HIGH INTENSITY MUON BEAM SOURCE FOR NEUTRINO BEAM EXPERIMENTS

Hisham Kamal Sayed^a

^aBrookhaven National Laboratory

Abstract

High intensity muon beams are essential for Muon accelerators like Neutrino Factories and Muon Colliders. In this study we report on a global optimization of the muon beam production and capture based on end-to-end simulations of the Muon Front End. The study includes the pion beam production target geometry, capture field profile, and forming muon beam into microbunches for further acceleration. The interplay between the transverse and longitudinal beam dynamics during the capture and transport of muon beam is evaluated and discussed. The goal of the optimization is to provide a set of design parameters that delivers high intensity muon beam that could be fit within the acceptance of a muon beam accelerator.

Keywords: muon beam, neutrino beam, high power target.

1. INTRODUCTION

High intensity muon beams are produced from decays of pions, which are produced as secondary beam from a primary high power proton beam hitting a pion production target. When protons hit the target material, the nucleon - nucleon interactions are responsible of pions production. The pion production is dominated by the single and double production processes given by (1). Later pions decay into muons and neutrinos.

The baseline design for the Front End of a Neutrino Factory and Muon Collider consists of a five major components, namely the Target System, Decay Channel, Buncher, Phase Rotator, and the Ionization Cooling Channel. Figures (1) and (2) show schematic drawings of the target station and the front end [1, 2]. Although each of the mentioned systems has a complex design which is optimized for the best performance with its own set of local objectives, the integration of all of them into one overall system requires a global optimization to insure the effectiveness of the local objectives and overall performance. This global optimization represents a highly constrained multi-objective optimization problem. The objectives are the number of muons captured into stable bunches of specified transverse and longitudinal emittances, as constrained by the momentum- and dynamic- acceptance of the subsequent acceleration systems in addition to the overall cost. A multi-objective global evolutionary algorithm is employed to address such a challenge.

The efficiency of the pion beam production has a strong dependence on the choice of the target material and target geometry. Whereas the efficiency of capturing the secondary pions and the subsequent muons depend on the capture field profile. In this study we report on a new target design parameters based on a global optimization of the target geometry and the capture

Figure 1: Schematic of a high power Muon Accelerator Target station.

field profile. The key parameters considered in this study include the proton beam and target geometrical parameters, the capture and transport field, the Buncher and Energy Phase Rotator phases, frequencies, and gradients, and finally the broadband match to the ionization cooling channel.

A parallelized evolutionary optimization algorithm was deployed on a CRAY based supercomputer to find the optimal design parameters. The goal of this study is to find the optimal parameters within the feasible limits of operation of high power targets and maximum achievable magnetic fields. In this study the statement of optimization strategy is discussed along with results of the optimization.

Email address: hsayed@bnl.gov (Hisham Kamal Sayed)

2. MUON BEAM PRODUCTION AND CAPTURE

The target station of the muon beam production is designed to accommodate a target rod made of graphite. The target is surrounded by a set of superconducting coils which provide a peak field of 15 T in addition to a normal conducting resistive magnet providing a 5 T field [3, 4]. Initial design of the Neutrino Factory target station proposed a Hg jet target. In this work we explore a staged approach that first uses graphite target for 1-2 MW and later uses mercury for higher beam power 3-4 MW. The proton driver is providing 10^{15} protons per second with bunch frequency of 60 Hz.

The advantage of solenoid capture over horn based collection is that they capture both positively and negatively charged pions at the same time. Target solenoid captures charged particles with broad momentum range, and its capture efficiency is only limited by the transverse momentum of captured pions. Target solenoids will capture any charged particle with transverse momentum $\sigma_{P_T} < eBa/2c$, where a is the aperture size at the target location. Most of the secondary pions are produced with $P_z \approx 100$ MeV/c and $P_T \approx 260$ MeV/c, which requires strong capture field in the range of 20 T. Figure 3 shows the transverse phase space distribution of secondary pions at the end of the target material.

Figure 2: Schematic of the Front end of the Muon Accelerator.

After the particles leave the tapered target solenoid they are transported in the Decay Channel, Buncher, and Energy Phase Rotator in a constant solenoid field. A high frequency Buncher and Energy Phase Rotator were first realized for muon beams on [5] and later adapted in [6].

