end{aligned} \quad (9)$$

Using (6), we derive $I \geq 0$. Consequently,

$$\sup_{t \in [0, T]} \|u\|_{L^2(\Omega)^d}^2 + 2\nu \int_0^T D_{ij}(u) D_{ij}(u) \, dt \leq C, \quad (10)$$

where C denotes a positive constant depending only on f , u_0 , b and T (see [34, Theorem 4]). This inequality is *useful*. It plays a crucial role in the construction of a solution of the Navier–Stokes equations (see [34]). Moreover, it is connected with the stability of numerical schemes from the view-point of numerical computation. That is, it is preferred that energy inequality *not be* spoiled after discretizations (see [33]). With (5), we do not know whether $I \geq 0$ or not. *Therefore*, energy inequality cannot be derived even for the continuous case.

There are a lot of strategies to treat the inequality condition (6) in numerical calculations. For example, we can use its penalty approximation

$$\tau_n(v, q) = \frac{1}{\varepsilon} [v_n]_-, \quad \tau_T(v) = 0 \quad \text{on } \Gamma, \quad (11)$$

where $0 < \varepsilon \ll 1$ and

$$[s]_{\pm} = \max\{0, \pm s\}, \quad s = [s]_+ - [s]_- \quad (s \in \mathbb{R}). \quad (12)$$

We also obtain energy inequality with (11) for a sufficiently small ε (see [34, Theorem 5]). Moreover, after introducing a C^1 regularization ϕ_δ of $[\cdot]_-$, we can solve (4) with (11) by using, for example, Newton's iteration.

Our *ultimate objective* is to develop the theory for the initial-boundary value problems for the Navier–Stokes equations (4) with (6) or with (11) from the *dual standpoints* of analysis and numerical computations. As a primary step, we studied the well-posedness of these problems in Ladyzhenskaya's class in [34]. That is, we studied the unique existence of a solution of

$$\begin{aligned} u_t + ((u + g) \cdot \nabla)u + (u \cdot \nabla)g - \nabla \cdot \sigma(u, p) &= F, & \nabla \cdot u &= 0 & \text{in } \Omega, \\ u &= 0 & & & \text{on } S_1 \cup S_2, \\ u_n + g_n &\geq 0, \quad \tau_n(u, p) + \tau_n(g, \pi) &\geq 0 & & \text{on } \Gamma, \\ (u_n + g_n)(\tau_n(u, p) + \tau_n(g, \pi)) &= 0, \quad \tau_T(u) + \tau_T(g) &\geq 0 & & \text{on } \Gamma, \end{aligned}$$

where $F = f - g_t - (g \cdot \nabla)g$.

For the analyses described herein, we devote our attention to the discretization of the space variable. Therefore, we study the finite element approximation using model Stokes problems. Consequently, we come to consider Problems (1) and (2).

As a matter of fact, (1) and (2) themselves are not new problems (see [4, 13, 15, 24] for example). In a classical monograph [24], Chapter 7 is devoted to similar problems, say Signorini's problem for incompressible materials. However, their problem includes the traction condition $\tau(u, p) = h$. More precisely, they assume that S_2 is divided into two parts S_{21}, S_{22} and consider

$$u = 0 \quad \text{on } S_{21}, \quad \tau(u, p) = h \quad \text{on } S_{22}$$

instead of (1b). Then, assuming

$$\bar{\Gamma} \cap \overline{(S_1 \cup S_{21})} = \emptyset, \quad (13)$$

we can prove that there exists a domain constant $C > 0$ satisfying

$$C \left[\|q\|_{L^2(\Omega)} + \|\tau\|_{H^{-\frac{1}{2}}(\Gamma)} \right] \leq \sup_{v \in H^1(\Omega)^d, v|_{S_1 \cup S_{21}} = 0} \frac{\int_{\Omega} q(\nabla \cdot v) \, dx + \int_{\Gamma} \tau v_n \, d\Gamma}{\|v\|_{H^1(\Omega)}} \quad (14)$$

for any $(q, \tau) \in L^2(\Omega) \times H^{-\frac{1}{2}}(\Gamma)$ (see [24, Theorem 7.2]). This inequality is usually designated as the coupled Babuška-Brezzi condition. The well-posedness and error estimates of the corresponding penalty problem (with no regularization) are direct consequences of this result from the general theory (see [5]). [13, 15] study similar unilateral problems and they also assume $S_{22} \neq \emptyset$. In contrast, we are interested in establishing a formulation without the traction boundary condition. Unfortunately, if $S_{22} = \emptyset$, then (14) is not available. It makes analysis somewhat difficult. Moreover, we do not prefer assuming (13). Consequently, we must develop a completely new method of analysis in this paper. Particularly, we offer a new device to treat the pressure part. As a result, we succeed in deriving the optimal-order error estimate for the finite element approximation for the penalty problem using the MINI (P1b/P1) element (see Theorem 18). Our method is different from that of [15] in which the optimal-order error estimates are proved for linear and quadratic finite element approximations for Signorini's contact problem (with no penalization). We first derive non-optimal error estimates and then apply them to improve several estimates. Our error estimates are novel even in the literature of elasticity theory (see [13]).

Finite element approximation of another class of unilateral boundary value problems for the Stokes equations, say unilateral problems of *friction type*, are discussed, for example, in several reports of the literature [1, 23, 25, 26].

This paper is composed of 7 sections. After introducing basic definitions and recalling some standard results in Section 2, we state two variational formulations, (PDE) and (VI), for (1) in Section 3. The equivalence of these formulations and well-posedness of (VI) are also established there (see Theorems 5 and 6). Section 4 is devoted to presentation of the finite element approximation for (2). We consider only the MINI (P1b/P1) element approximation. The well-posedness and error estimates are proved, respectively, in Sections 5 and 6 (see Theorems 9, 10 and 18). Finally, we confirm our results by numerical experiments in Section 7.

2. Preliminaries

Geometry. We recall that $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a bounded domain and the boundary $\partial\Omega$ comprises of three parts S_1 , S_2 and Γ . We address the following two cases:

(G1) S_1 , S_2 and Γ are smooth surfaces (curves) and Ω is a Lipschitz domain (see Fig. 1);

(G2) S_1 , S_2 are polygon (line segment) and Ω is a polyhedral (polygonal) domain.

For the following, we assume that Ω is given as (G1) or (G2) unless otherwise stated explicitly. Moreover, we set

$$S = S_1 \cup S_2$$

and assume that the $d - 1$ dimensional Lebesgue measure $|S|$ is positive.

Throughout this paper, C denotes various positive constants depending on Ω .

Remark 1. Although we mostly [address](#) the case [presented in](#) Fig. 1, our discussion is also valid for the case [in which](#) $\partial\Omega$ is smooth with $\bar{\Gamma} \cap \bar{S}_2 = \emptyset$, $\bar{S}_2 \cap \bar{S}_1 = \emptyset$, and $\bar{S}_1 \cap \bar{\Gamma} = \emptyset$.

Function spaces and forms. We use the standard Lebesgue and Sobolev spaces, for example, $L^2(\Omega)$, $H^1(\Omega)$, $L^2(\Gamma)$, $H^{\frac{1}{2}}(\Gamma)$. (We follow the notation of [29] as for function spaces and their norms.) The abbreviations

$$(v, w) = (v, w)_\Omega = (v, w)_{L^2(\Omega)}, \quad (v, w)_\Gamma = (v, w)_{0,\Gamma} = (v, w)_{L^2(\Gamma)},$$

$$\|v\| = \|v\|_\Omega = \|v\|_{0,\Omega} = \|v\|_{L^2(\Omega)}, \quad \|v\|_1 = \|v\|_{1,\Omega} = \|v\|_{H^1(\Omega)}, \quad \|v\|_\Gamma = \|v\|_{0,\Gamma} = \|v\|_{L^2(\Gamma)}$$

will be [used](#). Moreover,

$$|v|_m = |v|_{m,\Omega} = |v|_{H^m(\Omega)}, \quad |v|_{m,\Gamma} = |v|_{H^m(\Gamma)}$$

are the semi-norms of $H^m(\Omega)$, $H^m(\Gamma)$.

For a vector-valued function space, we use the same symbol to denote its norm [as](#)

$$\|v\| = \|v\|_{L^2(\Omega)^d} \quad (v \in L^2(\Omega)^d), \quad \|v\|_1 = \|v\|_{H^1(\Omega)^d} \quad (v \in H^1(\Omega)^d).$$

The basic function spaces of our consideration are

$$V = \{v \in H^1(\Omega)^d \mid v = 0 \text{ on } S\} \quad \text{and} \quad Q = L^2(\Omega).$$

They are, [respectively](#), Hilbert spaces equipped with the norms $\|v\|_1$ and $\|q\|$. We use closed subspaces of V ,

$$V^\sigma = \{v \in V \mid \nabla \cdot v = 0 \text{ in } \Omega\}, \quad V_0 = H_0^1(\Omega)^d, \quad V_0^\sigma = \{v \in V_0 \mid \nabla \cdot v = 0 \text{ in } \Omega\},$$

and that of Q [as](#)

$$Q_0 = \left\{ q \in Q \mid \int_\Omega q \, dx = 0 \right\}.$$

Convex subsets

$$K = \{v \in V \mid v_n + g_n \geq 0 \text{ on } \Gamma\} \quad \text{and} \quad K^\sigma = \{v \in V^\sigma \mid v_n + g_n \geq 0 \text{ on } \Gamma\}$$

of V and V^σ , respectively, play important roles.

We recall the so-called *Lions–Magenes space* $H_{00}^{\frac{1}{2}}(\Gamma)$. It is defined as ([see](#) [29, §11.5, Ch. 1])

$$H_{00}^{\frac{1}{2}}(\Gamma) = \{\mu \in H^{\frac{1}{2}}(\Gamma) \mid \rho^{-1/2} \mu \in L^2(\Gamma)\}$$

which is a Hilbert space equipped with the norm

$$\|\mu\|_{H_{00}^{\frac{1}{2}}(\Gamma)} = \left(\|\mu\|_{H^{\frac{1}{2}}(\Gamma)}^2 + \|\rho^{-1/2} \mu\|_\Gamma^2 \right)^{\frac{1}{2}}.$$

Here, $\rho \in C^\infty(\bar{\Gamma})$ denotes any positive function satisfying $\rho|_{\partial\Gamma} = 0$, and for $x_0 \in \partial\Gamma$,

$$\lim_{x \rightarrow x_0} \frac{\rho(x)}{\text{dist}(x, \partial\Gamma)} = d' > 0$$

with some $d' > 0$. Moreover, we know ([see](#) [29, Theorem 11.7, Ch. 1])

$$H_{00}^{\frac{1}{2}}(\Gamma) = (H_0^1(\Gamma), L^2(\Gamma))_{\frac{1}{2}, 2} \quad (\text{algebraically and topologically}), \quad (15)$$

where the right-hand side denotes the real interpolation space between $L^2(\Gamma)$ and $H_0^1(\Gamma)$ with the exponent $1/2$ and $p = 2$. Particularly $H_{00}^{\frac{1}{2}}(\Gamma)$ is *strictly* included in $H^{\frac{1}{2}}(\Gamma)$.

