An improved superconvergence error estimate for the LDG method [★]

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Abstract

In this manuscript we present an error analysis for the local discontinuous Galerkin method for a model elliptic problem on Cartesian meshes when polynomials of degree at most k and an appropriate approximation of the boundary condition are used. This special approximation allows us to achieve $k + 1$ order of convergence for both the potential and its gradient in the L^2 norm. Here we improve on existing estimates for the solution gradient by a factor $\sqrt{h}$.

Key words: Finite element method; Discontinuous Galerkin; Superconvergence; Elliptic problems.

1 Introduction

The discontinuous Galerkin (DG) finite element method was first used to solve the neutron equation [17]. Cockburn and Shu [11,13] extended the method to solve first-order hyperbolic partial differential equations of conservation laws. They also developed the Local Discontinuous Galerkin (LDG) method for convection-diffusion problems [12]. DG methods have been developed for other partial differential equations the reader may consult [10,14,15,16] and the references cited therein for a detailed discussion of the history of DG methods and their applications.

The success of the DG method is due to one or more of the following properties: (i) does not require continuity across element boundaries, (ii) is locally

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conservative, (iii) is well suited to solve problems on locally refined meshes with hanging nodes, (iv) exhibits strong superconvergence that can be used to estimate the discretization error, (v) has a simple communication pattern between elements with a common face that makes it useful for parallel computation and (vi) it can handle problems with complex geometries to higher order.

The LDG finite element method for solving one-dimensional convection-diffusion partial differential equations was introduced by Cockburn and Shu [12] and is based on the work by Bassi and Rebay [4]. Castillo *et al.* [6] presented the first *a priori* error analysis for the LDG method applied to a model elliptic problem. They considered arbitrary meshes with hanging nodes and elements of various shapes and studied general numerical fluxes. They showed that, for smooth solutions, the L^2 errors in u and ∇u , respectively, are of order $k + 1/2$ and k when polynomials of total degree not exceeding k are used. Later, Cockburn *et al.* [9] presented a superconvergence result for the LDG method for a model elliptic problem on Cartesian meshes. They identified a special numerical flux for which the L^2 norms of the solution gradient and the potential are of orders $k + 1/2$ and $k + 1$, respectively, when tensor product polynomials of degree at most k are used.

An optimal minimal dissipation LDG (md-LDG) method for one-dimensional convection-diffusion problems was first investigated in [7]. Later, the method was extended to two-dimensions on triangular meshes in [8] where the authors proved $\mathcal{O}(h^{k+1})$ and $\mathcal{O}(h^k)$ L^2 convergence rates, respectively, for the potential and solution gradient using the complete k -degree polynomial space P_k .

Cockburn *et al.* [9], at the end of section 4.3, stated the following: “ These experiments justify our contention that the optimal order of convergence in $\mathbf{q}$ can be reached if the boundary conditions are piecewise polynomials of degree k . Our theoretical analysis does not explain this phenomenon”. Computational results in section 3 and [2], respectively, for general LDG methods and the minimal dissipation LDG method suggest that the LDG solution of elliptic problems obtained by interpolating Dirichlet boundary conditions at Radau points is $\mathcal{O}(h^{k+1})$ superconvergent for both the solution gradient and potential. In this manuscript we prove that, by approximating Dirichlet boundary conditions with appropriate projections or interpolations, both the LDG solution and its gradient on Cartesian meshes are $\mathcal{O}(h^{k+1})$ convergent. The proof is carried out for the md-LDG method and its extension to other LDG methods is discussed in the next section.

Consider the following second-order elliptic boundary-value problem

$$-\Delta u = f, \quad \text{in } \Omega = [-1, 1]^2, \quad (2.1a)$$

$$u = g_D, \quad \text{on } \partial\Omega_D, \quad (2.1b)$$

$$\nabla u \cdot \mathbf{n} = \mathbf{g}_N \cdot \mathbf{n}, \quad \text{on } \partial\Omega_N, \quad (2.1c)$$

for some given functions g_D and $\mathbf{g}_N$, where $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$, $\mathbf{n}$ is the unit outer normal vector on $\partial\Omega$ and the measure of $\partial\Omega_D$ is nonzero.

In our analysis, we select the boundary conditions and the source term $f(x, y)$ such that the exact solution, $u(x, y)$, is a smooth function. In order to construct the LDG formulation of Cockburn *et al.* [9], we first introduce an auxiliary variable $\mathbf{q} = \nabla u$ and transform the elliptic problem (2.1) to the following system of first-order differential equations

$$\mathbf{q} = \nabla u, \quad \text{in } \Omega, \quad (2.2a)$$

$$-\nabla \cdot \mathbf{q} = f, \quad \text{in } \Omega, \quad (2.2b)$$

$$u = g_D, \quad \text{on } \partial\Omega_D, \quad (2.2c)$$

$$\mathbf{q} \cdot \mathbf{n} = \mathbf{g}_N \cdot \mathbf{n}, \quad \text{on } \partial\Omega_N. \quad (2.2d)$$

Next, we partition our domain Ω into a rectangular mesh $\mathcal{T}$ consisting of $N = n \times n$ elements $K_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, $i, j = 0, 1, \dots, n-1$, where $x_i = y_i = -1 + ih$, $i = 0, 1, \dots, n$ and $h = 2/n$. Now we multiply (2.2a) and (2.2b) by arbitrary smooth test functions v and $\mathbf{r}$, respectively, integrate over an arbitrary element K , and apply the divergence theorem to write

$$\int_K \mathbf{q} \cdot \mathbf{r} dx = - \int_K u \nabla \cdot \mathbf{r} dx + \int_{\partial K} u \mathbf{r} \cdot \mathbf{n} ds, \quad (2.3)$$

$$\int_K \mathbf{q} \cdot \nabla v dx = \int_K f v dx + \int_{\partial K} v \mathbf{q} \cdot \mathbf{n} ds. \quad (2.4)$$

Let $Q_k(K)$ denote the tensor product space on K consisting of polynomials where the degree in each variable does not exceed k , and define

$$M_N := \{\mathbf{q} \in (L^2(\Omega))^2 : \mathbf{q}|_K \in Q_k(K)^2\}, \quad (2.5)$$

$$V_N := \{u \in L^2(\Omega) : u|_K \in Q_k(K)\}. \quad (2.6)$$

The LDG formulation consists of finding $(\mathbf{q}_N, u_N) \in M_N \times V_N$ that satisfies

$$\int_K \mathbf{q}_N \cdot \mathbf{r} dx = - \int_K u_N \nabla \cdot \mathbf{r} dx + \int_{\partial K} \hat{u}_N \mathbf{r} \cdot \mathbf{n} ds, \quad (2.7a)$$

$$\int_K \mathbf{q}_N \cdot \nabla v dx = \int_K f v dx + \int_{\partial K} v \hat{\mathbf{q}}_N \cdot \mathbf{n} ds, \quad (2.7b)$$

for all $(\mathbf{r}, v) \in M_N \times V_N$ and for all $K \in \mathbf{T}$. Here $\mathbf{n}$ is the unit outward normal vector to the edges ∂K and the numerical fluxes $\hat{\mathbf{q}}_N$ and $\hat{u}_N$ are the discrete

approximations of the traces of $\mathbf{q}$ and u on element boundaries. In order to complete the definition of the md-LDG method we need to select the fluxes $\hat{\mathbf{q}}_N$ and $\hat{u}_N$ on ∂K . Let the mean value $\{\cdot\}$ and jump $\llbracket \cdot \rrbracket$ of a scalar function u_N and a vector $\mathbf{q}_N$ at (x, y) on an edge of ∂K be defined as

$$\{u\}(x, y) = \frac{1}{2}(u^+(x, y) + u^-(x, y)), \quad \{\mathbf{q}\}(x, y) = \frac{1}{2}(\mathbf{q}^+(x, y) + \mathbf{q}^-(x, y)), \quad (2.8)$$

$$\llbracket u \rrbracket(x, y) = (u^+(x, y) - u^-(x, y))\mathbf{n}, \quad \llbracket \mathbf{q} \rrbracket(x, y) = (\mathbf{q}^+(x, y) - \mathbf{q}^-(x, y)) \cdot \mathbf{n}, \quad (2.9)$$

where u^+ is the limit of the solution on K and u^- is the limit of the solution of an adjacent element sharing ∂K , *i.e.*, for $(x, y) \in \partial K$, and $\mathbf{n}$ is the outer unit normal on ∂K we have

$$u^+(x, y) = \lim_{\epsilon \rightarrow 0^-} u((x, y) + \epsilon \mathbf{n}), \quad u^-(x, y) = \lim_{\epsilon \rightarrow 0^+} u((x, y) + \epsilon \mathbf{n}). \quad (2.10)$$

50 We follow [9] to define the numerical fluxes $\hat{u}$ and $\hat{\mathbf{q}}$ on interior edges by
51 introducing an auxiliary vector $\mathbf{v}$ and write

$$\hat{\mathbf{q}} = \{\mathbf{q}\} - C_{11}\llbracket u \rrbracket - \mathbf{C}_{12}\llbracket \mathbf{q} \rrbracket, \quad (2.11)$$

$$\hat{u} = \{u\} + \mathbf{C}_{12} \cdot \llbracket \mathbf{q} \rrbracket. \quad (2.12)$$

