Transformed Contribution Ratio Test for the Number of Factors in [Static Approximate Factor Models](#)

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Abstract

Determining the number of factors (r) is of importance in static approximate factor models. Under some mild conditions, the r largest eigenvalues of the variance matrix of N response variables go to infinity as N increases, while the rest are bounded. Then, “Eigenvalue Ratio” (ER) and “Growth Ratio” (GR) estimators have been well exploited by maximizing the ratio of two adjacent eigenvalues in the literature. As a modification of ER and GR estimators, the new estimator named as “Transformed Contribution Ratio” (TCR) is obtained by maximizing the ratio of two adjacent transformed contribution of the eigenvalues. Under the same conditions of ER and GR estimators, the resulted estimator can be proved to be consistent. It can be further shown that, comparing with the competitors in the existing literature, the new method has desired performance on truly selecting the value of the number of latent common factors, [especially when both strong and weak factors or some dominant factors are in static approximate factor models](#). Monte Carlo simulation experiments and [one real data application](#) are carried out for illustration.

Keywords: Approximate factor model, Eigenvalues, Number of factors, Transformed contribution ratio

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1. Introduction

In the analysis of large dimensional factor models, one of fundamental issues is the consistent determination of the number of common factors. Recently, attempts in this direction have been focused on this issue in economic and financial data with both large cross-section dimension (N) and time series observations (T). Many examples include but not limited to Bai and Ng (2002), Stock and Watson (2002), Onatski (2006, 2010), Alessi et al. (2010), Fan et al. (2013) and Ahn and Horenstein (2013) for static approximate factor models of Chamberlain and Rothschild (1983). Pan and Yao (2008), Lam and Yao (2012), Xia et al. (2015) focused on the determination of the number of factors for high-dimensional time series. Much of the other related econometric research, such as Forni et al. (2000), Hallin and Liska (2007), Amengual and Watson (2007), Bai and Ng (2007), and Onatski (2009), among others, were papers on dynamic factor models.

These estimators, however, are not directly comparable as they are based on their different sets of hypothesis. Meanwhile, under the same predetermined conditions, various methods can lead to different estimation results for the number of factors. In this paper, we will improve eigenvalue-based ratio-type method for the number of common factors suggested by Ahn and Horenstein (2013) in static approximate factor models, the improved method can be expected to have desired performance comparing with the existing methods in the literature. Up to our knowledge, Bai and Ng (2002) proposed to estimate the number of factors in static approximate factor models by minimizing the information criteria of model selection, named PC and IC. But the criteria of Bai and Ng (2002) are known to overestimate in practice. Onatski (2006) obtained a consistent estimator of the $(r + 1)$ th largest eigenvalue and then the corresponding threshold value can be found easily only if it is slightly larger than the $(r + 1)$ th largest eigenvalue, where r is the true number of common factors. However, the method is available only under the assumption that the idiosyncratic errors are either autocorrelated or cross-sectionally correlated, but not both (Ahn and

Horenstein, 2013). Onatski (2010) considered the so-called “Edge Distribution” estimator by using the differenced eigenvalues. Alessi et al. (2010) introduced a tuning multiplicative constant in the penalty to improve the method of Bai and Ng (2002), an idea that was proposed by Hallin and Liska (2007) in the context of dynamic factor models. As reported by Ahn and Horenstein (2013), the methods proposed by Bai and Ng (2002) and Onatski (2010) have worse finite sample properties in the case with cross-sectional dependency although they do perform well in the case with independent idiosyncratic errors. Two eigenvalues-based ratio-type estimators of Ahn and Horenstein (2013) were proposed and shown to perform well even when the idiosyncratic errors are cross-sectionally dependent or serially correlated. Lam and Yao (2012) used the similar idea to deal with the factor modeling for high-dimensional time series based on the dimension-reduction viewpoint. As argued by Ahn and Horenstein (2013), almost all the above estimators need to predetermine the possible maximum of the number of factors, which may lead to overestimation or underestimation of the number when the maximum (hereafter, denoted by $kmax$) is too large or too small. But, the ER and GR estimators are not sensitive to $kmax$ by the simulation experiments of Ahn and Horenstein (2013).

This paper proposes a determination approach by improving eigenvalue-based ratio-type method for the number of common factors in static approximate factor models. According to Ahn and Horenstein (2013), the ER estimator performs well under the cases of strong factors, while the GR estimator outperforms the ER estimator in the case when both strong and weak factors or some dominant factors are in static approximate factor models. The new method named as “Transformed Contribution Ratio” (hereafter TCR) is based on the ratio values of adjacent transformed contribution of eigenvalues arranged in descending order, and expected to have both advantages of the ER and GR estimators. Under some mild conditions, the resulted estimator can be proved to be consistent. It can be further shown that, comparing with the competitors in the existing literature, the new estimator has desired performance on truly selecting the value of the number of latent common factors, especially when

both strong and weak factors or some dominant factors are in static approximate factor models. Monte Carlo simulation experiments and one real data application are carried out for illustration.

65 The rest of this paper is organized as follows. In Section 2, we introduce the new method of determining the number of common factors in the approximate factor models and state its asymptotic consistency. Section 3 carries out some simulation experiments to examine the finite sample performance of the new estimation. One real data analysis for illustration are given in Section 4.
70 Section 5 provides some concluding comments. Technical details on the proofs of theorems are described in the Appendix.

