CSR-Induced Emittance Growth in Achromats: Linear Formalism Revisited

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Abstract

We review the R-matrix formalism used to describe Coherent Synchrotron Radiation (CSR)-induced projected emittance growth in electron beam transport lines and establish the connection with a description in terms of the dispersion-invariant function.

Keywords: electron beam transport; coherent synchrotron radiation

1. Introduction

Coherent Synchrotron Radiation (CSR) is well known to represent a fundamental limitation to the preservation of electron beam brightness during transport or magnetic compression [1]. As an electron bunch travels through a dipole, synchrotron radiation is emitted coherently at wavelength comparable to the bunch length, and unless suppressed by a narrow vacuum chamber, it may introduce a significant modification of the beam particle energy depending on position along the bunch. The presence of dispersion correlates the energy change with a transverse offset in the bending plane (position, angle) resulting into emittance growth. Because of the correlated nature of the energy change the effect, unlike the one stemming from incoherent synchrotron radiation, is in principle reversible. This principle has long been

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19 realized and often exploited in the design of beamlines [2–11]. The purpose
 20 of this paper is to review aspects of the ‘R-matrix’ [2] used to model the
 21 CSR-induced emittance growth, restating some general conditions for the
 22 attainment of minimum emittance growth, and clarifying its connection with
 23 the dispersion-invariant (‘curly’- $\mathcal{H}$) formalism sometimes used to estimate
 24 the magnitude of the effect [8, 9].

25 **2. R-matrix formalism**

26 *2.1. Correlated emittance growth*

27 Consider a beamline extending from $s = s_i$ to $s = s_f$. The transverse
 28 coordinates of a particle having longitudinal coordinate z_s and experiencing
 29 a CSR-induced relative-energy change $\delta_s(z_s)$ at location $s \in [s_i, s_f]$ evolve
 30 according to

$$\begin{aligned} x(s_f) &= R_{11}^{s_i \rightarrow s_f} x_i + R_{12}^{s_i \rightarrow s_f} x'_i + R_{16}^{s \rightarrow s_f} \delta_s(z_s), \\ x'(s_f) &= R_{21}^{s_i \rightarrow s_f} x_i + R_{22}^{s_i \rightarrow s_f} x'_i + R_{26}^{s \rightarrow s_f} \delta_s(z_s), \end{aligned} \quad (1)$$

31 where R_{ij} are the entries of the linear transport matrix. Notice that the
 32 entries R_{11} , R_{12} , R_{21} , and R_{22} are for the transport matrix from s_i to s_f ,
 33 whereas the entries R_{16} and R_{26} are for transport starting from the point s
 34 within the beamline.

35 The notation emphasizes the dependence of the particle energy change
 36 on the arclength coordinate s and longitudinal coordinate z_s at s , with the
 37 latter in general also depending on s . Assuming a one-to-one correspondence

¹ between z_s and the particle longitudinal coordinate z_f at s_f , we can write $\delta_s(z_s) = \delta_s(z_s(z_f)) \equiv \delta_s(z)$, where to keep the notation simple we dropped the subscript from z_f .

Integrating over the whole dispersive region:

$$x(s_f) = x_\beta(s_f) + \int_{s_i}^{s_f} R_{16}^{s \rightarrow s_f} \frac{d\delta_s(z)}{ds} ds \equiv x_\beta(s_f) + \hat{x}(z), \quad (2)$$

$$x'(s_f) = x'_\beta(s_f) + \int_{s_i}^{s_f} ds R_{26}^{s \rightarrow s_f} \frac{d\delta_s(z)}{ds} \equiv x'_\beta(s_f) + \hat{x}'(z), \quad (3)$$

where $x_\beta(s_f) = R_{11}^{s_i \rightarrow s_f} x_i + R_{12}^{s_i \rightarrow s_f} x'_i$, $x'_\beta(s_f) = R_{21}^{s_i \rightarrow s_f} x_i + R_{22}^{s_i \rightarrow s_f} x'_i$, and the equalities on the RHS serve as definitions for $\hat{x}$ and $\hat{x}'$.