The parameters are defined on each edge of ∂K such that $C_{11} \geq 0$ and $\mathbf{C}_{12}$ is given by

$$\mathbf{C}_{12} \cdot \mathbf{n} = \frac{1}{2} \text{sign}(\mathbf{v} \cdot \mathbf{n}). \quad (2.13)$$

52 We note that the vector $\mathbf{v}$ is an arbitrary but fixed vector with nonzero com-
53 ponents as illustrated in Figure 2.1. Without loss of generality, from now on
54 we assume that $\mathbf{v}$ has strictly positive components, thus, $\hat{u}$ on a horizontal
55 edge is the limit of u from below while on a vertical edge it is the limit of u
56 from the left. $\hat{\mathbf{q}}$ on a horizontal edge is the limit of $\mathbf{q}$ from above while $\hat{\mathbf{q}}$ on
57 a vertical edge is the limit from the right. We also define

$$\partial\Omega^- = \{(x, y) \in \partial\Omega \mid \mathbf{v} \cdot \mathbf{n} < 0\}, \quad \partial\Omega^+ = \{(x, y) \in \partial\Omega \mid \mathbf{v} \cdot \mathbf{n} > 0\}. \quad (2.14)$$

58 If $\partial\Omega_D^\pm = \partial\Omega_D \cap \partial\Omega^\pm$, we let $\mathcal{E}$, $\mathcal{E}_D$, $\mathcal{E}_N$, $\mathcal{E}^\pm$, $\mathcal{E}_D^\pm$ respectively, denote the sets
59 of all edges in $\partial\Omega$, $\partial\Omega_D$, $\partial\Omega_N$, $\partial\Omega^\pm$, $\partial\Omega_D^\pm$. Finally the set of all interior edges
60 is defined as $\mathcal{E}_I = \mathcal{E} \setminus (\mathcal{E}_D \cup \mathcal{E}_N)$.

61 Letting C_{11} be a positive constant we define the numerical flux on the bound-
62 ary as

Fig. 2.1. An example of a Cartesian mesh $\mathbf{T}$ and a vector $\mathbf{v}$ to define mesh orientation with $\partial\Omega^\pm = \Gamma_1^\pm \cup \Gamma_2^\pm$.

$$\hat{\mathbf{q}}_N = \begin{cases} \mathbf{q}_N^+ - C_{11}(u_N^+ - g_D) & \text{if } e \in \mathcal{E}_D \\ \mathbf{g}_N & \text{if } e \in \mathcal{E}_N \end{cases}, \quad (2.15)$$

$$\hat{u}_N = \begin{cases} u_N^+ & \text{if } e \in \mathcal{E}_N \\ \mathcal{P}^+ g_D & \text{if } e \in \mathcal{E}_D \end{cases}, \quad (2.16)$$

63 where $\mathcal{P}^+ = \mathcal{P}_1^+$ if the edge e is parallel to the x_1 - axis and $\mathcal{P}^+ = \mathcal{P}_2^+$ if
 64 e is parallel to the x_2 - axis as defined below.

Let $I = [a^-, a^+]$ be an arbitrary interval, and let $P_k(I)$ be the space of polynomials of degree at most k on I , the Radau projection $\mathcal{P}^\pm w \in P_k(I)$ of w is determined by the following $k + 1$ conditions

$$\int_I [w(x) - \mathcal{P}^\pm w(x)] p(x) dx = 0 \quad \forall p \in P_{k-1}(I), \quad \mathcal{P}^\pm w(a^\pm) = w(a^\pm). \quad (2.17)$$

If $\mathcal{P}$ denotes the L^2 projection onto $P_k(I)$, then on a rectangle $K = I_1 \times I_2$ with $I_1 = [a^-, a^+]$ and $I_2 = [b^-, b^+]$ and $\mathbf{r} = (r_1, r_2)^t$, we define $\mathbf{\Pi}^\pm \mathbf{r} = (\mathcal{P}_1^\pm \otimes \mathcal{P}_2 r_1, \mathcal{P}_1 \otimes \mathcal{P}_2^\pm r_2)^t$ and $\pi^\pm u = \mathcal{P}_1^\pm \otimes \mathcal{P}_2^\pm u$. The subscripts in $\mathcal{P}$ and $\mathcal{P}^\pm$ indicate the one dimensional operators, for instance $\mathcal{P}_1$, $\mathcal{P}_1^+$, $\mathcal{P}_1^-$, respectively, denote the projections $\mathcal{P}$, $\mathcal{P}^+$, $\mathcal{P}^-$ onto $P_k(I_1)$ in the x_1 variable. Similarly, $\mathcal{P}_2$, $\mathcal{P}_2^+$, $\mathcal{P}_2^-$, respectively, denote the projections $\mathcal{P}$, $\mathcal{P}^+$, $\mathcal{P}^-$ onto $P_k(I_2)$ in the x_2 variable. Next we describe the two dimensional operator $\mathcal{P}_1^+ \otimes \mathcal{P}_2$ applied to a scalar function $u(x_1, x_2)$ defined on the rectangle $K = I_1 \times I_2$. Let $p_i^{(1)}(x_1)$ and $p_i^{(2)}(x_2)$, respectively, denote the i^{th} -degree Legendre

polynomials shifted to I_1 and I_2 and apply the L^2 projection operator $\mathcal{P}_2$ to u in the x_2 variable on I_2 to find $\mathcal{P}_2 u(x_1, x_2) = \sum_{i=0}^k c_i(x_1) p_i^{(2)}(x_2)$ such that

$$\int_{I_2} [u(x_1, x_2) - \mathcal{P}_2 u(x_1, x_2)] p_i^{(2)}(x_2) dx_2 = 0, \quad \forall i = 0, 1, \dots, k. \quad (2.18)$$

Then, we apply $\mathcal{P}_1^+$ to $\mathcal{P}_2 u$ in the x_1 variable on I_1 to find $\mathcal{P}_1^+ \otimes \mathcal{P}_2 u(x_1, x_2) = \sum_{i=0}^k \sum_{j=0}^k c_{ij} p_j^{(1)}(x_1) p_i^{(2)}(x_2) \in Q_k$ such that

$$\int_{I_1} [\mathcal{P}_2 u(x_1, x_2) - \mathcal{P}_1^+ \otimes \mathcal{P}_2 u(x_1, x_2)] p_j^{(1)}(x_1) dx_1 = 0, \quad \forall j = 0, 1, \dots, k-1, \quad (2.19)$$

and

$$\mathcal{P}_1^+ \otimes \mathcal{P}_2 u(a^+, x_2) = \mathcal{P}_2 u(a^+, x_2). \quad (2.20)$$

65 The other two dimensional operators are defined in a similar manner.

66

Since $\mathbf{v}$ has strictly positive components we use the projections

$$\mathbf{\Pi} \mathbf{q}|_K := \mathbf{\Pi}^- \mathbf{q}|_K, \quad \pi u|_K := \pi^+ u|_K, \quad \forall K \in \mathbf{T}, \quad (2.21)$$

having the properties

$$\int_K [\mathbf{r} - \mathbf{\Pi}^- \mathbf{r}] \cdot \nabla v dx = 0, \quad \forall v \in Q^k(K), \quad (2.22a)$$

$$\int_{\gamma_i^-} [\mathbf{r} - \mathbf{\Pi}^- \mathbf{r}] \cdot \mathbf{n} v ds = 0, \quad \forall v \in Q^k(\gamma_i^-), \quad i = 1, 2, \quad (2.22b)$$

$$(u - \pi^+ u)|_{\gamma_i^+} = u|_{\gamma_i^+} - \mathcal{P}_i^+ u|_{\gamma_i^+}, \quad i = 1, 2, \quad (2.22c)$$

67 where γ_i^+ , $i = 1, 2$ are the outflow edges and γ_i^- , $i = 1, 2$ are inflow edges of
68 element K as illustrated in Figure 2.1.

69 In our analysis we need the *a priori* error estimates stated in the following
70 Lemma.

Lemma 2.1 *Let $v \in H^{s+2}(K)$ and $\mathbf{r} \in H^{s+1}(K)^2$, $s \geq 0$. Then for $m = 0, 1$, we have*

$$|v - \pi^\pm v|_{m,K} \leq Ch^{\min\{s+1,k\}+1-m} \|v\|_{s+2,K}, \quad (2.23a)$$

$$\|v - \pi^\pm v\|_{0,e} \leq Ch^{\min\{s+1,k\}+\frac{1}{2}} \|v\|_{s+2,K}, \quad \forall e \in \partial K, \quad (2.23b)$$

$$|\mathbf{r} - \mathbf{\Pi}^\pm \mathbf{r}|_{m,K} \leq Ch^{\min\{s,k\}+1-m} \|\mathbf{r}\|_{s+1,K}, \quad (2.23c)$$

$$\|\mathbf{r} - \mathbf{\Pi}^\pm \mathbf{r}\|_{0,e} \leq Ch^{\min\{s,k\}+\frac{1}{2}} \|\mathbf{r}\|_{s+1,K}, \quad \forall e \in \partial K. \quad (2.23d)$$

71 Furthermore, for any edge e_i parallel to the x_i -axis, $i = 1, 2$, we have

72

$$\|w - \mathcal{P}_i^\pm w\|_{0,e_i} \leq Ch^{\min\{s+\frac{1}{2},k\}+1} \|w\|_{s+\frac{3}{2},e_i}, \quad \forall w \in H^{s+\frac{3}{2}}(e_i). \quad (2.23e)$$

If $v \in W^{s+1,\infty}(K)$, then

$$\|v - \pi^\pm v\|_{L^\infty(e)} \leq Ch^{\min\{s,k\}+1} \|v\|_{W^{s+1,\infty}(K)}, \quad \forall e \in \partial K. \quad (2.23f)$$

Let $\mathcal{R}^+ v$ be the k -degree polynomial interpolating v at the roots of the $(k+1)$ -degree right Radau polynomial shifted to the edge e and $v \in H^{k+2}(e)$ then

$$\|\mathcal{P}^+ v - \mathcal{R}^+ v\|_{0,e} \leq Ch^{k+2} \|v\|_{k+2,e}. \quad (2.24)$$

Proof. The proof of (2.23) can be found in [6,9]. The estimate (2.24) is proved in [1]. $\square$

For simplicity, the analysis is done for shape regular uniform Cartesian meshes and the extension to non uniform meshes is straight forward. Furthermore, the analysis will only consider domains Ω such that if $f \in L^2(\Omega)$ and the boundary data are zero, then $u \in H^2(\Omega)$ and $\|u\|_2 \leq C\|f\|_0$.