Throughout this paper, A' means the transpose of the matrix A , $\|A\| = [\text{trace}(A'A)]^{1/2}$ denotes the norm of the matrix A , and $\psi_i(\Omega)$ denotes the i th largest eigenvalue of a positive semi-definite matrix Ω .

75 2. Estimation of the Number of Factors

Consider the following static approximate factor model of Chamberlain and Rothschild (1983),

$$y_{\cdot t} = \Lambda F_t + u_{\cdot t}, \quad t = 1, 2, \dots, T, \quad (1)$$

where $y_{\cdot t} = (y_{1t}, \dots, y_{Nt})'$, F_t is an r -dimensional vector of common factors, $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)'$, λ_i is an r -dimensional vector of factor loadings, and $u_{\cdot t} =$
80 $(u_{1t}, \dots, u_{Nt})'$ is the vector of the idiosyncratic error. The number of factors (r) is unknown and needs to be estimated. For the sake of statement, we denote that $Y = (y_{\cdot 1}, \dots, y_{\cdot T})'$ is the $T \times N$ observation matrix with the element y_{it} , $F = (F_1, F_2, \dots, F_T)'$, and $U = (u_{\cdot 1}, \dots, u_{\cdot T})'$ is the $T \times N$ idiosyncratic error matrix with the element $u_{it}, i = 1, \dots, N, t = 1, \dots, T$. And then, model (1)
85 can be rewritten as the following matrix form

$$Y = F\Lambda' + U. \quad (2)$$

Let

$$\tilde{\mu}_{NT,i} =: \psi_i(Y'Y/(NT)) = \psi_i(YY'/(NT)) \quad (3)$$

denote the i -th largest eigenvalue of $Y'Y/(NT)$. Ahn and Horenstein (2013) found that the first r eigenvalues are $O_p(1)$ and the rest are at most $O_p(\frac{1}{m})$, and then proposed the eigenvalue-based ratio-type estimators, i.e., ER and GR, to determine the number of common factors as follows

$$\hat{r}_{ER} = \arg \max_{1 \leq i \leq kmax} \frac{\tilde{\mu}_{NT,i}}{\tilde{\mu}_{NT,i+1}}, \quad \hat{r}_{GR} = \arg \max_{1 \leq i \leq kmax} \frac{\ln(V_{i-1}) - \ln(V_i)}{\ln(V_i) - \ln(V_{i+1})} \quad (4)$$

where $m = \min\{N, T\}$ and $M = \max\{N, T\}$, $V_i = \sum_{j=i+1}^m \tilde{\mu}_{NT,j}$, and $kmax$ is the predetermined possible maximum value of the number of factors.

Ahn and Horenstein (2013) argued that the two objective functions in (4) at the true value point of the number of factors are $O_p(m)$ and the others $O_p(1)$, and then the eigenvalue ratio-based estimators can be proven to be consistent when m is sufficiently large. In practice, however, when not all factors are strong, the number of common factors is not detected accurately and can be underestimated. In order to avoid this problem, we can use the modified objective function to replace those of (4) suggested by Ahn and Horenstein (2013), and then obtain the estimator $\hat{r}_{TCR}$ as follows.

$$\hat{r}_{TCR} = \arg \max_{1 \leq i \leq kmax} \frac{\ln(V_{i-1} + \tilde{\mu}_{NT,i}) - \ln(V_{i-1})}{\ln(V_i + \tilde{\mu}_{NT,i+1}) - \ln(V_i)} \quad (5)$$

$$= \arg \max_{1 \leq i \leq kmax} \frac{\ln(1 + \tilde{\mu}_{NT,i}/V_{i-1})}{\ln(1 + \tilde{\mu}_{NT,i+1}/V_i)}, \quad (6)$$

Clearly, this transformation in the numerator of the objective function in (6) includes the main part $\tilde{\mu}_{NT,i}/V_{i-1}$, which measures $\tilde{\mu}_{NT,i}$'s influence on V_{i-1} , and can be called the contribution of $\tilde{\mu}_{NT,i}$ for V_{i-1} . Then, $\ln(1 + \tilde{\mu}_{NT,i}/V_{i-1})$ is the transformed contribution. Meanwhile, as a modification of ER and GR estimators in (5) or (6), we name it as ‘‘Transformed Contribution Ratio’’ (hereafter TCR) estimator.

