Title:

DEVELOPING A COSMIC RAY MUON SAMPLING CAPABILITY FOR MUON
TOMOGRAPHY AND MONITORING APPLICATIONS

Authors:

S. Chatzidakis^{*}, S. Chrysikopoulou, L.H. Tsoukalas

Purdue University, School of Nuclear Engineering, West Lafayette, IN 47907

***Contact Info:**

Tel: +1 765-496-6136

Fax: +1 765-494-9570

E-mail: schatzid@purdue.edu

Abstract

In this study, a cosmic ray muon sampling capability using a phenomenological model that captures the main characteristics of the experimentally measured spectrum coupled with a set of statistical algorithms is developed. The “muon generator” produces muons with zenith angles in the range 0-90° and energies in the range 1-100 GeV and is suitable for Monte Carlo simulations with emphasis on muon tomographic and monitoring applications. The muon energy distribution is described by the Smith & Duller (1959) phenomenological model. Statistical algorithms are then employed for generating random samples. The inverse transform provides a means to generate samples from the muon angular distribution, whereas the Acceptance-Rejection and Metropolis-Hastings algorithms are employed to provide the energy component. The predictions for muon energies 1 to 60 GeV and zenith angles 0-90° are validated with a series of actual spectrum measurements and with estimates from the software library CRY. The results confirm the validity of the phenomenological model and the applicability of the statistical algorithms to generate polyenergetic-polydirectional muons. The response of the algorithms and the impact of critical parameters on computation time and computed results were investigated. Final output from the proposed “muon generator” is a look-up table that contains the sampled muon angles and energies and can be easily integrated into Monte Carlo particle simulation codes such as Geant4 and MCNP.

Keywords: muon sampling, muon tomography, Monte Carlo, acceptance-rejection, Metropolis-Hastings, Geant4

1. Introduction

The applicability of cosmic muons for a number of monitoring and imaging applications has been investigated over the years and includes a large number of applications ranging from archaeology to volcano imaging, material identification and medical diagnosis [1-8]. It is worth noting the pioneering work of E. P. George [9] who used a similar method to infer rock depth covering underground tunnels and that of L. Alvarez et al., [10] who measured the cosmic ray muon flux attenuation to determine the presence and location of hidden chambers within the Egyptian pyramids. Applications of cosmic ray muons have been proposed for medical examination of comatose patients towards bone density monitoring and determination of the molten nuclear fuel location in nuclear reactors having suffered from the effects of a severe accident similar to the one happened in Chernobyl and Fukushima [6] and [11].

The role of cosmic ray muons becomes especially important since their use is extended to non-destructive assessment of nuclear materials, such as fuel pellets, fuel rods and fuel assemblies stored within sealed dense containers [12]. Energetic muons have the unique ability to penetrate high density materials allowing the distribution of the material within the object to be inferred from muon measurements [3]. The subsequent scattering and transmission of muons can provide a measurable signal about the structural and chemical composition of the stored materials [13]. Cosmic ray muon tomography is a potential next generation technology in non-destructive evaluation and muons can play a central role encompassing the experience accumulated from recent findings in proton radiography beyond conventional X-ray and gamma-ray techniques.

Successful application of muon technology necessitates the development of classification and imaging algorithms. Algorithm validation and testing is supported by large scale high fidelity Monte Carlo simulations. Simulations of muon-matter interactions are

commonly performed using Monte Carlo numerical codes such as Geant4 [14]. Customized Monte Carlo simulations are necessary for examination of the asymptotic behavior of signal processing and classification algorithms [15]. In all cases, accurate and efficient application of Monte Carlo simulations requires knowledge of the muon energy and angular distribution and the ability to generate repeatedly random samples from these distributions [6]. Muons produced in the atmosphere arrive on ground level with a wide range of energies and zenith angles best described by a non-standard differential spectrum. It is therefore desirable to develop the capability to generate muons having momentum and zenith angles distributed according to the actual experimentally measured spectrum.

In this paper, a cosmic ray muon sampling capability for use in Monte Carlo simulations is developed. The “muon generator” produces muons with zenith angles in the range $0\text{-}90^\circ$ and energies in the range $1\text{-}100$ GeV particularly suited for Monte Carlo simulations with emphasis on muon tomographic and monitoring applications. The muon energy distribution is described by a phenomenological model that captures the main characteristics of the experimentally measured spectrum. Statistical algorithms are then employed for generating random samples. The inverse transform provides a means to generate samples from the muon angular distribution, whereas the Acceptance-Rejection and Metropolis-Hastings algorithms are used to provide the energy component. The predictions are subsequently compared with actual spectrum measurements and with estimates from the software library CRY [16]. The main objective is: a) to develop a muon sampling capability using a phenomenological model coupled with a set of statistical algorithms, b) to perform benchmark analysis comparing the developed algorithms with experimental measurements and, c) to examine their response and identify the impact of critical parameters on the computation time and computed results. Main advantages of the “muon generator” include its ability to readily produce any number of muons for any altitude and any combination of

zenith angles and energies, along with the fact that it is based on a strong phenomenological model and validated statistical algorithms. A version of the “muon generator” developed for easy and fast integration to Geant4 is available online* [17].