We now present a superconvergence error analysis for the minimal dissipation LDG method where $C_{11} = \mathcal{O}(1)$ on all edges in $\mathcal{E}_D^+$ and $C_{11} = 0$ on all other edges. At the end of this section we will discuss how to extend this error analysis to other LDG methods.

Summing (2.7a) and (2.7b) over all elements the discrete LDG formulation consists of finding $(\mathbf{q}_N, u_N) \in M_N \times V_N$ such that

$$A(\mathbf{q}_N, u_N; \mathbf{r}, v) = \mathcal{F}(\mathbf{r}, v), \quad \forall (\mathbf{r}, v) \in M_N \times V_N, \quad (2.25a)$$

where

$$A(\mathbf{q}, u; \mathbf{r}, v) := a(\mathbf{q}, \mathbf{r}) + b(u, \mathbf{r}) - b(v, \mathbf{q}) + c(u, v), \quad \mathcal{F}(\mathbf{r}, v) := F(\mathbf{r}) + G(v), \quad (2.25b)$$

and

$$a(\mathbf{q}, \mathbf{r}) = \int_{\Omega} \mathbf{q} \cdot \mathbf{r} dx, \quad (2.25c)$$

$$b(u, \mathbf{r}) = \sum_{K \in \mathbf{T}} \int_K u \nabla \cdot \mathbf{r} dx - \sum_{e \in \mathcal{E}_I} \int_e (\{u\} + \mathbf{C}_{12} \llbracket u \rrbracket) \llbracket \mathbf{r} \rrbracket ds - \sum_{e \in \mathcal{E}_N} \int_e u \mathbf{r} \cdot \mathbf{n} ds, \quad (2.25d)$$

$$c(u, v) = \sum_{e \in \mathcal{E}_D^+} \int_e C_{11} u v ds \quad (2.25e)$$

$$F(\mathbf{r}) = \sum_{e \in \mathcal{E}_D} \int_e \mathcal{P}^+ g_D \mathbf{r} \cdot \mathbf{n} ds, \quad (2.25f)$$

$$G(v) = \int_K f v dx + \sum_{e \in \mathcal{E}_D^+} \int_e C_{11} v g_D ds + \sum_{e \in \mathcal{E}_N} \int_e v \mathbf{g}_N \cdot \mathbf{n} ds. \quad (2.25g)$$

We note the use of $\mathcal{P}^+ g_D$ in the right hand side F is essential for obtaining superconvergence of the solution gradient $\mathbf{q}$ everywhere. However, The use of either of $\mathcal{P}^+ g_D$ or g_D in the right hand side G yields superconvergence for $\mathbf{q}$. Furthermore, we note that the true solution satisfies

$$A(\mathbf{q}, u; \mathbf{r}, v) = a(\mathbf{q}, \mathbf{r}) + b(u, \mathbf{r}) - b(v, \mathbf{q}) + c(u, v) = \int_K f v dx + \int_{\partial\Omega_D^+} C_{11} g_D v ds + \int_{\partial\Omega_D} g_D \mathbf{r} \cdot \mathbf{n} ds + \int_{\partial\Omega_N} v \mathbf{g}_N \cdot \mathbf{n} ds, \quad \forall (\mathbf{q}, u) \in (H^1(\Omega))^2 \times H^2(\Omega). \quad (2.26)$$

Subtracting (2.25a) from (2.26), and letting $\xi_u = u - \pi u$, $\xi_{\mathbf{q}} = \mathbf{q} - \mathbf{\Pi} \mathbf{q}$, $e_u = u - u_N$, $\mathbf{e}_{\mathbf{q}} = \mathbf{q} - \mathbf{q}_N$, $\pi e_u = \pi u - u_N$ and $\mathbf{\Pi} \mathbf{e}_{\mathbf{q}} = \mathbf{\Pi} \mathbf{q} - \mathbf{q}_N$, the following LDG orthogonality condition holds for all $(\mathbf{r}, v) \in M_N \times V_N$

$$A(\mathbf{e}_{\mathbf{q}}, e_u; \mathbf{r}, v) - \sum_{e \in \mathcal{E}_D} \int_e (g_D - \mathcal{P}^+ g_D) \mathbf{r} \cdot \mathbf{n} ds = 0. \quad (2.27)$$

In the remainder of this manuscript we use the semi norm

$$|(\mathbf{q}, u)|_A^2 = A(\mathbf{q}, u; \mathbf{q}, u) = \|\mathbf{q}\|_0^2 + C_{11} \|u\|_{0, \partial\Omega_D^+}^2, \quad (2.28)$$

and assume

$$\partial\Omega_N \cap \partial\Omega^- = \emptyset. \quad (2.29)$$

⁸⁴ The next lemma contains preliminary results needed in our analysis.

Lemma 2.2 *Let $u \in H^{k+2}(\Omega)$, $s \geq 0$, and set $\mathbf{q} = \nabla u$. Assume further that $\mathbf{q} \in W^{k+1, \infty}$ and that C_{11} is a positive constant, and let $\mathbf{\Pi}$ and π be the operators defined in (2.21). Then, for all $(\mathbf{r}, v) \in M_N \times V_N$, we have*

$$|A(-\xi_{\mathbf{q}}, -\xi_u; \mathbf{r}, v) + \sum_{e \in \mathcal{E}_D} \int_e (g_D - \mathcal{P}^+ g_D) \mathbf{r} \cdot \mathbf{n} ds| \leq C h^{k+1} |(\mathbf{r}, v)|_A. \quad (2.30)$$

⁸⁵ Furthermore, if $\varphi \in H^2$ such that $\varphi|_{\partial\Omega_D} = 0$, and $\Phi = \nabla \varphi$ we have

$$|A(\mathbf{r}, v; \Phi - \mathbf{\Pi} \Phi, \varphi - \pi \varphi)| \leq C |(\mathbf{r}, v)|_A \|\varphi\|_2, \quad (2.31)$$

⁸⁶ where $C > 0$ is a generic constant independent of $h, v, \mathbf{r}$.

Proof. First, we write

$$A(-\xi_{\mathbf{q}}, -\xi_u; \mathbf{r}, v) + \sum_{e \in \mathcal{E}_D} \int_e (g_D - \mathcal{P}^+ g_D) \mathbf{r} \cdot \mathbf{n} ds$$

$$= a(-\xi_{\mathbf{q}}, \mathbf{r}) + b(-\xi_u, \mathbf{r}) - b(v, -\xi_{\mathbf{q}}) + c(-\xi_u, v) \quad (2.32a)$$

$$+ \sum_{e \in \mathcal{E}_D} \int_e (g_D - \mathcal{P}^+ g_D) \mathbf{r} \cdot \mathbf{n} ds \quad (2.32b)$$

$$= T_1 + T_2 + T_3 + T_4, \quad (2.32c)$$

where

$$T_1 = a(-\xi_{\mathbf{q}}, \mathbf{r}), \quad (2.33a)$$

$$T_2 = -b(v, -\xi_{\mathbf{q}}), \quad (2.33b)$$

$$T_3 = b(-\xi_u, \mathbf{r}) + \sum_{e \in \mathcal{E}_D} \int_e (g_D - \mathcal{P}^+ g_D) \mathbf{r} \cdot \mathbf{n} ds, \quad (2.33c)$$

$$T_4 = - \sum_{e \in \mathcal{E}_D^+} \int_e C_{11} \xi_u v ds. \quad (2.33d)$$

Applying Cauchy Schwarz inequality to $T_1 = a(-\xi_{\mathbf{q}}, r)$ leads to

$$|T_1| \leq \left| \sum_{K \in \mathbf{T}} \int_K \mathbf{r} \cdot \xi_{\mathbf{q}} dx \right| \leq \left(\sum_{K \in \mathbf{T}} \|\mathbf{r}\|_{0,K}^2 \right)^{1/2} \left(\sum_{K \in \mathbf{T}} \|\xi_{\mathbf{q}}\|_{0,K}^2 \right)^{1/2}$$

$$\leq C |(\mathbf{r}, v)|_A \left(\sum_{K \in \mathbf{T}} h^{2k+2} \|\mathbf{q}\|_{k+1,K}^2 \right)^{1/2} \quad (2.34)$$

Next, we integrate $b(v, -\xi_{\mathbf{q}})$ in (2.25d) by parts to write T_2

$$T_2 = -b(v, -\xi_{\mathbf{q}}) = - \sum_{K \in \mathbf{T}} \int_K \nabla v \cdot \xi_{\mathbf{q}} dx + \sum_{e \in \mathcal{E}_D} \int_e v \xi_{\mathbf{q}} \cdot \mathbf{n} ds + \sum_{e \in \mathcal{E}_I} \int_e \llbracket v \rrbracket \hat{\xi}_{\mathbf{q}} ds.$$

87 where $\hat{\xi}_{\mathbf{q}}$ on an interior edge is the limit of $\xi_{\mathbf{q}}$ from the right on a vertical edge
 88 and the limit from above on a horizontal edge.

