θ -Maruyama methods for nonlinear stochastic differential delay equations[☆]

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Abstract

In this paper, mean-square convergence and mean-square stability of θ -Maruyama methods are studied for nonlinear stochastic differential delay equations (SDDEs) with variable lag. Under global Lipschitz conditions, the methods are proved to be mean-square convergent with order $\frac{1}{2}$, and exponential mean-square stability of SDDEs implies that of the methods for sufficiently small step size $h > 0$. Further, the exponential mean-square stability properties of SDDEs and those of numerical methods are investigated under some non-global Lipschitz conditions on the drift term. It is shown in this setting that the θ -Maruyama method with $\theta = 1$ can preserve the exponential mean-square stability for any step size. Additionally, the θ -Maruyama method with $\frac{1}{2} \leq \theta \leq 1$ is asymptotically mean-square stable for any step size, provided that the underlying system with constant lag is exponentially mean-square stable. Applications of this work to some special problem classes show that the results are deeper or sharper than those in the literature. Finally, numerical experiments are included to demonstrate the obtained theoretical results.

Keywords: nonlinear stochastic differential delay equations, variable lag, θ -Maruyama methods, strong convergence, exponential mean-square stability, mean-square stability

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1. Introduction

Delay differential equations (DDEs) are widely utilized for mathematical models in physics and have been applied to various application areas [13, 14, 22, 33] with great success. The general DDEs with variable delay take the form

$$X'(t) = F(t, X(t), X(t - \tau(t))), \quad t \geq t_0. \quad (1)$$

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For accurate simulations of many engineering systems, in addition to the estimation error for system parameters, the environmental noise should also be considered seriously, which results in the wide application of stochastic differential delay equations (SDDEs) in plenty of fields [5, 9, 21, 27]. In this paper, a stochastic generalization of deterministic DDEs (1) in Itô's sense is considered as follows:

$$dX(t) = F(t, X(t), X(t - \tau(t)))dt + G(t, X(t), X(t - \tau(t)))dW(t), \quad t \geq t_0 \quad (2)$$

with the initial data $\psi \in L^2_{\mathcal{F}_{t_0}}([-\tau, 0]; \mathbb{R}^d)$. Here $\tau(t)$ is a continuous function satisfying $\tau(t) \geq 0$ and $-\tau := \inf\{t - \tau(t) - t_0 : t \geq t_0\}$. In general, there is no analytical closed-form solution for a SDDE, and a numerical approximation is to be sought, which necessitates the development of stable and efficient numerical schemes.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq t_0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq t_0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_{t_0}$ contains all the $\mathbb{P}$ -null sets). Let $W(t) := (W_1(t), \dots, W_m(t))^T$ be an m -dimensional Wiener process defined on the probability space. And $|\cdot|$ is used to denote both the Euclidean norm in $\mathbb{R}^d$ and the trace norm (F-norm) in $\mathbb{R}^{d \times m}$. Also, $\mathcal{C}([-\tau, 0]; \mathbb{R}^d)$ is used to represent the family of continuous functions ψ from $[-\tau, 0]$ to $\mathbb{R}^d$. Finally, $L^2_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^d)$ is used to denote the family of $\mathcal{F}_t$ -measurable, $\mathcal{C}([-\tau, 0]; \mathbb{R}^d)$ -valued random variables $\psi = \{\psi(u) : -\tau \leq u \leq 0\}$ such that $\|\psi\|_E^2 := \sup_{-\tau \leq u \leq 0} E|\psi(u)|^2 < \infty$.

It is assumed that both $F : [t_0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $G : [t_0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$ satisfy the global Lipschitz condition and linear growth condition as follows:

- (H1) (Global Lipschitz Condition) there exists a constant L such that

$$|F(t, x_2, y_2) - F(t, x_1, y_1)|^2 \vee |G(t, x_2, y_2) - G(t, x_1, y_1)|^2 \leq L(|x_2 - x_1|^2 + |y_2 - y_1|^2),$$

for all $t \geq t_0$ and $x_1, y_1, x_2, y_2 \in \mathbb{R}^d$.

- (H2) (Linear Growth Condition) there exists a constant K such that

$$|F(t, x, y)|^2 \vee |G(t, x, y)|^2 \leq K(1 + |x|^2 + |y|^2),$$

for all $t \geq t_0, x, y \in \mathbb{R}^d$.

Under the conditions of (H1) and (H2), there exists a solution that is path-wise unique on $[t_0 - \tau, \infty)$ for (2), which is discussed in [29].

In numerical analysis for stochastic differential equations (SDEs), convergence and stability are the two most important issues [7, 8, 15, 16, 17, 20, 24, 32, 34, 35]. For SDDEs, most of the existing works on numerical methods handle the cases which are of a constant lag τ and step size h being a fraction of τ [1, 3, 6, 19, 26, 31]. However, DDEs with time-varying lag (1) and their stochastic counterpart (2) play an important role in engineering modeling. For example, letting $\tau(t) = (1 - q)t, 0 < q < 1$ in (1) and (2) generates a deterministic pantograph equation

$$X'(t) = F(t, X(t), X(qt)), \quad t \geq 0, \quad X(0) = X_0, \quad (3)$$

and a stochastic pantograph equation

$$dX(t) = F(t, X(t), X(qt))dt + G(t, X(t), X(qt))dW(t), \quad t \geq 0, \quad X(0) = X_0. \quad (4)$$

Equation (3) is often used for modelling the dynamics of an overhead current collection system for an electric locomotive or modelling a one-dimensional wave motion [33, 11]. Concerning (4), Baker and Buckwar [2] established the strong convergence of θ -Maruyama methods for linear stochastic pantograph equations. Later on, Fan, Liu and Cao [10] presented the strong convergence of θ -Maruyama methods for general stochastic pantograph equation (4). For general SDDEs with variable lag, Mao and Sabanis [30] proved that the Euler-Maruyama method (corresponding to θ -Maruyama methods with $\theta = 0$) is convergent under local Lipschitz conditions. To the best of our knowledge, no analysis of the convergence and stability of semi-implicit numerical methods ($\theta > 0$) for SDDEs with general variable delays has been presented in the literature, which motivates the current work.

The main contributions of this work could be summarized as follows:

(a) This work first introduces θ -Maruyama methods with piecewise linear interpolation for SDDEs with variable lag. Given that the delay function $\tau(t)$ satisfies Lipschitz conditions, strong convergence of order $1/2$ is obtained for the schemes, for SDDEs with constant delays [1] and proportional delays [2, 10] as special cases.

(b) Based on the idea from [17, 31], that is, asymptotic property of numerical methods studied via finite-time convergence, we investigate the exponential mean-square stability of the numerical methods. Under global Lipschitz conditions, it is proved that, for sufficiently small step sizes, θ -Maruyama methods can reproduce the exponential mean-square stability of the SDDEs. This can be regarded as an extension of that in [31] to variable delay case.

(c) Provided that the drift term satisfies non-global Lipschitz conditions, sufficient conditions are obtained for exponential mean-square stability of SDDEs (2) with complex coefficients, for linear scalar SDDEs with constant delays [26] and SODE [34] as special cases. The sufficient conditions are weaker than those in [29]. Under some appropriate conditions (i.e. C2, C3, C4 in Section 4), the backward Euler method ($\theta = 1$) is proved to possess exponential mean-square stability without any constraint on step size $h > 0$. Furthermore, under slightly stronger conditions ((C2), (C3'), (C4) in Section 4), the θ -Maruyama methods with $0 \leq \theta < 1$ are able to inherit the exponential mean-square stability, but with a restriction on step size choice.

(d) When the exponential mean-square stability of the numerical methods is replaced by a weaker one, that is, asymptotic mean-square stability, it is shown that the latter relaxes the restriction on step size. In other words, for $\frac{1}{2} \leq \theta \leq 1$ and any step size $h > 0$, the θ -Maruyama method is asymptotically mean-square stable, provided that the SDDE with constant lag is exponentially mean-square stable. Although the stability (mean-square stability) of the θ -Maruyama methods is weaker than that (exponential mean-square stability) of the underlying system, specializing this work to the cases of linear scalar SDDEs, linear scalar SODEs and nonlinear deterministic DDEs, we have the following interesting observations. For linear scalar SDDEs, the result in this paper is sharper than that presented in [26]. For linear scalar SODEs with constant coefficients, the result is identical to that presented in [15]. For nonlinear deterministic DDEs, the result hasn't been presented in the literature.

The remainder of the paper is organized as follows. In Section 2, the strong convergence order of θ -Maruyama methods is proved to be $1/2$ under (H1)-(H2). In Section

3, the exponential mean-square stability of the methods is discussed under global Lipschitz conditions. In Section 4, under non-global Lipschitz condition on drift term, the exponential mean-square stability of the SDDEs are studied, the exponential mean-square stability and asymptotic mean-square stability of the θ -Maruyama methods are investigated. Finally, numerical experiments are reported in Section 5.

2. Finite-time convergence of θ -Maruyama methods

On a finite time interval $[t_0, T]$, a uniform partition is defined by

$$t_i = t_0 + ih, \quad i = 0, 1, \dots, N, \quad h = \frac{T - t_0}{N}.$$

The θ -Maruyama methods with piecewise linear interpolation applied to (2) yield

$$\begin{aligned} Y_{n+1} &= Y_n + \theta F(t_{n+1}, Y_{n+1}, \bar{Y}_{n+1})h \\ &\quad + (1 - \theta)F(t_n, Y_n, \bar{Y}_n)h + G(t_n, Y_n, \bar{Y}_n)\Delta W_n, \end{aligned} \quad (5)$$

where

$$\bar{Y}_n = \begin{cases} \psi(t_n - \tau(t_n)), & t_n - \tau(t_n) \leq t_0, \\ \gamma_n Y_{n-q_n-1} + (1 - \gamma_n)Y_{n-q_n}, & t_0 < t_n - \tau(t_n) = \gamma_n t_{n-q_n-1} + (1 - \gamma_n)t_{n-q_n} \end{cases} \quad (6)$$

for $n=0, 1, \dots, N$, $0 \leq q_n \in \mathbb{N}$, $0 \leq \gamma_n < 1$, where θ is a parameter with $0 \leq \theta \leq 1$, $h > 0$ is a step size defined by $h := (T - t_0)/N$ and $\Delta W_n := W(t_{n+1}) - W(t_n)$. Y_n denotes the solution of stochastic difference delay equations (5), which is $\mathcal{F}_{t_n}$ -measurable. Note that for $\theta > 0$, (5) becomes implicit. Under assumptions (H1)-(H2), for a sufficiently small step size h equation (5) admits a unique solution with probability one, based on the classic Banach contraction theorem.