Assuming for simplicity that the unperturbed beam is centered $\langle x_\beta \rangle = 0$, where $\langle \cdot \rangle$ is the average on the bunch population in the 6D phase space, the central 2^{nd} -moment of the beam distribution in the horizontal coordinate at s_f reads

$$\langle (x - \langle x \rangle)^2 \rangle = \langle (x_\beta - \langle x_\beta \rangle + \hat{x} - \langle \hat{x} \rangle)^2 \rangle = \langle (x_\beta + \Delta x)^2 \rangle = \langle x_\beta^2 \rangle + \langle (\Delta x)^2 \rangle, \quad (4)$$

having denoted $\Delta x = \hat{x} - \langle \hat{x} \rangle$, and exploited the lack of correlation between the CSR-induced energy kick and transverse coordinate: $\langle x_\beta \Delta \hat{x} \rangle = 0$, valid in a 1D approximation. We have

$$\begin{aligned} \langle (\Delta x)^2 \rangle &= \langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2 \\ &= \int dz \lambda \left[\int R_{16}^{s \rightarrow s_f} \frac{d\delta_s(z)}{ds} ds \right]^2 - \left[\int dz \lambda \int R_{16}^{s \rightarrow s_f} \frac{d\delta_s(z)}{ds} ds \right]^2 \end{aligned} \quad (5)$$

where $\lambda = \lambda(z)$ is the normalized beam longitudinal density at the observation point at the exit of the beamline, with similar expressions holding for

¹This requires that we neglect the longitudinal slippage due to the beam finite transverse emittance.

53 $\langle(\Delta x')^2\rangle$ and $\langle\Delta x\Delta x'\rangle$.

54 The rms projected (geometric) emittance is defined as the determinant
55 of the covariance matrix

$$\boldsymbol{\sigma} = \begin{pmatrix} \langle(x + \Delta x)^2\rangle & \langle(x + \Delta x)(x' + \Delta x')\rangle \\ \langle(x + \Delta x)(x' + \Delta x')\rangle & \langle(x' + \Delta x')^2\rangle \end{pmatrix}, \quad (6)$$

56 Observed at the exit s_f of the beamline the emittance reads

$$\varepsilon_x^2 = \varepsilon_{x0}^2 + \varepsilon_{x0}(\beta_x\langle\Delta x'^2\rangle + 2\alpha_x\langle\Delta x\Delta x'\rangle + \gamma_x\langle\Delta x^2\rangle) + \Delta\varepsilon_x^2, \quad (7)$$

57 where ε_{x0} is the beam unperturbed emittance, $\alpha_x = -\langle xx'\rangle/\varepsilon_{x0}$, $\beta_x =$
58 $\langle x^2\rangle/\varepsilon_{x0}$, $\gamma_x = (1 + \alpha_x^2)/\beta_x$ are the Twiss functions at s_f , and

$$\Delta\varepsilon_x^2 = \langle\Delta x^2\rangle\langle\Delta x'^2\rangle - \langle\Delta x\Delta x'\rangle^2. \quad (8)$$

59 We now specialize the model to the case of stationary longitudinal den-
60 sity and 1D steady-state CSR model [13], factoring out the s dependence
61 $d\delta_s(z)/ds \equiv g(s)\hat{\delta}'(z)$, where $g(s) = [R_0/R(s)]^{2/3}$ for s inside a bend with
62 radius of curvature $R(s)$, and $g(s) = 0$ outside the bends, (R_0 is a constant
63 introduced for convenience) and

$$\hat{\delta}'(z) = \frac{2mc^2r_cN_b}{3^{1/3}R_0^{2/3}} \int_z^\infty \frac{d\lambda(z')}{dz'} \frac{dz'}{(z' - z)^{1/3}}. \quad (9)$$

64 In the above expression, N_b is the bunch population, m and r_c the electron
65 mass and classical radius. Notice that we are adopting the convention that
66 a particle in the bunch head has $z < 0$ and $\delta'(z) > 0$ denotes energy gain.