Remark 1. Ahn and Horenstein (2013) argued that the ER and GR estimators are not sensitive to the choice of $kmax$ unless it is too large or too small. However, when both strong and weak factors or some dominant factors are existent, the ratio value of two adjacent eigenvalues may be not stable because the positive eigenvalue $\tilde{\mu}_{NT,i}$ is very small or very large. (5) is similar to the

ER estimator in fact, *only with the ln-difference versus the difference between V_{i-1} and V_i in the numerator of (5).* From (6), due to $\ln(1 + \tilde{\mu}_{NT,i}/V_{i-1}) < \tilde{\mu}_{NT,i}/V_{i-1}$ for $\tilde{\mu}_{NT,i}/V_{i-1} > 0$, when $\tilde{\mu}_{NT,i}$ is large, the use of the transformation $\ln(1 + \tilde{\mu}_{NT,i}/V_{i-1})$ is a shrinkage to the large values $\tilde{\mu}_{NT,i}/V_{i-1}$. *Moreover,*
115 *the main role of $\tilde{\mu}_{NT,i}/V_{i-1}$ is measuring $\tilde{\mu}_{NT,i}$'s effect, which is not only meaningful but also useful for TCR to avoid the underestimation of the number of common factors. Especially when the dominant factors or weak factors exist, TCR will perform well, see the simulation study in Section 3 for more details.*

120 By Ahn and Horenstein (2013), the above objective function in (5) or (6) can be expected to guarantee the consistency of the estimation under the same conditions of Ahn and Horenstein (2013), which are as follows.

Assumption A: Let $\mu_{NT,i} = \psi_i((\Lambda' \Lambda / N)(F' F / T))$ for $i = 1, \dots, r$. Then, for each $i = 1, \dots, r$, $\text{plim}_{m \rightarrow \infty} \mu_{NT,i} = \mu_i \in (0, +\infty)$. Moreover, the number r of
125 common factors is assumed to finite.

Assumption B:

- (i) $E\|F_t\|^4 \leq c_1$, $E\|\lambda_i\|^4 \leq c_1$, for all N and T .
- (ii) $E(\|N^{-\frac{1}{2}} \sum_{i=1}^T u_{it} \lambda_i\|^2) \leq c_1$ for all t .
- (iii) $E(N^{-1} \sum_{i=1}^N \|T^{-\frac{1}{2}} \sum_{t=1}^T u_{it} F_t\|^2) = E[(NT)^{-1} \|U' F\|] \leq c_1$.

130 **Assumption C:** $\psi_1(U' U / M) = O_p(1)$, and $\psi_{[d^c m]}(U' U / M) \geq c + o_p(1)$ for some positive and finite real number c and some $d^c \in (0, 1]$.

Remark 2. The above three conditions are the same as weak conditions of Ahn and Horenstein (2013). $\psi_1(U' U / M) = O_p(1)$ in Assumption C makes clear that the error matrix U does not contain common factors. $\psi_{[d^c m]}(U' U / M) \geq c + o_p(1)$
135 indicates that the first largest $[d^c m]$ eigenvalues of $U' U / M$ are bounded away from zero, which can guarantee the consistency of the TCR estimator.

Theorem 1. Suppose that Assumptions A-C hold *with $r > 1$, we then have*

$$\lim_{m \rightarrow \infty} \Pr(\hat{r}_{TCR} = r) = 1, \quad \text{for any } kmax \in (r, [d^c m - r - 1]) .$$

In the case with no factors, i.e., $r = 0$, we can slightly modify the above estimator by the same method as that of Ahn and Horenstein (2013). Specifically,
 140 we can define a mock eigenvalue, e.g., $\tilde{\mu}_{NT,0} = V_0/\ln(m)$ such that $\tilde{\mu}_{NT,0} \rightarrow 0$ and $m\tilde{\mu}_{NT,0} \rightarrow \infty$ as $m \rightarrow \infty$, although a finite multiple of the choice is also available.

Then, we have Corollary 1 as follows.

Corollary 1. *Suppose that Assumptions A-C hold, redefine $\hat{r}_{TCR}$ using $\tilde{\mu}_{NT,0}$ for $i = 0$, then it follows that*

$$\lim_{m \rightarrow \infty} Pr(\hat{r}_{TCR} = r) = 1,$$

$$\text{where } \hat{r}_{TCR} = \arg \max_{0 \leq i \leq kmax} \frac{\ln(1 + \tilde{\mu}_{NT,i}/V_{i-1})}{\ln(1 + \tilde{\mu}_{NT,i+1}/V_i)}.$$

145 3. Simulation Study

In this section, some simulation experiments are carried out to investigate the performance of the suggested estimator $\hat{r}_{TCR}$. To compare with the competitors, the foundation of our simulation exercises is the model of Ahn and Horenstein (2013):

$$y_{it} = \sum_{j=1}^r \lambda_{ij} F_{jt} + \sqrt{\theta} u_{it}, \quad u_{it} = \sqrt{\frac{1 - \rho^2}{1 + 2J\beta^2}} e_{it},$$

150 where $e_{it} = \rho e_{i,t-1} + (1 - \beta)v_{it} + \sum_{l=\max\{i-J,1\}}^{\min\{i+J,n\}} \beta v_{lt}$, and λ_{ij} and v_{lt} are both drawn from the standard normal distribution $N(0, 1)$. The factors F_{jt} are drawn from normal distribution with zero means.

In the simulation experiments, the settings of some parameters are important to show the various structure of data. The control parameter θ is the inverse
 155 of the signal to noise ratio (SNR) of each factor, i.e., $SNR = \frac{1}{\theta} = \frac{var(F_{jt})}{var(\sqrt{\theta}u_{it})}$ with $var(F_{jt}) = 1$. Then, we can adjust the value of θ while fixing variances of factors at 1. Moreover, ρ can show the serial correlation of the idiosyncratic errors over the time index, and β and J can control the cross-sectional dependency of the idiosyncratic errors over the individual index.