At the end of the Decay Channel, most pions have decayed into muons and the muon bunch length is about 15 m long. The beam is then bunched into a sequence of microbunches using a set of RF cavities that capture muons with kinetic energy ranging from 50-400 MeV. The bunching cavities RF voltage increases adiabatically along the channel where their frequencies decrease. The RF frequencies and gradients are used

Figure 3: Transverse phase space of secondary beam at the end of the target.

as free parameters in the optimization process. Following the RF Buncher an Energy Phase Rotation Channel is used, where lower energy muons are accelerated and high energy ones are decelerated, until at the end of the Rotator all the bunches have the same central momentum, and the original long bunch of muons of both signs has been formed into a series of microbunches with 21 bunches of μ^+ interleaved with 21 bunches of μ^- . The muon beam is then matched into the alternating, 2.8-T solenoid field in the Ionization Cooling Channel.

The ionization cooling channel consists of 70 cells. Each cell contains four 325 MHz RF cavities for acceleration and LiH blocks for ionization cooling. A series of focusing coils providing ± 2.8 T sinusoidal field. The cooling channel provides a transverse emittance reduction by a factor of 2.5 of the bunched muon beam.

3. OPTIMIZATION ALGORITHM WITH MULTI-LEVEL OF PARALLELISM

A multi-objective global differential evolution algorithm [7] was utilized to optimize the performance of the muon source Front End. Due to the stochastic nature of the pion beam production process and the energy loss in the Ionization Cooling Channel, a tracking of large number of initial particles ($> 10^6$) has to be carried out to limit the statistical fluctuations from influencing the optimization process.

In order to be able to perform such tracking the average evaluation of one run may take up to few hours. We utilized a parallel MPI [8, 9, 10] tracking code which reduces the running time of each run to 2 minutes. The heuristic optimization algorithms usually require large number of cost function evaluations before converging, the number of cost function evaluations has a dependence on the number of variables and initial population size. For a complicated optimization task that we are considering in this study it may take few days of running to reach a set of optimal solutions. To overcome this problem we developed a two layers of parallelism algorithm, where the optimization algorithm runs in parallel mode and each function evolution is evaluated using MPI - parallel Monte-Carlo tracking code.

Figure 4: (a,b) Impact of initial proton beam size and angle on the muon yield. (c,d) impact of the target length and radius on the muon yield.

In previous efforts [10] we were able to implement an integrated MPI-code where the control of parallel cores was managed by the MPI-optimization algorithm and the tracking code was called as an MPI function. Those efforts proved to be tedious in terms of code management and limited capability to run various codes for each runs (*e.g.*, running MARS [11, 12] or GEANT4 [13] for particle production and ICOOL for tracking).

To solve this problem, we separated the optimization task into three separate blocks: the first block is the optimization algorithm which generates an array of size $n \times m$, where n is the number of variables and m is the population size. Then this array is passed to the second block where it launches a set of m MPI jobs. Each job runs independently where the second block has to wait for all of the MPI-jobs to finish before collecting the results and sending an array of size m back to the optimization algorithm for processing and generating the second batch of n variables. For the algorithm to be robust we implemented a technique to detect failed function evaluations and either discard the result or if crucial it repeats the function evaluation.

The optimization process that we will adopt in the study consist of two parts, the local system optimization followed by a

global optimization of the whole front end.

We first start by setting an objective for every individual system of the Front End and optimize the performance without chaining it to the rest of the Front End. Then we optimize the whole front end using the key parameters of each channel to improve the integration between those channels.

4. LOCAL SYSTEM OPTIMIZATION

In this section we will consider each channel independently and optimize its performance. We will start by the proton driver beam and target parameters, followed by the capture magnetic field, Buncher/Rotator, and finally the broadband match to ionization cooling channel.

4.1. Optimization Of The Pion Production Target Geometry

The pion beam production off the graphite target was simulated using Geant4 [13] - QGSP-BERT [14]. A more detailed comparison of the production models and the experimental data is given in [15].

Table 1: Proton beam and graphite target geometrical parameters.

Target Parameter	Unit	Optimal Working Point
Target Rod Length	m	0.8
Target Radius	cm	1.0
Angle to solenoid axis	degree	2.4
Proton beam size	mm	0.2
Target Aperture size	cm	23.0
End Field	T	7
Taper Length	m	25.0

The optimization algorithm was launched to optimized the target geometrical parameters which included the target length, radius, and angle to solenoid axis. In addition to the proton beam size at the interaction point with the target was also considered as a parameter in the optimization. The interaction of a 6.75 GeV proton beam with the graphite target rod was simulated using Geant4 [13]. The simulation of pion beam production was done while most of the graphite target rod was surrounded by 20 T focusing solenoid field that tapers down to 2.0 T over a length of 5 m. The objective of the optimization was to increase the number of muons with longitudinal momenta $p_z < 150$ MeV/c and transverse momenta of $p_t < 450$ MeV/c at the end of Decay Channel 70 m downstream from the target. Both of those momenta values were chosen because they represent the limits of acceptance of most of the Muon accelerator front end designs [16].