67 Eq. (5) then simplifies to

$$\langle(\Delta x)^2\rangle = \left[\int ds g(s) R_{16}^{s \rightarrow s_f} \right]^2 \langle(\Delta \hat{\delta}')^2\rangle, \quad (10)$$

68 where $\langle(\Delta\hat{\delta}')^2\rangle = \langle(\hat{\delta}')^2\rangle - \langle\hat{\delta}'\rangle^2$. Similarly

$$\langle(\Delta x')^2\rangle = \left[\int ds g(s) R_{26}^{s \rightarrow s_f} \right]^2 \langle(\Delta\hat{\delta})^2\rangle, \quad (11)$$

$$\langle\Delta x \Delta x'\rangle = \left[\int ds g(s) R_{16}^{s \rightarrow s_f} \right] \left[\int ds g(s) R_{26}^{s \rightarrow s_f} \right] \langle(\Delta\hat{\delta})^2\rangle. \quad (12)$$

69 Notice that because position and angular displacement are correlated,
 70 $\langle\Delta x \Delta x'\rangle^2 = \langle\Delta x^2\rangle\langle\Delta x'^2\rangle$, in the expression for the rms emittance (7) the
 71 last term vanishes, $\Delta\epsilon_x^2 = 0$.

72 2.2. Minimization of emittance growth

73 Eq. (7) suggests two possible strategies for controlling emittance growth:
 74 *i)* choosing $\alpha_x, \beta_x, \gamma_x$ appropriately; *ii)* minimizing (ideally set to zero) the
 75 second moments $\langle\Delta x^2\rangle$, $\langle\Delta x'^2\rangle$ and $\langle\Delta x \Delta x'\rangle$. The case *i)* is briefly treated
 76 in [14].

77 Here we discuss *ii)*. Emittance growth is avoided if, by choosing the optics
 78 appropriately, the following two integrals vanish ($i = 1, 2$)

$$\int ds g(s) R_{i6}^{s \rightarrow s_f} = 0. \quad (13)$$

79 These integrals vanish, in particular, if we can decompose the beamline $[s_i, s_f]$
 80 into two subsets $S_1 \cup S_2 = [s_i, s_f]$ in such a way that for each s_1 in S_1 there
 81 exists a unique $s_2 > s_1$ in S_2 so that

$$g(s_1) R_{i6}^{s_1 \rightarrow s_f} = -g(s_2) R_{i6}^{s_2 \rightarrow s_f}. \quad (14)$$

82 Having defined the vectors

$$\mathbf{r}_s = \begin{pmatrix} R_{16}^{s \rightarrow s_f} \\ R_{26}^{s \rightarrow s_f} \end{pmatrix}, \quad \boldsymbol{\eta}_s = \begin{pmatrix} \eta(s) \\ \eta'(s) \end{pmatrix}, \quad (15)$$

83 where $\eta(s)$ and $\eta'(s)$ are the dispersion function and derivative at s , it can
 84 be shown (Appendix A) that they are related by

$$\mathbf{r}_s = -\mathbf{R}_{2 \times 2}^{s \rightarrow s_f} \boldsymbol{\eta}_s, \quad (16)$$

85 where $\mathbf{R}_{2 \times 2}^{s \rightarrow s_f}$ is the 2×2 transfer matrix for the betatron motion from s to
 86 the exit s_f of the beamline. Suppose $g(s_1) = g(s_2)$, so that we can cast (14)
 87 in the form

$$-\mathbf{R}_{2 \times 2}^{s_1 \rightarrow s_f} \boldsymbol{\eta}(s_1) = \mathbf{R}_{2 \times 2}^{s_2 \rightarrow s_f} \boldsymbol{\eta}(s_2), \quad (17)$$

88 and consider two subcases:

89 1. $\boldsymbol{\eta}_{s_2} = \boldsymbol{\eta}_{s_1}$; naturally occuring in a periodic lattice. Eq. (17) becomes

$$-\mathbf{R}_{2 \times 2}^{s_1 \rightarrow s_f} = \mathbf{R}_{2 \times 2}^{s_2 \rightarrow s_f}. \quad (18)$$

90 On the other hand, the matrix in the LHS of (18) can be decomposed
 91 as

$$\mathbf{R}_{2 \times 2}^{s_1 \rightarrow s_f} = \mathbf{R}_{2 \times 2}^{s_2 \rightarrow s_f} \mathbf{R}_{2 \times 2}^{s_1 \rightarrow s_2}. \quad (19)$$

92 Inserting (19) in (18) we find:

$$\mathbf{R}_{2 \times 2}^{s_1 \rightarrow s_2} = -\mathbf{1}, \quad (20)$$

93 *i.e.* compensation is achieved by separating CSR kicks by π in the
 94 horizontal betatron phase.