Using the properties (2.22) of the projection Π^- , we obtain

$$\int_K \nabla v \cdot \xi_{\mathbf{q}} dx = 0, \quad \forall K \in T, \quad \int_e v \xi_{\mathbf{q}} \cdot \mathbf{n} ds = 0, \quad \forall e \subset \partial\Omega^-,$$

$$\int_e \llbracket v \rrbracket \cdot \hat{\xi}_{\mathbf{q}} ds = 0, \quad \forall e \in \mathcal{E}_I.$$

Hence,

$$T_2 = \sum_{e \in \mathcal{E}_D^+} \int_e v \xi_{\mathbf{q}} \cdot \mathbf{n} ds. \quad (2.35)$$

Multiplying and dividing each term of the sum by $C_{11}^{1/2}$ and applying Cauchy Schwarz inequality, we obtain

$$\begin{aligned}
|T_2| &= \left| \sum_{e \in \mathcal{E}_D^+} \int_e v \xi_{\mathbf{q}} \cdot \mathbf{n} ds \right| \leq \left(\sum_{e \in \mathcal{E}_D^+} C_{11} \|v\|_{0,e}^2 \right)^{1/2} \left(\sum_{e \in \mathcal{E}_D^+} C_{11}^{-1} \|\xi_{\mathbf{q}}\|_{0,e}^2 \right)^{1/2} \quad (2.36) \\
&\quad (\text{by Lemma 2.1}) \leq C |(\mathbf{r}, v)|_A \left(\sum_{e \in \mathcal{E}_D^+} h^{2k+1} \|\mathbf{q}\|_{k+1,K}^2 \right)^{1/2} \\
&\leq C |(\mathbf{r}, v)|_A \left(\sum_{e \subset \partial \Omega_D^+} h^{2k+1} h^2 \|\mathbf{q}\|_{W^{k+1,\infty}}^2 \right)^{1/2} \\
&\leq C |(\mathbf{r}, v)|_A \left(\sum_{e \in \mathcal{E}_D^+} h^{2k+3} \right)^{1/2} \|\mathbf{q}\|_{W^{k+1,\infty}} \\
&= C |(\mathbf{r}, v)|_A h^{k+1} \|\mathbf{q}\|_{W^{k+1,\infty}}.
\end{aligned}$$

For T_3 we write

$$\begin{aligned}
T_3 &= b(-\xi_u, \mathbf{r}) + \sum_{e \in \mathcal{E}_D} \int_e (g_D - \mathcal{P}^+ g_D) \mathbf{r} \cdot \mathbf{n} ds = \\
&\quad - \left(\sum_{K \in \mathbf{T}} \xi_u \nabla \cdot \mathbf{r} dx - \sum_{K \in \mathbf{T}} \left(\sum_{\substack{e \in \mathcal{E}_I \\ e \subset \partial K}} \int_e \hat{\xi}_u \mathbf{r} \cdot \mathbf{n} ds \right) - \sum_{e \in \mathcal{E}_N} \int_e \xi_u \mathbf{r} \cdot \mathbf{n} ds \right) + \\
&\quad \sum_{e \in \mathcal{E}_D} \int_e (g_D - \mathcal{P} g_D) \mathbf{r} \cdot \mathbf{n} ds,
\end{aligned}$$

89 where $\hat{\xi}_u$ on an interior edge is the limit of ξ_u from the left if the edge is vertical
90 and from the bottom if the edge is horizontal.

91

92 Using the assumption (2.29) we write

$$\begin{aligned}
T_3 &= - \left(\sum_{K \in \mathbf{T}} \xi_u \nabla \cdot \mathbf{r} dx - \sum_{K \in \mathbf{T}} \left(\sum_{\substack{e \in \mathcal{E}_I \\ e \subset \partial K}} \int_e (u - \mathcal{P}^+ u) \mathbf{r} \cdot \mathbf{n} ds \right) \right) \\
&\quad + \sum_{e \subset \partial \Omega} \int_e (u - \mathcal{P}^+ u) \mathbf{r} \cdot \mathbf{n} ds \\
&= - \sum_{K \in \mathbf{T}} Z_K(\mathbf{r}, u), \quad (2.37)
\end{aligned}$$

where

$$Z_K(\mathbf{r}, u) = \int_K \xi_u \nabla \cdot \mathbf{r} dx - \int_{\gamma_1^- \cup \gamma_1^+} (u - \mathcal{P}_1^+ u) \mathbf{r} \cdot \mathbf{n} ds - \int_{\gamma_2^- \cup \gamma_2^+} (u - \mathcal{P}_2^+ u) \mathbf{r} \cdot \mathbf{n} ds,$$

93 with γ_i^+ being the outflow edges and γ_i^- the inflow edges of element K as
94 illustrated in Figure 2.1.

Applying Cauchy Schwarz inequality and Lemma 3.6 from [9] which proves that if $u \in H^{k+2}(\Omega)$ and $\mathbf{r} \in (H^1(\Omega))^2$, then $Z_K(\mathbf{r}, u)$ can be bounded as

$$|Z_K(\mathbf{r}, u)| \leq Ch^{k+1} \|u\|_{k+2,K} \|\mathbf{r}\|_{0,K}, \quad (2.38)$$

we write

$$\sum_{K \in \mathbf{T}} Z_K(\mathbf{r}, u) \leq C \sum_{K \in \mathbf{T}} h^{k+1} \|u\|_{k+2,K} \|\mathbf{r}\|_{0,K} \leq C |(\mathbf{r}, v)|_A h^{k+1} \|u\|_{k+2}.$$

Thus,

$$|T_3| \leq C |(\mathbf{r}, v)|_A h^{k+1} \|u\|_{k+2}.$$

Applying Cauchy Schwartz inequality we write

$$\begin{aligned} |T_4| &\leq \left| \sum_{e \in \mathcal{E}_D^+} \int_e C_{11} \xi_u v ds \right| \\ &\leq \left(\sum_{e \in \mathcal{E}_D^+} C_{11} \|v\|_{0,e}^2 \right)^{1/2} \left(\sum_{e \in \mathcal{E}_D^+} C_{11} \|\xi_u\|_{0,e}^2 \right)^{1/2}, \end{aligned}$$

leading to

$$|T_4| \leq |(\mathbf{r}, v)|_A \left(\sum_{e \in \mathcal{E}_D^+} C_{11} \|\xi_u\|_{0,e}^2 \right)^{1/2}, \quad (2.39)$$

which, by Lemma 2.1, becomes

$$\begin{aligned} |T_4| &\leq |(\mathbf{r}, v)|_A C \left(\sum_{e \in \mathcal{E}_D^+} h^{2k+1} \|u\|_{k+2,K}^2 \right)^{1/2} \\ &\leq |(\mathbf{r}, v)|_A Ch^{k+1} \|u\|_{W^{k+2,\infty}}. \end{aligned} \quad (2.40)$$

95 The proof of (2.30) is completed by combining the estimates for T_1, T_2, T_3 and
96 T_4 .

Similarly, we establish (2.31) by splitting

$$\begin{aligned} A(\mathbf{r}, v; \Phi - \Pi\Phi, \varphi - \pi\varphi) &= a(\mathbf{r}, \Phi - \Pi\Phi) + b(v, \Phi - \Pi\Phi) - b(\varphi - \pi\varphi, \mathbf{r}) \\ &\quad + c(v, \varphi - \pi\varphi) \\ &= \tilde{T}_1 + \tilde{T}_2 + \tilde{T}_3 + \tilde{T}_4, \end{aligned}$$

where

$$\tilde{T}_1 = a(\mathbf{r}, \xi_\Phi), \quad \tilde{T}_2 = -b(v, \xi_\Phi), \quad \tilde{T}_3 = b(\xi_\varphi, \mathbf{r}), \quad \tilde{T}_4 = \sum_{e \in \mathcal{E}_D^+} \int_e C_{11} \xi_\varphi v ds.$$

97 We establish the proof by bounding each of terms $\tilde{T}_i, i = 1, 2, 3, 4$.
 98 Following (2.34) for T_1 we apply Cauchy Schwarz inequality to $\tilde{T}_1$ and Lemma
 99 2.1 to obtain

$$|\tilde{T}_1| \leq |(\mathbf{r}, v)|_A \left(\sum_{K \in \mathbf{T}} \|\xi_{\Phi}\|_{0,K}^2 \right)^{1/2} \leq |(\mathbf{r}, v)|_A \|\xi_{\Phi}\|_0 \leq C |(\mathbf{r}, v)|_A \|\varphi\|_2. \quad (2.41)$$

