A Note on the stability of cut cells and cell merging

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Abstract

Embedded boundary meshes may have cut cells of arbitrarily small volume which can lead to stability problems in finite volume computations with explicit time stepping. We show that time step constraints are not as strict as often believed. We prove this in one dimension for linear advection and the first order upwind scheme. Numerical examples in two dimensions demonstrate that this carries over to more complicated situations. This analysis sheds light on the choice of time step when using cell merging to stabilize the arbitrarily small cells that arise in embedded boundary schemes.

Keywords: cut cell, cell merging, stability, embedded boundary

1. Introduction

Cell merging is a commonly used technique for cut cells in embedded boundary grids. It creates a cell with a larger volume, thus allowing an explicit difference scheme to use a larger time step and still remain stable on the small cut cells. A cell is usually merged with its neighbor in the direction most normal to the boundary [? ?]. Typically, cut cells whose volume is less than 0.5 of the regular cell volume are merged. Then a time step that is somewhat reduced to account for these smaller cell volumes is used for simulations. Some papers do not report the actual timestep used on their cut cell meshes [?]; others reduce the time step proportionately with the volume fraction of the cut cell [?]. The stability of cell merging is not well understood.

This note gives a theoretical analysis and some numerical examples of the stability of cut cells, to shed some light on the stability of cell merging. We use a one dimensional model problem and apply GKS stability theory. In two dimensions we show computations that suggest that the conclusions of this analysis also apply in more complicated situations, although not as clear cut. Related work was done in [?] but focused on discontinuous Galerkin methods. This note is hopefully just a beginning, and will serve to stimulate more research in this area.

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Figure 1: Uniform Cartesian mesh with one small cell at an inflow boundary. The dotted line indicates the stencil for the first order upwind scheme.

In more detail, we prove that one can take a full time step Δt if the cut cells at the boundary are at least half the regular cell volume for the first order upwind finite volume scheme. For the second order MUSCL scheme, the stability limit varies, depending on the details of the gradient computation. We will show that using central differences for the gradient has better stability properties than downwind gradients. This latter is equivalent to the Lax Wendroff scheme, whose stability limit is slightly worse than the first order case, suggesting that a cell's volume after merging should be somewhat larger than half. We also present numerical experiments for a linear and nonlinear problem in two dimensions on an embedded boundary grid to investigate the extent to which the one dimensional results carry over.

2. Cut cell model problem in one space dimension

We use the following model problem sketched in Fig. 1 to show stability for linear advection $u_t + au_x = 0$. The cut cell is at the boundary, to mimic the situation in higher dimensions. This is in contrast to some of our previous work where we put the cut cell in the middle of the grid [?]. The main tool we use is the theory of Gustafsson, Kreiss and Sundström (GKS) [?]. Consider $a > 0$ so that the cut cell is at an inflow boundary, where $u(x = 0) = u_b$. We look at both the first and second order case.

2.1. First Order Case

The explicit first order upwind scheme for this problem is

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{h}(u_j^n - u_{j-1}^n), \quad j \geq 1 \quad (1)$$

where the notation is illustrated in Fig. 1. The von Neumann stability condition for the first order upwind scheme is $\lambda \leq 1$, where the CFL number $\lambda = a\Delta t/h$. The small cell is the first cell, u_0 in the figure, with mesh width αh , $0 < \alpha \leq 1$. The conservative finite volume update for this cell is

$$u_0^{n+1} = u_0^n - \frac{a\Delta t}{\alpha h}(u_0^n - u_b^n) \quad (2)$$

For the stability analysis, it suffices to take $u_b = 0$.

GKS theory looks for solutions of the form

$$u_j^n = z^n v_j, \quad (3)$$

with the magnitude of the complex eigenvalue parameter $|z|$ determining stability. If $|z| < 1$ there is no growth in time; if $|z| > 1$ the growth is exponential. The borderline cases of $|z| = 1$ are more complicated and need further testing, but they are not an issue in this work since the numerical scheme is dissipative. (See [?] for two very readable presentations of GKS theory.) After substituting (3) into the difference scheme, one obtains a recurrence equation based on the finite difference scheme. If it has any l^2 solutions, this is an unstable solution v_j .