Although all of the parameters of the optimization was changed simultaneously during the optimization run, the dependence of the useful muon yield on each of the optimization parameters could be shown in Fig. 4. Table 1 presents the optimal parameters for the target geometry.

Figure 5: On-axis magnetic field profiles for the Target System for $B_i = 20$ T and $B_f = 15$ T.

4.2. Optimization Of The Capture Field And Adiabatic Match

The target capture solenoid field peaks at the target location then it should be adiabatically matched to a lower field solenoid channel for transport to the rest of the accelerator. The tapered field is generated by the function

$$B_z(0, z) = \frac{B_i}{1 + a_1(z - z_i) + a_2(z - z_i)^2 + a_3(z - z_i)^3} \quad (2)$$

$$a_1 = -B'_i/B_i \quad (3)$$

$$a_2 = 3 \frac{B_i - B_f}{B_f L_t^2} - \frac{2a_1}{L_t} \quad (4)$$

$$a_3 = -2 \frac{B_i - B_f}{B_f L_t^3} - \frac{a_1}{L_t^2}, \quad (5)$$

where L_t is the taper length, B_i is the field at longitudinal position $z = z_i$ and B_f is the field at $z = z_i + L_t$. This expression was only used for $z < z_i + L_t$; for $z \geq z_i + L_t$, the field is B_f . The longitudinal (B_z) and radial (B_r) field at a radius r is computed from the series expansion

$$B_z(r, z) = \sum_n (-1)^n \frac{B_z^{(2n)}(0, z)}{(n!)^2} \left(\frac{r}{2}\right)^{2n} \quad (6)$$

$$B_r(r, z) = \sum_n (-1)^{n+1} \frac{B_z^{(2n+1)}(0, z)}{(n+1)!n!} \left(\frac{r}{2}\right)^{2n+1}, \quad (7)$$

Where $B_z^{(k)}(0, z)$, is the k th derivative of $B_z(0, z)$ with respect to z .

The optimization included the end field strength and the field tapering length, see Fig. 5 for examples of such fields. In order to improve the capture efficiency of the particles at the target, the peak field has to be in order of ≈ 20.0 T or more. We will limit our studies to 20.0 T maximum peak field, as fields higher than that are beyond the current technological capabilities of operating super-conducting magnets in high radiation areas.

The acceptance of the Decay Channel depends on the physical aperture and field strength $\propto Ba^2$ given that we assume a constant field strength throughout the Decay Channel Fig. 6 shows the number of accepted muons (counted at the end of the Decay Channel) over a range of Decay Channel focusing fields for a fixed aperture of 30 cm. Notice that the number of captured muons saturates at 7 T. In this study both the target and the taper parameters were changed to find the optimal transmission.

In the simulation setting we used to study the dependence of the performance on the end field we have assumed a constant focusing field for the whole Decay Channel, Buncher, and Energy Phase Rotator. In reality we will use a series of coils that will produce a fluctuating field strength. A charged particle moving in a periodically varying magnetic field might suffer a non-stable motion for certain momenta *i.e.* stop bands. The equations of motion of a charged particle in solenoid field are given by

$$\ddot{r} - r\dot{\phi}^2 = \frac{er\dot{\phi}B_z}{m\gamma}, \quad (8)$$

$$\frac{m\gamma}{r}(r^2\ddot{\phi} + 2r\dot{r}\dot{\phi}) = -e\dot{r}B_z. \quad (9)$$

For a motion with neglected azimuthal acceleration $\dot{\phi} = -\frac{eB_z}{2m\gamma}$, and the radial equation of motion becomes

$$\ddot{r} + \frac{r}{4} \left(\frac{eB_z}{m\gamma} \right)^2 = 0. \quad (10)$$

Figure 6: Dependence of the muon yield on the end field. Notice that the capture efficiency saturates at 7 T. Muons are counted at the end of Decay Channel 70 m downstream of the target.