A brief description of the muon spectrum characteristics and available experimental measurements is given in Section 2. Sections 3 and 4 provide details related to the phenomenological model and the statistical algorithms developed for generating random samples, respectively. Finally, the results obtained along with the conclusions drawn are presented and discussed in Section 5.

*The muon generator is available on the MATLAB file exchange webpage:

[http://www.mathworks.com/matlabcentral/fileexchange/51203-muon-generator and](http://www.mathworks.com/matlabcentral/fileexchange/51203-muon-generator-and)
<http://www.mathworks.com/matlabcentral/fileexchange/51257-geant4-muon-energy-and-angular-histogram-generation>

2. Muon spectrum

Cosmic radiation, originating mainly from outside our solar system constantly bombards the upper layers of the Earth’s atmosphere [18]. The high energy cosmic rays that enter Earth’s atmosphere generate a cascade of secondary rays and relativistic particles. As a result of the processes occurring in the atmosphere, the hadronic component, the

electromagnetic component and the muonic component represent the spectrum of particles reaching the surface. Through absorption and decay processes the subatomic particles pions and kaons give rise to a considerable flux of muons that reaches sea level [19]. Cosmic ray muons are charged particles, having approximately 200 times the mass of electron, generated naturally in the atmosphere, and rain down upon the earth at an approximate rate of 10,000 particles $m^{-2} min^{-1}$.

Main properties characterizing muon spectrum are integral and differential intensity [19]. Intensity is defined as the number of particles per unit area dA , per unit time dt and per unit solid angle $d\Omega$:

$$I = \frac{dN}{dAdtd\Omega} (cm^{-2}s^{-1}sr^{-1}) \quad (1)$$

Differential intensity is a useful quantity to describe cosmic radiation and present experimental measurements, defined as the the number of particles per unit area dA , per unit time dt and per unit solid angle $d\Omega$ and unit energy dE :

$$D = \frac{dN}{dAdtd\Omega dE} (cm^{-2}s^{-1}sr^{-1}GeV^{-1}) \quad (2)$$

Differential intensity is a function of the particle energy, zenith and azimuthal angle, time and area. Integrating for energies higher than E , the integral energy spectrum can be obtained:

$$M(> E) = \int_E^{\infty} D dE = \frac{dN}{dAdtd\Omega} (cm^{-2}s^{-1}sr^{-1}) \quad (3)$$

The muon spectrum has been experimentally measured and shown to vary significantly with energy and zenith angle. The most representative experimental measurements are compiled in Table 1. The experiments cover a wide range of energies, from 0.2 to 20,000 GeV, zenith angles from from 0° to 89° and 10 m to 1270 m altitude. It appears that the majority of the experiments were performed in low altitude and vertical or near vertical direction and only few experiments were realized in higher zenith angles. The

experimental measurements of Nandi & Sinha [20], Tsuji et al. [21], Haino et al. [22] and Jokisch et al. [23] are shown in Figs 1 and 2. Above 10 GeV, the spectrum follows a power law profile which is practically independent of the zenith angle. For energies less than 10 GeV, the spectrum slope changes and zenith angle dependency increases. The influence of zenith angle is small up to 30° but becomes significant for larger angles. The muon spectrum is also influenced on geomagnetic effects which limit the primary proton flux and solar cycle which modulates the primary proton flux [18]. Geomagnetic effects filter the primary particles; this is equivalent to having a cut-off on the energy of the particles of ~ 15 GeV near the equator reducing to 0 GeV on the poles. Solar activity follows sinusoidal profile with a period of 11 years and is inversely proportional to the primary flux, i.e., maximum solar activity minimizes the primary proton flux. Less important are atmospheric temperature/pressure variations. The above dependencies become significant for muon energies less than 10 GeV. Less significant is the variation with respect to altitude [18].

