Managing Risk with a Realized Copula Parameter

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Abstract

A dynamic copula model is introduced, in which the copula structure is inferred from the realized covariance matrix estimated from within-day high-frequency data. The estimation is carried out in a method-of-moments fashion using Hoeffding's lemma. Applying this procedure day by day gives rise to a time series of daily copula parameters which can be approximated by an autoregressive time series model. This allows one to capture time-varying dependence. In an application to portfolio risk-management, it is found that this time-varying realized copula model exhibits very good forecasting properties for the one-day ahead value at risk.

Keywords: copula, multivariate dependence, realized covariance, realized variance, value at risk

1. Introduction

Realized variance (RV) and realized covariance (RCov) estimated from high-frequency intraday data have proved to be accurate ex-post measures for conditional variance and conditional covariance of daily returns. Nonparametric in nature, RV and RCov permit the econometrician to

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6 obtain proxies for financial (co)volatility without having to specify a priori
7 an explicit and potentially misspecified volatility model (Andersen et al.,
8 2001a,b). This insight has spurred intensive research in the field and has led
9 to widespread use of measures of RV and RCov in numerous applications in
10 finance, such as asset pricing, portfolio optimization, risk management, and
11 volatility forecasting.

12 The present article continues this agenda. We estimate RCov matrices
13 from high-frequency intraday data and take them as ex-post proxies for
14 daily conditional covariance. We complement these estimates by making
15 assumptions on the marginal distributions of daily returns and the copula
16 associated with their joint multivariate distribution. Based on these
17 assumptions, we estimate the copula shape parameters by means of a
18 covariance moment condition provided by Hoeffding's lemma. The procedure
19 yields estimates of daily copula shape parameters as materialized in RCov.
20 The resulting time series of RCov-implied copula shape parameters is
21 subsequently modeled by standard time series techniques, thereby allowing
22 the dependence structure to be time-varying with the business cycle. We
23 therefore call our approach the *realized copula (RCop)* model.

24 For risk-management purposes at a daily frequency, the benefits of using
25 copulae to capture salient features of multivariate dependence are widely
26 recognized; see Jin (2010) and references therein. Yet, purely RV-based
27 models often work with a conditional multivariate Gaussian structure. The
28 RCop model allows to drop this restrictive setting and offers a more realistic
29 description of the tails of the daily return distribution. It may therefore
30 yield more accurate estimates of the quantiles of a portfolio's profit and loss
31 distribution. Our application to forecasting the Value-at-Risk of two equity
32 portfolios confirms this expectation.

33 In this research, we build on several strands of literature. The first
34 strand is a series of studies in the RV literature extending the univariate
35 heterogeneous autoregressive (HAR) model to the multivariate level. The
36 HAR model, originally suggested by Corsi (2009), is a stationary, restricted
37 AR(22) model that captures the long-range dependence in RV data by means
38 of a cascade of volatility components, which are interpreted as a daily, weekly,
39 and monthly volatility component. Today, it is a standard benchmark for
40 modeling RV with unrivaled forecasting performance; see Corsi et al. (2012)
41 for a review. As an alternative to the HAR model, pure long-memory models
42 belonging to the ARFIMA class have been considered for modeling RV,
43 but their forecasting performance is close to that of HAR-type models; see

44 Baillie (1996), Baillie et al. (1996), Andersen et al. (2003), among others.
 45 A nontrivial challenge in constructing a multivariate HAR model for RV,
 46 i.e., a RCov model, is to ensure positive-definiteness of predicted covariance
 47 matrices. A number of avenues have been pursued concurrently. First, one
 48 considers modeling nonlinear transformations of RCov; examples include
 49 the Fisher transformation as in Audrino and Corsi (2010), the Cholesky
 50 factorization as in Chiriac and Voev (2011), or the matrix log transformation
 51 as in Bauer and Vorkink (2011). Second, one models the time series of
 52 realized covariances by means of a Wishart process; see Gouriéroux et al.
 53 (2009), Bonato et al. (2012), Golosnoy et al. (2012), Jin and Maheu (2012).
 54 Third, one may follow the routes of multivariate GARCH and DCC models
 55 such as suggested in Colacito et al. (2011), Hansen et al. (2011b), Noureldin
 56 et al. (2012), and Bauwens et al. (2012). A fourth strand employs classical
 57 factor models; see Hautsch and Kyj (2010) and Bannouh et al. (2012). The
 58 RCop model is in the spirit of this research, since the copula parameter,
 59 which we infer from RCov and subsequently describe by a time series model,
 60 defines – together with the assumptions on the marginals – a multivariate
 61 distribution and consequently a well-posed covariance matrix.

62 The second stream of research our work is related to is the growing
 63 literature of dynamic copula models, such as Dias and Embrechts (2004)
 64 and Patton (2004, 2006), Chen and Fan (2006), Jondeau and Rockinger
 65 (2006), Giacomini et al. (2009), Jin (2010), Christoffersen et al. (2011),
 66 Hafner and Manner (2012), Creal et al. (2013), and Härdle et al. (2013).
 67 Common to all these approaches is the notion of a copula structure that has
 68 time-varying parameters driven by past realizations of the underlying data
 69 generating process or by additional exogenous variables such as a latent state
 70 factor. By exploiting intra-day data, we uncover a daily time series of implied
 71 copula parameters which we model by formulating a time series model. We
 72 thus obtain a dynamic conditional copula model for daily returns, where
 73 time-variation is governed by the underlying dynamics of RCov measures.

74 Remarkably, the literature using copulae to model dependency in the
 75 context of high-frequency data is scarce. In two early studies, Breymann
 76 et al. (2003) and Dias and Embrechts (2004), a copula is directly applied to
 77 analyze intraday returns. This is not the purpose of the present investigation.
 78 As in De Lira Salvatierra and Patton (2013), our aim is to exploit the intraday
 79 information as condensed in the RV measure to improve the modeling of daily
 80 returns within a copula framework. In this sense, we follow recent suggestions
 81 by Engle and Gallo (2006), Shephard and Sheppard (2010), Hansen et al.