Below we set

$$M = H_{00}^{\frac{1}{2}}(\Gamma), \quad \|\mu\|_{\frac{1}{2}, \Gamma} = \|\mu\|_{H_{00}^{\frac{1}{2}}(\Gamma)}$$

and

$$M_0 = \left\{ \mu \in M \mid \int_{\Gamma} \mu \, d\Gamma = 0 \right\}.$$

In general, X' denotes the topological dual space of a Banach space X . The norm of X' is defined as

$$\|\varphi\|_{X'} = \sup_{v \in X, v \neq 0} \frac{\langle \varphi, v \rangle_{X', X}}{\|v\|_X},$$

where $\langle \cdot, \cdot \rangle_{X', X}$ is the duality pairing between X' and X . For a closed subspace Y of X and $\varphi \in Y'$, we mean by $\|\varphi\|_{Y'}$

$$\|\varphi\|_{Y'} = \sup_{v \in Y, v \neq 0} \frac{\langle \varphi, v \rangle_{X', X}}{\|v\|_X}.$$

Set

$$\begin{aligned} \langle \cdot, \cdot \rangle &= \langle \cdot, \cdot \rangle_{V', V} = \text{the duality pairing between } V' \text{ and } V, \\ [\cdot, \cdot] &= [\cdot, \cdot]_{M', M} = \text{the duality pairing between } M' \text{ and } M, \\ [[\cdot, \cdot]] &= [[\cdot, \cdot]]_{(M^d)', M^d} = \text{the duality pairing between } (M^d)' \text{ and } M^d. \end{aligned}$$

We use the following forms:

$$\begin{aligned} a(u, v) &= 2\nu \int_{\Omega} D_{ij}(u) D_{ij}(v) \, dx & (u, v \in H^1(\Omega)^d); \\ b(p, u) &= - \int_{\Omega} p(\nabla \cdot u) \, dx & (p \in Q, u \in H^1(\Omega)^d). \end{aligned}$$

Trace and lifting operators. Let $\text{Tr} = \text{Tr}(\Omega, \Gamma)$ be a trace operator from $H^1(\Omega)$ into $H^{\frac{1}{2}}(\Gamma)$. The meaning of $\text{Tr}(\Omega, S)$ is the same.

Lemma 2. *The trace operator $v \mapsto \mu = \text{Tr } v$ is linear and continuous of $V \rightarrow M^d$. Conversely, there exists a linear and bounded operator $\mathcal{E}$ of $M^d \rightarrow V$, which is called a lifting operator, such that $\mathcal{E}\mu = \mu$ on Γ for all $\mu \in M^d$.*

This result follows directly from [19, Theorem 2.5] and [20, Theorem 1.5.2.3]. A partial result is also reported in [30, Theorems 1.1 and 5.1]. As a consequence of Lemma 2, we obtain a lifting operator $\mathcal{E}_n : M \rightarrow V$ such that

$$(\mathcal{E}_n \mu)_n = \mu, \quad (\mathcal{E}_n \mu)_T = 0 \text{ on } \Gamma, \quad \|\mathcal{E}_n \mu\|_1 \leq C \|\mu\|_{\frac{1}{2}, \Gamma}$$

for any $\mu \in M$.

Below, we will often write as $v|_{\Gamma} = \text{Tr } v$ if there is no fear of confusion.

Remark 3. In view of Lemma 2 and the standard trace/lifting theorem, the zero extension $\hat{\mu}$ of $\mu \in M^d$ into $\partial\Omega$;

$$\hat{\mu} = \begin{cases} \mu & \text{on } \Gamma, \\ 0 & \text{on } \partial\Omega \setminus \Gamma \end{cases}$$

belongs to $H^{\frac{1}{2}}(\partial\Omega)^d$.

Remark 4. Another definition of $H_{00}^{\frac{1}{2}}(\Gamma)$ is given by Baiocchi and Capelo [2, Page 379]. That is,

$$H_{00}^{\frac{1}{2}}(\Gamma) = \{\text{Tr } v \mid v \in H^1(\Omega), \text{Tr}(\Omega, S)v = 0\}$$

which is a Hilbert space equipped with the norm

$$\|\mu\|_{H_{00}^{\frac{1}{2}}(\Gamma)} = \inf\{\|v\|_1 \mid v \in H^1(\Omega), \text{Tr}(\Omega, S)v = 0, \text{Tr } v = \mu\}.$$

Redefinition of traction vectors. Next we propose the redefinition of $\tau(u, p)$. If a smooth vector field u and scalar field p satisfy the Stokes equation

$$-\nu \Delta u + \nabla p = f, \quad \nabla \cdot u = 0 \quad \text{in } \Omega$$

for a given $f \in L^2(\Omega)^d$, they satisfy

$$a(u, v) + b(p, v) + \int_{\Gamma} \tau(u, p) \cdot v = (f, v) \quad (\forall v \in V) \quad (16)$$

and

$$a(u, v) + b(p, v) = (f, v) \quad (\forall v \in V_0). \quad (17)$$

(In (16), $\tau(u, p)$ is understood as a usual function defined on Γ .) Based on those identities, for functions $(u, p) \in V^\sigma \times Q$ satisfying (17), we redefine the traction vector $\tau(u, p)$ as a functional over M^d by

$$[[\tau(u, p), \mu]] = a(u, w_\mu) + b(p, w_\mu) - (f, w_\mu) \quad (\mu \in M^d), \quad (18)$$

where $w_\mu = \mathcal{E}\mu \in V$. Actually, the right-hand side of (18) is independent of the way of extension. Therefore, this definition is well-defined. Similarly, we redefine

$$[[\tau_T(u), \mu]] = a(u, w_\mu) + b(p, w_\mu) - (f, w_\mu) \quad (\mu \in M^d \text{ with } \mu_n = 0; w_\mu = \mathcal{E}\mu) \quad (19)$$

and

$$[\tau_n(u, p), \mu] = a(u, w_\mu) + b(p, w_\mu) - (f, w_\mu) \quad (\mu \in M; w_\mu = \mathcal{E}_n \mu). \quad (20)$$

Then, we deduce an expression

$$[[\tau(u, p), \mu]] = [\tau_n(u, p), \mu_n] + [[\tau_T(u), \mu_T]] \quad (\mu \in M^d). \quad (21)$$

3. Variational formulations and well-posedness

From this point forward in our discussion, we always assume

$$f \in Q^d, \quad b \in M^d, \quad \beta \equiv - \int_S b \cdot n \, dS > 0.$$

We take $g \in H^1(\Omega)^d$ satisfying

$$\nabla \cdot g = 0 \text{ in } \Omega, \quad g|_S = b, \quad g|_C = 0, \quad g|_\Gamma = \beta g_0,$$

where g_0 is the function defined as (8). Then, we have

$$g_n \geq 0 \text{ on } \Gamma, \quad g_n \in M, \quad \alpha \equiv 2\nu D(g)n \in (M^d)'$$

Under those assumptions and redefinitions in the previous section, we interpret (1) precisely as follows.

(PDE) Find $(u, p) \in V \times Q$ such that

$$a(u, v) + b(p, v) = (f, v) \quad (\forall v \in V_0), \quad (22a)$$

$$b(q, u) = 0 \quad (\forall q \in Q), \quad (22b)$$

$$u_n + g_n \geq 0 \quad \text{a.e. on } \Gamma, \quad (22c)$$

$$[\tau_n(u, p) + \alpha_n, \mu] \geq 0 \quad (\forall \mu \in M, \mu \geq 0), \quad (22d)$$

$$[\tau_n(u, p) + \alpha_n, u_n + g_n] = 0 \quad (22e)$$

$$[[\tau_T(u) + \alpha_T, \mu]] = 0 \quad (\forall \mu \in M^d, \mu_n = 0). \quad (22f)$$

If a solution (u, p) of (PDE) is sufficiently smooth, it solves (1) in the classical sense. Actually, (PDE) is equivalent to the following variational inequality.

(VI) Find $(u, p) \in K \times Q$ such that

$$a(u, v - u) + b(p, v - u) \geq (f, v - u) - [[\alpha, v - u]] \quad (\forall v \in K), \quad (23a)$$

$$b(q, u) = 0 \quad (\forall q \in Q). \quad (23b)$$

In this section, we prove the following two theorems.

Theorem 5. Problems (VI) and (PDE) are equivalent.

Theorem 6. There exists a unique solution $(u, p) \in K \times Q$ of (VI) and it holds that

$$\|u\|_1 + \|p\| \leq C_*, \quad (24)$$

where C_* denotes a positive constant depending only on Ω , $\|f\|$, $\|\alpha\|_{(M^d)^\vee}$ and $\|g\|_1$.

Remark 7. The boundary condition (22f) is nothing but one [alternative](#). One can pose

$$u_T + \alpha'_T = 0 \quad \text{a.e. on } \Gamma \quad (25)$$

with a prescribed α'_T instead of (22f). Actually, the discussion presented below remains true if we re-choose a suitable lifting function g and replace the original V [with](#)

$$V = \{v \in H^1(\Omega)^d \mid v = 0 \text{ on } \Gamma, v_T = 0 \text{ on } \Gamma\}.$$

Proof of Theorem 5. (PDE) $\Rightarrow$ (VI). Let $(u, p) \in V \times Q$ be a solution of (PDE). We verify (u, p) is a solution of (VI). First, we have $u \in K$ by (22c) and (22b). By using (18), (21), and (22c)–(22f), we have for any $v \in K$

$$\begin{aligned} & a(u, v - u) + b(p, v - u) - (f, v - u) + [[\alpha, v - u]] \\ &= [[\tau(u, p), v - u]] + [[\alpha, v - u]] \\ &= [\tau_n(u, p), v_n - u_n] + \underbrace{[[\tau_T(u) + \alpha_T, v_T - u_T]]}_{=0} + [\alpha_n, v_n - u_n] \\ &= \underbrace{[\tau_n(u, p) + \alpha_n, v_n + g_n]}_{\geq 0} - \underbrace{[\tau_n(u, p) + \alpha_n, u_h + g_n]}_{=0} \geq 0. \end{aligned}$$

(VI) $\Rightarrow$ (PDE). Let $(u, p) \in K \times Q$ be a solution of (VI). We now verify (u, p) actually satisfies (PDE). First, (22b) and (22c) are obvious.

Let $v \in V_0$ be arbitrary. Substituting $v = u \pm v \in K$ into (23a), we have (22a).

We recall $\tau(u, p)$ is defined as (18). [Consequently](#), (23a) implies

$$[[\tau(u, p), v - u]] \geq -[[\alpha, v - u]] \quad (\forall v \in K).$$

Moreover, by (21)

$$[\tau_n(u, p) + \alpha_n, v_n - u_n] + [[\tau_T(u) + \alpha_T, v_T - u_T]] \geq 0 \quad (\forall v \in K). \quad (26)$$

Let $\psi \in C_0^\infty(\Gamma)^d$ with $\psi_n = 0$ on Γ . Substituting $v = u \pm \mathcal{E}\psi \in K$ into (26), we have $[[\tau_T(u) + \alpha_T, \psi]] = 0$. By [density](#), this implies (22f).

Next, let $\psi \in C_0^\infty(\Gamma)^d$ with $\psi_n \geq 0$ and $\psi_T = 0$ on Γ . Substituting $v = u + \mathcal{E}\psi \in K$ into (26), we have $[\tau_n(u, p) + \alpha_n, \psi_n] \geq 0$. By [density](#), this implies (22d).

Combining (26) and (22f), we have

$$[\tau_n(u, p) + \alpha_n, v_n - u_n] \geq 0 \quad (\forall v \in K).$$

At this stage, we introduce $w^* \in V$ satisfying

$$w^* = g \text{ on } \Gamma, \quad \|w^*\|_1 \leq C\|g\|_1. \quad (27)$$

Since $g|_\Gamma \in M^d$, such w^* [truly](#) exists in view of the trace theorem. [However](#), it does not satisfy the divergence-free condition. [Consequently](#), $w^* \notin V^\sigma$. We now have $-w_n^* + g_n = -g_n + g_n \geq 0$ and $2u_n + w_n^* + g_n = 2(u_n + g_n) \geq 0$. [Therefore](#), we can choose as $v = -w^*$ and $v = 2u + w^*$ above and obtain (22e). $\square$

Proof of Theorem 6. Since a is a coercive bilinear form in $V^\sigma \times V^\sigma$ by virtue of Korn's inequality (see [24, Lemma 6.2]), we can apply Stampacchia's theorem (see [10, Theorem 5.6]) to conclude that there exists a unique $u \in K^\sigma$ satisfying the following:

$$a(u, v - u) \geq (f, v - u) - [[\alpha, v - u]] \quad (\forall v \in K^\sigma). \quad (28)$$

Taking $v = u \pm \varphi$ with $\varphi \in V_0^\sigma$ in (28), we deduce

$$a(u, \varphi) = (f, \varphi) \quad (\forall \varphi \in V_0^\sigma). \quad (29)$$

Therefore, according to [17, Lemma I.2.1], there exists a unique $\hat{p} \in Q_0$ satisfying

$$-b(\hat{p}, v) = a(u, v) - (f, v) \quad (\forall v \in V_0). \quad (30)$$

Now we set, for $k \in \mathbb{R}$,

$$p_k = \hat{p} + k \quad (31)$$

and verify, with an appropriate choice of k , that (u, p_k) is a solution of (VI). To this end, it suffices to check that (u, p_k) is a solution of (PDE).