For a periodically varying magnetic field $B_z(z) = A \cos(\omega_a z) + B \sin(\omega_b z)$ where A, B, a , and b are arbitrary coefficients, a stable periodic solution only exists for certain momenta. The instability of the particle motion in a varying field will eventually lead to a reduction of particle transmission through the channel. We optimize the design of the periodic solenoid channel to enable a good transmission for particles within momentum range of interest 100 - 500 MeV/c. The transmission of the periodic channel was assisted by tracking a number of particles with zero amplitudes and varying initial momenta. We considered a realistic triplet super-conducting coil configuration. The coils spacing and currents were optimized to provide the required on-axis field and eliminate the stop-bands losses within the momentum range of interest. Fig. 7 shows the super conduction coil triplets used in the Decay Channel, Buncher, and Energy Phase Rotator. The optimization of the coils settings reduced the particle losses due to resonances by 20%.

4.3. Formation of Microbunches

The initial pion beam off the target has a large momentum spread that is not suitable for RF acceleration. The Buncher section form the muon bunch into a series of microbunches matched to 325 MHz RF. After the target pions are left to drift in the Decay Channel where they decay into muons, and most importantly develop an energy time correlations, the muon bunch now has a high energy head and low energy tail. The temporal separation between two arbitrary particles with the same initial position $z=0$ and different velocities v_i and v_j after traveling a distance z is

Figure 7: A cell structure for the Buncher showing dimensions of three coils providing on axis field of 2-8 T.

$$\delta t = z \left(\frac{1}{v_j} - \frac{1}{v_i} \right) \quad (11)$$

The Buncher RF cavities phases are set to zero crossing and the voltage is increased gradually to form the long beam into a series of microbunches. The two selected particles will always remain at the zero crossing if the wavelength of each cavity is set to match the temporal difference between them *i.e.* $\lambda_{RF}(z) = z \delta \left(\frac{1}{\beta} \right)_{ij}$. This is accomplished by gradually decreasing the RF frequencies to maintain this condition.

The choice of the momentum band for the Buncher to capture and the required Energy - time correlations are subject to optimization. The initial pion beam is produced with non-homogenous distribution in longitudinal phase space see Fig. 8. We optimize the Decay Channel length, the choice of capture momentum band, and the RF gradient ramp to improve the microbunching efficiency.

After the Buncher the microbunches are formed with inherited difference in their central energies. The Energy Phase Rotator function is to set all the microbunches to equal central energies. This is accomplished by changing the RF wavelength to accelerate the low energy bunches and decelerate the high energy ones. At the end of the Phase Rotator all the bunches share the same central energy and matching to a fixed frequency Ionization Cooling Channel is now possible.

In order to automate the process of Buncher and Rotator optimization two quadratic generator functions (Eqs. 12, 13) were implemented to generate both the RF frequencies Φ and the gradient V . This provide a total of 5 parameters ($\kappa_1, \kappa_2, \tau_1, \tau_2, \tau_3$) that are used to optimize the captured muon momentum band. Here the third parameter of the frequency function κ_3 is fixed to provide the match to the 325 MHz Cooling Channel,

$$\Phi(n) = \kappa_1 \xi^2 + \kappa_2 \xi + \kappa_3 \quad (12)$$

$$V(n) = \tau_1 \xi^2 + \tau_2 \xi + \tau_3 \quad (13)$$

Figure 8: Longitudinal phase space of pion beam at the end of the target solenoid (left) and after formation of microbunches at the end of the Buncher (right).

Table 2: Optimized parameters for generating the Buncher and Rotator frequencies and gradients.

Parameter	Optimal Working Point
κ_1	0.2055×10^6
κ_2	-12.745×10^6
τ_1	0.00105×10^6
τ_2	1.6975×10^6
τ_3	-2.204×10^6

As the bunches are matched transversely and longitudinally to the ionization cooling channel, the optimal working point for the Buncher and energy phase Rotator will change to improve the performance of the ionization cooling channel. Table 2 show the locally optimized parameters for the buncher and phase rotator.

Figure 9: Muon beam momentum distribution at the end of the Phase Rotator.

4.4. Broadband Match to Ionization Cooling Channel

While the muons are confined with a constant solenoid field starting from the Decay Channel to the end of the Rotator, the field in the Cooling Channel is generated by alternating

Figure 10: Broad-band matching coils from constant solenoid channel (left) to a sinusoidal alternating ± 2.8 field (right).

solenoid coils of ± 2.8 T strength. The muon-beam phase-space ellipse would experience filamentation and emittance growth without proper matching between the regions of constant and alternating solenoid fields. The muon beam out of the Phase Rotator has a large momentum spread, as shown in Fig. 9, which makes the transverse matching of the muon beam more strenuous. A set of 9 solenoid coils (see Fig. 10) were introduced to perform the match between the Rotator and the Cooling Channel. The objective of this optimization was constructed as the sum of the emittance magnification factor [17], Equ. (14), at various momenta weighted by the number of muons within the vicinity δp of a central momentum $\Sigma_i n_i M_{\epsilon,i}$.

$$M_{\epsilon} = \frac{1}{2} \left[\frac{\beta_b}{\beta_s} + \frac{\beta_s}{\beta_b} + \left(\sqrt{\frac{\beta_b}{\beta_s}} \alpha_s - \sqrt{\frac{\beta_s}{\beta_b}} \alpha_b \right)^2 \right], \quad (14)$$

Where β_b and α_b are the beam ellipse parameters and β_s and α_s are calculated from the periodic solution for the motion in the Cooling Channel alternating field.