(2012), and Hansen et al. (2011b) that combine both low and high-frequency observations in the fashion of GARCH models to describe returns at daily frequency.

The paper is organized as follows. In Section 2, we introduce the RCop model and discuss estimation. The competitor models, which we use for the comparative risk management study, are presented in Section 3. In Section 4, we compare the empirical properties of all models out-of-sample by assessing their one-day ahead value at risk predictions on two portfolios of heavily traded U.S. stocks. Section 5 concludes.

2. The realized copula model

2.1. Estimation of copula parameter implied by realized variance

Copulae have emerged as a convenient way to construct multivariate distributions since they allow a strict separation of the marginal distributions from cross-sectional dependence, which is captured by the copula function; see Joe (1997), Nelsen (2006), and Jaworski et al. (2013) for an introduction to copulae. The main result due to Sklar (1959) states that if F is an arbitrary d -dimensional continuous distribution function of the random variables $X_1, \dots, X_d$, then the associated copula is unique and defined as a continuous function $C : [0, 1]^d \rightarrow [0, 1]$ satisfying the equality

$$C(u_1, \dots, u_d) = F\{F_1^{-1}(u_1), \dots, F_d^{-1}(u_d)\}, \quad u_1, \dots, u_d \in [0, 1], \quad (1)$$

where $F_1^{-1}, \dots, F_d^{-1}$ are the quantile functions of the corresponding marginal distributions $F_1, \dots, F_d$. If F belongs to the class of elliptical distributions, this results in a so-called elliptical copula. Most elliptical copulae, however, cannot be given explicitly, because the distribution function F and the inverse marginal distributions F_i usually have integral representations.

A class of copulae overcoming this drawback is the class of Archimedean copulae

$$C_\theta(u_1, \dots, u_k) = \phi_\theta\{\phi_\theta^{-1}(u_1) + \dots + \phi_\theta^{-1}(u_k)\}, \quad u_1, \dots, u_k \in [0, 1], \quad (2)$$

where $\phi_\theta : [0, \infty) \rightarrow [0, 1]$, with $\phi_\theta(0) = 1$, $\phi_\theta(\infty) = 0$. The function ϕ_θ is called the generator of the copula, and it usually depends on a single parameter θ . The generator ϕ_θ is required to be d -monotone, i.e., differentiable up to the order $d-2$, with $(-1)^j \phi_\theta^{(j)}(x) \geq 0$, $j = 0, \dots, d-2$, for any $x \in [0, \infty)$ and with $(-1)^{d-2} \phi_\theta^{(d-2)}(x)$ being nondecreasing and convex

family	ϕ_θ	ϕ_θ^{-1}	domain
Gumbel	$\exp\{-x^{1/\theta}\}$	$(-\log x)^\theta$	$\theta \in [1, \infty)$
Clayton	$(\theta x + 1)^{-1/\theta}$	$(x^{-\theta} - 1)/\theta$	$\theta \in (0, \infty)$
Nelsen (2006) (4.2.2)	$1 - x^{1/\theta}$	$(1 - x)^\theta$	$\theta \in [1, \infty)$
Nelsen (2006) (4.2.3)	$(1 - \theta)/(e^x - \theta)$	$\log \frac{1-\theta(1-x)}{x}$	$\theta \in [0, 1)$
Frank	$-\frac{1}{\theta} \log\{e^{-x}(e^{-\theta} - 1) + 1\}$	$-\log \frac{e^{-\theta x} - 1}{e^{-\theta} - 1}$	$\theta \in (0, \infty)$

Table 1: Generator function ϕ_θ and the inverse ϕ_θ^{-1} for selected Archimedean copulae; see Nelsen (2006), Table 4.1.

on $[0, \infty)$; see McNeil and Nešlehová (2009). We present some examples of Archimedean copulae and their generators in Table 1; see Nelsen (2006) for more details.

In the following, we focus our presentation on a setting with a single copula shape parameter such as the Archimedean copula. Yet, we emphasize that the RCop model is not limited to this class of copulae; for instance, the survival copula derived from an Archimedean copula by $C_{\text{rot}}(u, v) = C(1 - u, 1 - v) + u + v - 1$, such as the rotated Gumbel copula, which we use in our empirical part, is not Archimedean. For our empirical part, these copulae are very attractive as they display asymmetry in tail-dependence, which is a stylized fact which has been widely documented for equity data.

Suppose there are two random variables X_i and X_j with marginal distributions F_i and F_j and joint distribution F_{ij} and finite second moments. Hoeffding's lemma (Hoeffding, 1940), together with Sklar's theorem, states that the covariance between X_i and X_j is a function in the copula parameter θ , the marginals, and the joint distribution function:

$$\sigma_{ij}(\theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [F_{ij}\{x_i, x_j; \vartheta(\theta, \eta_i, \eta_j)\} - F_i(x_i; \eta_i)F_j(x_j; \eta_j)] dx_i dx_j \quad (3)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [C_\theta\{F_i(x_i; \eta_i), F_j(x_j; \eta_j)\} - F_i(x_i; \eta_i)F_j(x_j; \eta_j)] dx_i dx_j, \quad (4)$$

where θ is the parameter of the copula and η_i and η_j are the parameters of the marginal distributions F_i and F_j , respectively. The notation $\vartheta(\theta, \eta_i, \eta_j)$

is used to highlight that the parameters of the bivariate distribution are given by some functional relationship between the copula parameter and the parameters of the marginal distributions. For instance, if the marginal distributions F_i and F_j are mean-zero normals with variances η_i and η_j , respectively, and if the copula is the Gaussian copula, then θ corresponds to the correlation coefficient between X_i and X_j and

$$\vartheta(\theta, \eta_i, \eta_j) = \begin{pmatrix} \sqrt{\eta_i} & 0 \\ 0 & \sqrt{\eta_j} \end{pmatrix} \begin{pmatrix} 1 & \theta \\ \theta & 1 \end{pmatrix} \begin{pmatrix} \sqrt{\eta_i} & 0 \\ 0 & \sqrt{\eta_j} \end{pmatrix} \quad (5)$$