Our simulations are categorized into four parts with the same settings as Ahn and Horenstein (2013). The performances of TCR estimator ($\hat{r}_{TCR}$) are compared with those of the ER and GR estimators ($\hat{r}_{ER}$, $\hat{r}_{GR}$) of Ahn and Horenstein (2013), the BIC3 estimator ($\hat{r}_{BN}$) of Bai and Ng (2002), the ED estimator ($\hat{r}_{ED}$) of Onatski (2010), and the IC1 estimator ($\hat{r}_{ABC}$) of Alessi et al. (2010). The estimating procedure for the number r is replicated 1000 times, and the corresponding results are denoted by $x(y|z)$, where x is the mean of different estimators in the 1000 replications, the numbers y and z in parentheses ($y|z$) are the number of underestimation and overestimation in 1000 the replications, respectively, and then $1000 - y - z$ is the correct number of estimation.

The first part is designed to investigate four cases with different error structures in the following, which will influence the finite-sample performances of TCR estimator as well as the other estimators:

- (C_1) independent and identically distributed errors ($\rho = \beta = J = 0$);
- (C_2) serially correlated errors ($\rho = 0.7, \beta = J = 0$);
- (C_3) cross-sectionally correlated errors ($\rho = 0, \beta = 0.5, J = \max\{10, N/20\}$);
- (C_4) both serially and cross-sectionally correlated errors ($\rho = 0.5, \beta = 0.2, J = \max\{10, N/20\}$).

Table 1 reports the results of the six estimators when three factors ($r = 3$) are drawn from $N(0, 1)$ and θ is fixed at 1. In the cases with i.i.d. errors (C_1), the results show that $\hat{r}_{TCR}$ and $\hat{r}_{ABC}$ are slightly better than the other estimators with $N = T = 25$; when $N = T \geq 50$, all the six estimators $\hat{r}_{ER}$, $\hat{r}_{GR}$, $\hat{r}_{BN}$, $\hat{r}_{ABC}$, $\hat{r}_{ED}$ and $\hat{r}_{TCR}$ have the desired performance. For the data with serially correlated errors (C_2), $\hat{r}_{ED}$ is slightly better than the other estimators with $N = T = 25$; when $N = T \geq 50$, $\hat{r}_{TCR}$ performs equally to or better than the other estimators. For the data with cross-sectionally correlated errors (C_3), all estimators have the bad performance in the cases with $N = T = 25, 50$; when $N = T = 75$, $\hat{r}_{ER}$ outperforms the other estimators; when $N = T \geq 100$, all estimators excluding $\hat{r}_{BN}$ have the good performance. In the case with serially/ cross-sectionally correlated errors (C_4), $\hat{r}_{ER}$ is still better than the

190 other estimators with $N = T = 25, 50$; when $N = T \geq 75$, $\hat{r}_{TCR}$ performs as
 well as the $\hat{r}_{ER}$, $\hat{r}_{GR}$ estimators, which all outperform the other estimators. It
 appears that the performance of the $\hat{r}_{BN}$ estimator is much more sensitive to
 cross-sectional correlation than autocorrelation in the errors from the cases C_3 -
 C_4 , which are the same as the results reported by Ahn and Horenstein (2013).