Table 1. Compilation of muon spectrum experimental measurements

No.	Authors	Year	Altitude (m)	Momentum range (GeV/c)	Zenith angle range ($^\circ$)	Nominal zenith angle ($^\circ$)
1	Baber et al. [24]	1968	52	3-1000	0	0
2	Allkofer et al. [25]	1971	10	0.2-1000	0	0
3	Allkofer and Dau [26]	1972	10	0.3-26	0	0

4	Nandi et al. [20]	1972	70	5-1200	0-0.3	0
5	Ayre et al. [27]	1975	70	20-500	0-0.08	0
6	Baxendale et al. [28]	1975	70	7-500	0	0
7	Kellogg et al. [29]	1978	10	50-1700	25.9-34.1	30
				50-1700	70.9-79.1	75
8	Jokisch et al. [23]	1979	10	1-1000	68-82	75
9	Rastin [30]	1984	52	4-3000	0	0
10	Matsuno et al. [31]	1984	10	100-20000	86-90	89
11	De Pascale et al. [32]	1993	600	0.2-100	0	0
12	Tsuji et al. [21]	1998	5	1.5-250	0-1	0
				2-250	26-34	30
				3-250	59-61	60
				3-250	69-81	75
				3-150	79-81	80
13	Kremer et al. [33]	1999	360	0.2-120	0	0
			1270	0.2-120	0	0
14	Haino et al. [22]	2004	30	0.6-400	0	0
15	L3 collaboration [34]	2005	450	20-3000	0-58

Fig. 1. Experimentally measured muon differential intensity at 0° zenith angle

Fig. 2. Experimentally measured muon differential intensity at different zenith angles

3. Muon phenomenological model

A phenomenological model is used to fit the experimental measurements and provide the muon flux component at different altitudes, zenith angles and muon energies. The phenomenological model was proposed by Smith & Duller [35] and provides an overall description of the pion and muon interactions as they propagate through the atmosphere. The pion variation with atmospheric depth is described as the difference between the number of pions produced and the number of pions lost by absorption and decay processes. The diffusion equation as it is called can be written as follows:

$$\frac{dN_{\pi}(E_{\pi}, y, \theta)}{dy} = \frac{AE_{\pi}^{-\gamma}}{\cos\theta} \exp\left(\frac{-y}{\lambda_p \cos\theta}\right) - \frac{N_{\pi}(E_{\pi}, y, \theta)}{\cos\theta} \left[\frac{1}{\lambda_{\pi}} + \frac{dt}{\tau}\right] \quad (4)$$

where N_{π} is the differential pion intensity, E_{π} , y , θ are the pion energy, atmospheric depth and zenith angle, respectively. The first term in the equation represents the production term which is dominated by the proton absorption mean free path λ_p . The factor A is assumed to be a constant, although it varies slowly with energy [35]. The second term includes the absorption and decay mechanisms represented by the pion absorption mean free path λ_{π} and pion dilated mean lifetime τ . A summary of the parameters used, units and values is presented in Table 2. Applying the following substitutions (notation that follows was selected for consistency with reference [35]):

$$dt = \frac{dy}{c\rho(y)\cos\theta} \quad (5)$$

$$\tau = \frac{\tau_0}{\sqrt{1 - \left(\frac{u}{c}\right)^2}} = \frac{\tau_0 m}{m_{\pi}} = \frac{\tau_0 m c^2}{m_{\pi} c^2} = \frac{\tau_0 E_{\pi}}{m_{\pi} c^2} \quad (6)$$

$$\rho(y) = \frac{\rho_0 y}{y_0} \quad (7)$$

where ρ_0 , and y_0 are the density and depth at ground level, 0.00123 g/cm^3 and 1000 g/cm^2 , respectively. m_{π} and τ_0 are the pion mass and mean lifetime at rest. The isothermal

atmosphere is used as an approximation for $\rho(y)$. The pion kinetic energy E_π can safely be assumed approximately equal to the total particle energy, i.e., $E_\pi + m_\pi c^2 \approx E_\pi = mc^2$. Applying equations 5, 6 and 7, the initial equation is transformed to (Smith & Duller, 1959):

$$\frac{dN_\pi(E_\pi, y, \theta)}{dy} = \frac{AE_\pi^{-\gamma}}{\cos\theta} \exp\left(\frac{-y}{\lambda_p \cos\theta}\right) - \frac{N_\pi(E_\pi, y, \theta)}{\cos\theta} \left[\frac{1}{\lambda_\pi} + \frac{m_\pi c}{\rho(y)\tau_0 E_\pi} \right] \quad (8)$$

with initial conditions $N_\pi=0$ at $y=0$, the solution to the diffusion equation is:

$$N_\pi(E_\pi, y, \theta) = \frac{AE_\pi^{-\gamma} y \exp(-y/\lambda_\pi \cos\theta)}{\cos\theta} \sum_{n=0}^{\infty} \left(\frac{y}{\lambda' \cos\theta} \right)^n \left[n! \left(n + 1 + \frac{bj_\pi}{E_\pi \cos\theta} \right) \right]^{-1} \quad (9)$$

where:

$$\frac{1}{\lambda'} = \frac{1}{\lambda_\pi} - \frac{1}{\lambda_p} \quad (10)$$

and

$$j_\pi = \frac{m_\pi y_0 c}{\tau_0 \rho_0} \quad (11)$$

The quantity b has been included to improve the isothermal atmosphere. Truncating the solution after the first term:

$$N_\pi(E_\pi, y, \theta) = \frac{AE_\pi^{-\gamma+1} y \exp(-y/\lambda_\pi \cos\theta)}{E_\pi \cos\theta + bj_\pi} \quad (12)$$

Solutions as a function of pion momentum and atmospheric depth for the pion intensity are shown in Figs 3 and 4.