129 is the covariance matrix of the bivariate normal distribution F_{ij} . In this
 130 special example, integration shows that $\sigma_{ij} = \theta\sqrt{\eta_i\eta_j}$. In most cases,
 131 however, the integrals in (3) and (4) have no explicit form and must be
 132 approximated by numerical integration methods, such as adaptive quadrature
 133 techniques.

In order to infer the copula parameter from a daily aggregate of high-frequency data, we equate (4) with a realized covariance measure, i.e., given a full specification of the marginal distributions, we define the copula shape parameter θ implicitly through the equation

$$h_{ij,t} = f_{ij,t}(\theta_t) \quad (6)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [C_{\theta_t}\{F_i(x_i; \eta_{i,t}), F_j(x_j; \eta_{j,t})\} - F_i(x_i; \eta_{i,t})F_j(x_j; \eta_{j,t})] dx_i dx_j, \quad (7)$$

134 where $h_{ij,t}$ denotes an element of the RCov matrix estimated at day t . We
 135 then exploit Hoeffding's lemma in a method-of-moments fashion to estimate
 136 θ_t .

137 Consider the case $d = 2$, with one off-diagonal element $h_{12,t}$ in the RCov
 138 available. An estimate of θ_t is given by

$$\hat{\theta}_t^{\text{MM}} = f_{12,t}^{-1}(h_{12,t}), \quad (8)$$

139 where $f_{12,t}^{-1}$ denotes the inverse function of (6). In the general case for $d > 2$,
 140 define the moment condition $g_{ij,t}(\theta) = h_{ij,t} - f_{ij,t}(\theta)$, where $i < j$ and $i, j =$
 141 $1, \dots, d$. Stacking all $g_{ij,t}$ into a vector $\mathbf{g}_t$ of size $d(d-1)/2$, we define the
 142 estimate as the minimizer of the generalized moment criterion

$$\hat{\theta}_t^{\text{MM}} = \arg \min_{\theta} \mathbf{g}_t^{\top}(\theta) \mathbf{W} \mathbf{g}_t(\theta), \quad (9)$$

143 where $\mathbf{W}$ is a positive definite weight matrix. The simplest choice would be
144 $\mathbf{W} = \mathbf{I}_n$ with $\mathbf{I}_n$ denoting the n -dimensional unit matrix and $n = d(d-1)/2$.
145 This also corresponds to our implementation in the empirical applications.
146 One could think about two-step GMM estimation as well. In our applications,
147 however, the sampling variability is small in which case the two-step GMM
148 is not an option. For $d = 2$, (9) coincides with (8). Note that this approach
149 has much in common with the method-of-moments estimators of Genest and
150 Rivest (1993). In their study, the authors suggest estimating the copula
151 parameter of an Archimedean copula from Kendall's tau or Spearman's rho,
152 whereas we build on the covariance.

153 Obviously, $\hat{\theta}_t^{\text{MM}}$ in (9) is only identified if the dimension of the copula
154 shape parameters fulfills $\dim(\theta) \leq d(d-1)/2$. Student's t -copula, for
155 instance, would not be identified in this framework. If one works with
156 a single-parameter copula as we do, identification is not an issue. It
157 limits, however, the applicability of our approach to large dimensional
158 portfolios since one would impose a strong homogeneity assumption on the
159 dependence structure, which presumably one does not want to maintain in a
160 high-dimensional setting. This issue could be addressed by using richer, yet
161 still parsimoniously parametrized copulae. For instance, one could consider
162 a mixture of k single parameter copulae such that $2k - 1 \leq d(d-1)/2$.
163 Alternatively, one could work with hierarchical Archimedean copulae or vine
164 copulae, which would have $d-1$ parameters only if all sub-copulae have at
165 most one parameter. In the last two cases, however, one needs not only to
166 estimate the copula shape parameters, but also simultaneously to identify the
167 embedded copula structure. This appears to be challenging, and we leave it
168 for future research.

169 Given the assumptions on the copula family and the marginal
170 distributions, the structure

$$C_{\hat{\theta}_t} \{F_1(\cdot; \eta_{1,t}), \dots, F_d(\cdot; \eta_{d,t})\}, \quad (10)$$

171 where $\hat{\theta}_t$ is any estimator presented above, fully characterizes a multivariate
172 distribution, where the shape parameter is time-varying with the RCov
173 measure. In the next section, we complete the modeling by specifying the
174 dynamics of $\hat{\theta}_t$.

175 2.2. Model specification

176 For our portfolio risk management problem, we consider a model
177 framework which combines daily and within-day modeling frequencies. The

178 purpose is to exploit intra-day high-frequency data as an auxiliary source of
 179 information to make one-day ahead value at risk forecasts. In this sense, our
 180 approach is close to the mixed data sampling approach by Ghysels et al.
 181 (2006), Maheu and McCurdy (2011), and the multiplicative error model
 182 suggested by Engle and Gallo (2006) and Shephard and Sheppard (2010).
 183 It can also be embedded in the extensions of the multiplicative error models
 184 recently proposed by Hansen et al. (2011b) and Noureldin et al. (2012).