Definition 1. The solution Y_n for (2) is said to be mean-square convergent with order p , if

$$\max_{0 \leq n \leq N} (E(|X(t_n) - Y_n|^2))^{1/2} \leq Ch^p \quad \text{as } h \rightarrow 0,$$

where the constant C does not depend on h , but may depend on T and the initial data.

Theorem 2. Assume conditions (H1) and (H2) hold. Moreover, assume that

(i) $\psi(t)$ is Hölder continuous in mean-square sense with exponent $1/2$:

$$E|\psi(t) - \psi(s)|^2 \leq L_1|t - s|.$$

(ii) the delay function $\tau(t)$ is Lipschitz continuous, namely, there exists a Lipschitz constant L_2 such that

$$|\tau(t) - \tau(s)| \leq L_2|t - s| \quad \text{for all } t, s \geq t_0.$$

(iii) $|F(t, x, y) - F(s, x, y)|^2 \vee |G(t, x, y) - G(s, x, y)|^2 \leq K_0(|x|^2 + |y|^2)|t - s|$ for all $x, y \in \mathbb{R}^d$, $t, s \geq t_0$, $|t - s| < 1$.

Then the θ -Maruyama methods (5)-(6) for (2) are mean-square convergent with order $1/2$.

Before presenting the proof of Theorem 2, two lemmas go first.

Lemma 3. *Under the condition of (H2), the solution of (2) satisfies*

$$\sup_{t_0 - \tau \leq t \leq T} E(|X(t)|^2) \leq C_1 := (1 + \|\psi\|_E^2) e^{(6K+1)(T-t_0)}.$$

Moreover, for any $t_0 \leq s, t \leq T$

$$E|X(t) - X(s)|^2 \leq C_2 |t - s|,$$

where $C_2 = 2K(4C_1 + T + 1)$.

Proof. The proof can be obtained as in Lemma 5.5.2 in [29] and it is omitted here. $\square$

Lemma 4. *Assume conditions (H2) and (i),(ii) in Theorem 2 hold, then for $s \in [t_n, t_{n+1})$, $0 \leq n \leq N - 1$, we have*

$$E|X(s - \tau(s)) - X(t_n - \tau(t_n))|^2 \vee E|X(s - \tau(s)) - X(t_{n+1} - \tau(t_{n+1}))|^2 \leq \tilde{C}_2 h, \quad (7)$$

where $\tilde{C}_2 = 2(C_2 + L_1)(L_2 + 1)$.

Proof. In the case that $s - \tau(s) \geq t_0$ and $t_n - \tau(t_n) \geq t_0$, by Lemma 3 and condition (ii), we derive

$$E|X(s - \tau(s)) - X(t_n - \tau(t_n))|^2 \leq C_2 |s - \tau(s) - t_n + \tau(t_n)| \leq C_2 (L_2 + 1) |s - t_n|.$$

If both of them are less than t_0 , using conditions (i)-(ii), one sees

$$E|X(s - \tau(s)) - X(t_n - \tau(t_n))|^2 \leq L_1 |s - \tau(s) - t_n + \tau(t_n)| \leq L_1 (L_2 + 1) |s - t_n|.$$

Otherwise, either of them is less than t_0 , say, $t_n - \tau(t_n) < t_0$ and $s - \tau(s) \geq t_0$, then combining the elementary inequality and estimates obtained in the former cases yields

$$\begin{aligned} & E|X(s - \tau(s)) - X(t_n - \tau(t_n))|^2 \\ & \leq 2E|X(s - \tau(s)) - X(t_0)|^2 + 2E|X(t_0) - X(t_n - \tau(t_n))|^2 \\ & \leq 2(C_2 + L_1)(L_2 + 1) |s - t_n|. \end{aligned}$$

Hence

$$E|X(s - \tau(s)) - X(t_n - \tau(t_n))|^2 \leq \tilde{C}_2 h.$$

is always true.

Similarly, we have

$$E|X(s - \tau(s)) - X(t_{n+1} - \tau(t_{n+1}))|^2 \leq \tilde{C}_2 h. \quad \square$$

Armed with these two lemmas, we can now prove Theorem 2.

Proof of Theorem 2. We define the local error of θ -Maruyama methods (5)-(6) as follows:

$$\begin{aligned}
\delta_h(t_{n+1}) &= X(t_{n+1}) - X(t_n) - \theta h F(t_{n+1}, X(t_{n+1}), X(t_{n+1} - \tau(t_{n+1}))) \\
&\quad - (1 - \theta) h F(t_n, X(t_n), X(t_n - \tau(t_n))) \\
&\quad - G(t_n, X(t_n), X(t_n - \tau(t_n))) \Delta W_n \\
&= \int_{t_n}^{t_{n+1}} [\theta(F_s^X - F_{n+1}^X) + (1 - \theta)(F_s^X - F_n^X)] ds \\
&\quad + \int_{t_n}^{t_{n+1}} (G_s^X - G_n^X) dW(s),
\end{aligned} \tag{8}$$

where

$$\mathcal{D}_s^X = \mathcal{D}(s, X(s), X(s - \tau(s))), \quad \mathcal{D}_n^X = \mathcal{D}(t_n, X(t_n), X(t_n - \tau(t_n))), \quad \text{for } \mathcal{D} \equiv F, G.$$

Without loss of generality we assume $h < 1$. Using conditions (H1), (iii), properties of Itô's integral, Hölder's inequality and Jensen's inequality $|E(X|\mathcal{F})|^2 \leq E(|X|^2|\mathcal{F})$, we get from (8) that

$$\begin{aligned}
&E|E(\delta_h(t_{n+1})|\mathcal{F}_{t_n})|^2 \\
&\leq \theta h \int_{t_n}^{t_{n+1}} E|F_s^X - F_{n+1}^X|^2 ds + (1 - \theta) h \int_{t_n}^{t_{n+1}} E|F_s^X - F_n^X|^2 ds \\
&\leq 2\theta h K_0 \int_{t_n}^{t_{n+1}} E(|X(s)|^2 + |X(s - \tau(s))|^2) |s - t_{n+1}| ds \\
&\quad + 2\theta h L \int_{t_n}^{t_{n+1}} E|X(s) - X(t_{n+1})|^2 ds \\
&\quad + 2\theta h L \int_{t_n}^{t_{n+1}} E|X(s - \tau(s)) - X(t_{n+1} - \tau(t_{n+1}))|^2 ds \\
&\quad + 2(1 - \theta) h K_0 \int_{t_n}^{t_{n+1}} E(|X(s)|^2 + |X(s - \tau(s))|^2) |s - t_n| ds \\
&\quad + 2(1 - \theta) h L \int_{t_n}^{t_{n+1}} E|X(s) - X(t_n)|^2 ds. \\
&\quad + 2(1 - \theta) h L \int_{t_n}^{t_{n+1}} E|X(s - \tau(s)) - X(t_n - \tau(t_n))|^2 ds.
\end{aligned} \tag{9}$$

Hence, using Lemma 3, 4 and elementary inequalities, we have from (9) that

$$E|E(\delta_h(t_{n+1})|\mathcal{F}_{t_n})|^2 \leq \tilde{C} h^3, \quad \text{with } \tilde{C} := 2K_0 C_1 + LC_2 + 2L\tilde{C}_2. \tag{10}$$

Applying the same techniques, we can easily derive that

$$E|\delta_h(t_{n+1})|^2 \leq \bar{C} h^2, \quad \text{with } \bar{C} = 4\tilde{C}. \tag{11}$$

With the estimate of local error, we can make an estimate of global error

$$\varepsilon_n := X(t_n) - Y_n.$$

And we have

$$\varepsilon_{n+1} = \varepsilon_n + \delta_h(t_{n+1}) + u_n, \quad (12)$$

where $\delta_h(t_{n+1})$ is the local error defined above, and

$$\begin{aligned} u_n := & \theta h[F(t_{n+1}, X(t_{n+1}), X(t_{n+1} - \tau(t_{n+1}))) - F(t_{n+1}, Y_{n+1}, \bar{Y}_{n+1})] \\ & + (1 - \theta)h[F(t_n, X(t_n), X(t_n - \tau(t_n))) - F(t_n, Y_n, \bar{Y}_n)] \\ & + [G(t_n, X(t_n), X(t_n - \tau(t_n))) - G(t_n, Y_n, \bar{Y}_n)]\Delta W_n. \end{aligned}$$

It follows from (12) that

$$\begin{aligned} E|\varepsilon_{n+1}|^2 \leq & E|\varepsilon_n|^2 + 2E|\delta_h(t_{n+1})|^2 + 2E|u_n|^2 \\ & + 2|E\langle \varepsilon_n, \delta_h(t_{n+1}) \rangle| + 2|E\langle \varepsilon_n, u_n \rangle|. \end{aligned} \quad (13)$$

Here $E|\delta_h(t_{n+1})|^2$ is bounded by $\bar{C}h^2$. Now we shall estimate the last three terms in the right-hand side of (13). By using global Lipschitz condition (H1) and elementary inequalities, we have from definition of u_n that

$$\begin{aligned} 2E|u_n|^2 \leq & 4h^2LE|\varepsilon_{n+1}|^2 + 8hLE|\varepsilon_n|^2 + 4h^2LE|X(t_{n+1} - \tau(t_{n+1})) - \bar{Y}_{n+1}|^2 \\ & + 8hLE|X(t_n - \tau(t_n)) - \bar{Y}_n|. \end{aligned} \quad (14)$$

In the following estimate of $E|X(t_n - \tau(t_n)) - \bar{Y}_n|^2$, we always assume

$$t_n - \tau(t_n) > t_0, \quad n \in \mathbb{N}^+.$$

Otherwise, $t_n - \tau(t_n) \leq t_0$, $\bar{Y}_n := \psi(t_n - \tau(t_n))$ by (6). This is a trivial case, where by definition of $\bar{Y}_n$ we have $E|X(t_n - \tau(t_n)) - \bar{Y}_n|^2 = 0$.