95 2. $\boldsymbol{\eta}_{s_2} = -\boldsymbol{\eta}_{s_1}$; as, for example, in the FERMI dog-leg [9] at the end of
 96 the Linac consisting of two double-bend achromats with antisymmetric
 97 dispersion function with respect to the dogleg mid point. Repeating

the above steps, we conclude that the condition for vanishing emittance growth is now

$$\mathbf{R}_{2 \times 2}^{s_1 \rightarrow s_2} = \mathbf{1}. \quad (21)$$

In the FERMI transfer line consisting of a sequence of 4 dipoles, condition (21) is realized by setting a 2π phase advance from the 1st to 3rd and from the 2nd to 4th dipole.

We should point out that while these two subcases yield simple and practical recipes for designing transfer lines they do not exhaust all cases of interest. See Ref. [14] for an example of CSR compensation in a transport line that cannot be reduced to cancelation by set values of the betatron phase advances between binary kicks.

3. R-Matrix vs. ‘curly’- $\mathcal{H}$ formalism

Consider the simple scenario where the effect of CSR can be approximated as occurring as a single kick along the beamline localized over the short distance Δs at s as, for example, in the 4th bend of a standard bunch-compressor chicane where the beam is the shortest (and the CSR effect the largest) and approximately stationary.

We want show that the expression (7) for the emittance can be rephrased in terms of the dispersion invariant $\mathcal{H}$

$$\mathcal{H} = \eta_s^2 \gamma_x(s) + 2\alpha_x(s) \eta_s \eta'_s + \eta_s'^2 \beta_x(s). \quad (22)$$

Once again, we recognize that $\Delta \varepsilon_x^2 = 0$ and write the projected emittance as

$$\varepsilon_x^2 \simeq \varepsilon_{x0}^2 + \varepsilon_{x0} \left[\beta_x(s_f) [R_{26}^{s \rightarrow s_f}]^2 + 2\alpha_x(s_f) R_{16}^{s \rightarrow s_f} R_{26}^{s \rightarrow s_f} + \gamma_x(s_f) [R_{16}^{s \rightarrow s_f}]^2 \right] \sigma_{\Delta\delta}^2 \quad (23)$$

117 with $\sigma_{\Delta\delta}^2 = [g(s)\Delta s]^2 \langle (\Delta\hat{\delta}')^2 \rangle$ being the CSR-induced rms relative energy
 118 spread. In writing the above equation we emphasized that the relevant Twiss
 119 functions are those at the exit of the beam line at s_f . The second term on
 120 the RHS of (23) can be written in a more compact form as

$$\varepsilon_{x0} \mathbf{r}_s \cdot \mathbf{T}_{s_f}^{-1} \mathbf{r}_s \sigma_{\Delta\delta}^2 \quad (24)$$

121 where $\mathbf{r}_s$ was defined in (15) and $\mathbf{T}$ is the Twiss matrix

$$\mathbf{T} = \begin{pmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{pmatrix}, \quad (25)$$

122 with inverse

$$\mathbf{T}^{-1} = \begin{pmatrix} \gamma_x & \alpha_x \\ \alpha_x & \beta_x \end{pmatrix}. \quad (26)$$

123 We now use the relationship (16) to write

$$\begin{aligned} \mathbf{r}_s \cdot \mathbf{T}_{s_f}^{-1} \mathbf{r}_s &= \mathbf{R}_{2 \times 2}^{s \rightarrow s_f} \boldsymbol{\eta}_s \cdot \mathbf{T}_{s_f}^{-1} \mathbf{R}_{2 \times 2}^{s \rightarrow s_f} \boldsymbol{\eta}_s \\ &= \boldsymbol{\eta}_s \cdot [\mathbf{R}_{2 \times 2}^{s \rightarrow s_f}]^T \mathbf{T}_{s_f}^{-1} \mathbf{R}_{2 \times 2}^{s \rightarrow s_f} \boldsymbol{\eta}_s, \end{aligned} \quad (27)$$