185 For the daily level, denote the log-prices of a d -dimensional vector of assets
 186 by $P = (P_1, \dots, P_d)^\top$ and the associated daily returns by $\Delta P_t = P_t - P_{t-1} =$
 187 r_t , $t = 1, \dots, T$. The available information set is denoted by $\mathcal{F}_t$ and contains
 188 r_t and its history including intraday realizations.

189 We assume that the conditional distribution of daily returns r_t can be
 190 approximated by the mean-zero process

$$r_t = \epsilon_t, \quad (11)$$

191 where $\epsilon_t | \mathcal{F}_{t-1} \sim F(\cdot; \vartheta_t)$ denotes a multivariate distribution function, which
 192 is parametrized with an $\mathcal{F}_{t-1}$ -measurable parameter ϑ_t . In this specification,
 193 a volatility model of the multivariate GARCH type could be described by
 194 setting $F(\cdot; \vartheta_t)$ equal to the multivariate normal distribution and ϑ_t would
 195 comprise the variance-covariance matrix H_t . For the specification of the
 196 RCop model, however, we will replace ϑ_t by a forecast that relies on the
 197 finer information structure accumulated by intraday returns. In doing so,
 198 we exploit the fact that we can treat realized variances and the associated
 199 realized copula parameter as an observable quantity that can be modeled in
 200 its own right.

201 To be more specific, by Sklar's theorem and following the notion of a
 202 conditional copula as outlined in Patton (2006), we set

$$F(x; \vartheta_t) = C_{\hat{\theta}_{t|t-1}} \{F_{t,1}(x_1; \hat{h}_{1,t|t-1}), \dots, F_{t,d}(x_d; \hat{h}_{d,t|t-1})\}, \quad (12)$$

203 where $x = (x_1, \dots, x_d)^\top$ and $\vartheta_t = \vartheta(\hat{\theta}_{t|t-1}, \hat{h}_{1,t|t-1}, \dots, \hat{h}_{d,t|t-1})$ and C_θ
 204 denotes a copula belonging to some parametric family $\mathcal{C} = \{C_\theta, \theta \in \Theta\}$.
 205 In (12), $F(\cdot; \vartheta_t)$ is parametrized by the forecast of the copula parameter $\hat{\theta}_{t|t-1}$
 206 and forecasts of the RV process $\hat{h}_{j,t|t-1}$, $j = 1, \dots, d$. For the marginal
 207 distributions, we adopt two assumptions. First, we assume that $F_j(\cdot; \hat{h}_{j,t|t-1})$,
 208 $j = 1, \dots, d$, are $N(0, \hat{h}_{j,t|t-1})$, i.e., normal with mean zero and variance $\hat{h}_{j,t|t-1}$.
 209 Since the RCov benchmark models, described in Section 3.3, are based on

multivariate normality, this assumption allows us to assess the relevance of our selected copula vis-à-vis the Gaussian copula. Second, we adopt the standardized generalized error distribution denoted by $\text{GED}(\nu_j)$ where ν_j is a shape parameter. If $\nu_j < 2$, leptokurtic tails are obtained, while for $\nu_j = 2$ the GED is identical to the normal distribution. Since the GED has mean zero and unit variance, one simply scales the random variates by $\hat{h}_{j,t|t-1}^{1/2}$ to obtain the appropriate variance.

To obtain forecasts of RV and the copula parameter, we use the autoregressive system

$$\begin{pmatrix} \log h_{1,t+1} \\ \vdots \\ \log h_{d,t+1} \\ \theta_{t+1} \end{pmatrix} = \begin{pmatrix} \beta_0^1 + \beta_D^1 \log h_{1,t}^D + \beta_W^1 \log h_{1,t}^W + \beta_M^1 \log h_{1,t}^M + u_{1,t} \\ \vdots \\ \beta_0^d + \beta_D^d \log h_{d,t}^D + \beta_W^d \log h_{d,t}^W + \beta_M^d \log h_{d,t}^M + u_{d,t} \\ \beta_0^{d+1} + \beta_D^{d+1} \theta_t^D + \beta_W^{d+1} \theta_t^W + \beta_M^{d+1} \theta_t^M + u_{d+1,t} \end{pmatrix}, \quad (13)$$

where $\beta^j = (\beta_0^j, \beta_D^j, \beta_W^j, \beta_M^j)^\top$, for $j = 1, \dots, d+1$, is a parameter vector, and $x_t^D = x_t$ are daily, $x_t^W = \frac{1}{5} \sum_{i=0}^4 x_{t-i}$ weekly, and $x_t^M = \frac{1}{22} \sum_{i=0}^{21} x_{t-i}$ monthly averages of past realizations of x_t , and $u_t = (u_{1,t}, \dots, u_{d+1,t})^\top$ is an error term with mean zero and finite variance. We estimate (13) by least squares. After this, we obtain the estimates of the GED shape parameters from an ML estimation on the standardized asset returns.

The specification in (13) is owed to Corsi (2009) and based on the heuristic of the heterogeneous market hypothesis. Following this hypothesis, the observed variance process emerges as a cascade of variance components which is induced by market participants with daily, weekly, and monthly investment horizons. This heterogeneous autoregressive (HAR) model has found wide-spread use in the RV literature as it approximately captures the long-memory patterns typically observed in RV and RCov data in a parsimonious way and shows an excellent forecasting performance; for further refinements of this baseline model, see, e.g., Andersen et al. (2007), Corsi et al. (2008), Bollerslev et al. (2009), Corsi et al. (2010), and Audrino and Hu (2011). We extend the HAR specification to the copula parameter, since empirically the copula parameter θ_t exhibits the same patterns of long-range dependence as are widely documented for RV and RCov data. This can be seen in Figure 5, which shows the empirical autocorrelations of the Clayton RCop parameter for the portfolios considered in Section 4. The corresponding plots for the rotated Gumbel copula are akin and are therefore not reported. Given these observations, using the HAR model as an approximating model

240 is a natural choice. As a drawback, this specification does not guarantee the
241 copula parameters will remain in their allowed domain, e.g., $\theta > 0$ for the
242 Clayton copula. This could easily be achieved by applying suitable monotonic
243 transformations. In our empirical applications, however, violations never
244 turned out to be an issue, which is why we work with the simple specification
245 in Equation (13).