Let

$$\tau(t_n) = (q_n + \gamma_n)h, \quad 0 \leq q_n \in \mathbb{N}, \quad 0 \leq \gamma_n < 1.$$

Then $\bar{Y}_n$ defined in (6) reads

$$\bar{Y}_n = \gamma_n Y_{n-q_n-1} + (1 - \gamma_n)Y_{n-q_n}.$$

Based on (7), conditions (i), (ii) and elementary inequalities $|a + b|^2 \leq 2|a|^2 + 2|b|^2$, $|\gamma a + (1 - \gamma)b|^2 \leq \gamma|a|^2 + (1 - \gamma)|b|^2$, $0 \leq \gamma \leq 1$, we obtain that

$$\begin{aligned} & E|X(t_n - \tau(t_n)) - \bar{Y}_n|^2 \\ & = E|\gamma_n \varepsilon_{n-q_n-1} + (1 - \gamma_n)\varepsilon_{n-q_n} + \gamma_n[X(t_n - \tau(t_n)) - X(t_{n-q_n-1})] \\ & \quad + (1 - \gamma_n)[X(t_n - \tau(t_n)) - X(t_{n-q_n})]|^2 \\ & \leq 2\gamma_n E|\varepsilon_{n-q_n-1}|^2 + 2(1 - \gamma_n)E|\varepsilon_{n-q_n}|^2 + 2C_2h. \end{aligned} \quad (15)$$

Inserting (15) into (14) results in

$$\begin{aligned} 2E|u_n|^2 \leq & 4h^2LE|\varepsilon_{n+1}|^2 + 8hLE|\varepsilon_n|^2 + 8h^2L(\gamma_{n+1}E|\varepsilon_{n-q_{n+1}}|^2 \\ & + (1 - \gamma_{n+1})E|\varepsilon_{n+1-q_{n+1}}|^2 + C_2h) \\ & + 16hL(\gamma_n E|\varepsilon_{n-q_n-1}|^2 + (1 - \gamma_n)E|\varepsilon_{n-q_n}|^2 + C_2h). \\ \leq & 12hLE|\varepsilon_{n+1}|^2 + 8hLE|\varepsilon_n|^2 + 24hL \max_{0 \leq i \leq n} E|\varepsilon_i|^2 + 24LC_2h^2. \end{aligned} \quad (16)$$

Here the fact has been utilized that $h < 1$ and $E|\varepsilon_{n+1-q_{n+1}}|^2 \leq E|\varepsilon_{n+1}|^2 + \max_{0 \leq i \leq n} E|\varepsilon_i|^2$.

Noting that $E|E(\delta_h(t_{n+1})|\mathcal{F}_{t_n})|^2 \leq \tilde{C}h^3$, we have

$$\begin{aligned} 2|E\langle \varepsilon_n, \delta_h(t_{n+1}) \rangle| &= 2|E(E\langle \varepsilon_n, \delta_h(t_{n+1}) \rangle|\mathcal{F}_{t_n})| = 2|E\langle \varepsilon_n, E(\delta_h(t_{n+1})|\mathcal{F}_{t_n}) \rangle| \\ &\leq 2(hE|\varepsilon_n|^2)^{1/2}(h^{-1}E[E(\delta_h(t_{n+1})|\mathcal{F}_{t_n})]^2)^{1/2} \\ &\leq hE|\varepsilon_n|^2 + h^{-1}E[E(\delta_h(t_{n+1})|\mathcal{F}_{t_n})]^2 \\ &\leq hE|\varepsilon_n|^2 + \tilde{C}h^2. \end{aligned} \quad (17)$$

Now the only term left to be estimated in (13) is $2|E\langle \varepsilon_n, u_n \rangle|$. First, in a similar way as in deriving (14), we have

$$\begin{aligned} E|E(u_n|\mathcal{F}_{t_n})|^2 &\leq h^2LE|\varepsilon_{n+1}|^2 + h^2LE|\varepsilon_n|^2 + h^2LE|X(t_{n+1} - \tau(t_{n+1})) - \bar{Y}_{n+1}|^2 \\ &\quad + h^2LE|X(t_n - \tau(t_n)) - \bar{Y}_n|^2 \\ &\leq 3h^2LE|\varepsilon_{n+1}|^2 + h^2LE|\varepsilon_n|^2 + 4h^2L \max_{0 \leq i \leq n} E|\varepsilon_i|^2 + 4LC_2h^3. \end{aligned} \quad (18)$$

Thus

$$\begin{aligned} 2|E\langle \varepsilon_n, u_n \rangle| &= 2|E(E\langle \varepsilon_n, u_n \rangle|\mathcal{F}_{t_n})| = 2|E\langle \varepsilon_n, E(u_n|\mathcal{F}_{t_n}) \rangle| \\ &\leq hE|\varepsilon_n|^2 + h^{-1}E|E(u_n|\mathcal{F}_{t_n})|^2 \\ &\leq 3hLE|\varepsilon_{n+1}|^2 + h(L+1)E|\varepsilon_n|^2 \\ &\quad + 4hL \max_{0 \leq i \leq n} E|\varepsilon_i|^2 + 4LC_2h^2. \end{aligned} \quad (19)$$

Inserting (16), (17), (19) into (13), and requiring $h < h_0$ with $h_0C_3 < 1/2$, we get

$$(1 - hC_3)E|\varepsilon_{n+1}|^2 \leq (1 + hC_4)E|\varepsilon_n|^2 + hC_5 \max_{0 \leq i \leq n} E|\varepsilon_i|^2 + C_6h^2, \quad (20)$$

where $C_3 = 15L$, $C_4 = 9L + 2$, $C_5 = 28L$, $C_6 = 2\tilde{C} + 28LC_2 + \tilde{C}$.

Let $R_j = \max_{0 \leq i \leq j} E|\varepsilon_i|^2$ with $R_0 = 0$. From (20) we have

$$R_{n+1} \leq \left(\frac{1 + hC_4 + hC_5}{1 - hC_3} \right) R_n + 2C_6h^2 \leq (1 + C_7h)R_n + 2C_6h^2.$$

where $C_7 = 2(C_3 + C_4 + C_5)$. By induction, it can be achieved that

$$R_{n+1} \leq (1 + C_7h)^{n+1}R_0 + 2C_6h^2 \sum_{j=0}^n (1 + C_7h)^j \leq \frac{2C_6}{C_7}(e^{C_7T} - 1)h.$$

The assertion follows by taking the square root. This completes the proof. $\square$

Remark 5. Restricting Theorem 2 to the special cases: $\tau(t) = \tau$ and $\tau(t) = (1 - q)t$, we can obtain the corresponding convergence results as in [1] and [2, 10], respectively.

3. Exponential stability under global Lipschitz condition

In this section, we shall investigate whether θ -Maruyama methods can reproduce exponential mean-square stability under global Lipschitz conditions. We will give a positive result about that for sufficiently small step size h . From now on, we always assume the delay function $\tau(t)$ of (2) further satisfies

$$0 \leq \tau(t) \leq \tau,$$

where $\tau > 0$ is a constant and let $\tau = (\kappa - \delta)h$ with $1 \leq \kappa \in \mathbb{Z}^+$, $0 \leq \delta < 1$. Here and throughout the following context, we always assume the condition

$$(C1) \quad F(t, 0, 0) \equiv G(t, 0, 0) \equiv 0 \quad \text{on} \quad t \geq t_0$$

hold. Throughout this section we assume the global Lipschitz condition (H1) holds.

Condition (C1) implies that (2) has a null solution $X(t) \equiv 0$. Next the definitions of exponential stability in mean-square sense [31] follow.

Definition 6. *The equation (2) is said to be exponentially mean-square stable if there exist positive constants ν and M independent of t such that*

$$E(|X(t)|^2) \leq M \|\psi\|_E^2 e^{-\nu(t-t_0)} \quad \text{on} \quad t \geq t_0. \quad (21)$$

Without loss of generality, we always assume $M \geq 1$. Corresponding to the definition of exponential stability for continuous cases, we can therefore analogously introduce the definition for discrete cases.

Definition 7. *Given a step size $h > 0$, the scheme (5)-(6) is said to be exponentially mean-square stable if there exists a pair of constants ν_h and H such that*

$$E(|Y_n|^2) \leq H \|\psi\|_E^2 e^{-\nu_h n h} \quad \text{on} \quad n \geq 0, \quad (22)$$

where H and ν_h are positive constants independent of n .

In this section we use the same notations as in Section 2 when no confusion occurs. We also note that the global Lipschitz condition (H1) and (C1) imply the linear growth condition with $K = L$, i.e.,

$$|F(t, x, y)|^2 \vee |G(t, x, y)|^2 \leq L(|x|^2 + |y|^2). \quad (23)$$

Suppose that T is a constant satisfying $T = Nh > \tau$ and $i \geq 1$. Let $\hat{X}^{(i)}(t)$ be the solution of the SDDEs (2) for $t \in [t_0 + iT, \infty)$, with initial data

$$\psi(t) = \bar{Y}^{(i)}(t), \quad t_0 + iT - \tau \leq t \leq t_0 + iT. \quad (24)$$

Here $\bar{Y}^{(i)}(t)$ is the linear interpolation function over interval $[t_0 + iT - \tau, t_0 + iT]$ defined by

$$\bar{Y}^{(i)}(t) = \frac{t - t_j}{h} Y_{j+1} + \frac{t_{j+1} - t}{h} Y_j, \quad t \in [t_j, t_{j+1}).$$

Lemma 8. Assume that the equation (2) is exponentially mean-square stable and satisfies (21), then for $i \geq 1$ the following estimates hold

$$\sup_{t \geq t_0 + iT - \tau} E(|\hat{X}^{(i)}(t)|^2) \leq C_1 := M \sup_{iN - \kappa \leq j \leq iN} E|Y_j|^2, \quad (25)$$

and for any $t_0 + iT \leq s, t \leq t_0 + (i+1)T$

$$E|\hat{X}^{(i)}(t) - \hat{X}^{(i)}(s)|^2 \leq C_2|t - s|, \quad (26)$$

where $C_2 = 4L(T+1)M \sup_{iN - \kappa \leq j \leq iN} E|Y_j|^2$.