124 where we have made use of the properties of the scalar product to move
 125 $\mathbf{R}_{2 \times 2}^{s \rightarrow s_f}$ from the left to the right of ‘ $\cdot$ ’. Next, it is a matter of recognizing that
 126 $\mathbf{T}$ propagates like the beam co-variance matrix

$$\mathbf{T}_{s_f} = \mathbf{R}_{2 \times 2}^{s \rightarrow s_f} \mathbf{T}_s [\mathbf{R}_{2 \times 2}^{s \rightarrow s_f}]^T \quad (28)$$

127 OR

$$\mathbf{T}_{s_f}^{-1} = [\mathbf{R}_{2 \times 2}^{s \rightarrow s_f}]^{-T} \mathbf{T}_s^{-1} [\mathbf{R}_{2 \times 2}^{s \rightarrow s_f}]^{-1} \quad (29)$$

128 to write (27) as

$$\mathbf{r}_s \cdot \mathbf{T}_{s_f}^{-1} \mathbf{r}_s = \boldsymbol{\eta}_s \cdot \mathbf{T}_s^{-1} \boldsymbol{\eta}_s = \eta_s^2 \gamma_x(s) + 2\alpha_x(s) \eta_s \eta'_s + \eta_s'^2 \beta_x(s). \quad (30)$$

129 In conclusion, the beam emittance at the exit of the dispersive region is
 130 equivalent to

$$\varepsilon_x^2 = \varepsilon_{x0}^2 + \varepsilon_{x0} \mathcal{H}(s) \sigma_{\Delta\delta}^2. \quad (31)$$

131 It is worth noticing that in contrast to Eq. (23), where the Twiss functions
 132 are relative to the observation point s_f at the exit of the beamline, in (31)
 133 the Twiss functions entering into the expression for the dispersion invariant
 134 $\mathcal{H}$ are at the position s of the CSR kick.

135 In more general cases where CSR is not highly concentrated the expression
 136 of the emittance growth in terms of the dispersion invariant is less accurate
 137 and Eq. (23) should be used instead. We stress that one should not be
 138 tempted to integrate the RHS of (31) over s as a way to get an accurate
 139 estimate of CSR emittance growth contributed by the whole beamline of
 140 interest.

141 4. Conclusions

142 We have reviewed the linear R-matrix formalism describing CSR-induced
 143 emittance growth in transport lines and recovered lattice design prescriptions
 144 yielding vanishing emittance growth for lattice with periodic or antisymmet-
 145 ric dispersion functions. Finally, we established the connection between a
 146 description of the emittance growth in terms of the R-matrix and the one in
 147 terms of the dispersion invariant.

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153 **6. References**

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181 **Appendix A.**

182 The goal in this Appendix is to prove Eq. (16). The most general linear
183 matrix for transport through a sequence of bends, quads, and drifts has the
184 form

$$\mathbf{R} = \begin{pmatrix} R_{11} & R_{12} & 0 & R_{16} \\ R_{21} & R_{22} & 0 & R_{26} \\ R_{51} & R_{52} & 1 & R_{56} \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (\text{A.1})$$

185 Incidentally, notice that symplecticity implies

$$\begin{pmatrix} R_{16} \\ R_{26} \end{pmatrix} = - \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} R_{52} \\ -R_{51} \end{pmatrix} = -\mathbf{R}_{2 \times 2} \begin{pmatrix} R_{52} \\ -R_{51} \end{pmatrix}, \quad (\text{A.2})$$

186 where the second equality defines $\mathbf{R}_{2 \times 2}$. The inverse reads

$$\mathbf{R}^{-1} = \begin{pmatrix} R_{22} & -R_{12} & 0 & R_{52} \\ -R_{21} & R_{11} & 0 & -R_{51} \\ -R_{26} & R_{16} & 1 & -R_{56} \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (\text{A.3})$$

187 If the beamline $[s_i, s_f]$ is an achromat, by definition the entries $R_{16}^{s_i \rightarrow s} \equiv$
 188 $\eta(s)$ and $R_{26}^{s_i \rightarrow s} \equiv \eta'(s)$ where $s \in [s_i, s_f]$, identify the dispersion function and
 189 its derivative. We are interested in finding an expression for the entries $R_{16}^{s \rightarrow s}$,
 190 and $R_{26}^{s \rightarrow s}$ in terms of $\eta(s)$ and $\eta'(s)$.