We have by (18) and (28)

$$[\tau_n(u, p_k), v_n - u_n] + [[\tau_T(u), v_T - u_T]] \geq -[[\alpha, v - u]] \quad (v \in K^\sigma). \quad (32)$$

Letting $\psi \in C_0^\infty(\Gamma)^d$ with $\psi_n = 0$ on Γ , then since $\int_\Gamma \psi_n \, d\Gamma = 0$, there is a function $w \in V$ satisfying $w = \psi$ on Γ , $\nabla \cdot w = 0$ in Ω and $\|w\|_1 \leq C\|\psi\|_{M^d}$. Substituting $v = u \pm w \in K^\sigma$ into (32), one obtains $[[\tau_T(u) + \alpha_T, \psi_T]] = 0$. By density, this implies (22f).

Therefore, it follows from (32) that

$$[\tau_n(u, p_k) + \alpha_n, v_n - u_n] \geq 0 \quad (v \in K^\sigma). \quad (33)$$

We set

$$\gamma = \inf_{\mu \in Y} [\tau_n(u, \hat{p}) + \alpha_n, \mu],$$

where

$$Y = \left\{ \mu \in M \mid \mu \geq 0, \mu \not\equiv 0, \int_\Gamma \mu \, d\Gamma = 1 \right\}. \quad (34)$$

For any $\mu \in M$ with $\mu \geq 0$ and $\mu \not\equiv 0$, we have

$$[\tau_n(u, p_k) + \alpha_n, \mu] = [\tau_n(u, \hat{p}) + \alpha_n, \mu] - k \int_\Gamma \mu \geq \gamma \int_\Gamma \mu - k \int_\Gamma \mu$$

Therefore, we deduce (22d) if $k \leq \gamma$.

For the time being, we admit that

$$\gamma = \frac{[\tau_n(u, \hat{p}) + \alpha_n, u_n + g_n]}{\beta}, \quad (35)$$

When $u_n + g_n = 0$ on Γ , we have $\gamma = 0$, but this is impossible because $u_n + g_n \geq 0$ and $\int_\Gamma g_n \, d\Gamma > 0$. Therefore, we have by (35)

$$[\tau_n(u, \hat{p}) + \alpha_n, u_n + g_n] = \gamma\beta = \gamma \int_\Gamma g_n \, d\Gamma = \gamma \int_\Gamma (u_n + g_n) \, d\Gamma.$$

Hence, taking

$$k = \gamma,$$

we obtain

$$[\tau_n(u, p_k) + \alpha_n, u_n + g_n] = [\tau_n(u, \hat{p}) + \alpha_n, u_n + g_n] - \gamma \int_\Gamma (u_n + g_n) = 0.$$

Therefore, we have verified (22e).

To show (35), we use $w^* \in V$ defined as (27) again. From (33) with $k = 0$,

$$[\tau_n(u, \hat{p}) + \alpha_n, v_n + w_n^*] \geq [\tau_n(u, \hat{p}) + \alpha_n, u_n + w_n^*] \quad (v \in K^\sigma).$$

Since $w^* = g$ on Γ , this is expressed equivalently as

$$[\tau_n(u, \hat{p}) + \alpha_n, v_n + g_n] \geq [\tau_n(u, \hat{p}) + \alpha_n, u_n + g_n] \quad (v \in K^\sigma).$$

Moreover, we obtain

$$\left[\tau_n(u, \hat{p}) + \alpha_n, \frac{v_n + g_n}{\beta} \right] \geq \left[\tau_n(u, \hat{p}) + \alpha_n, \frac{u_n + g_n}{\beta} \right] \quad (v \in K^\sigma). \quad (36)$$

Now let $\mu \in Y$ be arbitrary and set $\tilde{\mu} = \beta\mu - g_n \in M$. Since $\int_\Gamma \tilde{\mu} \, d\Gamma = 0$, there exists $\tilde{v} \in V^\sigma$ such that $\tilde{v}_n = \tilde{\mu}$ on Γ according to Remark 3 and [17, Lemma I.2.2]. Then, the function $\tilde{v}$ satisfies that $\tilde{v}_n + g_n = \beta\mu \geq 0$ on Γ . Consequently, $\tilde{v} \in K^\sigma$. Therefore, we have by (36) that

$$\begin{aligned} [\tau_n(u, \hat{p}) + \alpha_n, \mu] &= \left[\tau_n(u, \hat{p}) + \alpha_n, \frac{\tilde{\mu} + g_n}{\beta} \right] \\ &= \left[\tau_n(u, \hat{p}) + \alpha_n, \frac{\tilde{v}_n + g_n}{\beta} \right] \geq \frac{1}{\beta} [\tau_n(u, \hat{p}) + \alpha_n, u_n + g_n], \end{aligned}$$

which implies (35).

It remains to derive (24). First, from (30), we have

$$\|\hat{p}\| \leq C \sup_{v \in V_0} \frac{|(f, v) - a(u, v)|}{\|v\|_1} \leq C(\|f\| + \|u\|), \quad (37)$$

where we have used the standard infsup (Babuška and Brezzi's) condition (see [17, Corollary I.2.4])

$$\inf_{q \in Q_0} \sup_{v \in V_0} \frac{b(q, v)}{\|q\| \|v\|_1} \geq C.$$

Equation (22e), together with (20), implies

$$a(u, u + g) + b(p, u + g) - (f, u + g) + [\alpha_n, u_n + g_n] = 0.$$

Therefore, by virtue of Korn's inequality (see [24, Lemma 6.2]),

$$C\|u + g\|_1^2 \leq C(\|f\| + \|\alpha\|_{(M^d)'} + \|g\|_1)\|u + g\|_1$$

and, consequently,

$$\|u + g\|_1 \leq C_*, \quad \|u\|_1 \leq C_*. \quad (38)$$

Finally, because of the expression (35), we can estimate as

$$|\gamma| \leq \frac{1}{\beta} \|\tau_n(u, \hat{p}) + \alpha_n\|_{M'} \|u_n + g_n\|_{\frac{1}{2}, \Gamma} \leq C(\|u\|_1 + \|\hat{p}\|) \|u + g\|_1. \quad (39)$$

Combining this with (37) and (38), we obtain (24). $\square$

4. Finite element approximation

While there are a lot of strategies for solving the variational inequality problem (VI), we concentrate our attention to its penalty approximation.

As a regularization of $[s]_-(s \in \mathbb{R})$, we introduce a function $\phi_\delta : \mathbb{R} \rightarrow \mathbb{R}$ that satisfies

$$\phi_\delta \text{ is a non-increasing } C^1(\mathbb{R}) \text{ function}; \quad (40a)$$

$$|\phi_\delta(s) - [s]_-| \leq C\delta \quad (s \in \mathbb{R}); \quad (40b)$$

$$\phi_\delta(s) = 0 \quad (s \geq 0), \quad 0 \leq \phi_\delta(s) \leq -s \quad (s < 0); \quad (40c)$$

$$\left| \frac{d}{ds} \phi_\delta(s) \right| \leq C \quad (s \in \mathbb{R}), \quad (40d)$$

where $\delta \in (0, 1]$ is a regularized parameter and C 's are independent of δ . As described in the Introduction, we can take, for example, the function $\phi_\delta(s)$ defined as (3).

For penalty parameter $\varepsilon \in (0, 1]$, we consider the following penalty problem,

($\mathbf{PE}_{\varepsilon, \delta}$) Find $(u, p) \in V \times Q$ such that

$$a(u, v) + b(p, v) - \frac{1}{\varepsilon} \int_{\Gamma} \phi_\delta(u_n + g_n) v_n \, d\Gamma = (f, v) - [[\alpha, v]] \quad (\forall v \in V), \quad (41a)$$

$$b(q, u) = 0 \quad (\forall q \in Q). \quad (41b)$$

This and the subsequent sections are devoted to the finite element approximation of ($\mathbf{PE}_{\varepsilon, \delta}$). To avoid unimportant difficulties related to the “curved boundary”, we consider only the case (G2). Consequently, the unit outer normal vector n to Γ is a constant vector over Γ .

We use the so-called **MINI** (P1b/P1) elements for discretization. Let $\{\mathcal{T}_h\}_h$ be a regular family of triangulations of Ω . As the granularity parameter, we used $h = \max\{h_T \mid T \in \mathcal{T}_h\}$, where h_T denotes the diameter of T . We introduce the following function spaces:

$$\begin{aligned} V_h &= \{v_h \in C^0(\bar{\Omega}) \mid v_h = 0 \text{ on } S, \, v_h|_T \in [\mathcal{P}_1^{(d)} \oplus \text{span}\{\varphi_T\}]^d \, (\forall T \in \mathcal{T}_h)\}, \\ V_{0h} &= V_h \cap H_0^1(\Omega)^d, \quad V_h^\sigma = \{v_h \in V_h \mid b(q_h, v_h) = 0 \, (\forall q_h \in Q_h)\}, \\ Q_h &= \{q_h \in C^0(\bar{\Omega}) \mid q_h|_T \in \mathcal{P}_1^{(d)} \, (\forall T \in \mathcal{T}_h)\}, \quad Q_{0h} = Q_h \cap Q_0, \\ M_h &= \{\mu_h = v_{hn}|_\Gamma \mid v_h \in V_h\}, \quad M_{0h} = \left\{ \mu_h \in M_h \mid \int_{\Gamma} \mu_h \, d\Gamma = 0 \right\}. \end{aligned}$$

Therein, $\mathcal{P}_k^{(d)}$ denotes the set of all polynomials in $x_1, \dots, x_d$ of degree $\leq k$, and $\varphi_T = \prod_{i=1}^{d+1} \lambda_{T,i}$, with $\lambda_{T,1}, \dots, \lambda_{T,d+1}$ the barycentric coordinates of T .

We denote by $\mathcal{S}_h$ the $d-1$ dimensional triangulation of Γ inherited from $\mathcal{T}_h$. We have

$$M_h = \{\mu_h \in C(\bar{\Gamma}) \mid \mu_h|_S \in \mathcal{P}_1^{(d-1)} \, (\forall S \in \mathcal{S}_h), \, \mu_h|_{\partial\Gamma} = 0\} \quad (\text{algebraically}). \quad (42)$$

Moreover, we introduce a projection operator $\Lambda : Q \rightarrow Q_0$ by

$$\Lambda q = q - m(q) \quad \text{with} \quad m(q) = \frac{1}{|\Omega|} \int_{\Omega} q \, dx \quad (q \in Q). \quad (43)$$

It seems readily apparent that $\|\Lambda q\| \leq C\|q\|$ for $q \in Q$ and $\Lambda q_h \in Q_{0h}$ for $q_h \in Q_h$.

Then, the finite element approximation for ($\mathbf{PE}_{\varepsilon, \delta}$) reads as follows.

($\mathbf{PE}_{\varepsilon, \delta, h}$) Find $(u_h, p_h) \in V_h \times Q_h$ such that

$$a(u_h, v_h) + b(p_h, v_h) - \frac{1}{\varepsilon} \int_{\Gamma} \phi_\delta(u_n + g_n) v_{hn} \, d\Gamma = (f, v_h) - [[\alpha, v_h]] \quad (\forall v_h \in V_h), \quad (44a)$$

$$b(q_h, u_h) = 0 \quad (\forall q_h \in Q_h). \quad (44b)$$

Before considering the well-posedness and error estimates, we recall here basic results on the finite element method.

Babuška-Brezzi condition. As is well-known, the Babuška-Brezzi condition is well-known to hold true in $V_{0h} \times Q_{0h}$. There is a constant $\gamma' > 0$, which is independent of h , such that

$$\inf_{q_h \in Q_{0h}} \sup_{v_h \in V_{0h}} \frac{b(q_h, v_h)}{\|v_h\|_1 \|q_h\|} \geq \gamma'. \quad (45)$$

Discrete lifting operators and discrete traction vectors. The following is a discrete analogue of Lemma 2.