246 The HAR model in (13) and the assumptions on the marginals allow us to
247 predict the entire multivariate distribution of r_t . Analogously to the “realized
248 GARCH” literature initiated by Hansen et al. (2012), we call the approach
249 the *realized copula* model. The RCop model provides well-defined covariance
250 matrices, since the copula parameter parametrizes the covariance matrix by
251 Hoeffding’s lemma. From this perspective, it is similar in spirit to the RCov
252 models owed to Bauer and Vorkink (2011) and Chiriac and Voev (2011).
253 Here, one subjects the RCov to nonlinear transformations, such as the matrix
254 logarithm or the Cholesky decomposition, to ensure positive-definiteness of
255 the predicted RCov; see Section 3 for further details.

256 The relation between the RCov models and the RCop model can be
257 exploited from yet another angle. Indeed, the direct dynamic description
258 of the copula parameter as in (13) is not the only possible way to arrive
259 at a specification for $F(\cdot; \vartheta_t)$. Alternatively, in a first step, one could use
260 one of the RCov models of Section 3 to obtain a forecast of the entire
261 matrix $\hat{H}_{t|t-1}$. In a second step, one implies the copula parameter from this
262 forecast, i.e., one sets the moment conditions to $g_{ij,t}(\theta) = \hat{h}_{ij,t|t-1} - f_{ij,t}(\theta)$,
263 where $i < j$ and $i, j = 1, \dots, d$. Minimization of (9) yields $\hat{\theta}_{t|t-1}^{\text{MM}}$, which
264 together with the diagonal elements in $\hat{H}_{t|t-1}$ can be used to parametrize
265 $C_{\hat{\theta}_{t|t-1}}\{F_1(x_1; \hat{h}_{1,t|t-1}), \dots, F_d(x_d; \hat{h}_{d,t|t-1})\}$. This could be advantageous since
266 forecasting RCov matrices first and then mapping into the copula parameter
267 could reduce the loss of information that may be associated with only
268 modeling the time series of the copula parameter. We include this second
269 modeling approach, which was suggested by a referee, in our set of test models
270 too.

271 As is discussed in Bauer and Vorkink (2011) and Chiriac and Voev
272 (2011), an unbiased prediction of the variables parameterizing the covariance
273 matrix will generally not yield unbiased covariance forecasts when the
274 transformation between both is nonlinear. This issue also applies to the
275 present model, since the relationship between the copula parameter and
276 covariance as presented by Hoeffding’s lemma is nonlinear. However, since

277 we consider one-day value at risk forecasts only, we imagine these biases to
 278 be small; see also Halbleib and Voev (2011) for corroborative evidence. As
 279 in Chiriac and Voev (2011), we therefore skip the bias adjustment.

280 2.3. Simulation study

281 In this section, we evaluate the performance of the (generalized) moment
 282 estimator in comparison with the usual ML estimators. This is a vital
 283 question as we exploit only second order moments, which might seem too
 284 few to estimate efficiently and accurately the copula shape parameter. The
 285 purpose is therefore not to simulate a RCop model, but to understand better
 286 how well a single-day estimate of the copula shape parameter performs if
 287 the underlying assumptions and in particular the second order moments are
 288 generated from the assumed population distribution.

We proceed as follows. Given an assumption on a copula family, we draw $n = 1,000$ vector-valued random variates from the copula based on standard normal margins (we consider dimensions $d = 2$ and $d = 3$). From these draws, the sample covariance matrix is estimated by the unbiased covariance estimator. Afterward, the method-of-moment estimators outlined in Section 2.1 are applied. For comparison, we use the maximum likelihood estimator

$$\hat{\vartheta}_{\text{ML}} = \arg \max_{\vartheta} \mathcal{L}(\vartheta) \quad (14)$$

$$= \arg \max_{\vartheta} \sum_{i=1}^n \log \left[c\{F_1(x_{1,i}; \eta_1), \dots, F_d(x_{d,i}; \eta_d); \theta\} \right] \quad (15)$$

$$+ \sum_{i=1}^n \sum_{j=1}^d \log f_j(x_{j,i}; \eta_j), \quad (16)$$

289 where $\vartheta = \vartheta(\theta, \eta_1, \dots, \eta_d)$ denotes the parameter vector of the multivariate
 290 distribution that splits into the parameters of the copula θ and those of
 291 the marginals η_j , $j = 1, \dots, d$. Furthermore, n is the sample size, $c(\cdot; \theta)$ is
 292 the copula density and $f_j(\cdot; \eta_j)$, $j = 1, \dots, d$, are the marginal densities.
 293 Optimization can be achieved either by a one-step maximization or by
 294 a two-step procedure, the so-called inference function for margins (IFM)
 295 method owed to Joe and Xu (1996). In the latter case, one first estimates
 296 the parameters of the marginals η_j , $j = 1, \dots, d$, and given these estimates,
 297 the parameter of the copula function θ . While less efficient than the one-step

Figure 1: Monte Carlo simulation. Top panel: two-dimensional case with Gumbel copula. Lower panel: three-dimensional case with Gumbel copula. Left: ML-IFM estimator. Right: moment estimator based on Hoeffding's lemma. Numbers are computed based on 1,000 estimates of the respective parameters. Each parameter estimate is based on a sample size of 1,000 randomly drawn observations. Red line is the mean of the differences between estimate and true value, blue the median (hardly visible). Shaded area is the 95% interval based on the 1,000 repetitions.

estimation, the IFM method is computationally more convenient. We will use the IFM method therefore.