195 Table 1: Effects of error covariance structure with three factors

Case	N	T	r	$\hat{r}_{ER}$	$\hat{r}_{GR}$	$\hat{r}_{BN}$	$\hat{r}_{ABC}$	$\hat{r}_{ED}$	$\hat{r}_{TCR}$
C_1	25	25	3	2.895(75 0)	2.974(21 0)	3.163(0 158)	2.994(7 6)	3.029(0 23)	2.991(9 0)
	50	50	3	3.000(0 0)	3.000(0 0)	3.000(0 0)	3.001(0 1)	3.011(0 9)	3.000(0 0)
	75	75	3	3.000(0 0)	3.000(0 0)	3.000(0 0)	3.000(0 0)	3.002(0 2)	3.000(0 0)
	100	100	3	3.000(0 0)	3.000(0 0)	3.000(0 0)	3.000(0 0)	3.009(0 5)	3.000(0 0)
	125	125	3	3.000(0 0)	3.000(0 0)	3.000(0 0)	3.000(0 0)	3.003(0 2)	3.000(0 0)
	150	150	3	3.000(0 0)	3.000(0 0)	3.000(0 0)	3.000(0 0)	3.005(0 5)	3.000(0 0)
	175	175	3	3.000(0 0)	3.000(0 0)	3.000(0 0)	3.000(0 0)	3.013(0 8)	3.000(0 0)
C_2	200	200	3	3.000(0 0)	3.000(0 0)	3.000(0 0)	3.000(0 0)	3.002(0 2)	3.000(0 0)
	25	25	3	2.163(765 40)	2.607(598 112)	6.800(0 1000)	2.833(456 157)	3.294(357 300)	2.826(517 183)
	50	50	3	2.997(3 0)	3.000(0 0)	4.494(0 955)	3.142(0 89)	3.072(0 46)	3.000(0 0)
	75	75	3	3.000(0 0)	3.000(0 0)	3.073(0 72)	3.048(0 36)	3.033(0 19)	3.000(0 0)
	100	100	3	3.000(0 0)	3.000(0 0)	3.000(0 0)	3.030(0 17)	3.008(0 8)	3.000(0 0)
	125	125	3	3.000(0 0)	3.000(0 0)	3.000(0 0)	3.009(0 9)	3.007(0 7)	3.000(0 0)
	150	150	3	3.000(0 0)	3.000(0 0)	3.000(0 0)	3.003(0 3)	3.009(0 5)	3.000(0 0)
C_3	175	175	3	3.000(0 0)	3.000(0 0)	3.000(0 0)	3.000(0 0)	3.005(0 5)	3.000(0 0)
	200	200	3	3.000(0 0)	3.000(0 0)	3.000(0 0)	3.000(0 0)	3.005(0 3)	3.000(0 0)
	25	25	3	4.942(1 998)	4.986(0 1000)	6.997(0 1000)	5.123(0 1000)	5.169(0 1000)	5.002(0 1000)
	50	50	3	6.615(1 976)	6.900(0 1000)	7.004(0 1000)	7.00(0 1000)	7.017(0 1000)	6.940(0 1000)
	75	75	3	3.113(303 157)	6.132(41 672)	8.000(0 1000)	3.963(213 330)	5.921(120 693)	6.338(31 711)
	100	100	3	2.999(1 0)	2.999(1 0)	8.000(0 1000)	3.046(0 15)	3.040(0 11)	3.000(0 0)
	125	125	3	3.000(0 0)	3.000(0 0)	8.000(0 1000)	3.015(0 6)	3.002(0 0)	3.000(0 0)
C_4	150	150	3	3.000(0 0)	3.000(0 0)	8.000(0 1000)	3.004(0 3)	3.001(0 1)	3.000(0 0)
	175	175	3	3.000(0 0)	3.000(0 0)	8.000(0 1000)	3.001(0 1)	3.001(0 1)	3.000(0 0)
	200	200	3	3.000(0 0)	3.000(0 0)	8.000(0 1000)	3.001(0 1)	3.001(0 1)	3.000(0 0)
	25	25	3	3.466(253 532)	4.177(103 747)	6.522(0 1000)	4.703(44 889)	4.749(11 934)	4.386(63 826)
	50	50	3	3.556(39 261)	4.571(2 555)	6.203(0 1000)	6.301(2 986)	5.787(7 873)	5.036(1 689)
	75	75	3	3.015(1 10)	3.142(0 39)	6.960(0 1000)	5.510(0 774)	5.120(0 464)	3.420(0 69)
	100	100	3	3.000(0 0)	3.009(0 2)	7.361(0 1000)	4.100(0 384)	3.162(0 46)	3.009(0 2)
C_4	125	125	3	3.000(0 0)	3.000(0 0)	7.442(0 1000)	3.380(0 177)	3.019(0 12)	3.000(0 0)
	150	150	3	3.000(0 0)	3.000(0 0)	7.287(0 1000)	4.106(0 111)	3.012(0 7)	3.000(0 0)
	175	175	3	3.000(0 0)	3.000(0 0)	7.133(0 1000)	3.155(0 71)	3.007(0 7)	3.000(0 0)
	200	200	3	3.000(0 0)	3.000(0 0)	6.984(0 1000)	3.017(0 28)	3.001(0 1)	3.000(0 0)

[†] C_1 : I.I.D. Errors with $r = 3, \theta = 1, kmax = 8, \rho = \beta = J = 0$; C_2 : Serially correlated errors with $r = 3, \theta = 1, kmax = 8, \rho = 0.7, \beta = J = 0$; C_3 : Cross-Sectionally correlated errors with $r = 3, \theta = 1, kmax = 8, \rho = 0, \beta = 0.5, J = \max\{10, N/20\}$; C_4 :

200 *Serially/ Cross-Sectionally correlated errors with $r = 3, \theta = 1, kmax = 8, \rho = 0.5, \beta =$*
 $0.2, J = \max\{10, N/20\}$.

In the second part, the effects of weak factors on the estimators are examined by considering two cases. The first is the case in which all three factors have weak explanatory power ($SNR = 0.17$). The second is the case in which two
205 factors are strong ($SNR = 1$) and one factor is weak ($SNR < 1$).

Table 2: Effects of all weak factors with serially/cross-sectionally correlated errors

N	T	r	$\hat{r}_{ER}$	$\hat{r}_{GR}$	$\hat{r}_{BN}$	$\hat{r}_{ABC}$	$\hat{r}_{ED}$	$\hat{r}_{TCR}$
25	25	3	2.902(439 384)	3.537(310 535)	6.413(0 1000)	2.528(570 327)	2.868(425 437)	3.750(256 582)
50	50	3	5.412(94 868)	5.912(44 943)	6.081(0 1000)	5.365(165 823)	4.168(404 591)	6.036(31 958)
75	75	3	5.755(67 893)	7.124(13 982)	6.797(0 1000)	4.195(424 572)	3.700(594 406)	7.185(10 988)
100	100	3	4.065(326 573)	5.888(152 798)	7.144(0 1000)	1.491(906 84)	1.084(984 6)	5.922(147 804)
125	125	3	2.910(469 275)	4.137(317 479)	7.323(0 1000)	1.452(888 63)	1.126(955 4)	4.253(295 498)
150	150	3	2.963(331 219)	3.563(244 336)	7.231(0 1000)	1.524(800 38)	1.221(914 3)	3.609(238 347)
175	175	3	2.692(338 62)	2.919(270 102)	7.030(0 1000)	2.284(457 38)	1.772(667 8)	2.965(253 111)
200	200	3	2.820(138 26)	2.897(109 39)	6.860(0 1000)	2.700(184 30)	2.334(342 6)	2.917(99 15)