Substituting (3) into the small cell boundary condition (2) gives

$$(z - 1)v_0 + \frac{a\Delta t}{\alpha h}v_0 = 0 \quad \text{or} \quad (4)$$

$$(z - 1 + \frac{\lambda}{\alpha})v_0 = 0. \quad (5)$$

From (5) we see that either $z - 1 + \lambda/\alpha = 0$ or $v_0 = 0$. The first case gives $z = 1 - \lambda/\alpha$, so for $|z| \leq 1$ we must have $\lambda/\alpha \leq 2$, or alternatively we need $\alpha \geq \lambda/2$ to guarantee stability. To put it a different way, in (2) we can take a small cell with mesh width αh at least half the uniform h and retain stability for all $\lambda \leq 1$. This means that the CFL number in the small cell $\frac{a\Delta t}{\alpha h}$ can exceed one!

The stability of the small cell scheme (1) and (2) is illustrated numerically in Fig. 2. We take 1000 steps using $\lambda = 0.9$ on a mesh with 100 cells, with exact solution $u(x, t) = \cos(2\pi(x - t))$, so the boundary condition $u_b(t) = \cos(2\pi t)$ is advecting into the domain, rather than the zero inflow conditions for analysis. We take the small cell parameter $\alpha = 0.46$ so that α is slightly above the stable limit of $\lambda/2 = 0.45$. After 1000 steps the computed solution is plotted on the left, with the 2 norm of the error 3.47×10^{-2} . Everything looks stable, as expected. On the right is an experiment with $\alpha = 0.40$, just slightly above the stability limit. The error after 40 steps is 0.16 in the 2 norm and 1.18 in the maximum norm. The instability is clearly visible emanating from the boundary.

Note that since the coefficients of the update for u_0 in (2) are no longer positive, a monotone solution is not guaranteed. But it is stable using the GKS definition, which essentially allows bounded growth independent of Δt . In fact this boundary condition admits an incoming oscillatory wave from the boundary that will not be damped if $\lambda = 1$ in the regular interior portion of the domain. When $\lambda = 1$, the solution shifts perfectly; it is simply translated exactly from one grid point to the next.

A similar calculation shows that this doubling of the stable allowable cell size for the one small cell at the boundary also occurs if the small cell is at the outflow boundary. This is not surprising however since there is no coupling back to the interior scheme in the first order case.

Figure 2: The numerical experiments have one small cell at the inflow boundary on the left. The figure on the left has $\alpha=0.46$, slightly above the stability limit $\lambda/2$. The solution is shown after 1000 steps. On the right we take $\alpha = 0.40$, so that the stability limit is slightly exceeded. After 40 time steps the growing instability is visible at the left boundary.

2.2. Second Order Case

Next we look at the stability of one small cell at an inflow boundary for a second order scheme. The algebra is more complicated here, so we demonstrate the stability limit numerically. First we consider the Lax Wendroff scheme, and later show the results carry over to related MUSCL scheme variants.

There is some variation in generalizing Lax Wendroff to non-uniform grids. The Lax Wendroff scheme on regular grids can be written in two step form as:

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{h}(u_{j+1/2}^{n+1/2} - u_{j-1/2}^{n+1/2}). \quad (6)$$

where the intermediate values $u^{n+1/2}$ are found using the predictor Lax Friedrichs at time n :

$$u_{j+1/2}^{n+1/2} = \frac{(u_j + u_{j+1})}{2} - \frac{a\Delta t}{2} \frac{(u_{j+1} - u_j)}{h}. \quad (7)$$

For the small cell we change (7) to use a weighted average in the predictor step, and take the right edge value for the small cell to be

$$u_{1/2}^{n+1/2} = \frac{(u_0^n + \alpha u_1^n)}{1 + \alpha} - \frac{a\Delta t}{2} \frac{(u_1^n - u_0^n)}{(1 + \alpha)h/2}. \quad (8)$$

At the left edge we use the inflow boundary condition at the half timestep.

We use numerical simulations in the interval $[0, 1]$ to find the stability limit for λ as a function of α . Figure 3 shows the results. We see that the Lax Wendroff scheme is less stable than the more diffusive upwind scheme.