Lemma 8 ([23, Lemma 2.1]). (i) *There is a continuous linear operator $\mathcal{E}_h$ from M_h^d to V_h such that $\mathcal{E}_h \mu_h = \mu_h$ on Γ and $\|\mathcal{E}_h \mu_h\|_1 \leq C \|\mu_h\|_{\frac{1}{2}, \Gamma}$ for any $\mu_h \in M_h^d$, where C is independent of h .*
(ii) *There is a continuous linear operator $\mathcal{E}_{nh}$ from M_h to V_h such that $(\mathcal{E}_{nh} \mu_h)_n = \mu_h$ and $(\mathcal{E}_{nh} \mu_h)_T = 0$ on Γ and $\|\mathcal{E}_{nh} \mu_h\|_1 \leq C \|\mu_h\|_{\frac{1}{2}, \Gamma}$ for any $\mu_h \in M_h$, where C is independent of h .*
(iii) *For $\mu_h \in M_{0h}$, the above $w_h = \mathcal{E}_{nh} \mu_h$ can be chosen in such way that $w_h \in V_h^\sigma$.*

As the continuous case, we define traction vectors $\tau(u_h, p_h) \in (M_h^d)'$, $\tau_T(u_h) \in (M_h^d)'$ and $\tau_n(u_h, p_h) \in M_h'$ for a solution $(u_h, p_h) \in V_h^\sigma \times Q_h$ of

$$a(u_h, v_h) + b(p_h, v_h) = (f, v_h) \quad (v_h \in V_{0h}) \quad (46)$$

as follows:

$$[[\tau(u_h, p_h), \mu_h]] = a(u_h, w_h) + b(p_h, w_h) - (f, w_h) \quad (\mu_h \in M_h^d, w_h = \mathcal{E}_h \mu_h); \quad (47a)$$

$$[\tau_T(u_h), \mu_h] = a(u_h, w_h) + b(p_h, w_h) - (f, w_h) \quad (\mu_h \in M_h^d \text{ with } \mu_{hn} = 0, w_h = \mathcal{E}_h \mu_h); \quad (47b)$$

$$[\tau_n(u_h, p_h), \mu_h] = a(u_h, w_h) + b(p_h, w_h) - (f, w_h) \quad (\mu_h \in M_h, w_h = \mathcal{E}_{nh} \mu_h) \quad (47c)$$

These definitions are independent of the way of extensions. In fact, for any μ_h , let $w_h \in V_h$ and $\tilde{w}_h \in V_h$ be both extension of λ_h ; $w_{hn} = \tilde{w}_{hn} = \lambda_h$ on Γ . Set $v_h = w_h - \tilde{w}_h$. Then, since $v_h \in V_{0h}$, we deduce, by (46),

$$\begin{aligned} a(u_h, w_h) + b(p_h, w_h) - (f, w_h) - [a(u_h, \tilde{w}_h) + b(p_h, \tilde{w}_h) - (f, \tilde{w}_h)] \\ = a(u_h, v_h) + b(p_h, v_h) - (f, v_h) = 0. \end{aligned}$$

Consequently, (47c) is well-defined.

5. Well-posedness of $(\text{PE}_{\varepsilon, \delta, h})$

In this section, we establish the well-posedness of $(\text{PE}_{\varepsilon, \delta, h})$. Thus, we shall prove the following two theorems. Recall that C_* denotes a positive constant depending only on Ω , $\|f\|$, $\|g\|_1$ and $\|\alpha\|_{(M^d)'}$.

Theorem 9. *There exists a unique solution $(u_h, p_h) \in V_h \times Q_h$ of $(\text{PE}_{\varepsilon, \delta, h})$, and we have*

$$\|u_h\|_1 + \|\hat{p}_h\| + \left\| \frac{1}{\varepsilon} \phi_\delta(u_{hn} + g_n) + k_h \right\|_{M_h'} \leq C_*, \quad (48)$$

where $\hat{p}_h = \Lambda p_h$ and $k_h = m(p_h)$.

Theorem 10. *Assume*

(A1) *the family $\{\mathcal{S}_h\}_h$ is **quasi-uniform**;*

(A2) *there exists $\Gamma_1 \subset \Gamma$ with $|\Gamma_1| > 0$ which is independent of h , ε , δ and Ω such that $u_{hn} + g_n > 0$ on Γ_1 .*

Then, the solution $(u_h, p_h) \in V_h \times Q_h$ of $(\text{PE}_{\varepsilon, \delta, h})$ admits the following estimates:

$$\|u_h\|_1 + \|p_h\| + \left\| \frac{1}{\varepsilon} \phi_\delta(u_{hn} + g_n) \right\|_{M_h'} \leq C_*; \quad (49a)$$

$$\left\| \frac{1}{\varepsilon} \phi_\delta(u_{hn} + g_n) \right\|_{M'} \leq C_* \left(1 + \frac{h}{\varepsilon} \right); \quad (49b)$$

$$\frac{1}{\sqrt{\varepsilon}} \|[u_{hn} + g_n]_-\|_\Gamma \leq C_* \left(1 + \frac{\delta}{\varepsilon} \right). \quad (49c)$$

Remark 11. Condition (A2) is not restrictive. It is natural to presume this condition if β is sufficiently large and h, ε, δ are suitably small.

Remark 12. If $\delta \leq c_0 \varepsilon$ with some $c_0 > 0$, we have, from (49c), $\|[u_{hn} + g_n]_-\|_\Gamma \rightarrow 0$ as $\varepsilon \rightarrow 0$.

To prove Theorem 9, we apply the following fundamental result.

Lemma 13 ([27, Theorem 2.1]). *Let X be a separable reflexive Banach space and let $T : X \rightarrow X'$ be a (possibly nonlinear) operator satisfying the following conditions:*

1. (boundness) *There exist $C, C', m > 0$ s.t. $\|Tu\|_{X'} \leq C\|u\|_X^m + C'$ for all $u \in X$;*
2. (monotonicity) *$\langle Tu - Tv, u - v \rangle_{X', X} \geq 0$ for all $u, v \in X$;*
3. (hemicontinuity) *For any $u, v, w \in X$, the function $\lambda \mapsto \langle T(u + \lambda v), w \rangle_{X', X}$ is continuous on $\mathbb{R}$;*
4. (coerciveness) *$\frac{\langle Tu, u \rangle_{X', X}}{\|u\|_X} \rightarrow \infty$ as $\|u\|_X \rightarrow \infty$.*

Then, for any $\varphi \in X'$, there exists $u \in X$ such that $Tu = \varphi$. Furthermore, if T is strictly monotone:

$$\langle Tu - Tv, u - v \rangle_{X', X} > 0 \quad (\forall u, v \in X, u \neq v),$$

then the solution is unique.

We set $\rho_\delta : V \rightarrow V'$ by

$$\langle \rho_\delta(u), v \rangle = - \int_\Gamma \phi_\delta(u_n + g_n) v_n \, d\Gamma \quad (v \in V).$$

Lemma 14. ρ_δ is a bounded, monotone and hemicontinuous operator from V to V' .

Proof. (boundness) By using (40c) and the trace theorem, we have

$$\langle \rho_\delta(u), v \rangle \leq \int_\Gamma |u_n + g_n| \cdot |v_n| \, d\Gamma \leq (\|u\|_1 + \|g_n\|_\Gamma) \|v\|_1$$

for $u, v \in V$. Hence,

$$\|\rho_\delta(u)\|_{V'} \leq \|u\|_1 + \|g_n\|_\Gamma \quad (u \in V).$$

(monotonicity) Since $-\phi_\delta(s)$ is non-decreasing function, we have

$$\langle \rho_\delta(u) - \rho_\delta(v), u - v \rangle = - \int_\Gamma (\phi_\delta(u_n + g_n) - \phi_\delta(v_n + g_n))(u_n + g_n - (v_n + g_n)) \, d\Gamma \geq 0$$

for $u, v \in V$.

(hemicontinuity) Let $u, v, w \in V$. Then, a real-valued function

$$\langle \rho_\delta(u + \lambda v), w \rangle = - \int_\Gamma \phi_\delta(u_n + \lambda v_n) w_n \, d\Gamma$$

of $\lambda \in \mathbb{R}$ is a continuous function, since the function ϕ_δ is continuous. □

Proof of Theorem 9. It is divided into three steps.

Step 1. First, we prove that there exists a unique $u_h \in V_h^\sigma$ satisfying

$$a(u_h, v_h) + \frac{1}{\varepsilon} \langle \rho_\delta(u_h), v_h \rangle = (f, v_h) - [[\alpha, v_h]] \quad (\forall v_h \in V_h^\sigma) \quad (50)$$

by using Lemma 13.

To do this, we introduce a nonlinear operator $A_\varepsilon : V_h \rightarrow V_h'$ by setting

$$\langle A_\varepsilon u_h, v_h \rangle = a(u_h, v_h) + \frac{1}{\varepsilon} \langle \rho_\delta(u_h), v_h \rangle \quad (u_h, v_h \in V_h).$$

and verify the conditions of Lemma 13.

(boundness) For $u_h \in V_h^\sigma$, we have immediately

$$\|A_\varepsilon u_h\|_{(V_h^\sigma)'} \leq \left(\|a\| + \frac{1}{\varepsilon} \right) \|u_h\|_1 + \frac{1}{\varepsilon} \|g_n\|_\Gamma.$$

(strictly monotonicity) By virtue of Korn's inequality and monotonicity of ρ_δ ,

$$\begin{aligned} \langle A_\varepsilon u_h - A_\varepsilon v_h, u_h - v_h \rangle &= a(u_h - v_h, u_h - v_h) + \frac{1}{\varepsilon} \langle \rho_\delta(u_h) - \rho_\delta(v_h), u_h - v_h \rangle \\ &\geq C \|u_h - v_h\|_1^2 > 0 \end{aligned}$$

for $u_h, v_h \in V_h^\sigma$, $u_h \neq v_h$.

(hemicontinuity) Let $u_h, v_h, w_h \in V_h^\sigma$. Then, a real-valued function

$$\langle A_\varepsilon(u_h + \lambda v_h), w_h \rangle = a(u_h + \lambda v_h, w_h) + \frac{1}{\varepsilon} \langle \rho_\delta(u_h + \lambda v_h), w_h \rangle$$

of $\lambda \in \mathbb{R}$ is continuous, since $a(\cdot, w_h)$ is continuous and $\rho_\delta(\cdot)$ is hemicontinuous.

(coerciveness) For $u_h \in V_h^\sigma$, we have by (40c)

$$\begin{aligned} \langle \rho_\delta(u_h), u_h \rangle &= - \int_\Gamma \phi_\delta(u_{hn} + g_n) u_{hn} \, d\Gamma \\ &= - \int_\Gamma \phi_\delta(u_{hn} + g_n) ([u_{hn} + g_n]_+ - [u_{hn} + g_n]_- - [g_n]_+ + [g_n]_-) \, d\Gamma \\ &\geq - \int_\Gamma \phi_\delta(u_n + g_n) [g_n]_- \, d\Gamma \\ &\geq -C (\|u_h\|_1 + \|g_n\|_\Gamma) \|g_n\|_\Gamma. \end{aligned}$$

This gives

$$\frac{\langle A_\varepsilon u_h, u_h \rangle}{\|u_h\|_1} = \frac{a(u_h, u_h)}{\|u_h\|_1} + \frac{1}{\varepsilon} \frac{\langle \rho_\delta(u_h), u_h \rangle}{\|u_h\|_1} \geq C \|u_h\|_1 - \frac{C (\|u_h\|_1 + \|g_n\|_\Gamma)}{\varepsilon} \|g_n\|_\Gamma,$$

and, hence,

$$\frac{\langle A_\varepsilon u_h, u_h \rangle}{\|u_h\|_1} \rightarrow \infty \quad \text{as} \quad \|u_h\|_1 \rightarrow \infty.$$

As a consequence, we can apply Lemma 13 to conclude that there exists a unique $u_h \in V_h^\sigma$ satisfying $A_\varepsilon u_h = F_h$, where $F_h \in (V_h^\sigma)'$ is defined as $\langle F, v_h \rangle = (f, v_h) - [[\alpha, v_h]]$ for $v_h \in V_h^\sigma$. **Therefore**, we have proved a unique existence of the solution $u_h \in V_h^\sigma$ of (50).