[†] Where $r = 3, \theta = 6, kmax = 8, \rho = 0.5, \beta = 0.2, J = \max\{10, N/20\}$, and
210 $F_1, F_2, F_3 \sim N(0, 1)$.

Table 3: Effects of strong and weak factors with serially/cross-sectionally correlated errors

SNR_3	r	$\hat{r}_{ER}$	$\hat{r}_{GR}$	$\hat{r}_{BN}$	$\hat{r}_{ABC}$	$\hat{r}_{ED}$	$\hat{r}_{TCR}$
0.45	3	2.992(4 0)	3.000(0 0)	7.329(0 1000)	4.064(0 394)	3.210(0 63)	3.000(0 0)
0.40	3	3.000(0 0)	3.000(0 0)	7.300(0 1000)	4.055(0 398)	3.126(0 49)	3.000(0 0)
0.35	3	2.998(1 0)	3.000(0 0)	7.289(0 1000)	4.043(0 358)	3.125(0 35)	3.000(0 0)
0.30	3	2.758(121 0)	2.990(5 0)	7.327(0 1000)	4.017(0 369)	3.181(0 54)	2.998(1 0)
0.25	3	2.996(2 0)	3.000(0 0)	7.315(0 1000)	3.989(0 363)	3.162(0 55)	3.000(0 0)
0.20	3	2.754(123 0)	2.998(1 0)	7.321(0 1000)	4.140(0 397)	3.189(0 56)	3.000(0 0)
0.15	3	2.324(338 0)	2.978(11 0)	7.360(0 1000)	4.100(0 389)	3.190(0 54)	3.000(0 0)
0.10	3	1.508(746 0)	2.878(61 0)	7.269(0 1000)	4.210(0 437)	3.163(0 51)	2.994(3 0)

[†] Where $N = T = 100, r = 3, \theta = 1, kmax = 8, \rho = 0.5, \beta = 0.2, J = \max\{10, N/20\}$,
215 $F_1, F_2 \sim N(0, 1)$, and $F_3 \sim N(0, SNR_3)$.

Table 2 reports the results from the first case of the second part of our simulations. In short, the $\hat{r}_{TCR}$, $\hat{r}_{ER}$ and $\hat{r}_{GR}$ estimators are better than the other estimators. However, **when $N = T \leq 100$, all the estimators are bad;**

when $N = T \geq 125$, the $\hat{r}_{TCR}$, $\hat{r}_{ER}$, $\hat{r}_{GR}$ estimators begin to perform well.

Moreover, with increasing $N(=T)$, $\hat{r}_{TCR}$ estimator outperforms $\hat{r}_{ER}$ and $\hat{r}_{GR}$ estimators. Table 3 reports the results from the second case of the second part of our simulations. With one factor turning weaker, the $\hat{r}_{TCR}$ and $\hat{r}_{GR}$ estimators perform better than the other estimators. In fact, the $\hat{r}_{TCR}$ estimator is also slightly better than the $\hat{r}_{GR}$ estimator.

Table 4: Estimation with different values of $kmax$ with three factors

$kmax$	r	$\hat{r}_{ER}$	$\hat{r}_{GR}$	$\hat{r}_{BN}$	$\hat{r}_{ABC}$	$\hat{r}_{ED}$	$\hat{r}_{TCR}$
8	3	3.000(0 0)	3.000(0 0)	7.215(0 1000)	3.198(0 106)	3.006(0 6)	3.000(0 0)
12	3	3.000(0 0)	3.000(0 0)	9.747(0 1000)	5.376(0 386)	3.157(0 37)	3.000(0 0)
16	3	3.000(0 0)	3.000(0 0)	10.92(0 1000)	10.08(0 790)	4.028(0 109)	3.000(0 0)
20	3	3.000(0 0)	3.000(0 0)	11.82(0 1000)	10.25(0 713)	4.173(0 125)	3.000(0 0)
25	3	3.000(0 0)	3.000(0 0)	12.77(0 1000)	10.28(0 800)	4.155(0 120)	3.000(0 0)
30	3	3.000(0 0)	3.000(0 0)	13.58(0 1000)	10.16(0 781)	4.298(0 135)	3.000(0 0)

[†] Where $N = T = 150, r = 3, \theta = 1, \rho = 0.5, \beta = 0.2, J = \max\{10, N/20\}$, and $F_1, F_2, F_3 \sim N(0, 1)$.