Proof. From (21), we have for $t \geq t_0 + iT$

$$E(|\hat{X}^{(i)}(t)|^2) \leq M \|\psi\|_E^2 e^{-\nu(t-t_0-iT)} \leq M \sup_{iN - \kappa \leq j \leq iN} E|Y_j|^2 = C_1. \quad (27)$$

Thus the first part is validated. For the second part, note that for any $t_0 + iT \leq s, t \leq t_0 + (i+1)T$, we have

$$\begin{aligned} \hat{X}^{(i)}(t) - \hat{X}^{(i)}(s) &= \int_s^t F(u, \hat{X}^{(i)}(u), \hat{X}^{(i)}(u - \tau(u))) du \\ &\quad + \int_s^t G(u, \hat{X}^{(i)}(u), \hat{X}^{(i)}(u - \tau(u))) dW(u), \end{aligned}$$

thus using linear growth condition (23), Hölder's inequality and the Itô's integral isometries gives

$$\begin{aligned} E|\hat{X}^{(i)}(t) - \hat{X}^{(i)}(s)|^2 &\leq 2L|t - s| \int_s^t (E|\hat{X}^{(i)}(u)|^2 + E|\hat{X}^{(i)}(u - \tau(u))|^2) du \\ &\quad + 2L \int_s^t (E|\hat{X}^{(i)}(u)|^2 + E|\hat{X}^{(i)}(u - \tau(u))|^2) du \\ &\leq 4L(T+1)M \sup_{iN - \kappa \leq j \leq iN} E|Y_j|^2 |t - s|, \end{aligned} \quad (28)$$

where the fact that $|t - s| \leq T$ was used. $\square$

Lemma 9. Assume that condition (ii) in Theorem 2 is satisfied, then for $s \in [t_{n+iN}, t_{n+1+iN})$, $0 \leq n \leq N-1$, we have

$$\begin{aligned} &E|\hat{X}^{(i)}(s - \tau(s)) - \hat{X}^{(i)}(t_{n+iN} - \tau(t_{n+iN}))|^2 \\ &\vee E|\hat{X}^{(i)}(s - \tau(s)) - \hat{X}^{(i)}(t_{n+1+iN} - \tau(t_{n+1+iN}))|^2 \leq \tilde{C}_2 h, \end{aligned} \quad (29)$$

where $\tilde{C}_2 = 8L[4(L_2 + 2)^2 + M(T+1)(L_2 + 1) + 4] \sup_{iN - \kappa \leq j \leq iN} E|Y_j|^2$.

Proof. In the case that $s - \tau(s) \geq t_0 + iT$ and $t_{n+iN} - \tau(t_{n+iN}) \geq t_0 + iT$, by Lemma 8 and condition (ii), we get

$$\begin{aligned} E|\hat{X}^{(i)}(s - \tau(s)) - \hat{X}^{(i)}(t_{n+iN} - \tau(t_{n+iN}))|^2 &\leq C_2 |s - \tau(s) - t_{n+iN} + \tau(t_{n+iN})| \\ &\leq 4LM(T+1)(L_2 + 1) \sup_{iN - \kappa \leq j \leq iN} E|Y_j|^2 h. \end{aligned}$$

If both $s - \tau(s)$ and $t_{n+iN} - \tau(t_{n+iN})$ are less than $t_0 + iT$, then by (24) we have

$$\hat{X}^{(i)}(s - \tau(s)) = \bar{Y}^{(i)}(s - \tau(s)), \quad \hat{X}^{(i)}(t_{n+iN} - \tau(t_{n+iN})) = \bar{Y}^{(i)}(t_{n+iN} - \tau(t_{n+iN})).$$

In this case, set $s - \tau(s) = (1 - \mu_1)t_{j_1} + \mu_1 t_{j_1+1}$, $t_{n+iN} - \tau(t_{n+iN}) = (1 - \mu_2)t_{j_2} + \mu_2 t_{j_2+1}$, $t_{j_1+1}, t_{j_2+1} \leq t_0 + iT$, where $j_1, j_2 \in \mathbb{Z}$ and $0 \leq \mu_1, \mu_2 < 1$. Without loss of generality we set $t_{j_2} \leq t_{j_1}$. Since $|s - \tau(s) - t_{n+iN} + \tau(t_{n+iN})| \leq (L_2 + 1)h$, one derives $j_1 - j_2 \leq L_2 + 2$. Denote

$$F_k := F(t_k, Y_k, \bar{Y}_k), \quad G_k := G(t_k, Y_k, \bar{Y}_k), \quad k = 0, 1, 2, \dots \quad (30)$$

Then it follows that

$$\begin{aligned} & \hat{X}^{(i)}(s - \tau(s)) - \hat{X}^{(i)}(t_{n+iN} - \tau(t_{n+iN})) \\ &= \mu_1 Y_{j_1+1} + (1 - \mu_1)Y_{j_1} - \mu_2 Y_{j_2+1} - (1 - \mu_2)Y_{j_2} \\ &= \mu_1 \sum_{k=j_2+1}^{j_1} [h\theta F_{k+1} + h(1 - \theta)F_k + G_k \Delta W_k] + (\mu_1 - \mu_2)Y_{j_2+1} \\ & \quad + (1 - \mu_1) \sum_{k=j_2}^{j_1-1} [h\theta F_{k+1} + h(1 - \theta)F_k + G_k \Delta W_k] - (\mu_1 - \mu_2)Y_{j_2} \\ &= \mu_1 \sum_{k=j_2+1}^{j_1} [h\theta F_{k+1} + h(1 - \theta)F_k + G_k \Delta W_k] \\ & \quad + (1 - \mu_1) \sum_{k=j_2}^{j_1-1} [h\theta F_{k+1} + h(1 - \theta)F_k + G_k \Delta W_k] \\ & \quad + (\mu_1 - \mu_2)[h\theta F_{j_2+1} + h(1 - \theta)F_{j_2} + G_{j_2} \Delta W_{j_2}]. \end{aligned}$$

Here the summations are defined to be zero when $j_1 = j_2$. Hence linear growth condition (23) and elementary inequalities together result in

$$\begin{aligned} & E|\hat{X}^{(i)}(s - \tau(s)) - \hat{X}^{(i)}(t_{n+iN} - \tau(t_{n+iN}))|^2 \\ & \leq 2(j_1 - j_2)\mu_1 \sum_{k=j_2+1}^{j_1} E|h\theta F_{k+1} + h(1 - \theta)F_k + G_k \Delta W_k|^2 \\ & \quad + 2(j_1 - j_2)(1 - \mu_1) \sum_{k=j_2}^{j_1-1} E|h\theta F_{k+1} + h(1 - \theta)F_k + G_k \Delta W_k|^2 \\ & \quad + 2E|h\theta F_{j_2+1} + h(1 - \theta)F_{j_2} + G_{j_2} \Delta W_{j_2}|^2 \\ & \leq 16(j_1 - j_2)^2 L \sup_{iN - \kappa \leq j \leq iN} E|Y_j|^2 h + 16L \sup_{iN - \kappa \leq j \leq iN} E|Y_j|^2 h \\ & \leq 16[(L_2 + 2)^2 + 1]L \sup_{iN - \kappa \leq j \leq iN} E|Y_j|^2 h. \end{aligned}$$

Here we used the fact that $j_1 - j_2 \leq L_2 + 2$.

Otherwise, either $s - \tau(s)$ or $t_{n+iN} - \tau(t_{n+iN})$ is less than $t_0 + iT$. Applying

elementary inequalities and the estimates obtained in the former two cases results in

$$\begin{aligned}
& E|\hat{X}^{(i)}(s - \tau(s)) - \hat{X}^{(i)}(t_{n+iN} - \tau(t_{n+iN}))|^2 \\
& \leq 2E|\hat{X}^{(i)}(s - \tau(s)) - \hat{X}^{(i)}(t_0 + iT)|^2 + 2E|\hat{X}^{(i)}(t_0 + iT) - \hat{X}^{(i)}(t_n - \tau(t_n))|^2 \\
& \leq 8L[4(L_2 + 2)^2 + M(T + 1)(L_2 + 1) + 4] \sup_{iN - \kappa \leq j \leq iN} E|Y_j|^2 h = \tilde{C}_2 h.
\end{aligned}$$

Similarly, we have

$$E|\hat{X}^{(i)}(s - \tau(s)) - \hat{X}^{(i)}(t_{n+1+iN} - \tau(t_{n+1+iN}))|^2 \leq \tilde{C}_2 h.$$

Therefore, (29) always holds. $\square$

Lemma 10. *Assume that the conditions (ii)-(iii) in Theorem 2 are satisfied, then the following estimate holds*

$$\sup_{(i+1)N - \kappa \leq j \leq (i+2)N - \kappa} E|Y_j - \hat{X}^{(i)}(t_j)|^2 \leq C_{2T} h \sup_{iN - \kappa \leq j \leq iN} E|Y_j|^2.$$

Proof. Applying Theorem 2 with $C_1, C_2, \tilde{C}_2$ replaced by constants obtained in Lemma 8, 9, we have in (20)

$$C_6 = 2\bar{C} + \tilde{C} + 28LC_2 = 18K_0C_1 + 18L\tilde{C}_2 + 37LC_2 = \hat{C} \sup_{iN - \kappa \leq j \leq iN} E|Y_j|^2,$$

with $\hat{C} = 18K_0M + 144L^2[4(L_2 + 2)^2 + M(T + 1)(L_2 + 1) + 4] + 148L^2M(T + 1)$. Note that in the previous section $C_7 = 2(C_3 + C_4 + C_5) = 104L + 4$, choosing $C_{2T} = \frac{2\hat{C}}{C_7}(e^{C_7T} - 1)$, then concludes the proof. $\square$

Theorem 11. *Assume that SDDE (2) is exponentially mean-square stable and satisfies (21) and the conditions (i),(ii),(iii) in Theorem 2. Let $T > 4(\ln M)/\nu + 4\tau$, and choose $h^* > 0$ such that for all step size $h < h^*$,*

$$(h + \sqrt{h})C_{2T} + (1 + \sqrt{h})e^{\nu h} \cdot e^{-(3/4)\nu T} \leq e^{-(1/2)\nu T}, \quad (31)$$

and

$$(h + \sqrt{h})C_{2T} + (1 + \sqrt{h})M \leq 2M. \quad (32)$$

Then for $h < h^*$ the scheme (5) is exponentially mean-square stable.