Fig. 3. Pion differential intensity vs. momentum

Fig. 4. Pion intensity vs. atmospheric depth

208 Muons are produced following pion decay and therefore depend directly upon the pion
 209 differential intensity. The muon intensity at depth y_0 is:

$$N_\mu(E_\pi, \theta) = \int_0^{y_0} P_\mu(E_\pi, y, \theta) N_\pi(E_\pi, y, \theta) \frac{dt}{\tau} \quad (13)$$

210 where N_π is the pion intensity and P_μ is the probability of survival to depth y_0 . The probability
 211 of survival has been shown to be:

$$P_\mu(E_\pi, y, \theta) = \left[\frac{yE_\mu}{y_0E_\mu^*} \right]^{(B_\mu / (E_\mu' \cos \theta))} \quad (14)$$

212 where:

$$B_\mu = \frac{b_\mu m_\mu y_0 c}{\tau_{\mu 0} \rho_0} \quad (15)$$

213 The solution to the above integral equation is:

$$N_\mu(E_\pi, \theta) = \frac{AE_\pi^{-\gamma+1} P_\mu \lambda_\pi b j_\pi}{\cos \theta} \sum_{n=0}^{\infty} \left(\frac{\lambda_p - \lambda_\pi}{\lambda_\pi} \right)^n \left(n + 1 + \frac{b j_\pi}{E_\pi \cos \theta} \right)^{-1} \quad (16)$$

214 Truncating after the first term, the muon intensity is:

$$N_\mu(E_\pi, \theta) = \frac{AE_\pi^{-\gamma} P_\mu \lambda_\pi b j_\pi}{E_\pi \cos \theta + b j_\pi} \quad (17)$$

215 The above derivation is based on the following assumptions: i) A is constant, ii) pions acquire
 216 a constant average fraction of the proton kinetic energy, iii) forward direction of the pion is
 217 always maintained, iv) isothermal atmosphere approximation, v) λ_π equal λ_p , vi) atmosphere
 218 curvature neglected and vii) muon production only from pions (kaons decay neglected).
 219 Kaons decay has been shown to be around 20% in some cases [30] but as will be
 220 demonstrated in the next section, these approximations do not significantly affect the
 221 accuracy of the model.

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Table 2. Phenomenological model parameters [30]

Parameter	Value
Depth of the atmosphere y_o (g/cm ²)	1000
Gas constant R (J/Kgmol)	8.314
Effective temperature T_e (K)	220
Effective molecular weight of air M (-)	28.966
Acceleration due to gravity g (cm/s ²)	981.3
Density of air at sea level ρ_o (g/cm ³)	0.00123
Speed of light c (cm/s)	3×10^{10}
Muon rate of energy loss in air α (MeV/gcm ²)	2.500
Mass of muon (at rest) m_μ (MeV/c ²)	105.659
Mass of pion (at rest) m_π (MeV/c ²)	139.580
Mass of kaon (at rest) m_K (MeV/c ²)	493.800
Muon mean lifetime (at rest) $\tau_{\mu o}$ (s)	2.20×10^{-6}
Pion mean lifetime (at rest) τ_o (s)	2.61×10^{-8}
Kaon mean lifetime (at rest) (s)	1.24×10^{-8}
Production depth of muons (g/cm ²)	100
Correction factor b (-)	0.771
Correction factor b_μ (-)	0.800
Muon absorption mean free path λ_μ (g/cm ²)	120
Pion absorption mean free path λ_π (g/cm ²)	120
Kaon absorption mean free path λ_K (g/cm ²)	120
j_π (GeV)	148.16
j_K (GeV)	1105.80

The above phenomenological model was fitted to the experimental measurements of Nandi & Sinha [20] using non-linear least squares regression fitting. The fitting minimized the sum-squared error function and the resulting parameters A and γ with 95% confidence intervals are shown in Table 3. The fitted curve with 95% prediction intervals is shown in Fig. 5. The phenomenological model using the above parameters was tested and evaluated against the experimental datasets of Tsuji et al. [21], Haino et al. [22] and Jokisch et al. [23] which cover a large range of energies, from 0.5 GeV to 1100 GeV and a wide range of zenith angles (0° , 30° , 60° , 75°). The graphical representation of the present model and its behavior against various experimental measurements for different levels of energy and zenith angle is shown in Figs 6 to 9. The dotted lines represent the 95% prediction intervals of the model. Overall, the phenomenological model appears to capture correctly the different features of the muon spectrum even in large zenith angles. In Fig. 10 the residuals plot is shown. There is an increase in the residual magnitude for energies below 10 GeV. This can be attributed to the lack of the model to capture the increasing dependencies of the muon spectrum on latitude, solar and temperature variations which increasingly affect the muon spectrum at low energies. However, taking into account the simplicity of the model, its accuracy and prediction capability are remarkable and within the limits of the proposed application.