We follow (2.35) and (2.36) for T_2 and use projection properties (2.22) and Lemma 2.1 to write

$$\begin{aligned} |\tilde{T}_2| &= \left| \sum_{e \in \mathcal{E}_D^+} \int_e v \xi_{\Phi} \cdot \mathbf{n} ds \right| \\ &\leq C |(\mathbf{r}, v)|_A \left(\sum_{e \in \mathcal{E}_D^+} C_{11}^{-1} \|\xi_{\Phi}\|_{0,e}^2 \right)^{1/2} \\ &\leq C |(\mathbf{r}, v)|_A \|\varphi\|_2. \end{aligned}$$

Next we will find a bound for $\tilde{T}_3$ as

$$|\tilde{T}_3| = \left| - \left(\sum_{K \in \mathbf{T}} \xi_{\varphi} \nabla \cdot \mathbf{r} dx - \sum_{K \in \mathbf{T}} \left(\sum_{e \in \mathcal{E}_T, e \subset \partial K} \int_e \hat{\xi}_{\varphi} \mathbf{r} \cdot \mathbf{n} ds \right) - \sum_{e \in \mathcal{E}_N} \int_e \xi_{\varphi} \mathbf{r} \cdot \mathbf{n} ds \right) \right|, \quad (2.42)$$

100 where $\hat{\xi}_{\varphi}$ is the limit of ξ_{φ} from the left on a vertical edge and from the bottom
 101 on a horizontal edge.

Since $\varphi = 0$ on $\partial\Omega_D$ and $\partial\Omega_N \subset \partial\Omega^+$, $\tilde{T}_3$ can be written as

$$\begin{aligned} \tilde{T}_3 &= - \sum_{K \in \mathbf{T}} Z_K(\mathbf{r}, \varphi) \\ &\leq C |(\mathbf{r}, v)|_A \|\xi_{\varphi}\|_1 \leq C |(\mathbf{r}, v)|_A \|\varphi\|_2. \end{aligned}$$

Finally for $\tilde{T}_4$ we follow the same reasoning as for T_4 to write

$$\begin{aligned} \tilde{T}_4 &\leq |(\mathbf{r}, v)|_A \left(\sum_{e \in \mathcal{E}_D^+} C_{11} \|\xi_{\varphi}\|_{0,e}^2 \right)^{1/2} \\ &\leq C |(\mathbf{r}, v)|_A \|\varphi\|_2. \end{aligned} \quad (2.43)$$

102 Combining the bounds for $\tilde{T}_i, i = 1, 2, 3, 4$ completes the proof of the lemma.
 103 $\square$

104 The main result of this work is summarized in the following theorem where
 105 the estimate for $\mathbf{q} - \mathbf{q}_N$ is improved by $\sqrt{h}$ while that for $u - u_N$ is the same
 106 as in [9].

Theorem 2.1 *Assume that the solution $(\mathbf{q}, u)$ of the problem (2.2) is in $H^{k+1}(\Omega) \times H^{k+2}(\Omega)$ for $k \geq 0$. If $(\mathbf{q}_N, u_N)$ satisfies the LDG weak equations (2.25), then*

$$|(\mathbf{q} - \mathbf{q}_N, u - u_N)|_A \leq Ch^{k+1}, \quad (2.44)$$

and

$$||u - u_N||_0 \leq Ch^{k+1} \quad (2.45)$$

107 where the constant $C > 0$ is independent of the mesh size h .

108 *Proof.* We first prove (2.44) which is an improvement of the result in [9] by a
 109 factor of $\sqrt{h}$.

Since $|(\cdot, \cdot)|_A$ is a semi norm we write

$$|(\mathbf{e}_q, e_u)|_A = |(\Pi \mathbf{e}_q + \xi_q, \pi e_u + \xi_u)|_A \leq |(\xi_q, \xi_u)|_A + |(\Pi \mathbf{e}_q, \pi e_u)|_A, \quad (2.46)$$

where the projection error term

$$|(\xi_q, \xi_u)|_A = \mathcal{O}(h^{k+1}). \quad (2.47)$$

Next, we establish that $|(\Pi \mathbf{e}_q, \pi e_u)|_A = \mathcal{O}(h^{k+1})$ to prove (2.44).

Let us write

$$\begin{aligned} |(\Pi \mathbf{e}_q, \pi e_u)|_A^2 &= A(\Pi \mathbf{e}_q, \pi e_u; \Pi \mathbf{e}_q, \pi e_u) \\ &= A(\Pi \mathbf{q} - \mathbf{q} + \mathbf{e}_q, \pi u - u + e_u; \Pi \mathbf{e}_q, \pi e_u) \\ &= A(\Pi \mathbf{q} - \mathbf{q}, \pi u - u; \Pi \mathbf{e}_q, \pi e_u) + A(\mathbf{e}_q, e_u; \Pi \mathbf{e}_q, \pi e_u) \end{aligned}$$

110 Applying the LDG orthogonality condition (2.27) with $\mathbf{r} = \Pi \mathbf{e}_q$ and $v = \pi e_u$
 111 we obtain

$$|(\Pi \mathbf{e}_q, \pi e_u)|_A^2 = A(-\xi_q, -\xi_u; \Pi \mathbf{e}_q, \pi e_u) + \sum_{e \in \mathcal{E}_D} \int_e (g_D - \mathcal{P}^+ g_D) \Pi \mathbf{e}_q \cdot \mathbf{n} ds. \quad (2.48)$$

Applying Lemma 2.2 yields

$$|A(-\xi_q, -\xi_u; \Pi \mathbf{e}_q, \pi e_u) + \sum_{e \in \mathcal{E}_D} \int_e (g_D - \mathcal{P}^+ g_D) \Pi \mathbf{e}_q \cdot \mathbf{n} ds| \leq Ch^{k+1} |(\Pi \mathbf{e}_q, \pi e_u)|_A, \quad (2.49)$$

which establishes

$$|(\Pi \mathbf{e}_q, \pi e_u)|_A \leq Ch^{k+1}. \quad (2.50)$$

112 We complete the proof of (2.44) by combining (2.46), (2.47) and (2.50).

113 Next, for $\lambda \in L^2$ we let φ be the solution of the adjoint problem

114

$$\begin{aligned} -\Delta\varphi &= \lambda, & \text{in } \Omega, \\ \varphi &= 0, & \text{on } \partial\Omega_D, \\ \frac{\partial\varphi}{\partial n} &= 0, & \text{on } \partial\Omega_N. \end{aligned}$$

115 If $\Phi = -\nabla\varphi$ from standard partial differential equations theory we have

116 $\|\Phi\|_1 < C\|\lambda\|_0$.

One can verify that for $\mathbf{s} \in (L^2)^2$ and $w \in L^2$ the following holds

$$A(-\Phi, \varphi; -\mathbf{s}, w) = (\lambda, w) = \int_{\Omega} \lambda w dx.$$

Letting $\mathbf{s} = \mathbf{e}_q$, and $w = \lambda = e_u$ we obtain

$$A(-\Phi, \varphi; -\mathbf{e}_q, e_u) = \|e_u\|_0^2,$$

which leads to

$$\|e_u\|_0^2 = A(\mathbf{e}_q, e_u; \Phi, \varphi).$$

117 Applying the orthogonality condition (2.27) yields

$$\|e_u\|_0^2 = A(\mathbf{e}_q, e_u; \Phi - \Pi\Phi, \varphi - \pi\varphi) + \sum_{e \in \mathcal{E}_D} \int_e (g_D - \mathcal{P}^+ g_D) \Pi\Phi \cdot \mathbf{n} ds.$$

118 Again we split $\mathbf{e}_q = \mathbf{q} - \mathbf{q}_N = \mathbf{q} - \Pi\mathbf{q} + \Pi\mathbf{q} - \mathbf{q}_N$ and $e_u = u - u_N =$

119 $u - \pi u + \pi u - u_N$ to obtain

$$\begin{aligned} \|e_u\|_0^2 &= A(\Pi\mathbf{e}_q, \pi e_u; \Phi - \Pi\Phi, \varphi - \pi\varphi) + A(\xi_q, \xi_u; \Phi - \Pi\Phi, \varphi - \pi\varphi) \\ &\quad + \sum_{e \in \mathcal{E}_D} \int_e (g_D - \mathcal{P}^+ g_D) \Pi\Phi \cdot \mathbf{n} ds. \end{aligned}$$

which can be written as

$$\begin{aligned} \|e_u\|_0^2 &= A(\Pi\mathbf{e}_q, \pi e_u; \Phi - \Pi\Phi, \varphi - \pi\varphi) \\ &\quad - \left[A(-\xi_q, -\xi_u; \Phi - \Pi\Phi, \varphi - \pi\varphi) + \sum_{e \in \mathcal{E}_D} \int_e (g_D - \mathcal{P}^+ g_D) (\Phi - \Pi\Phi) \cdot \mathbf{n} ds \right] \\ &\quad + \sum_{e \in \mathcal{E}_D} \int_e (g_D - \mathcal{P}^+ g_D) \Phi \cdot \mathbf{n} ds \\ &= H_1 + H_2 + H_3, \end{aligned} \tag{2.51}$$

where

$$\begin{aligned}
H_1 &= A(\Pi \mathbf{e}_{\mathbf{q}}, \pi e_u; \Phi - \Pi \Phi, \varphi - \pi \varphi), \\
H_2 &= -[A(-\xi_q, -\xi_u; \Phi - \Pi \Phi, \varphi - \pi \varphi) + \sum_{e \in \mathcal{E}_D} \int_e (g_D - \mathcal{P}^+ g_D)(\Phi - \Pi \Phi) \cdot \mathbf{n} ds], \\
H_3 &= \sum_{e \in \mathcal{E}_D} \int_e (g_D - \mathcal{P}^+ g_D) \Phi \cdot \mathbf{n} ds.
\end{aligned}$$