Proof. For any $i \geq 1, \alpha > 0$, applying Young's inequality results in

$$\begin{aligned}
\sup_{(i+1)N - \kappa \leq j \leq (i+2)N - \kappa} E|Y_j|^2 & \leq (1 + \frac{1}{\alpha}) \sup_{(i+1)N - \kappa \leq j \leq (i+2)N - \kappa} E|Y_j - \hat{X}^{(i)}(t_j)|^2 \\
& + (1 + \alpha) \sup_{(i+1)N - \kappa \leq j \leq (i+2)N - \kappa} E|\hat{X}^{(i)}(t_j)|^2.
\end{aligned} \quad (33)$$

Choosing $\alpha = \sqrt{h}$, using Lemma 10 and (31), and noticing that $Me^{\nu\tau} \cdot e^{-\nu T} < e^{-(3/4)\nu T}$, we derive that

$$\begin{aligned}
& \sup_{(i+1)N-\kappa \leq j \leq (i+2)N-\kappa} E|Y_j|^2 \\
& \leq [(h + \sqrt{h})C_{2T} + (1 + \sqrt{h})Me^{-\nu(N-\kappa)h}] \sup_{iN-\kappa \leq j \leq iN} E|Y_j|^2 \\
& \leq [(h + \sqrt{h})C_{2T} + (1 + \sqrt{h})Me^{\nu(\tau+h)}e^{-\nu T}] \sup_{iN-\kappa \leq j \leq iN} E|Y_j|^2 \\
& \leq [(h + \sqrt{h})C_{2T} + (1 + \sqrt{h})e^{\nu h} \cdot e^{-\frac{3}{4}\nu T}] \sup_{iN-\kappa \leq j \leq iN} E|Y_j|^2 \\
& \leq e^{-\frac{1}{2}\nu T} \sup_{iN-\kappa \leq j \leq (i+1)N-\kappa} E|Y_j|^2 \leq \dots \\
& \leq e^{-\frac{1}{2}i\nu T} \sup_{N-\kappa \leq j \leq 2N-\kappa} E|Y_j|^2. \tag{34}
\end{aligned}$$

Similarly, using (32) we obtain

$$\sup_{N-\kappa \leq j \leq 2N-\kappa} E|Y_j|^2 \leq [(h + \sqrt{h})C_{2T} + (1 + \sqrt{h})M]\|\psi\|_E^2 \leq 2M\|\psi\|_E^2. \tag{35}$$

Hence inserting (35) into (34) yields

$$\begin{aligned}
\sup_{(i+1)N-\kappa \leq j \leq (i+2)N-\kappa} E|Y_j|^2 & \leq 2Me^{-\frac{1}{2}i\nu T}\|\psi\|_E^2 \\
& \leq 2Me^{\nu T - \nu\tau/2}e^{-\frac{\nu}{2}[(i+2)N-\kappa]h}\|\psi\|_E^2. \tag{36}
\end{aligned}$$

Thus, setting $H = 2Me^{\nu T - \nu\tau/2}$ and $\nu_h = \frac{\nu}{2}$ in (22) completes the proof. $\square$

Theorem 11 shows that under global Lipschitz conditions exponential mean-square stability of the SDDE (2) implies the exponential mean-square stability of the θ -Maruyama methods (5) for sufficiently small h .

Remark 12. *Theorem 11 can be regarded as an extension of those in both [17], where the relationship between the stability of the SDEs and that of θ -Maruyama methods is established under global Lipschitz condition, and [31], where similar results are proved for Euler-Maruyama method and SDDEs with constant lag.*

4. Stability under non-global Lipschitz condition

In the previous section, it was proved that under global Lipschitz conditions the θ -Maruyama methods (5) reproduce exponential mean-square stability of the underlying problem for a sufficiently small h , it did not indicate how the methods with different θ differ in inheriting stability. In order to investigate the influence of θ on the stability, we consider a kind of nonlinear SDDE with complex coefficients, that is, $F : [t_0, \infty) \times \mathbb{C}^d \times \mathbb{C}^d \rightarrow \mathbb{C}^d$, $G : [t_0, \infty) \times \mathbb{C}^d \times \mathbb{C}^d \rightarrow \mathbb{C}^{d \times m}$. Suppose that the differential equation (2) possesses a unique solution.

In this section, the assumption (ii), i.e., the delay function $\tau(t)$ is Lipschitz continuous, is not required.

4.1. Stability of SDDEs

To begin with, we present a generalized Halanay-type inequality (see Lemma 2.1 in [28])

Lemma 13. *Let $\tau \geq 0$ be a given real constant. Assume that $\alpha(t)$ and $\beta(t)$ are two continuous functions on $[t_0, \infty)$, and $u(t)$ is a continuous positive-valued function on $[t_0 - \tau, \infty)$. Assume further that*

$$0 < \alpha_0 \leq \alpha(t), \quad 0 < \beta(t) \leq q\alpha(t), \quad \text{for all } t \geq t_0$$

with $0 \leq q < 1$, and

$$D^+u(t) \leq -\alpha(t)u(t) + \beta(t) \sup_{s \in [t-\tau, t]} u(s),$$

where the Dini derivative $D^+u(t)$ is defined by

$$D^+u(t) := \limsup_{\delta \rightarrow 0^+} \frac{u(t + \delta) - u(t)}{\delta}.$$

Then for all $t \geq t_0$

$$u(t) \leq \left(\sup_{s \in [t_0 - \tau, t_0]} u(s) \right) \exp\{-u^*(t - t_0)\},$$

where u^* is defined as

$$u^* = \inf_{t \geq t_0} \{v(t) : v(t) - \alpha(t) + \beta(t)e^{v(t)\tau} = 0\}.$$

Theorem 14. *Assume that*

(1) *there are nonnegative continuous functions $\lambda(t), \mu(t), \beta_0(t), \beta_1(t)$ such that*

$$\begin{aligned} \Re \langle x, F(t, x, 0) \rangle &\leq -\lambda(t)|x|^2, \\ |F(t, x, 0) - F(t, x, y)| &\leq \mu(t)|y|, \\ |G(t, x, y)|^2 &\leq \beta_0(t)|x|^2 + \beta_1(t)|y|^2, \end{aligned} \tag{37}$$

for all $t \geq t_0, x, y \in \mathbb{C}^d$, and

(2) *there exist numbers $0 < q < 1, \rho_0 > 0$ such that*

$$2\lambda(t) - \beta_0(t) - \mu(t) \geq \rho_0, \quad 2q\lambda(t) - (q+1)\mu(t) - q\beta_0(t) - \beta_1(t) \geq 0, \tag{38}$$

then (2) is exponentially mean-square stable.

Proof. Let

$$\begin{aligned} X(t) &= X_R(t) + iX_I(t), \\ F(t, X(t), X(t - \tau(t))) &= F_R(t, X(t), X(t - \tau(t))) + iF_I(t, X(t), X(t - \tau(t))), \\ G(t, X(t), X(t - \tau(t))) &= G_R(t, X(t), X(t - \tau(t))) + iG_I(t, X(t), X(t - \tau(t))), \end{aligned}$$

then the real part $X_R(t)$ and the imaginary part $X_I(t)$ satisfy

$$dX_R(t) = F_R(t, X(t), X(t - \tau(t)))dt + G_R(t, X(t), X(t - \tau(t)))dW(t)$$

and

$$dX_I(t) = F_I(t, X(t), X(t - \tau(t)))dt + G_I(t, X(t), X(t - \tau(t)))dW(t),$$

respectively. Using Itô's formula and taking expectation gives

$$\begin{aligned} & E|X_R(t + \delta)|^2 - E|X_R(t)|^2 \\ &= \int_t^{t+\delta} 2E\langle X_R(s), F_R(s, X(s), X(s - \tau(s))) \rangle ds \\ & \quad + \int_t^{t+\delta} E|G_R(s, X(s), X(s - \tau(s)))|^2 ds \end{aligned}$$

and

$$\begin{aligned} & E|X_I(t + \delta)|^2 - E|X_I(t)|^2 \\ &= \int_t^{t+\delta} 2E\langle X_I(s), F_I(s, X(s), X(s - \tau(s))) \rangle ds \\ & \quad + \int_t^{t+\delta} E|G_I(s, X(s), X(s - \tau(s)))|^2 ds \end{aligned}$$

Thus we have

$$\begin{aligned} & E|X(t + \delta)|^2 - E|X(t)|^2 = E|X_R(t + \delta)|^2 + E|X_I(t + \delta)|^2 - E|X_R(t)|^2 - E|X_I(t)|^2 \\ &= \int_t^{t+\delta} 2E\Re\langle X(s), F(s, X(s), X(s - \tau(s))) \rangle ds + \int_t^{t+\delta} E|G(s, X(s), X(s - \tau(s)))|^2 ds \\ &\leq \int_t^{t+\delta} (-2\lambda(s) + \beta_0(s))E|X(s)|^2 ds + \int_t^{t+\delta} \beta_1(s)E|X(s - \tau(s))|^2 ds \\ & \quad + 2 \int_t^{t+\delta} E\Re\langle X(s), F(s, X(s), X(s - \tau(s))) - F(s, X(s), 0) \rangle ds \\ &\leq \int_t^{t+\delta} -\alpha_1(s)E|X(s)|^2 ds + \int_t^{t+\delta} \alpha_2(s) \sup_{r \in [s-\tau, s]} E|X(r)|^2 ds, \end{aligned}$$

where $\alpha_1(s) = 2\lambda(s) - \beta_0(s) - \mu(s)$, $\alpha_2(s) = \mu(s) + \beta_1(s)$. Letting $u(t) = E|X(t)|^2$ yields

$$D^+u(t) \leq -\alpha_1(t)u(t) + \alpha_2(t) \sup_{s \in [t-\tau, t]} u(s). \quad (39)$$

Further, condition (38) implies that

$$\alpha_1(t) = 2\lambda(t) - \beta_0(t) - \mu(t) \geq \rho_0 > 0, \quad (40)$$

$$0 \leq \alpha_2(t) \leq q\alpha_1(t). \quad (41)$$

Taking into account (39)-(41) and applying the generalized Halanay-type inequality, we can obtain the desired result, i.e., (21) is true with $M = 1$ and $v > 0$ defined by

$$v = \inf_{t \geq t_0} \{v(t) : v(t) - \alpha_1(t) + \alpha_2(t)e^{v(t)\tau} = 0\}. \quad \square$$

Now we consider an important special case of Theorem 14, i.e., all of $\lambda(t), \mu(t), \beta_0(t), \beta_1(t)$ are independent of t .

Corollary 15. *Assume there exist nonnegative real $\lambda, \mu, \beta_0, \beta_1$ such that for all $t \geq 0, x, y \in \mathbb{C}^d$,*

$$(C2) \quad \Re \langle x, F(t, x, 0) \rangle \leq -\lambda |x|^2,$$

$$(C3) \quad |F(t, x, 0) - F(t, x, y)| \leq \mu |y|,$$

$$(C4) \quad |G(t, x, y)|^2 \leq \beta_0 |x|^2 + \beta_1 |y|^2,$$

and

$$2\lambda - 2\mu - \beta_0 - \beta_1 > 0, \quad (42)$$

then the problem (2) is exponentially mean-square stable.

Observe that conditions (C2)-(C4) are weaker than those in [29, Corollary 6.5 in Chapter 5] with one delay and $p = 2$. As a consequence, we have

Corollary 16. *Assume that conditions in Corollary 15 are fulfilled with only (C3) replaced by*

$$(C3') \quad |F(t, \bar{x}, 0) - F(t, x, y)| \leq \hat{\mu} |\bar{x} - x| + \mu |y|, \text{ for all } t \geq 0, x, \bar{x}, y \in \mathbb{R}^d.$$

Then (2) is exponentially mean-square stable.

Next we observe applications of Corollary 15 to special problem classes including linear SDDEs and nonlinear SODEs.