The present fitted phenomenological model was also compared to a recently developed parametric model proposed by Reyna [36]. The Reyna model includes energy and angular dependencies and is described by the following form:

$$I(p_\mu, \theta) = \cos^3(\theta) I_v(\zeta) \quad (18)$$

where p_μ and θ are the muon energy and zenith angle, respectively. $I_v(\zeta)$ is the differential intensity at $\zeta = p_\mu \cos(\theta)$. This relationship implies that the differential intensity at any momentum and zenith angle can be expressed as the product of two functions, the cosine

cubed at the specific zenith angle and the intensity $\zeta=p_\mu\cos(\theta)$. For zero zenith angles, the intensity relationship used by Reyna [36] is:

$$I(p_\mu) = c_1 p_\mu^{-1(c_2+c_3\log(p_\mu)+c_4\log^2(p_\mu)+c_5\log^3(p_\mu))} \quad (19)$$

The parameters of the Reyna model were calculated following a chi-square fit to several experimental datasets. The parameters are:

$$c_1=0.00253, c_2=0.2455, c_3=1.288, c_4=-0.2555, c_5=0.0209$$

This parameterization is valid for $p_\mu > 1$ GeV and $p_\mu < 2000$ GeV/cos θ . To evaluate the fitting accuracy and compare the two models the following metrics have been used: the Mean Absolute Percentage Error (MAPE):

$$MAPE = \frac{\sum_{i=1}^n \left| \frac{O_i - E_i}{O_i} \right|}{n} \times 100\% \quad (20)$$

and the Symmetric Mean Absolute Percentage Error (MAPE_{sym}):

$$MAPE_{sym} = \frac{1}{n} \sum_{i=1}^n \left| \frac{O_i - E_i}{(O_i + E_i)/2} \right| \times 100\% \quad (21)$$

where n is the number of measurements, O_i is the observed experimental measurement and E_i is the estimated value. MAPE and MAPE_{sym} are relative measures that range from 0 to 200%. MAPE and MAPE_{sym} were selected instead of the commonly used Mean Square Error (MSE) Root Mean Square Error (RMSE) since they are influenced by extreme values to a much lesser extent. The comparison between the present fitted phenomenological model and Reyna model is shown in Table 4 and Fig. 11. It appears that the present model performs slightly better in most cases.

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Table 3. Phenomenological model fitting parameters

Parameter	A (95% CI)	γ (95% CI)
Value	0.002382 (0.002075, 0.002688)	2.645 (2.594, 2.697)

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Table 4. Comparison of present model and Reyna model with experiments

	Present model		Reyna model	
	MAPE (%)	MAPE _{sym} (%)	MAPE (%)	MAPE _{sym} (%)
Nandi et al., 1972	8.08	8.72	8.71	8.82
Ayre et al., 1975	7.72	7.66	2.49	2.48
Tsuji et al., 1998	17.80	14.70	19.09	15.85
Haino et al., 2004	13.95	12.86	17.66	16.12
Jokisch et al., 1979	22.11	19.13	28.06	27.13
All of the above	13.55	12.34	14.75	13.63

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Fig. 5. Best fit curve on muon measurements

Fig. 6. Fitted phenomenological model vs. experimental measurements (0° zenith angle)

Fig. 7. Phenomenological model vs experimental measurements at 75° zenith angle

Fig. 8. Phenomenological model vs. experimental measurements (0° zenith angle)

Fig. 9. Phenomenological model vs. experimental measurements for different zenith angles

Fig. 10. Residuals plot

Fig. 11. Phenomenological model compared to Reyna model and the experimental measurements of Table 4.

4. Statistical sampling algorithms

The muon spectrum when normalized to unit area represents a probability density function. The normalized muon spectrum is a non-standard probability distribution that follows a power law profile for energies larger than 10 GeV with a slope change for energies lower than 10 GeV . In principle, it is not straightforward to sample from a non-standard distribution. Commercial statistical software packages generate random samples mainly from standard distributions such as uniform, exponential or Gaussian distributions. To generate random samples (i.e., observations) from a non-standard probability density function statistical methods have to be applied. Three approaches are used in the present work. First, the inverse transform method is used to obtain random samples from the muon angular distribution. Then, the Acceptance-Rejection (A-R) and the Metropolis-Hastings (M-H) algorithms are used to generate samples from the muon energy distribution. These methods assume the availability of a uniform random number generator that provides independent and identically distributed random numbers uniformly distributed on the interval (0, 1).