Since this second order method has a lower stability limit, a larger cell volume is needed for the merged cells. If a CFL of 0.9 for example is used, the cell volume should be at least 0.615 of the regular cell size. This is larger than the

Figure 3: Stability of Lax Wendroff with one small cell at an inflow boundary, compared to the stability limit of upwind differences, and a simple reference line.

0.45 stable volume fraction in the first order case. This result is also observed when the small cell is in the middle of the grid.

The MUSCL scheme can be viewed as a variant of Lax Wendroff. On a regular grid, Lax Wendroff can be derived by reconstructing a gradient in each cell using downwind slopes of the solution at time n , predicting the solution at the half-step $n + 1/2$, and then using upwind differencing to get to time $n + 1$. This gives a second order method with a 3 point stencil. MUSCL is commonly implemented using centered differences for the linear reconstruction, which adds a 4th point to the stencil. This has a somewhat better stability limit than Lax Wendroff, with both methods sensitive to the slope reconstruction in the first small cell. Not surprisingly, it is more stable to use a centered slope, obtained using a ghost cell whose value is set using the boundary condition at the inflow boundary, than to use a downwind difference, which for this example is a forward difference, since the solution travels to the right when $a > 0$.

For example, with small cell parameter $\alpha = 0.4$, using central differences over the entire domain has a stability limit of $\lambda \approx 0.81$, larger than the first order case, whereas Lax Wendroff has a limit of ≈ 0.64 . This is a much more satisfactory situation in practice, since if a reconstructed gradient is limited, the scheme degenerates to the first order method, so it is necessary to use a time step that is stable for both the first and second order schemes.

The case of linear systems of equations follows directly from the stability of linear advection with both inflow and outflow conditions [?]. In the next section we consider two dimensional nonlinear examples.

3. Two dimensional examples

In two dimensions the situation is not as clear cut. There is no equivalent to GKS theory for two dimensional stability on non-coordinate-aligned grids. Instead, we perform numerical experiments using a special embedded boundary test problem where we can control the cut cell volume fractions. We will show

that in more complex situations, the factor of two improvement in time step is (mostly) retained, and in any case the stability limit in all cases is much larger than the volume fraction would suggest. These experiments will hopefully entice others who are interested in cell merging to examine the stability question further in higher dimensions.

3.1. Some Linear Examples

We first study the simple test case of linear advection

$$u_t + au_x + bu_y = 0 \quad (9)$$

on a channel that is rotated 45 degrees from the coordinate directions. The setup is illustrated in Fig. 4. We will use the first order upwind scheme to approximate (9), which on a regular Cartesian grid is

$$u_{ij}^{n+1} = u_{ij}^n - \frac{\Delta t}{V_{ij}} (F(u_{i,j}^n, u_{i+1,j}^n) \Delta y - F(u_{i-1,j}^n, u_{i,j}^n) \Delta y + G(u_{i,j}^n, u_{i,j+1}^n) \Delta x - G(u_{i,j-1}^n, u_{i,j}^n) \Delta x), \quad (10)$$

where F and G are the numerical fluxes, and V_{ij} is the cell volume. For cut cells some terms may be zero when the cut cell edge length is zero, and there is an additional flux H_b from the cut cell boundary segment. For a regular periodic Cartesian grid in two space dimensions, the first order upwind approximation to (9) has a stability limit

$$\frac{|a|\Delta t}{\Delta x} + \frac{|b|\Delta t}{\Delta y} \leq 1. \quad (11)$$

First we take $a = b = -1$, approximating $u_t - u_x - u_y = 0$, so the flow is tangential to the boundary, not normal to it as in one dimension. The boundary flux $H_b \equiv 0$ in this case. We will adjust the channel boundary to generate cut cells with prescribed volume fractions. For example, the top boundary in Fig. 4 shows cut cells with volume fraction ≈ 0.5 . We take a solution that is linear in the direction normal to the boundary, and constant parallel to the channel. The inflow boundary is on the right side of the domain and outflow is on the left; the upwind solution at each edge is taken from the cell on the right.