191 As the longitudinal dynamics is inconsequential in this calculation, we
 192 rewrite the transfer matrix $\mathbf{R}$ from s_i to s as a 3×3 matrix

$$\mathbf{R}^{s_i \rightarrow s} = \begin{pmatrix} \mathbf{R}_{2 \times 2}^{s_i \rightarrow s} & \boldsymbol{\eta}_s \\ 0 & 1 \end{pmatrix} \quad (\text{A.4})$$

193 where $\boldsymbol{\eta}_s = (\eta(s), \eta'(s))$ is the vector defined in (15). The inverse of this
 194 matrix $\mathbf{R}^{s_i \rightarrow s}$ is [see (A.2) and (A.3)]:

$$[\mathbf{R}^{s_i \rightarrow s}]^{-1} = \begin{pmatrix} [\mathbf{R}_{2 \times 2}^{s_i \rightarrow s}]^{-1} & -[\mathbf{R}_{2 \times 2}^{s_i \rightarrow s}]^{-1} \boldsymbol{\eta}_s \\ 0 & 1 \end{pmatrix}. \quad (\text{A.5})$$

195 Next, we write the matrix from s to s_f ; using the definition of $\mathbf{r}_s$ from
 196 Eq. (15) we have

$$\mathbf{R}^{s \rightarrow s_f} = \begin{pmatrix} \mathbf{R}_{2 \times 2}^{s \rightarrow s_f} & \mathbf{r}_s \\ 0 & 1 \end{pmatrix}. \quad (\text{A.6})$$

197 Combining the two matrices from s_i to s and from s to s_f we have $\mathbf{R}^{s_i \rightarrow s_f} =$
 198 $\mathbf{R}^{s \rightarrow s_f} \mathbf{R}^{s_i \rightarrow s}$ and therefore

$$\mathbf{R}^{s \rightarrow s_f} = \mathbf{R}^{s_i \rightarrow s_f} [\mathbf{R}^{s_i \rightarrow s}]^{-1}$$

$$\begin{aligned}
&= \begin{pmatrix} \mathbf{R}_{2 \times 2}^{s_i \rightarrow s_f} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} [\mathbf{R}_{2 \times 2}^{s_i \rightarrow s}]^{-1} & -[\mathbf{R}_{2 \times 2}^{s_i \rightarrow s}]^{-1} \boldsymbol{\eta}_s \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{R}_{2 \times 2}^{s_i \rightarrow s_f} [\mathbf{R}_{2 \times 2}^{s_i \rightarrow s}]^{-1} & -\mathbf{R}_{2 \times 2}^{s_i \rightarrow s_f} [\mathbf{R}_{2 \times 2}^{s_i \rightarrow s}]^{-1} \boldsymbol{\eta}_s \\ 0 & 1 \end{pmatrix}. \tag{A.7}
\end{aligned}$$

199 Comparing the two RHSs of (A.6) and (A.7) we conclude

$$\begin{aligned}
\mathbf{r}(s) &= -\mathbf{R}_{2 \times 2}^{s_i \rightarrow s_f} [\mathbf{R}_{2 \times 2}^{s_i \rightarrow s}]^{-1} \boldsymbol{\eta}_s \\
&= -\mathbf{R}_{2 \times 2}^{s \rightarrow s_f} \mathbf{R}_{2 \times 2}^{s_i \rightarrow s} [\mathbf{R}_{2 \times 2}^{s_i \rightarrow s}]^{-1} \boldsymbol{\eta}_s \\
&= -\mathbf{R}_{2 \times 2}^{s \rightarrow s_f} \boldsymbol{\eta}_s, \tag{A.8}
\end{aligned}$$

200 the same as Eq. (16).