4.1 Inverse transform

The inverse transform involves inversion of the Cumulative Distribution Function (CDF) to obtain random samples $X=F^{-1}(U)$ where U is the uniform probability density function [37]. Let X be a continuous random variable and $F_X(x) = P(X \leq x)$, $x \in \mathbb{R}$ the non-negative non-decreasing cumulative distribution function. To generate random variables X distributed as CDF F_X the following steps are needed:

1. Invert CDF F_X to obtain the expression $F_X^{-1}(U)$
2. Generate random numbers uniformly distributed $U \sim U(0,1)$
3. Obtain random variable X from $X= F_X^{-1}(U)$

The inverse transform takes a uniformly generated random number and transforms it to a random observation x distributed as F_X . To demonstrate that X is distributed as the CDF F_X , let F_X be strictly increasing and thus invertible over all x . Then, if x is the observation:

$$P(X \leq x) = P(F_X^{-1}(U) \leq x) = P(U \leq F_X(x)) = F_X(x)$$

The muon angular distribution follows a squared cosine distribution:

$$f_{\theta}(\theta) = A \cos^2(\theta) \quad (22)$$

where A is the normalization constant. Correcting for the solid angle effect, i.e., take into account that fewer particles will be accepted at zenith angle 0° than at zenith angle 30° due to larger area, the normalized density is given as:

$$f_{\theta}(\theta) = 2C\pi \sin(\theta) \cos^2(\theta) \quad (23)$$

where C is the normalization constant. The CDF is then calculated:

$$F_{\theta}(\theta) = \int_0^{\theta} 2C\pi \sin(\theta) \cos^2(\theta) d\theta = 1 - \cos^3(\theta) \quad (24)$$

The inverse transform is then:

$$\theta = \arccos\left(\sqrt[3]{1 - F_{\theta}(U)}\right) \quad (25)$$

100,000 zenith angles were randomly selected using the inverse transform. The results are shown in Fig. 12 together with the curve obtained using Eq. 23. The Rice rule [38] was used to calculate the optimal number of bins for distribution graphing:

$$\text{number of bins} = 2N^{1/3} \quad (26)$$

where N is the number of muons.

4.2 Acceptance-Rejection method

The inverse transform method requires knowledge of the CDF which for non-standard distributions may not be easily derived. In addition, even if the CDF is known, an explicit formula for the inverse function may not be always possible. This is the case for generating

muons using the phenomenological model described previously. Instead, the A-R and M-H methods [39] provide an alternative way to circumvent this difficulty. To describe the A-R method, let X be a continuous random variable with probability density function $f_X(x)$. A proposal probability density function $g_X(x)$ is selected such that the ratio $f_X(x)/g_X(x)$ is bounded by a constant $c>0$ for all x . As a result, $g_X(x)$ is always greater or equal to $f_X(x)$ creating an envelope around the target distribution:

$$g_X(x) \geq f_X(x) \quad (27)$$

The area under the proposal density $g_X(x)$ is then:

$$c = \int_{-\infty}^{\infty} g_X(x) dx \geq 1 \quad (28)$$

Therefore the $g_X(x)$ is not a density function. However, the function:

$$h_X(x) = \frac{g_X(x)}{c} \quad (29)$$

has area equal to one, hence it is a probability density function and can be used to generate samples from the desired pdf $f_X(x)$. In the present work, the proposal distribution was selected to be uniformly distributed. The steps for generating random variables X distributed as $f_X(x)$ are:

1. generate Y distributed as $Y \sim h_Y(y) = g_Y(y)/c$
2. find a constant c that satisfies the bounded criterion
3. generate random numbers uniformly distributed $U \sim U(0,1)$
4. if $u \leq f_Y(y)/(cg_Y(y))$ then $X=Y$ ("accept"); else repeat ("reject")

Points are generated from $cg_Y(y)$ and those that are inside the curve $f_X(y)$ are accepted as belonging to the target distribution. The accepted points are distributed as $X \sim f_X(x)$. Accepted and rejected samples from the target and proposal distributions are shown in Fig. 13. It is noted that these distributions are not normalized to one. It can be seen that the

uniform distribution is not very close to the target distribution and as such the area between that separates the two distributions is large. As a result, a large number of samples is rejected reducing the efficiency of the method. The probability of acceptance is:

$$P(accepted) = \frac{1}{c}$$

The acceptance rate, the accepted samples and the execution time are shown in Table 5. The acceptance rate remains practically constant independent of the iterations but the execution time increases as $O(N)$. Therefore, the A-R execution time depends strongly on the number of muons sampled. Samples drawn using this method agree very well with experimental measurements, as shown in Fig. 14.