For illustration, we choose the Gumbel copula and repeat the exercise 1,000 times; the simulation results of other copulae, such as the Clayton copula, are similar and are therefore not reported. For both estimators, we compute the differences of the estimates from the true parameter value. In Figure 1, the mean (red) and the median (blue) difference are shown as

functions of the underlying Gumbel copula parameter. The shaded areas are the 95% pointwise confidence intervals computed from the 1,000 repetitions. The top panel in Figure 1 features the two-dimensional case, the lower panel the three-dimensional case. As is apparent, the moment-based estimator is unbiased and only slightly less efficient than the ML estimator. For the two cases, both estimators perform very similarly.

3. Competitor models

To assess the empirical significance of the RCop model, we will study its ability to forecast a one-day ahead value at risk (VaR). Here, we present the benchmark models we will use. We choose classical representatives that make use of only daily data, only high-frequency data, or both. They can be classified into three groups. First, we study constant copula models estimated in a rolling window fashion and in a locally adaptive manner based on a local change point detection algorithm. Second, we include three dynamic conditional copula models with GARCH margins. Third, we consider two standard RCov models. With regard to different copulae and margins, we consider 26 models altogether.

For these models, we assume that the d -dimensional return vector r_t is given by

$$r_t = \mu_t + \epsilon_t, \quad (17)$$

where $\mu_t = (\mu_{1,t}, \dots, \mu_{d,t})^\top$ is an $\mathcal{F}_{t-1}$ -measurable mean process and $\epsilon_t|_{\mathcal{F}_{t-1}} \sim F_t$, where F_t denotes a multivariate distribution function that may depend on time-varying parameters. In line with the assumptions on the RCop model in (11), we will assume for all competitor models that $\mu_t = 0$, the exception being the Patton (2004) model which requires a nonzero mean to model variation in the copula parameter.

3.1. Rolling window and local change point detection for copulae

The rolling window approach estimates the copula parameter on a fixed window of size w , while the locally adaptive change point (LCP) detection algorithm allows for a time-varying window width. We only informally describe the LCP algorithm here; see Härdle et al. (2013) for more details on the implementation. The underlying theory and further applications in volatility modeling and risk management can be found in Spokoiny (1998), Härdle et al. (2003), Mercurio and Spokoiny (2004), Chen et al. (2008), Čížek et al. (2009), Giacomini et al. (2009), Spokoiny (2009), and Chen et al. (2010).

339 The rolling window model as well as the local change point detection
 340 approach rest on the assumption that the copula structure is constant
 341 (homogeneous) on the window of estimation. For the rolling window
 342 approach, this window is assumed to be constant and equal to $w = 200$ days.
 343 In the LCP method, the window width may vary and its length depends on
 344 the outcome of a sequential testing procedure. On their respective intervals
 345 of homogeneity, both models are estimated by the IFM method as described
 346 in Section 2.3. For the marginal processes, we assume $\mu_{i,t} = 0$, $i = 1, \dots, d$
 347 and the innovations $\epsilon_{i,t}$ are assumed to be driven by $\mathbf{N}(0, h_i)$ variates, where
 348 the variance h_i , $i = 1, \dots, d$, is estimated from the (daily) returns on the
 349 respective homogeneous time interval.

350 To explain the LCP method in more detail, let θ_t denote the time varying
 351 but otherwise unknown copula parameter. Locally adaptive estimation
 352 selects for each time point t_0 an interval I during which θ_t is reasonably well
 353 approximated by a constant θ^* . A possible measure of discrepancy between
 354 two copulae $C_\theta(\cdot)$ and $C_{\theta'}(\cdot)$ is the Kullback-Leibler divergence denoted by

$$\mathcal{K}\{C_{\theta'}(\cdot), C_\theta(\cdot)\} = \mathbb{E}_{\theta'} \log\{c(\cdot; \theta')/c(\cdot; \theta)\} , \quad (18)$$

355 where $c(\cdot; \theta)$ is the copula density. The aim is to select I as close as possible
 356 to the so-called “oracle” choice interval I_{k^*} , defined as the largest interval
 357 $I = [t_0 - m_{k^*}; t_0]$, for which the small modeling bias condition

$$\Delta_I(\theta) = \sum_{t \in I} \mathcal{K}\{C_{\theta_t}(\cdot), C_\theta(\cdot)\} \leq \Delta, \text{ for some } \Delta \geq 0, \theta , \quad (19)$$

358 is fulfilled. The LCP is based on sequentially testing the hypotheses of
 359 homogeneity on intervals I_k . We select I_k with $k = -1, 0, 1, \dots$ as the
 360 sequence of intervals $I_k \subset I_{k+1}$, starting with $k = 1$. If there are no change
 361 points in $\mathcal{T}_k \subset I_k \setminus I_{k-1}$, we accept I_k as an interval with a constant copula
 362 structure. At the next step, we take $\mathcal{T}_{k+1}$ and test it for homogeneity.
 363 We repeat these steps until rejection or until the largest possible interval
 364 I_K is accepted, leading to an interval $I_{\hat{k}}$. The smallest possible interval
 365 of homogeneity is set to 40 days; for this choice and more details on the
 366 implementation see Giacomini et al. (2009) and Härdle et al. (2013).

367 To obtain forecasts of the estimated parameters in the rolling window
 368 and LCP approach, we extrapolate the current estimates to the following
 369 day, i.e., we hold them constant. The logic of this “degenerated prediction”
 370 is that both approaches assume that the parameters involved are estimated

371 on a local interval of homogeneity. It is therefore natural to assume that this
 372 interval of homogeneity continues to hold on the following day. As another
 373 benefit, we avoid fitting time series models on estimates obtained from
 374 overlapping return data, which may invalidate standard statistical inference.