- Linear scalar SDDEs

$$dX(t) = (aX(t) + bX(t - \tau(t)))dt + (cX(t) + dX(t - \tau(t)))dW(t), \quad t \geq t_0, \quad (43)$$

where a, b, c and d are complex coefficients. In this case, (42) is reduced to

$$\Re a < -|b| + \frac{1}{2}(|c| + |d|)^2. \quad (44)$$

Therefore, Corollary 15 implies the following proposition:

Proposition 17. *If $\Re a < -|b| + \frac{1}{2}(|c| + |d|)^2$, then linear scalar SDDE (43) with bounded variable lag is exponentially mean-square stable.*

- Nonlinear SODEs

$$dX(t) = f(t, X(t))dt + g(t, X(t))dW(t), \quad t \geq t_0, \quad (45)$$

in this case, the conditions (C2)-(C4) are reduced to

$$\Re x^T f(t, x) \leq -\lambda |x|^2, \quad |g(t, x)|^2 \leq \beta_0 |x|^2. \quad (46)$$

Consequently, by Corollary 15, we have

Proposition 18. *Assume that (46) and $2\lambda - \beta_0 > 0$ are satisfied, then the SODE (45) is exponentially mean-square stable.*

Remark 19. *Note that Propositions 17-18 are consistent with related exponential mean-square stability results in [29], where the drift and diffusion coefficients of SDDEs only take real values. Therefore, Propositions 17-18 are extensions of those in the literature to complex coefficients.*

When Theorem 14 is applied to the special case, i.e. deterministic DDEs, some interesting observation will also be found.

- Deterministic DDEs

$$dX(t) = F(t, X(t), X(t - \tau(t)))dt, \quad t \geq t_0, \quad (47)$$

in this case, the conditions (37) and (38) are reduced to

$$\begin{aligned} \Re \langle x, F(t, x, 0) \rangle &\leq -\lambda(t)|x|^2, \\ |F(t, x, 0) - F(t, x, y)| &\leq \mu(t)|y| \end{aligned} \quad (48)$$

and

$$\lambda(t) - \mu(t) \geq \rho_1 > 0, \quad (49)$$

respectively, where ρ_1 is a constant. Therefore, Theorem 14 implies that

Corollary 20. *Assume that conditions (48) and (49) are satisfied, then the deterministic DDEs (47) with bounded variable lag is exponentially asymptotically stable.*

Remark 21. *We mention that the conditions (48) are weaker than the following ones [4, 23]*

$$\begin{aligned} \Re \langle x_2 - x_1, F(t, x_2, y) - F(t, x_1, y) \rangle &\leq -\lambda_1(t)|x_2 - x_1|^2, \\ |F(t, x, y_2) - F(t, x, y_1)| &\leq \mu_1(t)|y_2 - y_1|. \end{aligned} \quad (50)$$

In fact, conditions (50) imply conditions (48), provided that $F(t, 0, 0) \equiv 0$. The majority of numerical stability analysis for nonlinear DDEs in the literature is based on the condition (50). It should be very interesting to investigate the stability of numerical methods for nonlinear DDEs under the conditions (48).

4.2. Stability of θ -Maruyama methods

Now we are in a position to make sure whether the methods can inherit the stability of the test problem which obeys the conditions of Corollary 15. In general, the Euler-Maruyama method fails to preserve exponential mean-square stability under non-global Lipschitz conditions. In [17], the authors gave a counterexample to illustrate the phenomena for SDEs. However, the backward Euler method can excellently preserve the stability in that situation. Here we extend the result to the case with delay.

Theorem 22. Suppose that the conditions of Corollary 15 are fulfilled, then for any $h > 0$, the backward Euler method (θ -Maruyama method with $\theta = 1$) for (2) is exponentially mean-square stable, that is, (22) is satisfied with

$$H = (\bar{\nu}_h)^{-1/(\kappa+1)}, \quad \nu_h = \frac{-\ln \bar{\nu}_h}{(\kappa+1)h}. \quad (51)$$

Here $\bar{\nu}_h$ is defined as in (59).

Proof. From (5) with $\theta = 1$ we have

$$|Y_{n+1} - hF(t_{n+1}, Y_{n+1}, \bar{Y}_{n+1})|^2 = |Y_n + G(t_n, Y_n, \bar{Y}_n)\Delta W_n|^2, \quad (52)$$

which leads to

$$\begin{aligned} & |Y_{n+1}|^2 \\ &= |Y_n|^2 - h^2|F(t_{n+1}, Y_{n+1}, \bar{Y}_{n+1})|^2 + 2h\Re\langle Y_{n+1}, F(t_{n+1}, Y_{n+1}, \bar{Y}_{n+1}) \rangle \\ & \quad + |G(t_n, Y_n, \bar{Y}_n)\Delta W_n|^2 + 2\Re\langle Y_n, G(t_n, Y_n, \bar{Y}_n)\Delta W_n \rangle. \end{aligned} \quad (53)$$

Note that $\Delta W_n = W(t_{n+1}) - W(t_n)$ is independent of $\mathcal{F}_{t_n}$ and $Y_n, \bar{Y}_n$ are $\mathcal{F}_{t_n}$ -measurable, and the following equalities hold

$$\begin{aligned} E\Re\langle Y_n, G(t_n, Y_n, \bar{Y}_n)\Delta W_n \rangle &= 0, \\ E|G(t_n, Y_n, \bar{Y}_n)\Delta W_n|^2 &= hE|G(t_n, Y_n, \bar{Y}_n)|^2. \end{aligned} \quad (54)$$

Taking expectation of both sides of equality (53) and using (54) we arrive at

$$\begin{aligned} & E|Y_{n+1}|^2 \\ & \leq E|Y_n|^2 + 2hE\Re\langle Y_{n+1}, F(t_{n+1}, Y_{n+1}, \bar{Y}_{n+1}) \rangle + hE|G(t_n, Y_n, \bar{Y}_n)|^2 \\ & \leq E|Y_n|^2 - 2h\lambda E|Y_{n+1}|^2 + h\beta_0 E|Y_n|^2 + h\beta_1 E|\bar{Y}_n|^2 \\ & \quad + 2hE\Re\langle Y_{n+1}, F(t_{n+1}, Y_{n+1}, \bar{Y}_{n+1}) - F(t_{n+1}, Y_{n+1}, 0) \rangle \\ & \leq E|Y_n|^2 - 2h\lambda E|Y_{n+1}|^2 + h\beta_0 E|Y_n|^2 + h\beta_1 E|\bar{Y}_n|^2 \\ & \quad + h\mu E|Y_{n+1}|^2 + h\mu E|\bar{Y}_{n+1}|^2, \end{aligned} \quad (55)$$

where the Cauchy-Schwarz inequality and conditions (C2)-(C4) were used.

Let us consider two possible cases:

- Case a) $0 \leq \tau(t_{n+1}) < h$.

In this case, we set $\tau(t_{n+1}) = \gamma h$, $0 \leq \gamma < 1$. Then $\bar{Y}_{n+1} = \gamma Y_n + (1 - \gamma)Y_{n+1}$. Using elementary inequality $[\gamma x + (1 - \gamma)y]^2 \leq \gamma x^2 + (1 - \gamma)y^2$, (55) becomes

$$\begin{aligned} [1 + 2h\lambda - (2 - \gamma)h\mu]E|Y_{n+1}|^2 & \leq [1 + h\beta_0 + h\gamma\mu]E|Y_n|^2 + h\beta_1 E|\bar{Y}_n|^2 \\ & \leq [1 + h\beta_0 + h\beta_1 + h\gamma\mu] \max_{n-\kappa \leq i \leq n} E|Y_i|^2. \end{aligned} \quad (56)$$

- Case b) $\tau(t_{n+1}) \geq h$.

In this case, we have from (55) that

$$[1 + 2h\lambda - h\mu]E|Y_{n+1}|^2 \leq [1 + h\beta_0 + h\beta_1 + h\mu] \max_{n-\kappa \leq i \leq n} E|Y_i|^2. \quad (57)$$

Because of the fact $2\lambda - 2\mu - \beta_0 - \beta_1 > 0$ and

$$\frac{1 + h\beta_0 + h\beta_1 + h\gamma\mu}{1 + 2h\lambda - (2 - \gamma)h\mu} < \frac{1 + h\beta_0 + h\beta_1 + h\mu}{1 + 2h\lambda - h\mu},$$

both (56) and (57) lead to

$$E|Y_{n+1}|^2 \leq \bar{\nu}_h \max_{n-\kappa \leq i \leq n} E|Y_i|^2, \quad (58)$$

where

$$\bar{\nu}_h = \frac{1 + h\beta_0 + h\beta_1 + h\mu}{1 + 2h\lambda - h\mu} < 1. \quad (59)$$

By recursive calculations, we obtain from (58)

$$E|Y_n|^2 \leq \bar{\nu}_h^{\lfloor \frac{n-1}{\kappa+1} \rfloor + 1} \|\psi\|_E^2 \leq \bar{\nu}_h^{\frac{n-1}{\kappa+1}} \|\psi\|_E^2 \leq H \|\psi\|_E^2 \exp\{-\nu_h n h\}, \quad (60)$$

where $\lfloor x \rfloor$ denotes the integer part of x , and $H = (\bar{\nu}_h)^{-1/(\kappa+1)}$, $\nu_h = \frac{-\ln \bar{\nu}_h}{(\kappa+1)h}$. $\square$

Corollary 23. *Suppose that the condition (44) is fulfilled, then for any $h > 0$, the backward Euler method (θ -Maruyama method with $\theta = 1$) for linear scalar SDDEs with bounded variable lag (43) is exponentially mean-square stable.*

Corollary 24. *Suppose that the conditions (46) and $2\lambda - \beta_0 > 0$ are fulfilled, then for any $h > 0$, the backward Euler method (θ -Maruyama method with $\theta = 1$) for SODEs (45) is exponentially mean-square stable.*

Corollary 25. *Suppose that the conditions (48) and (49) with $\lambda(t) \equiv \lambda$, $\mu(t) \equiv \mu$ are fulfilled, then for any $h > 0$, the backward Euler method (θ -method with $\theta = 1$) for DDEs with bounded variable lag (47) is exponentially asymptotically stable.*

Theorem 22 shows that the backward Euler method can reproduce exponential mean-square stability of the system satisfying (42), without any constraint on step size h . Next we discuss the stability properties of θ -Maruyama methods ($0 \leq \theta < 1$) under slightly stronger conditions than those in Theorem 22, that is, under conditions of Corollary 16.