This method, besides the availability of a uniform random number generator, requires the ability to generate samples from the proposal distribution $g_X(x)$ also exists. For this, it is necessary to have a density $g_X(x)$ that is easy to sample from. It is also desirable to have the constant c as close to 1 as possible to achieve high efficiency. In the present work, a uniform distribution was used which is very easy to sample from but resulted in a value of $c=19$. Therefore, the A-R sampling algorithm has a low efficiency of $\sim 5\%$. As a result, to generate 10,000 muons, a total of 500,000 must be generated. If a different proposal distribution was used that would have been closed to the target distribution then c would have been lower. However, an appropriate method would have to be devised to sample from this distribution, for example inverse transform cancelling the main advantage of the A-R method, which is its simplicity.

4.3 Metropolis-Hastings algorithm

The Metropolis-Hastings (M-H) algorithm belongs to the family of Markov Chain Monte Carlo algorithms (MCMC). The MCMC exploits the fact that a Markov Chain starting from an initial distribution will eventually reach a stationary distribution that is independent

408 of the starting point [39]. Similarly to the A-R algorithm, the M-H algorithm requires the
 409 selection of a proposal distribution. Let $X^{(0)}$ be the initial state and define the proposal
 410 distribution from which draw random sample Y :

$$Y \sim q(y|X^{(t)}) \quad (30)$$

411 Then calculate the following parameter:

$$a = \frac{f_X(y) q(X^{(t)}|y)}{f_X(x) q(y|X^{(t)})} \quad (31)$$

412 Acceptance probability is then given by:

$$r = \min\{1, a\} \quad (32)$$

413 Set with probability r (acceptance):

$$X^{(t+1)} = Y \quad (33)$$

414 Set with probability $1-r$ (rejection):

$$X^{(t+1)} = X^{(t)} \quad (34)$$

415 In the present work, the proposal distribution was selected to be a Gaussian which is easy to
 416 sample from:

$$q(y|X^{(t)}) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y-X^{(t)})^2} \quad (35)$$

417 The proposal distribution is symmetric:

$$q(y|X^{(t)}) = q(X^{(t)}|y) \quad (36)$$

418 And therefore the acceptance ratio simplifies to:

$$a = \frac{f_X(y)}{f_X(x)} \quad (37)$$

419 The convergence rate is controlled by the variance σ^2 of the proposal distribution. A small
 420 variance covers the whole distribution very slowly whereas a large variance results in a small
 421 acceptance probability. The acceptance rate, the accepted samples and the execution time for
 422 $\sigma=10$ are shown in Table 6. The execution time scales with $O(N)$ but it appears that M-H

algorithm ten times more efficient than the A-R. As such, fewer iterations are required to obtain the same number of samples reducing the computational load. The acceptance rates for different values of σ are shown in Table 7. For smaller values of σ the acceptance rate is even higher as can be seen from Table 7 where 10^6 iterations were used for different values of σ . To demonstrate the agreement between the model and the experimental data, the distribution of the samples is compared against experimental measurements in Fig. 15. The samples agree very well with the experimental measurements even for large zenith angles.

4.4 Comparison with CRY

CRY is a software library developed by LLNL [16] that provides pre-computed data tables with muon energies and zenith angles at three altitudes. CRY allows for Geant4 and MCNP coupling offering an alternative way to generate random muon samples although the random sample generation methodology is considerably different than the one described herein. It is noted that typical CRY computational time is equivalent to the proposed methodology, i.e., in the order of minutes, however, CRY does not offer the capability to generate monoenergetic or monodirectional muons and therefore post-processing is required which can significantly increase the required computational time. 100,000 muons were generated from CRY at 0 altitude, 50° latitude out of which $\sim 30,000$ vertical muons were selected ($-0.9 < \cosine < 1.1$) with energies from 1 GeV to 60 GeV. Similarly, 1500 muons were selected with zenith angles in the range of $70-80^\circ$ with energies from 1 GeV to 60 GeV. To facilitate comparison of experimental measurements, A-R sampling and muons generated from CRY, all data were normalized to area=1. The Rice rule was used to calculate the optimal number of bins for the distribution graphing. 60 bins were used for vertical muons and 20 bins for muons in the range of $70-80^\circ$. The results are shown in Fig. 16.