Applying Lemma 2.2 and the estimate (2.50) we prove that

$$|H_1| \leq C|(\Pi \mathbf{e}_{\mathbf{q}}, \pi e_u)|_A \|\varphi\|_2 \leq Ch^{k+1} \|e_u\|_0. \quad (2.52)$$

By Lemma 2.2

$$|H_2| \leq Ch^{k+1} |(\Phi - \Pi \Phi, \varphi - \pi \varphi)|_A \leq Ch^{k+1} \|\varphi\|_2 \leq Ch^{k+1} \|e_u\|_0. \quad (2.53)$$

Applying Cauchy Schwarz inequality, the trace theorem [5] and Lemma 2.1 we write

$$\begin{aligned}
|H_3| &\leq C \|g_D - \mathcal{P}^+ g_D\|_{0, \partial \Omega_D} \|\Phi\|_{0, \partial \Omega_D} \leq C \|g_D - \mathcal{P}^+ g_D\|_{0, \partial \Omega_D} \|\varphi\|_2 \\
&\leq Ch^{k+1} \|e_u\|_0.
\end{aligned} \quad (2.54)$$

120 Combining (2.51), (2.52), (2.53) and (2.54) completes the proof of the theorem.
121 $\square$

Instead of the projection $\mathcal{P}^+ g_D$, one could also use the interpolation $\mathcal{R}^+ g_D$ of the boundary condition g_D in (2.16) and still have $\mathcal{O}(h^{k+1})$ superconvergence rates for both u and $\mathbf{q}$. The proof will follow the same line of reasoning to prove Lemma 2.2 and Theorem 2.1. The term T_3 in the proof of Lemma 2.2 now contains

$$\sum_{e \in \mathcal{E}_D} \int_e (g_D - \mathcal{R}^+ g_D) \mathbf{r} \cdot \mathbf{n} ds.$$

Adding and subtracting $\sum_{e \in \mathcal{E}_D} \int_e \mathcal{P}^+ g_D \mathbf{r} \cdot \mathbf{n} ds$, we write T_3 as

$$T_3 = - \sum_{K \in \mathbf{T}} Z_k(\mathbf{r}, v) + \sum_{e \in \mathcal{E}_D} \int_e (\mathcal{P}^+ g_D - \mathcal{R}^+ g_D) \mathbf{r} \cdot \mathbf{n} ds.$$

122 Applying the bounds for Z_K used above, the superconvergence result (2.24)
123 and the bound $\|\mathbf{r}\|_{0,e} \leq C \|\mathbf{r}\|_{0,K} / \sqrt{h}$ we establish that $|T_3| \leq C |(\mathbf{r}, v)|_A h^{k+1}$.

Letting $\mathcal{E}_S \subset (\mathcal{E} \setminus \mathcal{E}_D^+)$ contain $\mathcal{O}(n)$ edges of a Cartesian mesh having $N = n \times n$ elements, our proof can be extended to more general LDG methods with $C_{11} = \mathcal{O}(1)$ on all edges in $\mathcal{E}_D^+ \cup \mathcal{E}_S$ and $C_{11} = 0$ elsewhere. Hence, The terms

T_1 , T_2 and T_3 in (2.33) are unchanged while we obtain a new term T_4 as

$$T_4 = - \sum_{e \in \mathcal{E}_S} \int_e C_{11} [\xi_u] [v] ds - \sum_{e \in \mathcal{E}_D^+} \int_e C_{11} \xi_u v ds.$$

Using the same line of reasoning we can establish a bound similar to (2.40). Moreover, one can easily prove the superconvergence of $\mathbf{q}_N$ and u_N if $C_{11} = \mathcal{O}(1)$ on all edges in $\mathcal{E}_D^+ \cup \mathcal{E}_S$ and $C_{11} = h$ elsewhere. Numerical computations suggest that the superconvergence result holds for the case $C_{11} = 1$ on all edges, however, we are not able to extend our error analysis to this case.

Finally, we note that this error analysis can be easily extended to three dimensional problems on hexahedral meshes.

3 Numerical results

In order to verify the sharpness of our L^2 *a priori* error estimates in Theorem 2.1 we conduct several computational experiments on the standard linear diffusion problem

$$-\Delta u = f, \quad \in (-1, 1)^2,$$

with Dirichlet boundary conditions and a source term $f(x, y)$ such that the true solution is

$$u(x, y) = \cos(\pi x) + \cos(\pi y).$$

In all numerical experiments we use uniform Cartesian meshes having $N = n^2 = 16, 36, 64, 100, 144, 196, 256, 324, 400$ square elements with spaces Q_k , $k = 1, 2, 3, 4$ to compute numerical solutions by several versions of LDG methods with the auxiliary vector $\mathbf{v} = [1, 1]^T$.

First, we use the md-LDG method with stabilization parameter $C_{11} = 1$, the projection $\mathcal{P}^+ g_D$ in (2.16) and g_D in (2.15). In Table 3.1 we present the md-LDG L^2 errors for u and $\mathbf{q}$ which are in full agreement with Theorem 2.1 and yield $\mathcal{O}(h^{k+1})$ convergence rates. We repeat the previous computational experiment using the true Dirichlet boundary condition g_D in (2.16) and $\mathcal{P}^+ g_D$ in (2.15) and present the md-LDG L^2 errors for u and the first component of $\mathbf{q}$ in Table 3.2. In this case we obtain the suboptimal $\mathcal{O}(h^{k+1/2})$ convergence rates for $\mathbf{q}$ which confirms our error analysis that states that $\mathcal{P}^+ g_D$ is only needed in (2.16). Next, we use the LDG method with $C_{11} = 1$ on all edges in $\mathcal{E}_D^+$ and $C_{11} = h$ elsewhere with the projection $\mathcal{P}^+ g_D$ used in (2.16) and (2.15). We present the L^2 errors in Table 3.3 which are in full agreement with our theory. We repeat this experiment using the LDG method with $C_{11} = 1$ on all edges in $\mathcal{E}_D^+ \cup \mathcal{E}_S$ where $\mathcal{E}_S$ is the set of all edges on the lines $x = -1 + 6/n$

149 and $y = -1 + 6/n$ and $C_{11} = 0$ elsewhere and show L^2 errors in Table 3.4.
 150 Again, these results are in full agreement with our theory.

151 The final case is not covered by our theory and consists of the LDG method
 152 with $C_{11} = 1$ on all edges with the projection $\mathcal{P}^+g_D$ applied in (2.16) and
 153 (2.15). The L^2 errors and their orders of convergence shown in Table 3.5 also
 154 suggest optimal $\mathcal{O}(h^{k+1})$ convergence rates for both u and $\mathbf{q}$.

155 4 Conclusion

156 We proved that using a suitable projection or interpolation of the Dirichlet
 157 boundary data the LDG solutions for both u and its gradient $\mathbf{q}$ converge at
 158 $\mathcal{O}(h^{k+1})$ on regular Cartesian meshes with k -degree tensor product polynomial
 159 spaces. This is an improvement over existing $\mathcal{O}(h^{k+1/2})$ results for $\mathbf{q}$. Extensive
 160 numerical computations suggest that similar $\mathcal{O}(h^{k+1})$ convergence rates also
 161 hold the LDG method with $C_{11} = 1$ on all edges for elliptic problems and the
 162 md-LDG for convection-diffusion problems [3], however, our analysis does not
 163 extend to these cases and will be subject to further investigation. Moreover,
 164 numerical experiments [2,3] for modified spaces $\mathcal{V}_k = P_{k+1} \cap Q_k$ also exhibit
 165 $\mathcal{O}(h^{k+1})$ superconvergence rates for both u and $\mathbf{q}$ which is yet to be proved.