Taking $\Delta x = \Delta y$, and using $a = b = -1$, the regular stability limit according to (11) is $\Delta t/\Delta x \leq 1/2$. If the Cartesian cells are all cut in half, producing isosceles triangles, it is shown in [?] that the stability limit is

$$\Delta t/\Delta x \leq \frac{1}{2\sqrt{2}}. \quad (12)$$

First we set the channel boundaries to both the top and bottom cut cells have volume fractions of at least 0.499. For concreteness, the channel boundaries are at $y = x + 0.119049$ and $y = x + 0.78571$, and we use 42 points in the unit interval to start. The time step is chosen to get a CFL number (defined for the

Figure 4: The upwind scheme blows up when $\Delta t \approx l$, not $l/2$ as might be expected. Zooms of the top and bottom boundary show a blowup of a Cartesian cell intersected by the channel edge (indicated by the black line). The volume fraction of the triangular cell in the bottom zoom is $\approx .0037$.

moment as $\Delta t/\Delta x$) of 0.49896, so that it is very close to the limit of 0.5 (again for concreteness $\Delta t = .01188$). We take 400,000 time steps, starting with the exact solution as the initial conditions; the solution remains bounded. Refining the mesh to 84 points and then 168 points, keeping the CFL constant, does not change the results. So in this case we can take a full time step even though the cut cells have half the volume fraction.

Next we lower the bottom boundary to produce cut cells with smaller volume fractions and see if we can take twice the “allowable” time step for these smaller cut cells. However it is not clear if we should take twice the time step given by these cells if they were half the regular Cartesian cells, or twice the stable limit given by (12). We look for the stability limit numerically, and see where it falls.

We take the bottom channel boundary to be the line $y = x + 0.117$. Look in detail at a small triangular cell cut by the boundary (see Fig. 4, bottom zoom showing the dark blue triangle next to the red one). These are the most unstable cells in the mesh. Their cell volumes are all

$$V_{ij} = \frac{1}{2} l \cdot l$$

where l is the length of the cut edge of the triangle, and the lengths are equal since the channel is at 45 degrees and $\Delta x = \Delta y$. For the cut cells in the mesh shown in Fig. 4, the volume fraction is $\approx .0037$, ($l = .0020476$, and $\Delta x = 1/42 \approx .0238$.) The cell update (10) for these triangular cells simplifies to

$$u_{ij}^{n+1} = u_{ij}^n \left(1 - \frac{\Delta t}{\frac{1}{2}l}\right) + \frac{\Delta t}{\frac{1}{2}l} u_{i,j+1}^n. \quad (13)$$

For a monotone scheme one would take $\Delta t < \frac{1}{2}l$. But using our previous results we try taking double the time step and experimentally use $\Delta t \approx l$. Again we take 400,000 steps ($\Delta t = .00203$), and the solution looks as expected. However if we take $\Delta t = .00204$, the solution blows up. The unstable solution after 2000 steps is shown in Fig. 4. The growing instability is evident in the checkerboarding at the bottom left boundary of the channel. The top boundary of the channel still has cells with volume fraction 0.5, and has no problem.

Looking in more detail at (13), when $\Delta t/l$ is 1 instead of 1/2, the update for a triangular cell becomes

$$u_{ij}^{n+1} = 2u_{i,j+1}^n - u_{ij}^n.$$

For large n , if the solution reaches steady-state, the limiting solution is $u_{ij} = u_{i,j+1}$. In fact we see this for the entire cut cell boundary, on both sides. The triangular cells whose stencil depends only on itself and its upwind neighbor converge to the neighboring upwind value. (Those cells with more than one upwind cell in their stencil have a more complicated formula at steady state.) This scheme has converged to the results of cell merging without having to actually merge cells!

For this bottom boundary, the mesh can be refined twice more (taking 84 cells and 168 cells in the unit interval) without changing the smallest cut cell volume. The volume fraction goes up from .0037, to .0148 to .0592 as the full cell volume shrinks. Unfortunately the stable time step does need to be reduced somewhat, from .00203 to .001875. It is hard to detect a pattern for the dependence of the CFL limit on the cell size or edge length, aside from the observation that it is still much larger than the volume fraction of the small cells would give. If the full time step were reduced by the volume fraction .0592, the resulting time step would be .000175, over an order of magnitude smaller. On the finest grid we tested, the stable regular time step was reduced by ≈ 1.08 , less than the factor of $\sqrt{2}$ in (12).