375 3.2. Dynamic copula models

376 3.2.1. The copula DCC model

377 An important benchmark model for VaR predictions with daily data is
 378 the copula dynamic conditional correlation (DCC) model; see Jin (2010).
 379 This model combines the multivariate covariance dynamics of the DCC-type
 380 with marginals that are driven by jointly dependent innovations. In our
 381 implementation of the model, we use a GJR-GARCH(1,1) specification
 382 for the marginal variance processes and the correlation dynamics of the
 383 symmetric DCC model owed to Engle (2002). The mean process is $\mu_t = 0$.
 384 The standardized residuals $\varepsilon_{i,t} = r_{i,t} / \sqrt{h_{i,t}}$, $i = 1, \dots, d$, are assumed to be
 385 generated from a time-varying Student's t -copula such that

$$\left(\varepsilon_{1,t}, \dots, \varepsilon_{d,t} \right)^\top \sim C_{\nu, R_t} \{ F_{1,t}, \dots, F_{d,t} \},$$

386 where R_t denotes the conditional correlation matrix of the DCC model and ν
 387 the degrees of freedom parameter. The copula varies only in the correlation
 388 matrix, while the degrees of freedom parameter is constant. It should be
 389 noted, however, that this does not imply that the model has constant tail
 390 dependence; this is because the coefficients of tail-dependence of the t -copula
 391 also depend on the elements of R_t . The margins are assumed to be normal
 392 and GED distributed. The model is estimated by three-stage ML estimation
 393 using the R-package `rmgarch`; see Ghalanos (2014).

394 3.2.2. The Patton (2004) model

Our next benchmark model is the dynamic conditional copula model
 suggested by Patton (2004). Here, each marginal return series follows an
 AR(1)-GARCH(1,1) model. For this model, one cannot assume $\mu_{i,t} = 0$
 since the dynamics of the copula parameter depend on the mean equation.
 Specifically, this model assumes that the standardized residuals $\varepsilon_{i,t} = (r_{i,t} -$
 $\mu_{i,t}) / \sqrt{h_{i,t}}$, $i = 1, \dots, d$, are drawn from a time-varying copula such that

$$\left(\varepsilon_{1,t}, \dots, \varepsilon_{d,t} \right)^\top \sim C_{\theta_t} \{ F_{1,t}, \dots, F_{d,t} \}, \quad (20)$$

and

$$\theta_t = \Lambda \left(\sum_{i=0}^d \gamma_i \mu_{i,t} \right), \quad (21)$$

where for $i = 0, \dots, d$, γ_i and $\mu_{i,t}$ (with $\mu_{0,t} = 1$), denote unknown parameters and the conditional means of the return processes, respectively. Thus, the copula shape parameter is allowed to fluctuate with the conditional means of the processes. Furthermore, $\Lambda(x)$ is a function ensuring that θ_t is in the allowed domain of the copula: we use $\Lambda(x) = \exp(x)$ for the Clayton copula and $\Lambda(x) = \exp(x) + 1$ for the rotated Gumbel copula. This model can also be estimated by means of a multi-stage ML approach. The Patton model assumes GED margins.

3.2.3. The GAS model

The third dynamic copula model we consider is in autoregressive score (GAS) form as suggested in Creal et al. (2013). The specification we choose is as follows. As for the Patton model, the marginal returns are assumed to be driven by a GARCH(1,1) with GED innovations. The mean process is $\mu_t = 0$. Define standardized residuals as $\varepsilon_{i,t} = r_{i,t} / \sqrt{h_{i,t}}$, $i = 1, \dots, d$ and denote their probability integral transform by $u_{i,t} = F_{i,t}(\varepsilon_{i,t})$, $i = 1, \dots, d$. Set $u_t = (u_{1,t}, \dots, u_{d,t})^\top$. The dynamics of the copula parameter are modeled by the autoregressive updating equation

$$\Lambda(\theta_t) = \omega + \beta \Lambda(\theta_{t-1}) + \alpha s_{t-1}, \quad (22)$$

$$\text{where} \quad s_{t-1} = S_{t-1} \nabla_{t-1} \quad (23)$$

$$\text{and} \quad \nabla_{t-1} = \frac{\partial \log c(u_{t-1}; \theta_{t-1})}{\partial \theta_{t-1}}, \quad (24)$$

and $\Lambda(\cdot)$ is a monotonic mapping ensuring that the copula parameter θ_t remains in its allowed domain (see Section 3.2.2 for our selections), S_{t-1} is a scaling matrix, and $c(\cdot; \theta)$ denotes the copula density. In the GAS model, the lagged score of the copula model acts as the forcing variable. As shown by Creal et al. (2013), for suitable selections of the scaling matrix (e.g., the inverse of the Hessian, its square root, or simply the unit matrix), this specification nests many familiar models discussed in the literature. Since we find very similar estimates for the copula parameters for various selections of S_t , we set S_t equal to the identity matrix for the sake of simplicity. The models are estimated with the rotated Gumbel and the Clayton copula.

414 3.2.4. The GRAS model

The GRAS model has the same structure and is based on the same assumptions as the GAS model above, but one augments the autoregressive updating equation (22) with a realized measure RM_t estimated from high-frequency data (whence De Lira Salvatierra and Patton (2013) called it GRAS). The updating equation reads as

$$\Lambda(\theta_t) = \omega + \beta\Lambda(\theta_{t-1}) + \alpha s_{t-1} + \gamma RM_{t-1} . \quad (25)$$

415 In our specification, we use as realized measure an average of realized
416 correlation, i.e., $RM_t = \frac{2}{d(d-1)} \sum_{i=1}^d \sum_{j>i}^d \hat{\rho}_{t,ij}$, where $\hat{\rho}_{t,ij}$ denotes an
417 estimate of realized correlations obtained from high frequency data. Similar
418 suggestions are made in De Lira Salvatierra and Patton (2013).