5. GLOBAL OPTIMIZATION OF THE MUON FRONT END

The performance of the whole Muon Front End after putting the pieces together is now to be optimized by tracking a large number of particle ($> 10^6$) and evaluating the performance in terms of the total number of muons that was captured, bunched and cooled to fit within the acceptance of the Muon Accelerator complex. A global optimization of the Front End parameters was launched which includes: the proton beam driver and target geometrical parameters, capture field parameters, phases of

(a) Buncher and Rotator RF frequencies. The first cavity has a frequency ≈ 500 MHz and the frequencies are gradually decreased to match to the 325 MHz cavities in the ionization cooling channel.

(b) Buncher and Rotator RF gradients. The RF gradients are gradually increased in the Buncher to adiabatically capture the muon beam into a series of 325 MHz bunches. In the energy phase rotation all cavities have a fixed gradient of 20.0 MV/m

Figure 11: The Buncher and Energy Phase Rotator RF cavities frequencies and gradients.

Table 3: Pion Production Target and Capture Field global optimal parameters.

Target Parameter	Unit	Optimal Working Point
Target Aperture size	cm	12.5
End Field	T	4.3
Taper Length	m	5.15

the Buncher/Rotator RF cavities, matching coils to Ionization Cooling Channel, and finally the phase of Ionization Cooling Channel RF cavities. The results parameterized in terms of the end field throughout the Front End are shown in Fig. 12.

In the following we will discuss only the systems which yielded a different optimal parameters than the ones which were given in the local system optimization.

The initial aperture at the target location was found to hit an asymptote at 12.5 cm. Which implies that particles that were captured with larger amplitude at the target were not transported

Table 4: Optimized parameters for generating the Buncher and Rotator frequencies and gradients.

Parameter	Optimal Working Point
κ_1	0.2344×10^6
κ_2	-12.821×10^6
τ_1	0.15798452×10^6
τ_2	0.0265232×10^6
τ_3	4.1428628×10^6

to the downstream channels. End field higher than 4 T did not turn to improve the yield. This is explained as the higher end field will increase the transverse - longitudinal coupling in addition the higher end field will confine large amplitude particles which cannot be cooled to within the required acceptance in the ionization cooling channel.

The transverse - longitudinal coupling due to the helical motion in solenoid channel leads to an increase in the bunch length. This effect is more prominent for large amplitude particles (which is the case for pions coming off the target).

The bunch lengthening due to the helical motion in strong focusing field will work against the improved acceptance of the higher field and eventually the optimal working parameter for both the end field and the capture field tapering length was found, see Table 3.

The optimized parameters for generating the Buncher and Energy Phase Rotator frequencies and gradients are given in Table 4 and the resulting frequencies and gradients are shown in Fig. 11. It should be noted that the optimal Buncher and Energy Phase Rotation gradients and frequencies are different from the locally optimized design, as it is integrated to ionization cooling channel.

Figure 12: Number of Positive muons within the acceptance cuts of the Muon accelerator with various end fields through the Front End.

6. CONCLUSION

We presented a detailed study of High intensity Muon Source optimization based on end-to-end simulation. A global optimization algorithm with multi-level of parallelism was developed and discussed. A first application of this robust algorithm was adopted for each separate channel of the Muon Front End for local system optimization. Then a global optimization based on end-to-end tracking with large number of particles was launched. The overall performance of the Front End was discussed, and the distinction between optimal parameters of local versus global optimization was explained.

The advantage of the global optimization technique is that it found an optimal working parameters which was not achievable by local optimizations of various channels before integration. The global optimization algorithm took about 10000 CPU computing hours to converge. Using the parallel processing features made the Wall time (real time) within 10 hours on NERSC supercomputer.

Human led optimization processes used to take weeks of scanning various parameters and running the Monte- Carlo simulations. The differential evolution optimization algorithm with multilayers of parallelism and coupled with Monte - Carlo simulation reduced the optimization process to within 10 hours.

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