Step 2. We verify the unique existence of $p_h \in Q_h$ such that (u_h, p_h) is a solution of $(\text{PE}_{\varepsilon, \delta, h})$. In view of (45), there exists a unique $\hat{p}_h \in Q_{0h}$ satisfying

$$a(u_h, v_h) + b(\hat{p}_h, v_h) = (f, v_h) \quad (v_h \in V_{0h}). \quad (51)$$

Now, we find a constant k_h such that (u_h, p_h) is a solution of $(\text{PE}_{\varepsilon, \delta, h})$, where $p_h = \hat{p}_h + k_h$. (**In fact**, with any $k_h \in \mathbb{R}$, (u_h, p_h) also solves (46).) To do this, we first rewrite (44a) as by using (47a)–(47c),

$$[\tau_n(u_h, p_h) - \varepsilon^{-1} \phi_\delta(u_{hn} + g_n) - \alpha_n, v_{hn}] + [[\tau_T(u_h) - \alpha_T, v_{hT}]] = 0 \quad (v_h \in V_h).$$

Consequently, in view of Lemma 8, it suffices to prove the following two equations:

$$[\tau_n(u_h, p_h) - \varepsilon^{-1} \phi_\delta(u_{hn} + g_n) - \alpha_n, \mu_h] = 0 \quad (\mu_h \in M_h); \quad (52a)$$

$$[[\tau_T(u_h) - \alpha_T, \mu_h]] = 0 \quad (\mu_h \in M_h^d \text{ with } \mu_{hn} = 0). \quad (52b)$$

It might be readily apparent that $v_h = \mathcal{E}_h \mu_h \in V_h$ belongs to V_h^σ for any $\mu_h \in M_h^d$ with $\mu_{hn} = 0$. Hence, (50) and (47b) immediately implies (52b).

On the other hand, combining (50) and (47c), we have

$$[\tau_n(u_h, p_h) + \alpha_n - \varepsilon^{-1} \phi_\delta(u_{hn} + g_n), \lambda_h] = 0 \quad (\forall \lambda_h \in M_{0h}). \quad (53)$$

At this stage, let us take

$$\tilde{\mu}_h \in Y_h = \left\{ \mu_h \in M_h \mid \mu_h \geq 0, \mu_h \neq 0, \int_\Gamma \mu_h d\Gamma = 1 \right\}.$$

Then, for any $\mu_h \in M_h$, the function $\mu_h - \kappa_h \tilde{\mu}_h$ belongs to M_{0h} , where $\kappa_h = \int_\Gamma \mu_h d\Gamma$.

Therefore, for any $\mu_h \in M_h$,

$$\begin{aligned} [\tau_n(u_h, p_h) + \alpha_n - \varepsilon^{-1} \phi_\delta(u_{hn} + g_n), \mu_h] &= [\tau_n(u_h, p_h) + \alpha_n - \varepsilon^{-1} \phi_\delta(u_{hn} + g_n), \mu_h - \kappa_h \tilde{\mu}_h] \\ &\quad + [\tau_n(u_h, p_h) + \alpha_n - \varepsilon^{-1} \phi_\delta(u_{hn} + g_n), \kappa_h \tilde{\mu}_h] \\ &= \kappa_h [\tau_n(u_h, p_h) + \alpha_n - \varepsilon^{-1} \phi_\delta(u_{hn} + g_n), \tilde{\mu}_h] \end{aligned} \quad (54)$$

Now, choosing

$$k_h = [\tau_n(u_h, \hat{p}_h) - \varepsilon^{-1} \phi_\delta(u_{hn} + g_n), \tilde{\mu}_h], \quad (55)$$

we have

$$\begin{aligned} [\tau_n(u_h, p_h) + \alpha_n - \varepsilon^{-1} \phi_\delta(u_{hn} + g_n), \tilde{\mu}_h] &= [\tau_n(u_h, \hat{p}_h) + \alpha_n - \varepsilon^{-1} \phi_\delta(u_{hn} + g_n), \tilde{\mu}_h] - k_h \\ &= 0. \end{aligned}$$

Hence, we get (52a) by (54).

It remains to be verified that (55) is independent of the choice of $\tilde{\mu}_h$. We let $\tilde{\mu}_h, \tilde{\mu}'_h \in Y_h$ with $\tilde{\mu}_h \neq \tilde{\mu}'_h$ and let the corresponding k_h be denoted by $\tilde{k}_h, \tilde{k}'_h$, respectively. Then, since $\lambda_h = \mu_h - \mu'_h$ satisfies $\int_\Gamma \lambda_h d\Gamma = 0$, we have by (53),

$$\tilde{k}_h - \tilde{k}'_h = [\tau_n(u_h, \hat{p}_h) + \alpha_n - \varepsilon^{-1} \phi_\delta(u_{hn} + g_n), \lambda_h] = 0,$$

which means that k_h is uniquely determined by (55).

Step 3. Finally, we derive the stability result (48). Substituting $v_h = u_h \in V_h^\sigma$ into (50), we obtain,

$$a(u_h, u_h) - \frac{1}{\varepsilon} \int_\Gamma \phi_\delta(u_{hn} + g_n) u_{hn} d\Gamma = (f, u_h) - [[\alpha, u_h]]. \quad (56)$$

Noting that, by (40c)

$$\begin{aligned} -\frac{1}{\varepsilon} \int_\Gamma \phi_\delta(u_{hn} + g_n) u_{hn} d\Gamma &= -\frac{1}{\varepsilon} \int_\Gamma \phi_\delta(u_{hn} + g_n)(u_{hn} + g_n) d\Gamma + \frac{1}{\varepsilon} \int_\Gamma \phi_\delta(u_{hn} + g_n) g_n d\Gamma \\ &\geq \frac{1}{\varepsilon} \int_\Gamma \phi_\delta(u_{hn} + g_n)[u_{hn} + g_n]_- d\Gamma \geq 0, \end{aligned} \quad (57)$$

we get

$$a(u_h, u_h) \leq (f, u_h) - [[\alpha, u_h]].$$

Hence, by virtue of Korn's inequality ([24, Lemma 6.2]),

$$\|u_h\|_1 \leq C(\|f\| + \|\alpha\|_{(M^d)'}). \quad (58)$$

Moreover, according to (45) and (51),

$$\|\hat{p}_h\| \leq \sup_{v_h \in V_{0h}} \frac{b(\hat{p}_h, v_h)}{\|v_h\|_1} = \sup_{v_h \in V_{0h}} \frac{(f, v_h) - a(u_h, v_h)}{\|v_h\|_1} \leq C(\|f\| + \|u_h\|_1). \quad (59)$$

Since (44a) is expressed as

$$\begin{aligned} \int_\Gamma (\varepsilon^{-1} \phi_\delta(u_{hn} + g_n) + k_h) \mu_h d\Gamma \\ = a(u_h, v_h) + b(\hat{p}_h, v_h) - (f, v_h) + [[\alpha, v_h]] \quad (\forall \mu_h \in M_h, v_h = \mathcal{E}_{nh} \mu_h \in V_h), \end{aligned}$$

we deduce

$$\left\| \frac{1}{\varepsilon} \phi_\delta(u_{hn} + g_n) + k_h \right\|_{M'_h} \leq C(\|u_h\|_1 + \|\hat{p}_h\| + \|f\| + \|\alpha\|_{(M^d)'}). \quad (60)$$

Summing up (58), (59) and (60), we obtain (48). $\square$

We proceed to the proof of Theorem 10. We use the standard Lagrange interpolation operator $i_h : C(\bar{\Gamma}) \rightarrow M_h$ defined by

$$i_h \mu(P) = \mu(P) \quad (\text{every node } P \text{ of } \mathcal{S}_h) \quad (61)$$

and the L^2 projection operator $\pi_h : L^2(\Gamma) \rightarrow M_h$ defined by

$$\int_{\Gamma} (\pi_h \mu - \mu) \mu_h \, d\Gamma = 0 \quad (\mu_h \in M_h). \quad (62)$$

The following results are well-known.

$$\mu \geq 0 \Rightarrow i_h \mu \geq 0, \quad (63a)$$

$$\|i_h \mu - \mu\|_{\Gamma} + h \|i_h \mu - \mu\|_{1,\Gamma} \leq Ch^2 \|\mu\|_{2,\Gamma} \quad (\mu \in H^2(\Gamma) \cap H_0^1(\Gamma)), \quad (63b)$$

$$\|\pi_h \mu\|_{\Gamma} \leq C \|\mu\|_{\Gamma} \quad (\mu \in L^2(\Gamma)), \quad (63c)$$

$$\|\pi_h \mu\|_{1,\Gamma} \leq C \|\mu\|_{1,\Gamma} \quad (\mu \in H_0^1(\Gamma)), \quad (63d)$$

$$\|\pi_h \mu - \mu\|_{\Gamma} \leq Ch \|\mu\|_{1,\Gamma} \quad (\mu \in H_0^1(\Gamma)). \quad (63e)$$

In fact, (63a), (63b), (63c), (63e) are standard. On the other hand, (63d) holds true if $\{\mathcal{S}_h\}_h$ is quasi-uniform (see [22, 14, 6]).

Remark 15. According to (63b), $\|i_h \mu\|_{1,\Gamma}$ is bounded by a positive constant depending only on μ if $\mu \in C_0^\infty(\Gamma)$.

Lemma 16. $\|\pi_h \mu - \mu\|_{M'} \leq Ch \|\mu\|_{\frac{1}{2},\Gamma}$ for any $\mu \in M$.

Proof. It follows from (63c) that $\|\pi_h \mu - \mu\|_{\Gamma} \leq C \|\mu\|_{\Gamma}$. Combining this with (63e), (15) and applying the interpolation theorem (see [29, Theorem 5.1, Ch. 1]), we obtain

$$\|\pi_h \mu - \mu\|_{\Gamma} \leq Ch^{\frac{1}{2}} \|\mu\|_{\frac{1}{2},\Gamma} \quad (\mu \in M).$$

We can use this in the following way. That is, noting (62),

$$\begin{aligned} \|\pi_h \mu - \mu\|_{M'} &= \sup_{\lambda \in M} \frac{[\pi_h \mu - \mu, \lambda]}{\|\lambda\|_{\frac{1}{2},\Gamma}} = \sup_{\lambda \in M} \frac{[\pi_h \mu - \mu, \pi_h \lambda - \lambda]}{\|\lambda\|_{\frac{1}{2},\Gamma}} \\ &\leq \sup_{\lambda \in M} \frac{\|\pi_h \mu - \mu\|_{\Gamma} \|\pi_h \lambda - \lambda\|_{\Gamma}}{\|\lambda\|_{\frac{1}{2},\Gamma}} \leq Ch \|\mu\|_{\frac{1}{2},\Gamma}. \end{aligned}$$

$\square$

Lemma 17. $\|\phi_\delta(\mu)\|_{\frac{1}{2},\Gamma} \leq C \|\mu\|_{\frac{1}{2},\Gamma}$ for any $\mu \in M$.

Proof. Using (40c) and (40d), we have $\|\phi_\delta(\mu)\|_{\Gamma} \leq C \|\mu\|_{\Gamma}$ for $\mu \in Q$ and $\|\phi_\delta(\mu)\|_{1,\Gamma} \leq C \|\mu\|_{1,\Gamma}$ for $\mu \in H_0^1(\Gamma)$. Hence, we can apply the (nonlinear) interpolation theorem (see [28, Theorem 3.1]) and (15) to get the desired result. $\square$

Proof of Theorem 10. First, we derive an estimation for k_h . We take $\tilde{\mu} \in C_0^\infty(\Gamma)$ satisfying

$$\tilde{\mu} \geq 0, \quad \tilde{\mu} \not\equiv 0 \text{ in } \Gamma, \quad \text{supp } \tilde{\mu} \subset \Gamma_1.$$

Then, setting $\tilde{\mu}_h = i_h \mu \in M_h$, we have

$$\tilde{\mu}_h \geq 0, \quad \tilde{\mu}_h \not\equiv 0 \text{ in } \Gamma, \quad \tilde{\mu}_h = 0 \text{ in } \Gamma \setminus \Gamma_1, \quad \|\tilde{\mu}_h\|_M \leq C, \quad \left| \int_{\Gamma} \tilde{\mu}_h \, d\Gamma - \int_{\Gamma} \tilde{\mu} \, d\Gamma \right| \leq Ch^2, \quad (64)$$

where those C 's depend on μ .