Theorem 26. *Suppose that conditions of Corollary 16 are fulfilled, then for $0 \leq \theta < 1$, the θ -Maruyama method (5)-(6) is exponentially mean-square stable under $h < h_0(\theta) := \min\{h_1(\theta), h_2(\theta)\}$, where $h_1(\theta)$ and $h_2(\theta)$ are defined as follows:*

$$\begin{aligned} h_1(\theta) &= \begin{cases} \infty, & \hat{\mu} = \mu = 0, \\ \frac{2\lambda - 2\mu - \beta_0 - \beta_1}{(1-\theta)^2(\hat{\mu} + \mu)^2}, & \hat{\mu} + \mu > 0, \end{cases} \\ h_2(\theta) &= \inf\{h > 0 : Q(\theta, h) := (1-\theta)^2(\hat{\mu}^2 + \hat{\mu}\mu)h^2 \\ &\quad + \beta_0 h + (1-\theta)\mu h - 2(1-\theta)\lambda h + 1 < 0\}. \end{aligned}$$

Here $Q(\theta, h)$ is a quadratic polynomial with respect to h . If $Q(\theta, h) \geq 0$ is always true, then $h_2(\theta)$ is defined as ∞ . Otherwise, noticing that $Q(\theta, 0) = 1 > 0$, there must exist $h_2(\theta) > 0$ such that $h < h_2(\theta)$ promises $Q(\theta, h) > 0$.

Proof. For simplicity, here and in the following context we use the notation F_k, G_k in (30). From (5) we have

$$|Y_{n+1} - \theta h F_{n+1}|^2 = |Y_n + (1 - \theta)h F_n + G_n \Delta W_n|^2. \quad (61)$$

Using the same techniques as in Theorem 22, we can readily obtain

$$\begin{aligned} E|Y_{n+1}|^2 &\leq E|Y_n|^2 + (1 - \theta)^2 h^2 E|F_n|^2 + h E|G_n|^2 \\ &\quad + 2\theta h E\Re\langle Y_{n+1}, F_{n+1} \rangle + 2(1 - \theta)h E\Re\langle Y_n, F_n \rangle. \end{aligned} \quad (62)$$

Since conditions (C1) and (C3') imply $|F(t, x, y)| \leq \hat{\mu}|x| + \mu|y|$, one sees that

$$\begin{aligned} |F(t, x, y)|^2 &\leq \hat{\mu}^2|x|^2 + \mu^2|y|^2 + 2\hat{\mu}\mu|x||y| \\ &\leq (\hat{\mu}^2 + \hat{\mu}\mu)|x|^2 + (\mu^2 + \hat{\mu}\mu)|y|^2. \end{aligned} \quad (63)$$

Using conditions (C2), (C3'), (C4) and (63), we get from (62) that

$$\begin{aligned} &E|Y_{n+1}|^2 \\ &\leq E|Y_n|^2 + (1 - \theta)^2 h^2 (\hat{\mu}^2 + \hat{\mu}\mu) E|Y_n|^2 + (1 - \theta)^2 h^2 (\mu^2 + \hat{\mu}\mu) E|\bar{Y}_n|^2 \\ &\quad + h\beta_0 E|Y_n|^2 + h\beta_1 E|\bar{Y}_n|^2 - 2\theta h \lambda E|Y_{n+1}|^2 + \theta \mu h E|Y_{n+1}|^2 \\ &\quad + \theta \mu h E|\bar{Y}_{n+1}|^2 - 2(1 - \theta) \lambda h E|Y_n|^2 + (1 - \theta) \mu h E|Y_n|^2 \\ &\quad + (1 - \theta) \mu h E|\bar{Y}_n|^2, \end{aligned} \quad (64)$$

which leads to

$$\begin{aligned} &(1 + 2\theta h \lambda - \theta h \mu) E|Y_{n+1}|^2 \\ &\leq Q(\theta, h) E|Y_n|^2 + [(1 - \theta)^2 h^2 \mu^2 + (1 - \theta)^2 h^2 \hat{\mu}\mu \\ &\quad + h\beta_1 + (1 - \theta) \mu h] E|\bar{Y}_n|^2 + \theta \mu h E|\bar{Y}_{n+1}|^2. \end{aligned} \quad (65)$$

Subsequent calculations depend on the magnitude of the delay $\tau(t_{n+1})$. We also divide it into two cases.

- Case a) $0 \leq \tau(t_{n+1}) < h$.

In this case, we set $\tau(t_{n+1}) = \gamma h$, $0 \leq \gamma < 1$. Then $\bar{Y}_{n+1} = \gamma Y_n + (1 - \gamma)Y_{n+1}$. Using elementary inequality $[\gamma x + (1 - \gamma)y]^2 \leq \gamma x^2 + (1 - \gamma)y^2$, (65) becomes

$$\begin{aligned} [1 + 2\theta h \lambda - (2 - \gamma)\theta h \mu] E|Y_{n+1}|^2 &\leq [Q(\theta, h) + \gamma \theta \mu h] E|Y_n|^2 \\ &\quad + [(1 - \theta)^2 h^2 \mu^2 + (1 - \theta)^2 h^2 \hat{\mu}\mu + h\beta_1 + (1 - \theta) \mu h] E|\bar{Y}_n|^2. \end{aligned} \quad (66)$$

- Case b) $\tau(t_{n+1}) \geq h$.

In this case, we set $\tau(t_{n+1}) = (q + \gamma)h$, $0 \leq \gamma < 1, 1 \leq q \in \mathbb{N}^+$. Then $\bar{Y}_{n+1} = \gamma Y_{n-q} + (1 - \gamma)Y_{n-q+1}$. Hence (65) becomes

$$\begin{aligned} &(1 + 2\theta h \lambda - \theta h \mu) E|Y_{n+1}|^2 \\ &\leq Q(\theta, h) E|Y_n|^2 + [(1 - \theta)^2 h^2 \mu^2 + (1 - \theta)^2 h^2 \hat{\mu}\mu + h\beta_1 \\ &\quad + (1 - \theta) \mu h] E|\bar{Y}_n|^2 + \theta \mu h [\gamma E|Y_{n-q}|^2 + (1 - \gamma) E|Y_{n-q+1}|^2]. \end{aligned} \quad (67)$$

We require $h < h_2(\theta)$, which guarantees $Q(\theta, h) > 0$, then the coefficients in the right-hand side of (66), (67) are positive. Consequently, from (66), we have

$$[1 + 2\theta h\lambda - (2 - \gamma)\theta h\mu]E|Y_{n+1}|^2 \leq [1 + S(\theta, h) + \gamma\theta h\mu] \max_{n-\kappa \leq i \leq n} E|Y_i|^2.$$

From (67), we have

$$(1 + 2\theta h\lambda - \theta h\mu)E|Y_{n+1}|^2 \leq [1 + S(\theta, h) + \theta h\mu] \max_{n-\kappa \leq i \leq n} E|Y_i|^2,$$

where

$$S(\theta, h) := (1 - \theta)^2 h^2 (\hat{\mu} + \mu)^2 + h(\beta_0 + \beta_1) + 2(1 - \theta)h\mu - 2(1 - \theta)h\lambda,$$

provided that $h < h_1(\theta)$, which promises $1 + 2\theta h\lambda - 2\theta h\mu > 1 + S(\theta, h)$ and

$$\frac{1 + S(\theta, h) + \gamma\theta h\mu}{1 + 2\theta h\lambda - (2 - \gamma)\theta h\mu} < \frac{1 + S(\theta, h) + \theta h\mu}{1 + 2\theta h\lambda - \theta h\mu}.$$

Thus in any case, it is always true that

$$E|Y_{n+1}|^2 \leq \bar{\nu}_h \max_{n-\kappa \leq i \leq n} E|Y_i|^2, \quad (68)$$

where

$$\bar{\nu}_h = \frac{1 + S(\theta, h) + \theta h\mu}{1 + 2\theta h\lambda - \theta h\mu} < 1.$$

Continuing in the same way as in Theorem 22, we can deduce from (68) that if $h < h_0(\theta) := \min\{h_1(\theta), h_2(\theta)\}$, for $n \geq 0$, then

$$E|Y_n|^2 \leq H \|\psi\|_E^2 \exp\{-\nu_h n h\}, \quad (69)$$

where $H = (\bar{\nu}_h)^{-1/(\kappa+1)}$, $\nu_h = \frac{-\ln \bar{\nu}_h}{(\kappa+1)h}$. The proof is completed. $\square$

From Theorem 26, we observe that, if F behaves linearly as in condition (C3'), the θ -Maruyama methods with $0 \leq \theta < 1$ can preserve stability of the underlying system under step size restriction, and with implicitness increasing (i.e., parameter θ increasing), the range of permitted h ensuring the inheritance of stability grows large. Note that $h_1(\theta)$ and $h_2(\theta)$ both tend to infinity as θ approaches 1, that is, restriction on h disappears as $\theta = 1$, which is a straight forward derivation from Theorem 22.

Theorem 22 and 26 show that only backward Euler method can reproduce exponential mean-square stability of the underlying system without any constraint on step size h . If the exponential mean-square stability of the numerical methods is replaced by a weaker one, for example, asymptotic mean-square stability, is the restriction to step size less? The answer is positive.

In order to investigate the asymptotic mean-square stability of the θ -Maruyama method, the delay function $\tau(t)$ is restricted to a special case, i.e., $\tau(t) \equiv \tau$. Let $\tau = (\kappa - \delta)h$ with $1 \leq \kappa \in \mathbb{Z}^+$, $0 \leq \delta < 1$, thus (6) is reduced to

$$\bar{Y}_n = \begin{cases} \psi(t_n - \tau), & t_n - \tau \leq t_0, \\ (1 - \delta)Y_{n-\kappa} + \delta Y_{n-\kappa+1}, & t_0 < t_n - \tau = t_0 + (n - \kappa)h + \delta h. \end{cases} \quad (70)$$

Theorem 27. Suppose that the conditions of Corollary 15 are fulfilled, then for any $h > 0$ and $\frac{1}{2} \leq \theta \leq 1$, the θ -Maruyama method for (2) with constant lag is asymptotically mean-square stable.