In the third part, we investigate how the choice of large $kmax$ may influence the performances of estimators when the eigenvalues $\tilde{\mu}_{NT,i}$ are close to zero for some large $i(< m)$. Six different values are used for $kmax$, and the data generating process is the same as the case C_4 of the first part with $N = T = 150$, the estimated results are recorded in Table 4. The results show that the performances of the TCR estimator are the same as those of the ER and GR estimators, which are insensitive to $kmax$. The ED estimator is less sensitive to $kmax$ than the ABC and BIC3 estimators, but the BIC3 estimator is easily sensitive to $kmax$, because the mean of the estimations in the BIC3 estimator increases with $kmax$ obviously.

The fourth part of our simulations considers the case in which both factors have strong explanatory power, but one factors power is increasingly dominant. We generate data using two factors with different $SNRs$ ($SNR_1 = 1, SNR_2 > 1$), i.e., the two factors are drawn from $N(0, 1)$ and $N(0, SNR_2)$, respectively. We investigate four cases C_5 : $N = T = 75$; C_6 : $N = T = 100$; C_7 : $N = T = 150$; C_8 : $N = T = 200$. Also, we can examine how large a difference in the

245 explanatory power of two factors is needed to make the estimators underestimate
the true number of factors.

Table 5 reports the results of the six estimators when there is one dominant
factor. For such a case with large SNR_2 , the TCR and GR estimators are always
outperforming the other estimators, [the peak of ER's estimations may be often](#)
250 [at 1](#). Further, the TCR estimator perform better than the GR estimator in
most cases with large SNR_2 and $N = T \leq 150$.

Table 5: Effects of dominant factor with two factors

	SNR_2	r	$\hat{r}_{ER}$	$\hat{r}_{GR}$	$\hat{r}_{BN}$	$\hat{r}_{ABC}$	$\hat{r}_{ED}$	$\hat{r}_{TCR}$
C_5	1	2	2.009(8 6)	2.136(2 36)	6.162(0 1000)	6.391(0 973)	5.045(0 644)	2.397(1 95)
	3	2	1.874(130 1)	2.005(21 7)	6.170(0 1000)	6.264(0 960)	5.133(0 664)	2.231(5 58)
	7	2	1.652(348 0)	1.985(15 0)	6.184(0 1000)	6.396(0 969)	5.025(0 634)	2.022(0 5)
	10	2	1.449(551 0)	1.975(25 0)	6.155(0 1000)	6.325(0 981)	4.975(0 631)	2.024(0 5)
	15	2	1.004(996 0)	1.506(494 0)	6.170(0 1000)	6.3270 960)	5.015(0 630)	2.084(57 35)
	20	2	1.000(1000 0)	1.254(746 0)	6.136(0 1000)	6.290(0 966)	5.034(0 641)	1.981(89 18)
C_6	1	2	2.000(0 0)	2.000(0 0)	6.768(0 1000)	4.320(0 613)	2.858(0 175)	2.000(0 0)
	3	2	1.944(56 0)	1.965(35 0)	6.761(0 1000)	4.460(0 620)	2.751(0 149)	2.000(0 0)
	7	2	1.368(632 0)	1.999(1 0)	6.759(0 1000)	4.390(0 599)	2.751(0 148)	1.999(1 0)
	10	2	1.092(908 0)	1.934(66 0)	6.819(0 1000)	4.234(0 555)	2.755(0 145)	1.998(2 0)
	15	2	1.004(996 0)	1.850(150 0)	6.813(0 1000)	4.281(0 586)	2.737(0 149)	1.999(1 0)
	20	2	1.000(1000 0)	1.741(259 0)	6.800(0 1000)	4.232(0 569)	2.773(0 154)	1.998(2 0)
C_7	1	2	2.000(0 0)	2.000(0 0)	7.058(0 1000)	2.334(0 134)	2.019(0 17)	2.000(0 0)
	3	2	2.000(0 0)	2.000(0 0)	7.058(0 1000)	2.353(0 135)	2.018(0 14)	2.000(0 0)
	7	2	1.511(489 0)	1.999(1 0)	7.037(0 1000)	2.318(0 137)	2.013(0 37)	2.000(0 0)
	10	2	1.047(953 0)	1.994(4 0)	7.046(0 1000)	2.284(0 111)	2.021(0 16)	2.000(0 0)
	15	2	1.000(1000 0)	1.797(203 0)	7.029(0 1000)	2.286(0 110)	2.016(0 11)	2.000(0 0)
	20	2	1.000(1000 0)	1.963(37 0)	6.997(0 1000)	2.357(0 125)	2.016(0 14)	2.000(0 0)
C_8	1	2	2.000(0 0)	2.000(0 0)	6.750(0 1000)	2.046(0 27)	2.007(0 7)	2.000(0 0)
	3	2	2.000(0 0)	2.000(0 0)	6.725(0 1000)	2.073(0 44)	2.011(0 10)	2.000(0 0)
	7	2	1.411(589 0)	2.000(0 0)	6.704(0 1000)	2.054(0 37)	2.003(0 3)	2.000(0 0)
	10	2	1.460(540 0)	2.000(0 0)	6.705(0 1000)	2.049(0 34)	2.010(0 4)	2.000(0 0)
	15	2	1.558(442 0)	2.000(0 0)	6.703(0 1000)	2.036(0 25)	2.005(0 5)	2.000(0 0)
	20	2	1.047(953 0)	2.000(0 0)	6.731(0 1000)	2.061(0 39)	2.015(0 15)	2.000(0 0)

255 [†] Where $r = 2, \theta = 1, kmax = 8, \rho = 0.5, \beta = 0.2, J = \max\{10, N/20\}, F_1 \sim N(0, 1),$
and $F_2 \sim N(0, SNR_2)$. C_5 : $N = T = 75$; C_6 : $N = T = 100$; C_7 : $N = T = 150$; C_8 :
 $N = T = 200$.