Since (A2) gives

$$\phi_\delta(u_{hn} + g_n) = 0 \quad \text{on } \Gamma_1,$$

we deduce from (44a) and (64)

$$\begin{aligned} k_h \int_\Gamma \tilde{\mu}_h d\Gamma &= a(u_h, \tilde{v}_h) + b(\hat{p}_h, \tilde{v}_h) - \frac{1}{\varepsilon} \int_\Gamma \phi_\delta(u_{hn} + g_n) \tilde{\mu}_h d\Gamma - (f, \tilde{v}_h) + [[\alpha, \tilde{v}_h]] \\ &= a(u_h, \tilde{v}_h) + b(\hat{p}_h, \tilde{v}_h) - (f, \tilde{v}_h) + [[\alpha, \tilde{v}_h]], \end{aligned} \quad (65)$$

where $\tilde{v}_h = \mathcal{E}_{nh} \tilde{\mu}_h \in V_h$.

This leads to

$$|k_h| \leq C_*, \quad \|p_h\| \leq C_*. \quad (66)$$

Hence, we have proved (49a).

Using (62) and (44a), we can make the following calculation.

$$\begin{aligned} \int_\Gamma \varepsilon^{-1} \pi_h \phi_\delta(u_{hn} + g_n) \mu_h d\Gamma &= \int_\Gamma \varepsilon^{-1} \phi_\delta(u_{hn} + g_n) \mu_h d\Gamma \\ &= a(u_h, v_h) + b(p_h, v_h) - (f, v_h) + [[\alpha, v_h]] \quad (\forall \mu_h \in M_h, v_h = \mathcal{E}_{nh} \mu_h \in V_h). \end{aligned}$$

Therefore,

$$\|\varepsilon^{-1} \pi_h \phi_\delta(u_{hn} + g_n)\|_{M'_h} \leq (\|u_h\|_1 + \|\hat{p}_h\| + \|f\| + \|\alpha\|_{(M^d)'} + \|g\|_1). \quad (67)$$

We write as

$$\begin{aligned} \sup_{\mu \in M} \frac{[\varepsilon^{-1} \phi_\delta(u_{hn} + g_n), \mu]}{\|\mu\|_{\frac{1}{2}, \Gamma}} &= \underbrace{\frac{1}{\varepsilon} \sup_{\mu \in M} \frac{[\phi_\delta(u_{hn} + g_n) - \pi_h \phi_\delta(u_{hn} + g_n), \mu]}{\|\mu\|_{\frac{1}{2}, \Gamma}}}_{=I_1} + \underbrace{\sup_{\mu \in M} \frac{[\varepsilon^{-1} \pi_h \phi_\delta(u_{hn} + g_n), \mu]}{\|\mu\|_{\frac{1}{2}, \Gamma}}}_{=I_2}. \end{aligned}$$

Using Lemmas 16 and 17,

$$\begin{aligned} \|\pi_h \phi_\delta(u_{hn} + g_n) - \phi_\delta(u_{hn} + g_n)\|_{M'} &\leq Ch \|\phi_\delta(u_{hn} + g_n)\|_{\frac{1}{2}, \Gamma} \\ &\leq Ch \|u_{hn} + g_n\|_{\frac{1}{2}, \Gamma} \\ &\leq Ch (\|u_h\|_1 + \|g\|_1). \end{aligned}$$

Consequently,

$$|I_1| \leq C \frac{h}{\varepsilon} (\|u_h\|_1 + \|g\|_1).$$

On the other hand, by virtue of (62), (63d) and (67), we have

$$\begin{aligned} I_2 &= \sup_{\mu \in M} \frac{[\varepsilon^{-1} \pi_h \phi_\delta(u_{hn} + g_n), \mu]}{\|\mu\|_{\frac{1}{2}, \Gamma}} \leq C \sup_{\mu \in M} \frac{[\varepsilon^{-1} \pi_h \phi_\delta(u_{hn} + g_n), \pi_h \mu]}{\|\pi_h \mu\|_{\frac{1}{2}, \Gamma}} \\ &\leq C \sup_{\mu_h \in M_h} \frac{[\varepsilon^{-1} \pi_h \phi_\delta(u_{hn} + g_n), \mu_h]}{\|\mu_h\|_{\frac{1}{2}, \Gamma}} \end{aligned}$$

Therefore, from (67),

$$|I_2| \leq C (\|u_h\|_1 + \|p_h\| + \|f\| + \|\alpha\|_{(M^d)'} + \|g\|_1)$$

Summing up those estimates, we get (49b).

Finally, using (40b),

$$\begin{aligned} -\frac{1}{\varepsilon} \int_\Gamma \phi_\delta(u_{hn} + g_n) u_{hn} d\Gamma &\geq \frac{1}{\varepsilon} \int_\Gamma \phi_\delta(u_{hn} + g_n) [u_{hn} + g_n]_- d\Gamma \geq 0 \\ &\geq \frac{1}{\varepsilon} \int_\Gamma ([u_{hn} + g_n]_-^2 - C \delta [u_{hn} + g_n]_-) d\Gamma \geq 0. \end{aligned}$$

We apply this to (56) and obtain

$$\begin{aligned} \frac{1}{\varepsilon} \int_{\Gamma} [u_{hn} + g_n]_-^2 d\Gamma &\leq (f, u_h) - [[\alpha, u_h]] + C \frac{\delta}{\varepsilon} \int_{\Gamma} [u_{hn} + g_n]_- d\Gamma \\ &\leq \|f\| \cdot \|u_h\|_1 + \|\alpha\|_{(M^d)'} \|u_h\|_1 + C \frac{\delta}{\varepsilon} (\|u_h\|_1 + \|g\|_1), \end{aligned}$$

which implies (49c). $\square$

6. Error estimate

We are now ready to state the error estimates between (PDE) and $(\text{PE}_{\varepsilon, \delta, h})$.

Theorem 18. Assume that (A1) and (A2) are satisfied. Let (u, p) and (u_h, p_h) be solutions of (PDE) and $(\text{PE}_{\varepsilon, \delta, h})$, respectively, and suppose that $(u, p) \in H^2(\Omega)^d \times H^1(\Omega)$ and $\tau_n(u, p) + \alpha_n \in M$. Moreover, assume that h, ε, δ are sufficiently small and $h \leq c_1 \varepsilon$ with a constant $c_1 > 0$, then we have

$$\|u - u_h\|_1 + \|p - p_h\| \leq C_{**} \left(h + \varepsilon + \sqrt{\frac{\delta^2}{\varepsilon}} \right) \quad (68)$$

and

$$\begin{aligned} \left\| \tau_n(u, p) + \alpha_n - \frac{1}{\varepsilon} \phi_{\delta}(u_{hn} + g_n) \right\|_{M'} + \sqrt{\varepsilon} \left\| \tau_n(u, p) + \alpha_n - \frac{1}{\varepsilon} \phi_{\delta}(u_{hn} + g_n) \right\|_{\Gamma} \\ \leq C_{**} \left(h + \varepsilon + \sqrt{\frac{\delta^2}{\varepsilon}} \right), \quad (69) \end{aligned}$$

where C_{**} denotes a positive constant depending only on $c_1, \Omega, |u|_2, |p|_1, \|\tau_n(u, p) + \alpha_n\|_M, \|f\|, \|g\|_1$ and $\|\alpha\|_{(M^d)'}$. Particularly, if taking as $c_2 \varepsilon \leq h$ and $\delta \leq c_3 h^{\frac{3}{2}}$ with constants $(c_1 >) c_2, c_3 > 0$, we have the optimal-order error estimate

$$\|u - u_h\|_1 + \|p - p_h\| \leq C_{**} h.$$

We use the standard Lagrange interpolation operator $I_h : C(\Omega)^d \rightarrow V_h$ and the L^2 projection operator $\Pi_h : Q \rightarrow Q_h$. Actually, the following are well known.

$$\|v - I_h v\|_1 \leq Ch |v|_2 \quad (v \in [H^2(\Omega) \cap H_0^1(\Omega)]^d), \quad (70a)$$

$$\|q - \Pi_h q\| \leq Ch |q|_1 \quad (q \in H^1(\Omega)). \quad (70b)$$

Proof of Theorem 18. It is divided into three steps.

Step 1. Let us show that the following non-optimal error estimate holds true:

$$\|u - u_h\|_1 + \|\hat{p} - \hat{p}_h\| \leq C_{**} \left(\sqrt{\varepsilon} + \sqrt{\delta} + \sqrt{h} \right), \quad (71)$$

where $\hat{p} = \Lambda p, \hat{p}_h = \Lambda p_h$.

We recall that (22) together with (20) give

$$a(u, v) + b(p, v) - [\tau_n(u, p), v_n] = (f, v) - [[\alpha_T, v_T]] \quad (v \in V). \quad (72)$$

Hence, errors $u - u_h$ and $p - p_h$ satisfy

$$a(u - u_h, v_h) + b(p - p_h, v_h) - [\tau_n(u, p) + \alpha_n - \varepsilon^{-1} \phi_{\delta}(u_{hn} + g_n), v_{hn}] = 0 \quad (v_h \in V_h).$$

Setting $\hat{p} = \Lambda p, \hat{p}_h = \Lambda p_h, k = m(p)$ and $k_h = m(p_h)$, we can write as

$$\begin{aligned} a(u - u_h, v_h) &= \underbrace{-b(\hat{p} - \hat{p}_h, v_h)}_{=J_1(v_h)} \\ &\quad + \underbrace{[\tau_n(u, p) + \alpha_n - \varepsilon^{-1} \phi_{\delta}(u_{hn} + g_n) + k - k_h, v_{hn}]}_{=J_2(v_h)} \quad (v_h \in V_h). \end{aligned} \quad (73)$$

Particularly we have

$$b(\Lambda q_h - \hat{p}_h, v_h) = -a(u - u_h, v_h) - b(\hat{p} - \Lambda q_h, v_h) \quad (v_h \in V_{0h}, q_h \in Q_h).$$

and, by application of (45),

$$\begin{aligned} \|\Lambda q_h - \hat{p}_h\| &\leq C \sup_{v_h \in V_{0h}} \frac{-a(u - u_h, v_h) - b(\hat{p} - \Lambda q_h, v_h)}{\|v_h\|_1} \\ &\leq C(\|u - u_h\|_1 + \|\hat{p} - \Lambda q_h\|) \\ &\leq C(\|u - u_h\|_1 + \|p - q_h\|) \quad (q_h \in Q_h). \end{aligned} \quad (74)$$

At this stage, we set

$$v_h = I_h u - u_h \in V_h, \quad q_h = \Pi_h p \in Q_h, \quad \hat{q}_h = \Lambda q_h \in Q_{0h}.$$

Then,

$$\begin{aligned} \|\hat{p} - \hat{p}_h\| &\leq \|\hat{p} - \hat{q}_h\| + \|\hat{q}_h - \hat{p}_h\| \\ &\leq \|p - \Pi_h p\| + C(\|u - u_h\|_1 + \|p - \Pi_h p\|) \\ &\leq C_{**}h + C\|u - u_h\|_1. \end{aligned} \quad (75)$$

Using (70), (74) and $\|\hat{p} - \hat{q}_h\| \leq C\|p - q_h\|$, we estimate as

$$\begin{aligned} |J_1(I_h u - u_h)| &\leq |b(\hat{p} - \hat{p}_h, I_h u - u_h)| + |b(\hat{p} - \hat{q}_h, u - u_h)| + |b(\hat{q}_h - \hat{p}_h, u - u_h)| \\ &\leq \|b\| \cdot \|\hat{p} - \hat{p}_h\| \cdot \|I_h u - u_h\|_1 + \|b\| \cdot \|\hat{p} - \hat{q}_h\| \cdot \|u - u_h\|_1 + 0 \\ &\leq C(C_{**}h + \|u - u_h\|_1) \cdot h\|u\|_2 + Ch\|p\|_1 \|u - u_h\|_1 \\ &\leq C_{**}h^2 + C_{**}h\|u - u_h\|_1 \\ &\leq C_{**}h^2 + C_{**}h\|I_h u - u_h\|_1. \end{aligned} \quad (76)$$