Table 5. Efficiency of A-R algorithm

Iterations	Acceptance rate	Number of accepted samples	Execution time (sec)
10^4	5.22%	522	0.0096
10^5	5.18%	5178	0.0871
10^6	5.15%	51535	0.8563
10^7	5.18%	518460	8.5252
10^8	5.18%	5176799	84.9990

Table 6. Efficiency of M-H algorithm for $\sigma=10$

Iterations	Acceptance rate	Number of accepted samples	Execution time (sec)
10^4	28.64%	2863	0.0803
10^5	28.40%	28402	0.7139
10^6	28.49%	284941	7.2081
10^7	28.45%	2844970	71.2345

Table 7. Efficiency of M-H algorithm for 10^6 iterations

Sigma σ	Acceptance rate	Number of accepted samples	Execution time (sec)
1	79.04%	790400	8.1213
5	43.89%	438947	7.4653
10	28.49%	284941	7.2081

Fig. 12. Inverse transform sampling of muon zenith angles

Fig. 13. Visualization of accepted and rejected samples

Fig. 14. A-R sampling vs. experimental measurements (data normalized)

Fig. 15. M-H sampling vs. experimental measurements

Fig. 16. Comparison of A-R sampling against CRY (data normalized)

4.5 Application of muon generator

The muon generator can be very useful for validating classification and imaging algorithms. The asymptotic behavior of classification algorithm developed for monitoring spent nuclear fuel was investigated by Chatzidakis and Tsoukalas [15]. This required extensive testing of the classification algorithm using 200 training and 1000 testing data that can only be obtained with a muon generator coupled with in-house Monte Carlo simulations. A Geant4-CRY approach would provide the muon tracks but performing the simulation 1,000 times to obtain the necessary data for algorithm validation would not have been possible due to time limitations. Another use of the muon generator would be the estimation of quantities related to signals obtained from muon detectors. A quantity of interest is the scattering density, proposed by Schultz [3]. For monoenergetic muons the scattering density can be calculated analytically from the theory of Multiple Coulomb Scattering. However, for polyenergetic muons, the scattering density is the convolution of the scattering densities and the analytical solution is not tractable. Instead, a muon generator can provide polyenergetic muons that can subsequently be used for estimation of the scattering density. Table 8 shows the scattering density results obtained from 100,000 samples generated using the muon generator for monoenergetic-monodirectional muons at 4 GeV, polyenergetic-monodirectional and polyenergetic-polydirectional muons.

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Table 8. Scattering density (mr²/cm) at different energies

	Density (g/cm ³)	Radiation length (g/cm ²)	4 GeV (0° zenith angle)	1-60 GeV (0° zenith angle)	1-60 GeV (0° - 80° zenith angle)
Air	0.00123	36.62	$3.88 \cdot 10^{-4}$	$1.20 \cdot 10^{-3}$	$9.35 \cdot 10^{-4}$
Oxygen	1.14	34.23	0.38	1.19	0.92
Concrete	2.30	26.57	1.00	3.11	2.41
Iron	7.87	13.83	6.57	20.48	15.84
UO ₂	10.3	6.65	17.90	55.71	43.10
Uranium	19.0	6.00	36.60	113.88	88.26

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5. Conclusions

The main purpose of this paper is to develop a cosmic ray muon sampling capability *using a phenomenological model that captures the main characteristics of the experimentally measured spectrum coupled with a set of statistical algorithms* for muon tomographic and monitoring applications. Increasing application of muon technology necessitates the development of classification and imaging algorithms and Monte Carlo simulations incorporating the capability to generate muon samples are necessary for algorithm validation and testing. However, the muon spectrum follows a non-standard probability distribution with a power law profile for energies larger than 10 GeV whereas the angular distribution has a squared cosine distribution.

The proposed muon generator provides a means to overcome the difficulty of sampling from a non-standard distribution. The muon energy distribution is described by the Smith & Duller [35] phenomenological model fitted to experimental measurements of using non-linear least squares regression fitting. The accuracy of the fitting was investigated by extensive comparison with experimental measurements and the recently proposed parametric model of Reyna [36] with error in the order of 12%. Overall, it appears that the fitted model captures correctly the different features of the muon spectrum even in large zenith angles and energies.

Statistical algorithms are then employed for generating random samples. The inverse transform provides a means to generate samples from the muon angular distribution, whereas the Acceptance-Rejection and Metropolis-Hastings algorithms are employed to provide the energy component. The predictions for muon energies 1 to 60 GeV and zenith angles 0-90° are validated with a series of actual spectrum measurements and estimates from the software library CRY. The results confirm the validity of the phenomenological model and the applicability of the statistical algorithms to generate polyenergetic-polydirectional muons.

The response of the algorithms and the impact of critical parameters on computation time and computed results were investigated demonstrating that the A-R algorithm appears to have low efficiency requiring a large number of samples to be rejected reducing the efficiency of the method.

Final output from the proposed “muon generator” is a look-up table that contains the sampled muon angles and energies and can be integrated into Monte Carlo particle simulation codes such as Geant4 and MCNP. The “muon generator” can be easily expanded to incorporate solar activity and latitude dependence. Main advantages of the “muon generator” include its ability to readily produce any number of muons for any altitude and any combination of zenith angles and energies, along with the fact that it is based on a strong phenomenological model and validated statistical algorithms.

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