166 References

- 167 [1] S. Adjerid, M. Baccouch, Asymptotically exact *a posteriori* error estimates for
 168 a one-dimensional linear hyperbolic problem, Appl. Numer. Math. 60 (2010)
 169 903–914.
 170 URL <http://dx.doi.org/10.1016/j.apnum.2010.04.014>
- 171 [2] S. Adjerid, M. Baccouch, A superconvergent local discontinuous Galerkin
 172 method for elliptic problems, J. Sci. Comput. 52 (2012) 113–152.
 173 URL <http://dx.doi.org/10.1007/s10915-011-9537-8>
- 174 [3] S. Adjerid, M. Baccouch, A posteriori local discontinuous Galerkin error
 175 estimation for two-dimensional convection-diffusion problems, J. Sci. Comput. In
 176 press.
 177 URL <http://dx.doi.org/10.1007/s10915-014-9861-x>
- 178 [4] F. Bassi, S. Rebay, A high-order accurate discontinuous finite element method
 179 for the numerical solution of the compressible Navier-Stokes equations, J.
 180 Comput. Phys. 131 (1997) 267–279.
 181 URL <http://dx.doi.org/10.1006/jcph.1996.5572>
- 182 [5] S. Brenner, R. Scott, The Mathematical Theory of Finite Element Methods,
 183 Springer, New York, 2008.

k	N	$\ u - u_N\ _{0,\Omega}$	Order	$\ q_1 - q_{1,N}\ _{0,\Omega}$	Order
$k = 1$	16	4.4127e-01	NA	8.0770e-01	NA
	36	1.4973e-01	2.6656	3.2685e-01	2.2313
	64	7.5654e-02	2.3729	1.7566e-01	2.1583
	100	4.6725e-02	2.1596	1.0996e-01	2.0992
	144	3.2141e-02	2.0521	7.5482e-02	2.0636
	196	2.3605e-02	2.0026	5.5094e-02	2.0425
	256	1.8119e-02	1.9805	4.2016e-02	2.0295
	324	1.4365e-02	1.9712	3.3115e-02	2.0211
	400	1.1675e-02	1.9680	2.6780e-02	2.0155
$k = 2$	16	8.2880e-02	NA	1.0980e-01	NA
	36	2.5288e-02	2.9277	3.5270e-02	2.8009
	64	1.0205e-02	3.1542	1.4991e-02	2.9741
	100	4.9612e-03	3.2324	7.6106e-03	3.0380
	144	2.7357e-03	3.2649	4.3513e-03	3.0664
	196	1.6503e-03	3.2788	2.7064e-03	3.0805
	256	1.0645e-03	3.2838	1.7919e-03	3.0878
	324	7.2300e-04	3.2841	1.2450e-03	3.0916
	400	5.1164e-04	3.2819	8.9873e-04	3.0935
$k = 3$	16	6.1615e-03	NA	1.1106e-02	NA
	36	8.5780e-04	4.8628	1.8783e-03	4.3829
	64	2.3538e-04	4.4952	5.5323e-04	4.2490
	100	9.1811e-05	4.2190	2.1883e-04	4.1565
	144	4.3638e-05	4.0796	1.0358e-04	4.1021
	196	2.3498e-05	4.0158	5.5314e-05	4.0698
	256	1.3797e-05	3.9874	3.2210e-05	4.0496
	324	8.6387e-06	3.9751	2.0022e-05	4.0366
	400	5.6856e-06	3.9704	1.3098e-05	4.0277
$k = 4$	16	6.3759e-04	NA	8.0053e-04	NA
	36	8.4088e-05	4.9963	1.1493e-04	4.7869
	64	1.8723e-05	5.2215	2.7300e-05	4.9968
	100	5.7411e-06	5.2974	8.8037e-06	5.0716
	144	2.1728e-06	5.3294	3.4706e-06	5.1055
	196	9.5340e-07	5.3436	1.5756e-06	5.1229
	256	4.6672e-07	5.3494	7.9397e-07	5.1325
	324	2.4852e-07	5.3507	4.3350e-07	5.1379
	400	1.4144e-07	5.3494	2.5221e-07	5.1409

Table 3.1

L^2 errors for the md-LDG method on uniform meshes having N elements and Q_k with $\mathcal{P}^+g_D$ in (2.16) and g_D in (2.15).

- [6] P. Castillo, B. Cockburn, I. Perugia, D. Schötzau, An a priori error analysis of the local discontinuous Galerkin method for elliptic problems, SIAM J. Numer. Anal. 38 (2000) 1676–1706.
URL <http://dx.doi.org/10.1137/S0036142900371003>
- [7] P. Castillo, B. Cockburn, D. Schötzau, C. Schwab, Optimal a priori error

k	N	$\ u - u_N\ _{0,\Omega}$	Order	$\ q_1 - q_{1,N}\ _{0,\Omega}$	Order
$k = 1$	16	4.6592e-01	NA	8.1159e-01	NA
	36	1.5476e-01	2.7182	3.5707e-01	2.0250
	64	7.6531e-02	2.4478	2.0960e-01	1.8518
	100	4.6716e-02	2.2121	1.4160e-01	1.7575
	144	3.1965e-02	2.0811	1.0373e-01	1.7072
	196	2.3425e-02	2.0165	8.0097e-02	1.6771
	256	1.7969e-02	1.9858	6.4200e-02	1.6568
	324	1.4245e-02	1.9716	5.2911e-02	1.6419
	400	1.1581e-02	1.9656	4.4560e-02	1.6303
$k = 2$	16	7.9648e-02	NA	1.3297e-01	NA
	36	2.4826e-02	2.8750	4.4042e-02	2.7252
	64	1.0088e-02	3.1302	1.9528e-02	2.8271
	100	4.9201e-03	3.2180	1.0381e-02	2.8317
	144	2.7179e-03	3.2550	6.2156e-03	2.8131
	196	1.6414e-03	3.2714	4.0433e-03	2.7895
	256	1.0596e-03	3.2780	2.7946e-03	2.7661
	324	7.2008e-04	3.2794	2.0227e-03	2.7446
	400	5.0979e-04	3.2779	1.5178e-03	2.7254
$k = 3$	16	6.3200e-03	NA	1.2346e-02	NA
	36	8.7191e-04	4.8852	2.4734e-03	3.9652
	64	2.3655e-04	4.5346	8.4950e-04	3.7149
	100	9.1681e-05	4.2477	3.7923e-04	3.6142
	144	4.3454e-05	4.0951	1.9772e-04	3.5722
	196	2.3373e-05	4.0228	1.1436e-04	3.5521
	256	1.3720e-05	3.9895	7.1269e-05	3.5411
	324	8.5911e-06	3.9745	4.7002e-05	3.5343
	400	5.6554e-06	3.9683	3.2404e-05	3.5298
$k = 4$	16	6.2182e-04	NA	1.1241e-03	NA
	36	8.3110e-05	4.9634	1.7174e-04	4.6337
	64	1.8585e-05	5.2065	4.4120e-05	4.7242
	100	5.7104e-06	5.2883	1.5392e-05	4.7191
	144	2.1636e-06	5.3231	6.5369e-06	4.6972
	196	9.5007e-07	5.3389	3.1802e-06	4.6742
	256	4.6532e-07	5.3457	1.7083e-06	4.6538
	324	2.4786e-07	5.3476	9.8947e-07	4.6364
	400	1.4111e-07	5.3468	6.0803e-07	4.6218

Table 3.2

L^2 errors for the md-LDG method on uniform meshes having N elements and Q_k with g_D in (2.16) and $\mathcal{P}^+g_D$ in (2.15).

estimates for the hp -version of the local discontinuous Galerkin method for
convection-diffusion problems, Math. Comp. 71 (2002) 455–478.
URL <http://dx.doi.org/10.1090/S0025-5718-01-01317-5>

[8] B. Cockburn, B. Dong, An analysis of the minimal dissipation local
discontinuous Galerkin method for convection-diffusion problems, J. Sci.

k	N	$\ u - u_N\ _{0,\Omega}$	Order	$\ q_1 - q_{1,N}\ _{0,\Omega}$	Order
$k = 1$	16	4.4073e-01	NA	8.3145e-01	NA
	36	1.4997e-01	2.6586	3.3132e-01	2.2692
	64	7.6020e-02	2.3619	1.7715e-01	2.1764
	100	4.6956e-02	2.1591	1.1061e-01	2.1104
	144	3.2275e-02	2.0563	7.5814e-02	2.0720
	196	2.3684e-02	2.0078	5.5280e-02	2.0491
	256	1.8168e-02	1.9854	4.2128e-02	2.0348
	324	1.4397e-02	1.9755	3.3187e-02	2.0255
	400	1.1696e-02	1.9716	2.6827e-02	2.0191
$k = 2$	16	8.7706e-02	NA	1.2221e-01	NA
	36	2.6002e-02	2.9985	3.7367e-02	2.9225
	64	1.0340e-02	3.2056	1.5457e-02	3.0685
	100	4.9957e-03	3.2599	7.7479e-03	3.0949
	144	2.7469e-03	3.2804	4.4011e-03	3.1021
	196	1.6547e-03	3.2882	2.7274e-03	3.1041
	256	1.0664e-03	3.2899	1.8018e-03	3.1043
	324	7.2396e-04	3.2883	1.2501e-03	3.1037
	400	5.1216e-04	3.2849	9.0155e-04	3.1026
$k = 3$	16	6.2541e-03	NA	1.1422e-02	NA
	36	8.6330e-04	4.8838	1.8950e-03	4.4304
	64	2.3623e-04	4.5048	5.5624e-04	4.2608
	100	9.2042e-05	4.2241	2.1978e-04	4.1614
	144	4.3719e-05	4.0833	1.0395e-04	4.1066
	196	2.3530e-05	4.0188	5.5471e-05	4.0741
	256	1.3812e-05	3.9897	3.2284e-05	4.0537
	324	8.6460e-06	3.9770	2.0060e-05	4.0402
	400	5.6895e-06	3.9719	1.3118e-05	4.0309
$k = 4$	16	7.0512e-04	NA	9.6592e-04	NA
	36	8.8496e-05	5.1186	1.2736e-04	4.9969
	64	1.9184e-05	5.3145	2.8846e-05	5.1621
	100	5.8154e-06	5.3489	9.0935e-06	5.1735
	144	2.1891e-06	5.3588	3.5430e-06	5.1699
	196	9.5789e-07	5.3616	1.5978e-06	5.1659
	256	4.6819e-07	5.3610	8.0194e-07	5.1626
	324	2.4907e-07	5.3586	4.3672e-07	5.1599
	400	1.4167e-07	5.3550	2.5364e-07	5.1575

Table 3.3

L^2 errors for LDG method on uniform meshes having N elements and Q_k with $C_{11} = 1$ on edges in $\mathcal{E}_D^+$ and $C_{11} = h$ elsewhere and using the projection $\mathcal{P}^+_{gD}$ in (2.16) and (2.15).