4. A Real Data Example

The real data set we use is the FRED-MD dataset updated to the more recent date, which can be found at [http : //research.stlouisfed.org/econ/mccracken/fred - md/](http://research.stlouisfed.org/econ/mccracken/fred-md/), and includes 128 monthly time series in total 692 spanning the period of 1959.1 – 2016.10. The variables in the data set are first transformed into stationary form, and the outliers are removed from the transformed data as described in McCracken and Ng (2015). Then, the determination of the number of common factors was based on this transformed data subset. Meanwhile, we want to check whether the estimation results are sensitive to the choice of $kmax$. For $kmax = 8, 10, 15, 20$, the resulted estimators are always the same, i.e., $\hat{r}_{ER} = \hat{r}_{GR} = \hat{r}_{BN} = \hat{r}_{ED} = \hat{r}_{ABC} = \hat{r}_{TCR} = 3$, respectively. But when $kmax = 30$, $\hat{r}_{BN}$ is influenced by $kmax$, because it finds 4 factors in this time, while the estimated results of the other estimators are still 3 factors.

5. Conclusion

In this paper, we propose a method modifying ER and GR estimators to determine the number of common factors in the approximate factor models. The new method has many desired properties through the simulation experiments and real data applications, which are conducted to compare the performances of the estimators with those of the estimators by Bai and Ng (2002), Onatski (2010), Alessi et al. (2010), and Ahn and Horenstein (2013). The numerical results indicate that the TCR estimator will be almost performing equally to or better than the other estimators, while the BN estimator is clearly the one overestimating more often in the case with the cross-sectionally correlated errors. Moreover, the TCR estimator outperforms the other estimators for the cases in which some factors are strongly dominant, and the cases in which both strong and weak factors are existent.

Appendix: Technical Details

The proof of Theorem 1 Under Assumptions A-C, it follows from Lemma A.11 of Ahn and Horenstein (2013) that, in probability, for any $i \leq r$, $\tilde{\mu}_{NT,i} \rightarrow$

$\mu_{NT,i} > c_0$ for some positive constant c_0 .

Also, it follows from Lemma A.9 of Ahn and Horenstein (2013) that, for any $i \geq r+1$, $\tilde{\mu}_{NT,i} \leq \tilde{\mu}_{NT,r+1} = O_p(\frac{1}{m})$.

290 Meanwhile, by Lemma A.12 of Ahn and Horenstein (2013), it holds that $V_{r+1} = O_p(1)$.

According to the inequality

$$\frac{c}{1+c} < \ln(1+c) < c, \quad \text{for } c \in (0, \infty),$$

for one hand, we have that

$$\begin{aligned} \frac{\ln(1 + \tilde{\mu}_{NT,i}/V_{i-1})}{\ln(1 + \tilde{\mu}_{NT,i+1}/V_i)} &< \frac{\tilde{\mu}_{NT,i}/V_{i-1}}{(\tilde{\mu}_{NT,i+1}/V_i)/(1 + \tilde{\mu}_{NT,i+1}/V_i)} \\ &= \frac{\tilde{\mu}_{NT,i}}{\tilde{\mu}_{NT,i+1}} \cdot \frac{V_i + \tilde{\mu}_{NT,i+1}}{V_i + \tilde{\mu}_{NT,i}} = O_p(1), \end{aligned}$$

for $i = 1, 2, \dots, r-1, r+1, \dots, kmax$. For another,

$$\frac{V_r}{V_{r-1} + \tilde{\mu}_{NT,r}} = \frac{V_{r+1} + \tilde{\mu}_{NT,r+1}}{V_{r+1} + 2\tilde{\mu}_{NT,r} + \tilde{\mu}_{NT,r+1}} = O_p(1),$$

295 then, we have

$$\begin{aligned} \frac{\ln(1 + \tilde{\mu}_{NT,r}/V_{r-1})}{\ln(1 + \tilde{\mu}_{NT,r+1}/V_r)} &> \frac{(\tilde{\mu}_{NT,r}/V_{r-1})/(1 + \tilde{\mu}_{NT,r}/V_{r-1})}{\tilde{\mu}_{NT,r+1}/V_r} \\ &= \frac{\tilde{\mu}_{NT,r}}{\tilde{\mu}_{NT,r+1}} \cdot \frac{V_r}{V_{r-1} + \tilde{\mu}_{NT,r}} = O_p(m). \end{aligned}$$

It then follows that

$$\lim_{m \rightarrow \infty} Pr(\hat{r}_{TCR} = r) = 1.$$

This means the consistency of the estimator $\hat{r}_{TCR}$, and then the proof of Theorem 1 is completed. $\square$

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