To perform an estimation for J_2 , we divide it as

$$\begin{aligned} J_2(I_h u - u_h) &= \underbrace{[\tau_n(u, p) + \alpha_n - \varepsilon^{-1}\phi_\delta(u_{hn} + g_n) + k - k_h, (I_h u)_n - u_n]}_{=J_{21}} \\ &\quad + \underbrace{[\tau_n(u, p) + \alpha_n - \varepsilon^{-1}\phi_\delta(u_{hn} + g_n) + k - k_h, u_n - u_{hn}]}_{=J_{22}}. \end{aligned} \quad (77)$$

According to stability results (48) and (49b), we deduce

$$\begin{aligned} |J_{21}| &\leq (\|\tau_n(u, \hat{p})\|_{M'} + \|\varepsilon^{-1}\phi_\delta(u_{hn} + g_n)\|_{M'} + \|\alpha_n\|_{M'} + |k_h|) \|(I_h u)_n - u_n\|_{\frac{1}{2}, \Gamma} \\ &\leq C_* \|I_h u - u\|_1 \leq C_{**}h. \end{aligned}$$

Noting

$$\int_{\Gamma} (u_n - u_{hn}) \, d\Gamma = \int_{\Omega} \nabla \cdot (u - u_h) \, dx = 0$$

and using (22c), (22d), (22e), (40b), (40c) and (49b), we can calculate as:

$$\begin{aligned}
J_{22} &= [\tau_n(u, p) + \alpha_n - \varepsilon^{-1} \phi_\delta(u_{hn} + g_n), u_n + g_n - (u_{hn} + g_n)] \\
&= -[\varepsilon^{-1} \phi_\delta(u_{hn} + g_n), u_n + g_n] - [\tau_n(u, p) + \alpha_n, u_{hn} + g_n] \\
&\quad + [\varepsilon^{-1} \phi_\delta(u_{hn} + g_n), u_{hn} + g_n] \\
&= \underbrace{-[\varepsilon^{-1} \phi_\delta(u_{hn} + g_n), u_n + g_n]}_{\leq 0} \underbrace{-[\tau_n(u, p) + \alpha_n, [u_{hn} + g_n]_+]}_{\leq 0} \\
&\quad + [\tau_n(u, p) + \alpha_n, [u_{hn} + g_n]_-] \underbrace{-[\varepsilon^{-1} \phi_\delta(u_{hn} + g_n), [u_{hn} + g_n]_-]}_{\leq 0} \\
&\leq [\tau_n(u, p) + \alpha_n, [u_{hn} + g_n]_- - \phi_\delta(u_{hn} + g_n)] + \varepsilon[\tau_n(u, p) + \alpha_n, \varepsilon^{-1} \phi_\delta(u_{hn} + g_n)] \\
&\leq \|\tau_n(u, p) + \alpha_n\|_\Gamma \| [u_{hn} + g_n]_- - \phi_\delta(u_{hn} + g_n) \|_\Gamma \\
&\quad + \varepsilon \|\tau_n(u, p) + \alpha_n\|_M \|\varepsilon^{-1} \phi_\delta(u_{hn} + g_n)\|_{M'} \\
&\leq C_* \left[\delta + \varepsilon \left(1 + \frac{h}{\varepsilon} \right) \right] \leq C_*(\delta + \varepsilon + h). \tag{78}
\end{aligned}$$

Summing up those estimates, we obtain

$$\begin{aligned}
C \|I_h u - u_h\|_1^2 &\leq a(I_h u - u_h, I_h u - u_h) \\
&= a(I_h u - u, I_h u - u_h) + a(u - u_h, I_h u - u_h) \\
&= a(I_h u - u, I_h u - u_h) + J_1(I_h u - u_h) + \textcolor{red}{J}_2(I_h u - u_h) \\
&\leq C_{**} h \|I_h u - u_h\|_1 + C_{**} h^2 + C_{**} h \|I_h u - u_h\|_1 + C_{**} h + C_*(\delta + \varepsilon + \textcolor{red}{h}). \tag{79}
\end{aligned}$$

Therefore, we deduce

$$\|I_h u - u_h\|_1 \leq C_{**} (\sqrt{h} + \sqrt{\varepsilon} + \sqrt{\delta})$$

and

$$\|u - u_h\|_1 \leq \|u - I_h u\|_1 + \|I_h u - u_h\|_1 \leq C_{**} (\sqrt{h} + \sqrt{\varepsilon} + \sqrt{\delta}).$$

This, together with (75), implies (71).

Step 2. We derive an estimation for $|k_h - k|$ by using (71). For sufficiently small ε, h, δ with $h \leq c_1 \varepsilon$, according to (71) and (A2), there exists $\Gamma_0 \subset \Gamma$ with $|\Gamma_0| > 0$ such that $u_n + g_n > 0$ and $u_{hn} + g_n > 0$ on Γ_0 . As in the proof of Theorem 10 (see (64)), we take $\tilde{\mu} \in C_0^\infty(\Gamma)$ satisfying $\tilde{\mu} \geq 0$, $\tilde{\mu} \not\equiv 0$ in Γ and $\text{supp } \tilde{\mu} \subset \Gamma_0$. Then, setting $\tilde{\mu}_h = i_h \mu \in M_h$, we have

$$\tilde{\mu}_h \geq 0, \quad \tilde{\mu}_h \not\equiv 0 \text{ in } \Gamma, \quad \tilde{\mu}_h = 0 \text{ in } \Gamma \setminus (\Gamma_0), \quad \|\tilde{\mu}_h\|_M \leq C, \quad \left| \int_\Gamma \tilde{\mu}_h \, d\Gamma - \int_\Gamma \tilde{\mu} \, d\Gamma \right| \leq Ch^2.$$

Since $u_n + g_n > 0$ on Γ_0 , we have $\tau_n(u, p) + \alpha_n = 0$ on Γ_0 in view of (22e). Substituting $\tilde{v}_h = \mathcal{E}_{nh} \tilde{\mu}_h \in V_h \subset V$ into (72) and using (22d), we have

$$\begin{aligned}
k \int_\Gamma \tilde{\mu}_h \, d\Gamma &= a(u, \tilde{v}_h) + b(\hat{p}, \tilde{v}_h) - (f, \tilde{v}_h) + [[\alpha, \tilde{v}_h]] - \int_\Gamma (\tau_n(u, p) + \alpha_n) \tilde{\mu}_h \, d\Gamma \\
&= a(u, \tilde{v}_h) + b(\hat{p}, \tilde{v}_h) - (f, \tilde{v}_h) + [[\alpha, \tilde{v}_h]].
\end{aligned}$$

This, together with (65) and (75), gives

$$\begin{aligned}
|k_h - k| &\leq |a(u_h - u, \tilde{v}_h)| + |b(\hat{p}_h - \hat{p}, \tilde{v}_h)| \\
&\leq C(\|u_h - u\|_1 + \|\hat{p}_h - \hat{p}\|) \leq C_{**}(\|I_h u - u_h\|_1 + h). \tag{80}
\end{aligned}$$

Step 3. We proceed to the proof of (68) and (69). We will prove

$$J_2(I_h u - u_h) + \frac{\varepsilon}{2} \|\lambda\|_\Gamma^2 \leq C_{**} \left[h^2 + \frac{h^3}{\varepsilon} + \varepsilon h + \frac{\delta^2}{\varepsilon} + (\varepsilon + h) \|I_h u - u_h\|_1 \right], \tag{81}$$

where

$$\lambda = \tau_n(u, p) + \alpha_n - \frac{1}{\varepsilon} \phi_\delta(u_{hn} + g_n).$$

Recall that $J_2(I_h u - u_h)$ is divided into $J_{21} + J_{22}$ as in (77). We have, by (80) and the standard trace theorem,

$$J_{21} \leq (\|\lambda\|_{M'} + |k - k_h|) \|(I_h u)_n - u_n\|_{\frac{1}{2}, \Gamma} \leq C_{**} h (\|\lambda\|_{M'} + h + \|I_h u - u_h\|_1). \quad (82)$$

We derive an estimation for $\|\lambda\|_{M'}$. First,

$$\|\lambda\|_{M'} = \sup_{\mu \in M} \left(\frac{[\lambda, \mu - \mu_h]}{\|\mu\|_{\frac{1}{2}, \Gamma}} + \frac{[\lambda, \mu_h]}{\|\mu\|_{\frac{1}{2}, \Gamma}} \right) \leq C \|\lambda\|_\Gamma \sup_{\mu \in M} \frac{\|\mu - \mu_h\|_\Gamma}{\|\mu\|_{\frac{1}{2}, \Gamma}} + \sup_{\mu \in M} \frac{[\lambda, \mu_h]}{\|\mu\|_{\frac{1}{2}, \Gamma}}, \quad (83)$$

where μ_h is an arbitrary element of M_h .

In order to set μ_h appropriately, we use Scott and Zhang's projection $\tilde{\Pi}_h : H^1(\Omega)^d \rightarrow Q_h^d$ ([31]). The projection $\tilde{\Pi}_h$ satisfies the same stability and interpolation error estimates as Π_h . For example, we have $\|\tilde{\Pi}_h v\| \leq C\|v\|$, $\|\tilde{\Pi}_h v\|_1 \leq C\|v\|_1$ and $\|v - \tilde{\Pi}_h v\| \leq Ch\|v\|_1$ (see [9, §4.8]). Particularly, we have $\|v - \tilde{\Pi}_h v\|_{H^{\frac{1}{2}}(\Omega)} \leq Ch^{\frac{1}{2}}\|v\|_1$ by the interpolation. Furthermore, $\tilde{\Pi}_h$ preserves the boundary condition; $\tilde{\Pi}_h v \in V_h$ for $v \in V$. At this stage, for $\mu \in M$, we set

$$\mu_h = (w_h \cdot n)|_\Gamma, \quad w_h = \tilde{\Pi}_h \mathcal{E}_n \mu.$$

Then, again by the standard trace theorem,

$$\|\mu - \mu_h\|_\Gamma \leq C \|\mathcal{E}_n \mu - \tilde{\Pi}_h \mathcal{E}_n \mu\|_1 \leq Ch^{\frac{1}{2}} \|\mathcal{E}_n \mu\|_1 \leq Ch^{\frac{1}{2}} \|\mu\|_{\frac{1}{2}, \Gamma}.$$

In view of (73),

$$[\lambda, \mu_h] = a(u - u_h, w_h) + b(\hat{p} - \hat{p}_h, w_h) - (k - k_h) \int_\Gamma \mu_h \, d\Gamma$$

and, by (75) and (80),

$$\begin{aligned} [\lambda, \mu_h] &\leq C(\|u - u_h\|_1 + \|\hat{p} - \hat{p}_h\| + |k - k_h|) \|w_h\|_1 \\ &\leq C_{**}(h + \|I_h u - u_h\|_1) \|\mu\|_{\frac{1}{2}, \Gamma}. \end{aligned}$$

Putting those estimates together in (83), we obtain

$$\|\lambda\|_{M'} \leq C_{**}(h^{\frac{1}{2}} \|\lambda\|_\Gamma + h + \|I_h u - u_h\|_1). \quad (84)$$

Substituting this into (82) and applying Schwarz's inequality, we deduce

$$\begin{aligned} J_{21} &\leq C_{**}(h^2 + Ch^{\frac{3}{2}} \|\lambda\|_\Gamma + h \|I_h u - u_h\|_1) \\ &\leq C_{**} \left(h^2 + \frac{h^3}{\varepsilon} + h \|I_h u - u_h\|_1 \right) + \frac{\varepsilon}{6} \|\lambda\|_\Gamma^2. \end{aligned} \quad (85)$$

Next, we derive a sharp estimate for J_{22} . According to the third equality of (78), we have

$$J_{22} \leq [\lambda, [u_{hn} + g_n]_-] = \underbrace{[\lambda, [u_{hn} + g_n]_- - \phi_\delta(u_{hn} + g_n)]}_{=J_{221}} - \varepsilon [\lambda, \lambda] + \underbrace{\varepsilon [\lambda, \tau_n(u, p) + \alpha_n]}_{=J_{222}}.$$

From (40b),

$$J_{221} \leq \|\lambda\|_\Gamma \|[u_{hn} + g_n]_- - \phi_\delta(u_{hn} + g_n)\|_\Gamma \leq C\delta \|\lambda\|_\Gamma \leq C \frac{\delta^2}{\varepsilon} + \frac{\varepsilon}{6} \|\lambda\|_\Gamma^2.$$

Using (84), we get the estimation of J_{222} as follows:

$$\begin{aligned} J_{222} &\leq \varepsilon \|\lambda\|_{M'} \|\tau_n(u, p) + \alpha_n\|_{\frac{1}{2}, \Gamma} \\ &\leq C_{**} \varepsilon (h^{\frac{1}{2}} \|\lambda\|_\Gamma + h + \|I_h u - u_h\|_1) \leq C_{**} (\varepsilon h + \varepsilon \|I_h u - u_h\|_1) + \frac{\varepsilon}{6} \|\lambda\|_\Gamma^2. \end{aligned}$$

Summing up those estimates, we obtain

$$J_{22} \leq C_{**} \left(\varepsilon h + \frac{\delta^2}{\varepsilon} + \varepsilon \|I_h u - u_h\|_1 \right) - \frac{2\varepsilon}{3} \|\lambda\|_\Gamma^2.$$

Combing this with (85), we deduce (81).