194 Comput. 32 (2007) 233–262.

195 URL <http://dx.doi.org/10.1007/s10915-007-9130-3>

196 [9] B. Cockburn, G. Kanschat, I. Perugia, D. Schötzau, Superconvergence of the
197 local discontinuous Galerkin method for elliptic problems on Cartesian grids,

k	N	$\ u - u_N\ _{0,\Omega}$	Order	$\ q_1 - q_{1,N}\ _{0,\Omega}$	Order
$k = 1$	16	4.4408e-01	NA	8.9148e-01	NA
	36	1.5330e-01	2.6232	3.3464e-01	2.4166
	64	8.0870e-02	2.2231	1.8162e-01	2.1242
	100	4.9246e-02	2.2229	1.1413e-01	2.0819
	144	3.3047e-02	2.1877	7.8337e-02	2.0642
	196	2.3915e-02	2.0983	5.7179e-02	2.0424
	256	1.8232e-02	2.0319	4.3607e-02	2.0293
	324	1.4413e-02	1.9954	3.4359e-02	2.0236
	400	1.1702e-02	1.9781	2.7767e-02	2.0217
$k = 2$	16	8.9470e-02	NA	1.3378e-01	NA
	36	2.6581e-02	2.9934	3.8551e-02	3.0686
	64	1.0327e-02	3.2865	1.5810e-02	3.0983
	100	4.9811e-03	3.2673	7.9147e-03	3.1008
	144	2.7521e-03	3.2613	4.4963e-03	3.1011
	196	1.6626e-03	3.2694	2.7811e-03	3.1166
	256	1.0725e-03	3.2830	1.8320e-03	3.1263
	324	7.2803e-04	3.2893	1.2672e-03	3.1293
	400	5.1475e-04	3.2901	9.1140e-04	3.1281
$k = 3$	16	6.3133e-03	NA	1.2287e-02	NA
	36	8.8244e-04	4.8530	1.9812e-03	4.5005
	64	2.6889e-04	4.1309	5.9925e-04	4.1566
	100	1.0247e-04	4.3233	2.3454e-04	4.2039
	144	4.6281e-05	4.3597	1.0934e-04	4.1857
	196	2.4122e-05	4.2270	5.7944e-05	4.1193
	256	1.3945e-05	4.1040	3.3657e-05	4.0685
	324	8.6757e-06	4.0295	2.0913e-05	4.0400
	400	5.6972e-06	3.9916	1.3683e-05	4.0264
$k = 4$	16	7.0512e-04	NA	9.6592e-04	NA
	36	8.8496e-05	5.1186	1.2736e-04	4.9969
	64	1.9184e-05	5.3145	2.8846e-05	5.1621
	100	5.7722e-06	5.3548	9.3721e-06	5.1318
	144	2.1891e-06	5.3031	3.5430e-06	5.1248
	196	9.5789e-07	5.3616	1.5978e-06	5.1659
	256	4.6819e-07	5.3610	8.0194e-07	5.1626
	324	2.4907e-07	5.3586	4.3672e-07	5.1599
	400	1.4167e-07	5.3550	2.5364e-07	5.1575

Table 3.4

L^2 errors for the LDG method on uniform meshes having N elements and Q_k with $C_{11} = 1$ on edges in $\mathcal{E}_D^+ \cup \mathcal{E}_S$, $C_{11} = 0$ elsewhere and using the projection $\mathcal{P}^+_{gD}$ in (2.16) and (2.15).

198 SIAM J. Numer. Anal. 39 (2001) 264–285.

199 URL <http://dx.doi.org/10.1137/S0036142900371544>

200 [10] B. Cockburn, G. Karniadakis, C.-W. Shu, Discontinuous Galerkin Methods:
201 Theory, Computation and Applications, vol. 11 of Lecture Notes in

k	N	$\ u - u_N\ _{0,\Omega}$	Order	$\ q_1 - q_{1,N}\ _{0,\Omega}$	Order
$k = 1$	16	4.4571e-01	NA	8.8583e-01	NA
	36	1.5242e-01	2.6464	3.5325e-01	2.2674
	64	7.8217e-02	2.3191	1.9103e-01	2.1368
	100	4.8570e-02	2.1353	1.2092e-01	2.0495
	144	3.3389e-02	2.0556	8.3758e-02	2.0139
	196	2.4453e-02	2.0207	6.1531e-02	2.0006
	256	1.8710e-02	2.0047	4.7134e-02	1.9961
	324	1.4788e-02	1.9973	3.7263e-02	1.9952
	400	1.1986e-02	1.9939	3.0196e-02	1.9957
$k = 2$	16	1.0181e-01	NA	1.5434e-01	NA
	36	3.1291e-02	2.9098	5.1499e-02	2.7070
	64	1.2166e-02	3.2836	2.1387e-02	3.0547
	100	5.7263e-03	3.3772	1.0580e-02	3.1539
	144	3.0811e-03	3.3994	5.9172e-03	3.1873
	196	1.8246e-03	3.3989	3.6140e-03	3.1986
	256	1.1602e-03	3.3904	2.3569e-03	3.2012
	324	7.7929e-04	3.3791	1.6168e-03	3.2000
	400	5.4655e-04	3.3671	1.1544e-03	3.1972
$k = 3$	16	6.5114e-03	NA	1.2343e-02	NA
	36	8.9963e-04	4.8816	2.0119e-03	4.4739
	64	2.4500e-04	4.5213	5.9422e-04	4.2394
	100	9.5249e-05	4.2339	2.3911e-04	4.0796
	144	4.5131e-05	4.0968	1.1492e-04	4.0184
	196	2.4221e-05	4.0371	6.2043e-05	3.9989
	256	1.4177e-05	4.0109	3.6395e-05	3.9945
	324	8.8517e-06	3.9992	2.2733e-05	3.9955
	400	5.8112e-06	3.9941	1.4918e-05	3.9981
$k = 4$	16	8.7887e-04	NA	1.3507e-03	NA
	36	1.1802e-04	4.9517	2.0218e-04	4.6840
	64	2.5051e-05	5.3878	4.6586e-05	5.1024
	100	7.3412e-06	5.5006	1.4528e-05	5.2219
	144	2.6792e-06	5.5285	5.5639e-06	5.2641
	196	1.1425e-06	5.5289	2.4655e-06	5.2800
	256	5.4675e-07	5.5193	1.2173e-06	5.2854
	324	2.8585e-07	5.5062	6.5313e-07	5.2860
	400	1.6026e-07	5.4920	3.7428e-07	5.2843

Table 3.5

L^2 errors for the LDG method on uniform meshes having N elements and Q_k with $C_{11} = 1$ on all edges and using the projection $\mathcal{P}^+_{gD}$ in (2.16) and (2.15).

202 Computational Science and Engineering, Springer, Berlin, 2000.

203 [11] B. Cockburn, C.-W. Shu, TVB Runge-Kutta local projection discontinuous
204 Galerkin methods for scalar conservation laws II: General framework, Math.
205 Comp. 52 (1989) 411–435.

206 URL <http://dx.doi.org/10.2307/2008474>

- 207 [12] B. Cockburn, C.-W. Shu, The local discontinuous Galerkin finite element
208 method for time dependent convection-diffusion systems, SIAM J. Numer. Anal.
209 35 (1998) 2240–2463.
210 URL <http://dx.doi.org/10.1007/s10915-007-9130-3>
- 211 [13] B. Cockburn, C.-W. Shu, The Runge-Kutta discontinuous Galerkin finite
212 element method for conservation laws V: Multidimensional systems, J. Comput.
213 Phys. 141 (1998) 199–224.
214 URL <http://dx.doi.org/10.1006/jcph.1998.5892>
- 215 [14] B. Cockburn, C.-W. Shu, Foreword for the special issue on discontinuous
216 Galerkin methods, J. Sci. Comput. 22/23 (2005) 1–2.
217 URL <http://dx.doi.org/10.1007/s10915-004-4131-y>
- 218 [15] B. Cockburn, C.-W. Shu, Foreword, proceedings of the first international
219 symposium on dg methods, J. Sci. Comput. 40 (2009) 1–3.
220 URL <http://dx.doi.org/10.1007/s10915-009-9298-9>
- 221 [16] C. Dawson, Foreword for the special issue on discontinuous Galerkin methods,
222 Comput. Methods Appl. Mech. Engrg. 195 (2006) 3183–3183.
223 URL <http://dx.doi.org/10.1016/j.cma.2005.06.010>
- 224 [17] W. Reed, T. Hill, Triangular mesh methods for the neutron transport equation,
225 Tech. Rep. LA-UR-73-479, Los Alamos Scientific Laboratory, Los Alamos
226 (1973).