Proof. It follows from (5) that

$$Y_{n+1} - \theta h F_{n+1} = Y_n - \theta h F_n + h F_n + G_n \Delta W_n,$$

which yields

$$\begin{aligned} |Y_{n+1} - \theta h F_{n+1}|^2 &= |Y_n - \theta h F_n|^2 + h^2(1 - 2\theta)|F_n|^2 \\ &\quad + 2h\Re\langle Y_n, F_n \rangle + |G_n \Delta W_n|^2 + 2\Re\langle Y_n + (1 - \theta)hF_n, G_n \Delta W_n \rangle \\ &\leq |Y_n - \theta h F_n|^2 + 2h\Re\langle Y_n, F_n \rangle + |G_n \Delta W_n|^2 \\ &\quad + 2\Re\langle Y_n + (1 - \theta)hF_n, G_n \Delta W_n \rangle. \end{aligned}$$

Consequently,

$$\begin{aligned} E|Y_{n+1} - \theta h F_{n+1}|^2 &\leq E|Y_n - \theta h F_n|^2 + 2hE\Re\langle Y_n, F_n \rangle + E|G_n \Delta W_n|^2 \\ &= E|Y_n - \theta h F_n|^2 + 2hE\Re\langle Y_n, F(t, Y_n, 0) \\ &\quad + F(t_n, Y_n, \bar{Y}_n) - F(t, Y_n, 0) \rangle + E|G_n \Delta W_n|^2 \\ &\leq E|Y_n - \theta h F_n|^2 - 2h\lambda|Y_n|^2 + h\mu(E|Y_n|^2 + E|\bar{Y}_n|^2) \\ &\quad + h(\beta_0 E|Y_n|^2 + \beta_1 E|\bar{Y}_n|^2) \\ &\leq \dots \\ &\leq E|Y_0 - \theta h F_0|^2 + h(-2\lambda \sum_{i=0}^n E|Y_i|^2 + (\mu + \beta_0) \sum_{i=0}^n E|Y_i|^2) \\ &\quad + h(\mu + \beta_1) \sum_{i=0}^n E|\bar{Y}_i|^2. \end{aligned} \tag{71}$$

It follows from (70) that

$$E|\bar{Y}_i|^2 = E|(1 - \delta)Y_{i-\kappa} + \delta Y_{i-\kappa+1}|^2 \leq (1 - \delta)E|Y_{i-\kappa}|^2 + \delta E|Y_{i-\kappa+1}|^2. \tag{72}$$

Hence,

$$\sum_{i=0}^n E|\bar{Y}_i|^2 \leq \sum_{i=0}^n E|Y_i|^2 + \kappa \max_{-\kappa \leq j \leq -1} E|Y_j|^2. \tag{73}$$

By (71) and (73), we have

$$\begin{aligned} E|Y_n - \theta h F_n|^2 &\leq E|Y_0 - \theta h F_0|^2 + h(-2\lambda + 2\mu + \beta_0 + \beta_1) \sum_{i=0}^{n-1} E|Y_i|^2 \\ &\quad + (\mu + \beta_1)\kappa h \max_{-\kappa \leq j \leq -1} E|Y_j|^2, \end{aligned} \tag{74}$$

which yields

$$\begin{aligned} &E|Y_n - \theta h F_n|^2 + h(2\lambda - 2\mu - \beta_0 - \beta_1) \sum_{i=0}^{n-1} E|Y_i|^2 \\ &\leq E|Y_0 - \theta h F_0|^2 + (\mu + \beta_1)\kappa h \max_{-\kappa \leq j \leq -1} E|Y_j|^2. \end{aligned} \tag{75}$$

(74) and (42) imply that

$$\lim_{n \rightarrow \infty} E|Y_n|^2 = 0.$$

That is, for $\frac{1}{2} \leq \theta \leq 1$, the θ -Maruyama methods are asymptotically mean-square stable. $\square$

Theorem 27 shows that, for $\frac{1}{2} \leq \theta \leq 1$, the θ -Maruyama method is asymptotically mean-square stable, provided that the underlying system is exponentially mean-square stable. Although the stability of the numerical methods is weaker than that of the underlying system, specializing Theorem 27 to the cases of linear scalar SDDEs, nonlinear SODEs and nonlinear deterministic DDEs, respectively, we have the following interesting observations.

Corollary 28. *For $\frac{1}{2} \leq \theta \leq 1$ and any $h > 0$, the θ -Maruyama method is mean-square stable for linear scalar SDDEs (43)-(44) with constant lag.*

Corollary 29. *For $\frac{1}{2} \leq \theta \leq 1$ and any $h > 0$, the θ -Maruyama method is asymptotically mean-square stable for nonlinear SODEs (45)-(46) with $2\lambda - \beta_0 > 0$.*

Corollary 30. *For $\frac{1}{2} \leq \theta \leq 1$ and any $h > 0$, the θ -method is asymptotically stable for nonlinear DDEs (47)-(49) with constant lag provided $\lambda(t) \equiv \lambda, \mu(t) \equiv \mu$.*

It should be pointed out that the three corollaries above are different from those in the literature.

Remark 31. *Corollary 28 shows that, for $\frac{1}{2} \leq \theta \leq 1$, the θ -Maruyama methods are mean-square stable without any constraint on step size, whereas a constraint is required in [26].*

Remark 32. *There are differences between Corollary 29 and that in [15]. The former is valid for linear and nonlinear systems, whereas the latter only for linear scalar problems.*

Remark 33. *To be our best knowledge, there is no work on stability analysis of numerical methods for nonlinear deterministic DDEs (47) under conditions (48). The majority of numerical stability analysis for nonlinear deterministic DDEs in the literature is based on the conditions (50).*

Therefore, all the three corollaries above are new results.

Remark 34. *We note that (5) is semi-implicit, a natural question is the solvability of it. We know of no general existence and uniqueness theorem for the form (5), (37) [18]. If we impose a one-sided Lipschitz condition of $F(t, x, y)$ in x , the condition and the second inequality of (37) ensure that there exists a unique solution of (5) with appropriate step size restriction [12]. The main focus of this section is to find conditions for stability of solutions whenever they exist, rather than to concentrate on the practical construction of solution. Therefore, we always assume that there exists a unique solution of (5) without taking into account any constraint on step size.*

5. Numerical experiments

This section is devoted to presenting numerical experiments to confirm the obtained theoretical results.

Example 1. Consider the Mackey-Glass equation [13] with a multiplicative noise input

$$dX(t) = [-aX(t) + \frac{bX(t-\tau)}{1 + [X(t-\tau)]^n}]dt + \sigma X(t)dW(t), \quad (76)$$

where $a > 0$, b and σ are real parameters and n is an even positive integer. In the following, we use θ -Maruyama methods proposed in this article to solve the problem with parameters

$$a = 3, \quad b = 1, \quad \sigma = 1, \quad n = 2, \quad \tau = 1, \quad X(t) = 1, \quad t \in [-1, 0].$$

We identify the numerical solution of Euler Maruyama method ($\theta = 0$) with a suf-

Figure 1: Simulations with different θ . Left: $\theta=0.5$, right: $\theta=1$.

ficiently small step size (here $h = \Delta t = 2^{-10}$) as the "true solution". In Figure 1, we draw the numerical solution obtained from the θ -Maruyama method with step size $h = 1/32$ together with the "true solution" for one sample path. There one can see the convergence of θ -Maruyama method.

Next we test the stability. Using the same notations as in the preceding section, one can readily verify that equation (76) obeys conditions in Corollary 15 with $\lambda = a$, $\mu = |b|$, $\beta_0 = |\sigma|^2$, $\beta_1 = 0$. Therefore $\lambda - \mu - \frac{\beta_0 + \beta_1}{2} = 1.5 > 0$ and (76) is exponentially mean-square stable. In Figure 2, we show stability behavior of different methods with parameters $\theta = 0.45, 0.5, 0.75$. The figure shows that θ -Maruyama method is unstable at $\theta = 0.45$ for large step size $h = 6$. But $\theta \geq 0.5$, i.e., $0.5, 0.75$ remain stable for any large step size. Note that Theorem 27 in the previous section tells us that, for $\frac{1}{2} \leq \theta \leq 1$, the numerical method is mean-square stable for any step size $h > 0$. The numerical results here are consistent with our established results concerning convergence and stability.

Example 2. Consider a nonlinear SDDE with a time-varying delay as follows

$$dX(t) = [-4X(t) - 3X^3(t) + X(t - \tau(t))]dt + [X(t) + X(t - \tau(t))]dW(t), \quad t > 0, \quad (77)$$

Figure 2: Stability behavior with different θ . Left: $\theta=0.45$, middle: $\theta=0.5$, right: $\theta=0.75$.

where $\tau(t) = \frac{1}{1+t^2}$, $X(t) = 0.5, t \in [-1, 0]$. Similarly, we identify numerical solution by EM with a sufficiently small step size (here $\Delta t = 2^{-10}$) as the "true solution" for the error-comparison. We simulate the solution in the spirit of [16], and $M = 5000$ sample trajectories are simulated for the step size $h = 4\Delta t, 8\Delta t, 16\Delta t, 32\Delta t$. The mean-square errors $\epsilon^r = \epsilon_{2^{1+r}\Delta t}$, $r = 1, 2, 3, 4$, measured at a fixed terminal time $T = 1$, are estimated by trajectory averaging, i.e., $\epsilon^r \cong \frac{1}{M} \sum_{j=1}^M |X_{r,T}(\omega_j) - Y_{r,N}(\omega_j)|^2$. To provide an intuitive understanding of the strong order $1/2$, we plot mean-square errors $\sqrt{\epsilon^r}$ against h on a log-log scale, which is shown in Figure 3, where asterisks are connected with solid lines. For reference, a dashed line of slope one-half is added. It is very clear that the slopes of two curves match very well, which indicates a strong convergence order of $1/2$.

Obviously, equation (77) satisfies conditions (C2)-(C4) in Corollary 15, with $\lambda = 4, \mu = 1, \beta_0 = \beta_1 = 2$. Thus $2\lambda - 2\mu - \beta_0 - \beta_1 = 2 > 0$ and the problem is exponentially mean-square stable. As is shown in Figure 4, the backward Euler method can well reproduce stability for large step size $h = 2, 5, 10$.

Figure 3: $\log \sqrt{\epsilon^r}$ versus $\log h$. Left: $\theta = 0$, middle: $\theta = 0.5$, right: $\theta = 1$.

6. Conclusions

This work focuses on mean-square convergence and mean-square stability of θ -Maruyama methods for nonlinear SDDEs with variable lag. It is first proved that the methods are mean-square convergent with order $\frac{1}{2}$ under global Lipschitz conditions. Further, mean-square stability properties of SDDEs and those of θ -Maruyama methods are studied under both global Lipschitz and non-global Lipschitz settings. The stability

Figure 4: Simulations for (77) with $\theta=1$.

results are shown to be deeper or sharper than those in existing works. Also, the stability properties of θ -Maruyama methods indicate that different choice of method parameter θ results in different stability performance. In particular, the θ -Maruyama methods with $\frac{1}{2} \leq \theta \leq 1$ possess excellent asymptotic mean-square stability properties. Therefore, the θ -Maruyama methods with $\frac{1}{2} \leq \theta \leq 1$ can be a good candidate for solving stiff SDDs, where the explicit scheme like the Euler-Maruyama method faces severe time step reduction due to its bad stability property.

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