We next test channel flow where the flow is not parallel to boundary, but flows into a cut cell from the embedded boundary – inflow on one side, outflow on the other. This is more akin to the one dimensional situation. For linear advection (9) with (a, b) pointing into the second quadrant, the cut boundary on the bottom of our 45 degree channel is an inflow boundary, and the top is outflow. Taking $a = -1, b = 1$, the first order upwind formula (10) for the cut cells at the bottom becomes

$$\begin{aligned} u_{ij}^{n+1} &= u_{ij}^n - \frac{\Delta t}{V_{ij}} (F(u_{i,j-1}, u_{i,j}) + G(u_{i,j}, u_{i,j+1}) + H_B \\ &= u_{ij}^n (1 - \frac{\Delta t}{\frac{1}{4}l}) + \frac{\Delta t}{\frac{1}{4}l} u_b^n. \end{aligned} \tag{14}$$

where H_B includes the inflow boundary condition ($u = u_B$ on the boundary). For monotonicity we would require $0 \leq 1 - \frac{4\Delta t}{l} \leq 1$, but for the coefficient to have a norm of less than 1 we can take $-1 \leq 1 - \frac{4\Delta t}{l} \leq 1$. In other words, again

Figure 5: Results after 2000 steps with cosine boundary conditions at the cut cell inflow boundary, on the mesh with small triangular cut cells with volume fraction .0037, and a time step of .001. With $l = .00204$ this corresponds to a CFL number for the smallest cut cells of $\Delta t/l \approx 0.49$

we can take twice the time step, although in this case the stable Δt is half the previous case, since the coefficient is now 4 instead of 2.

Figure 5 shows the result of taking 2000 steps on the 42 cell mesh, with cosine inflow boundary conditions. This experiment uses $\Delta t = .001$. The solution is nicely bounded. At $\Delta t = .0011$, which slightly exceeds $l/2$, the solution blows up.

3.2. A Nonlinear Example

We next try the Euler equations, in the same 45 degree channel, to see if any of this carries over to a nonlinear problem. We use the Sod shock tube problem initial conditions, with velocity rotated by 45 degrees so the jump is normal to the boundary. We also change the boundary conditions at the left and right side of the domain (i.e. not the cut cells) to reflecting boundary conditions. We implement this using ghost cells, where the ghost cell values are copied from the cell just inside the domain that shares a face; then the component of the velocity normal to the face is negated.

Using the first order Godunov method the stability limit is $\frac{\Delta t \max(|u|+c)}{\Delta x} + \frac{\Delta t \max(|v|+c)}{\Delta y} \leq 1.0$, where c is the sound speed. We take 0.95 for the CFL limit. For these experiments we either compute the maximum wave speed over only the full cells, not the cut cells, to adjust the time step, or use a fixed time step. We use the mesh with 168 cells in a unit interval, i.e. $\Delta x = \Delta y \approx .00595$.

In the first test we use the geometry that creates cut cells with half the cell volume at the top and bottom. In this test the CFL = 0.95, so that the time step varies with the velocity and sound speed (of the full cells only). Fig. 6 shows the solution at several times during the simulation. We also extracted the solution along the bottom boundary at the same times. The solution remains nicely bounded.

We repeat the test with the bottom boundary shifted so that the bottom cut cells have the smaller triangles with side $\approx .00204$ and volume fraction .0592, as before in the linear case. With a fixed time step of .0008 we run for 1000 steps,

Figure 6: Euler solution to rotated shock tube problem in a 45 degree channel. The line plots on the bottom show the solution taken from the cut cells at the bottom boundary.

and the solution looks good. Using .0009 is unstable. This gives a CFL number computed as $\Delta t(\max(|u| + c) + \max(|v| + c))/\Delta x$ that ranges from 0.5 to 0.67, again computing the maximum over full cells. Again this is much larger than the volume fraction would suggest is stable.

4. Conclusions

In this note we have investigated how the stability limit is affected by the size of the cut cells at an embedded boundary. In one dimension we proved a stability result, and in higher dimensions we performed both linear and nonlinear numerical experiments, showing that the time step could be roughly twice what was expected based on the cut cell volume alone. There are many more investigations we could perform, for example we did not test a second order finite volume scheme in two dimensions, but we do not pursue this further. By themselves, our results are not of practical utility, since in real calculations the cut cells could have arbitrarily small cell volumes. Nevertheless, these results are intriguing, and are presented here as a start of an investigation into the stability of cell merging.

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