Now, instead of (79), we have by (76) and (81)

$$\begin{aligned} C \|I_h u - u_h\|_1^2 &\leq a(I_h u - u, I_h u - u_h) + J_1(I_h u - u_h) + J_2(I_h u - u_h) \\ &\leq C_{**}(h + \varepsilon) \|I_h u - u_h\|_1 + C \left(h^2 + \frac{h^3}{\varepsilon} + \varepsilon h + \frac{\delta^2}{\varepsilon} \right) - \frac{\varepsilon}{2} \|\lambda\|_\Gamma^2. \end{aligned}$$

This yields

$$\|I_h u - u_h\|_1 \leq C_{**} \left(h + \sqrt{\frac{h^3}{\varepsilon}} + \sqrt{\varepsilon h} + \sqrt{\frac{\delta^2}{\varepsilon}} \right) \leq C_{**} \left(h + \varepsilon + \sqrt{\frac{\delta^2}{\varepsilon}} \right),$$

since we chose as $h \leq c_1 \varepsilon$.

Finally, we obtain (68) and (69) by combining (75), (80), (81) and (84). This completes the proof of Theorem 18. $\square$

Remark 19. In addition to the basic assumption of Theorem 18, we suppose that

$$u_{hn} + g_n > 0 \quad \text{on } \Gamma. \quad (86)$$

Then, for sufficiently small h, ε, δ with $c_2 \varepsilon \leq h \leq c_1 \varepsilon$, we have

$$\|u - u_h\|_1 + \|p - p_h\| \leq C_{**} h. \quad (87)$$

In particular, we do not need to choose as $\delta \leq c_3 h^{\frac{3}{2}}$. Inequality (87) is derived by noting $J_{22} = 0$ under (86).

7. Numerical examples

In this section, we present some results of numerical experiments [to confirm](#) our theoretical results. We prefer the original setting (4) with (6), (11) to (1) and (2). [Therefore](#), we consider a model Stokes problem with a nonlinear Robin condition [as](#)

$$-\nu \Delta v + \nabla q = f, \quad \nabla \cdot v = 0 \quad \text{in } \Omega, \quad (88a)$$

$$v = b \quad \text{on } S_1, \quad (88b)$$

$$v = 0 \quad \text{on } S_2, \quad (88c)$$

$$\tau_n(v, q) = \frac{1}{\varepsilon} \phi_\delta(v_n), \quad v_T = 0 \quad \text{on } \Gamma, \quad (88d)$$

where ϕ_δ [represents](#) the regularized function defined as (3).

Remark 20. As [described](#) in Introduction, we are interested in computing v and q in (4). The unknown functions u and p in (1) and (2) are introduced as “perturbations” of those target variables. They [clarify analysis](#). Moreover, the reference flow (g, π) plays an important role in theoretical considerations, [although it is not readily apparent that](#) it is always available in actual computations.

Remark 21. In (88), we take $v_T = 0$ instead of $\tau_T(u) = 0$ as a boundary condition for the tangential component of v on Γ . See Remark 7.

The finite element approximation for (88) reads as follows.

($\mathbf{PE}'_{\varepsilon, \delta, h}$) Find $(v_h, q_h) \in W_h \times Q_h$ such that $v_h = i_h b$ on S_1 and

$$a(v_h, w_h) + b(q_h, w_h) - \frac{1}{\varepsilon} \int_\Gamma \phi_\delta(v_n) w_{hn} \, d\Gamma = (f, w_h) \quad (\forall w_h \in V_h),$$

$$b(r_h, v_h) = 0 \quad (\forall r_h \in Q_h),$$

where

$$W_h = \{v_h \in C^0(\bar{\Omega}) \mid v_h = 0 \text{ on } S_2, v_h|_T = 0 \text{ on } \Gamma, v_h|_T \in [\mathcal{P}_1^{(d)} \oplus \text{span}\{\varphi_T\}]^d \ (\forall T \in \mathcal{T}_h)\}.$$

First, we deal with a simple example, [setting](#) $\Omega = \{(x, y) \mid 0 \leq x \leq L, -R \leq y \leq R\}$, $S_1 = \{0\} \times [-R, R]$, and $\Gamma = \{L\} \times [-R, R]$, we impose

$$b(x, y) = (C_0(R^2 - y^2), 0), \quad f \equiv 0 \quad (89)$$

with $C_0 > 0$. Then, (88) has the exact solution, which is given [explicitly](#) as

$$v(x, y) = (C_0(R^2 - y^2), 0), \quad q(x, y) = 2\nu C_0 L \left(1 - \frac{x}{L}\right). \quad (90)$$

This is [the well-known](#) Poiseuille flow.

[Details](#) of our computation are [the following](#). Set $L = 15$, $R = 5$, $\nu = 1/50$, and $C_0 = 5/(\nu L)$. For the triangulation of Ω , we use a uniform mesh composed of $12N^2$ congruent right-angle triangles; The rectangle is divided into $3N \times 2N$ squares. Then, each small square is decomposed into two equal triangles by a diagonal. Consequently, $h = \sqrt{2}/N$. Since we have employed the C^1 regularization ϕ_δ , Newton's method is available for computing the nonlinear equation $(\text{PE}'_{\varepsilon, \delta, h})$. Penalty parameters are chosen as $\varepsilon = \delta = h/20$. [Hence, it is ensured by Theorem 18 and Remark 19 that](#)

$$\|v - v_h\|_1 + \|q - q_h\| \leq Ch. \quad (91)$$

[To verify this point](#), we set

$$E_h^{(1)} = \|v - v_h\|, \quad E_h^{(2)} = \|v - v_h\|_1, \quad E_h^{(3)} = \|q - q_h\|,$$

and observe [that](#)

$$\rho_h^{(i)} = \frac{\log E_{h'}^{(i)} - \log E_h^{(i)}}{\log h' - \log h} \quad (i = 1, 2, 3)$$

with $h' \approx 2h$.

The result is reported in Tab. 1 [and support our theoretical result \(91\)](#). [We were unable to derive the \$L^2\$ error for \$v_h\$. From Tab. 1, we observe that second-order convergence actually occurs.](#)

h	$E_h^{(1)}$	$\rho_h^{(1)}$	$E_h^{(2)}$	$\rho_h^{(2)}$	$E_h^{(3)}$	$\rho_h^{(3)}$
1.0743	13.9	—	$1.20 \cdot 10^2$	—	$2.07 \cdot 10^{-1}$	—
0.5371	3.47	2.001	$5.96 \cdot 10^1$	1.010	$6.57 \cdot 10^{-2}$	1.656
0.2685	0.87	2.000	$2.97 \cdot 10^1$	1.003	$2.18 \cdot 10^{-2}$	1.594
0.1342	0.21	2.000	$1.48 \cdot 10^1$	1.001	$7.42 \cdot 10^{-3}$	1.553
0.0665	0.052	2.000	7.17	1.000	$2.56 \cdot 10^{-3}$	1.527

Table 1: Numerical convergence rates of $(\text{PE}'_{\varepsilon, \delta, h})$ for (90).

Next, we consider a two-dimensional branched pipe as [portrayed in](#) Fig. 2. Since this Ω is not a polygon, we approximate it by a polygon Ω_h [with vertices located](#) on $\partial\Omega$. On S_1 , we impose a parabolic inflow similarly to (89). Fig. 3 shows the state of a numerical flow velocity v_h .

As before, we observe $\rho_h^{(1)}$, $\rho_h^{(2)}$ and $\rho_h^{(3)}$. Since, in this case, we [are unable to](#) obtain the (explicit) exact solution, we [use](#) numerical solutions with extra fine mesh. Tab. 2 [presents](#) the results. We observe that [convergence rates](#) of the H^1 error for v_h and the L^2 error for q_h are close to [unity](#) even in the curved domain. Moreover, [that of](#) the L^2 error for v_h is close to 2.

[Finally, we examine the convergence in terms of the penalty parameter \$\varepsilon\$. We now consider a non-trivial external force \$f\$ and the resulting flow velocity is showed in Fig. 4.](#)

Figure 2: A branched pipe and an example of triangulation.

Figure 3: Velocity and pressure field in branched pipe.

Letting $\delta = \varepsilon^{\frac{3}{2}}$ (see Theorem 18), we set

$$E_{\varepsilon}^{(1)} = \|v - v_h\|, \quad E_{\varepsilon}^{(2)} = \|v - v_h\|_1, \quad E_{\varepsilon}^{(3)} = \|q - q_h\|,$$

and observe

$$\rho_{\varepsilon}^{(i)} = \frac{\log E_{\varepsilon'}^{(i)} - \log E_{\varepsilon}^{(i)}}{\log \varepsilon' - \log \varepsilon} \quad (i = 1, 2, 3),$$

where $\varepsilon' \approx 2\varepsilon$ and (v, q) is the numerical solution with extra small ε .

Results are reported in Tab. 3 and 4, respectively, for $h = 0.6928$ and $h = 0.1757$. We observe from these tables that the first order convergence with respect to ε actually occurs.

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h	$E_h^{(1)}$	$\rho_h^{(1)}$	$E_h^{(2)}$	$\rho_h^{(2)}$	$E_h^{(3)}$	$\rho_h^{(3)}$
0.69279	$2.497 \cdot 10^{-1}$	—	5.941	—	$1.786 \cdot 10^{-1}$	—
0.33353	$7.767 \cdot 10^{-2}$	1.552	3.359	0.780	$5.909 \cdot 10^{-2}$	1.513
0.17571	$2.044 \cdot 10^{-2}$	2.083	1.768	1.001	$3.069 \cdot 10^{-2}$	1.022

Table 2: Numerical convergence rates of $(PE'_{\varepsilon,\delta,h})$ for branched pipe.

Figure 4: Velocity and pressure fields.

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ε	$E_\varepsilon^{(1)}$	$\rho_\varepsilon^{(1)}$	$E_\varepsilon^{(2)}$	$\rho_\varepsilon^{(2)}$	$E_\varepsilon^{(3)}$	$\rho_\varepsilon^{(3)}$
0.034639	7.315	—	22.400	—	9.359	—
0.017319	3.649	1.003	11.177	1.002	4.668	1.003
0.008659	1.749	1.060	5.358	1.060	2.238	1.060

Table 3: Numerical convergence rate of $(PE'_{\varepsilon,\delta,h})$ for $h = 0.6928$.

ε	$E_\varepsilon^{(1)}$	$\rho_\varepsilon^{(1)}$	$E_\varepsilon^{(2)}$	$\rho_\varepsilon^{(2)}$	$E_\varepsilon^{(3)}$	$\rho_\varepsilon^{(3)}$
0.0087854	2.088	—	6.421	—	2.469	—
0.0043927	1.024	1.028	3.149	1.027	1.210	1.028
0.0021964	0.486	1.073	1.497	1.072	0.575	1.073

Table 4: Numerical convergence rate of $(PE'_{\varepsilon,\delta,h})$ for $h = 0.1757$.

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