419 3.3. Realized covariance models

420 While at the univariate level the HAR formulation as described in
421 Section 2.2 is an undisputed baseline model (Corsi et al., 2012), the literature
422 has yet not found agreement on its most competitive multi-variate extension.
423 We therefore consider two approaches.

424 In Bauer and Vorkink (2011), RCov is modeled by means of the matrix
425 exponential and its inverse function, the matrix logarithm. The matrix
426 exponential is a function of a square matrix A and is given by the series
427 representation

$$H = \expm(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k . \quad (26)$$

428 As the most important property of (26), if A is a real and symmetric, $H =$
429 $\expm(A)$ is a real, positive-definite matrix. Since the converse is true as well,
430 the inverse function of the matrix exponential, the matrix logarithm,

$$A = \logm(H) , \quad (27)$$

431 is a useful device to guarantee that predicted covariance matrices are
432 positive-definite.

433 Given a time series of RCov matrices H_t , $t = 1, \dots, T$, of size $d \times d$, Bauer
434 and Vorkink (2011) suggest applying the matrix logarithm, $A_t = \logm(H_t)$.
435 Now, A_t , $t = 1, \dots, T$, forms a time series of symmetric $d \times d$ matrices. As
436 a next step, the vech-operator is applied

$$a_t = \text{vech}(A_t) , \quad (28)$$

which stacks the upper triangular of A_t columnwise into a $\frac{1}{2}d(d+1) \times 1$ vector. The vector time series a_t is now modeled along the lines of the univariate HAR model, i.e., by forming elementwise weekly and monthly aggregates of daily components. The resulting forecasting models for these aggregates take exactly the same form as presented in (13), i.e., one considers

$$a_t = \beta_0 + \beta_D a_{t-1}^D + \beta_W a_{t-1}^W + \beta_M a_{t-1}^M + u_t, \quad (29)$$

where the β 's are scalar parameters and $a_t^D = a_t$ are daily, $a_t^W = \frac{1}{5} \sum_{i=0}^4 a_{t-i}$ weekly, and $a_t^M = \frac{1}{22} \sum_{i=0}^{21} a_{t-i}$ monthly averages of past realizations of the vector-valued factors a_t , and u_t is an error term with mean zero and finite variance. Based on an estimate of this model, one forms predictions $\hat{a}_{t|t-1}$. By first applying the reverse vech-operator and then the matrix exponential to these predictions, the predicted covariance matrix is obtained by means of $\hat{H}_{t|t-1} = \expm(\hat{A}_{t|t-1})$, which is positive-definite as long as the elements in $\hat{a}_{t|t-1}$ are real.

In Chiriac and Voev (2011), the time series of covariance matrices is decomposed into a series of Cholesky factors, i.e., (27) is replaced by

$$AA^\top = H, \quad (30)$$

where A is a real upper triangular $d \times d$ matrix with positive diagonal elements. As before, applying first (30) and subsequently the vech-operator to a time series H_t gives rise to the vector-valued time series a_t which is modeled as in (29). Predictions $\hat{a}_{t|t-1}$ are converted to positive-definite predicted covariance matrices by applying the reverse vech-operator, which yields an upper triangular matrix $\hat{A}_{t|t-1}$, and by computing the matrix product $\hat{H}_{t|t-1} = \hat{A}_{t|t-1} \hat{A}_{t|t-1}^\top$.

For the VaR predictions of both models, we set $\mu_t = 0$. The return distribution is assumed to be multivariate normal parametrized with the forecast of the RCov matrix, i.e., $\hat{H}_{t|t-1}$. This corresponds to the usual assumptions for these models.

4. Empirical part

After describing the data, we study the in-sample estimates of the copula parameters of the models under consideration. The discussion of the out-of-sample backtesting results follows in Section 4.3.

4.1. Data description, data filtering, and realized (co)variance estimation

The empirical part of this work is based on high-frequency stock price data obtained from the TAQ database for the period from March 1, 2006 to May 31, 2013. The stocks we consider are IBM, Google, Oracle, Pfizer, and Exxon, which are among the most heavily traded names on NASDAQ and NYSE. For our empirical analysis, we form two three-dimensional portfolios: a homogeneous technologies sector portfolio consisting of Google-IBM-Oracle and a heterogeneous portfolio that comprises stocks from various sectors, namely IBM-Pfizer-Exxon. We choose these two types of test portfolios in order to be able to judge whether the reduction of the covariances into a single copula parameter comes at the cost of accuracy for heterogeneous portfolios.

High frequency data are known to be noisy such that the accuracy of the RV and RCov estimates can be impaired. We therefore subject the data to the filtering procedure established by Barndorff-Nielsen et al. (2009) for TAQ data, which comprises the following steps:

1. Delete entries outside 9:30-16:00 and with zero transaction price.
2. Delete entries with corrected trades or abnormal sale condition.
3. Replace multiple trades for the same time stamp by the median price.
4. Delete entries with prices above ask plus bid-ask spread or below bid minus bid-ask spread.
5. Delete thin trading days.

After applying this cleaning procedure we estimate RCov matrices by the realized kernel estimator due to Barndorff-Nielsen et al. (2011); see Appendix A for all relevant details on the procedure. The realized kernel estimator warrants a positive-definite estimate of RCov and is robust to market micro structure noise, such as non-synchronous trading and the bid-ask-bounce. Descriptive statistics on estimated RCov are displayed in the upper panels of Tables 2 and 3. Between the two tables, the entries for RV of IBM differ. This is due to refresh time sampling of the realized kernel estimator. Since both covariance matrices are estimated separately, refresh time sampling for both RV series differs for the two portfolios implying slightly differing estimates. An alternative would be to increase dimension and to estimate directly the six-dimensional RCov. However, as a consequence of refresh time sampling, fewer data observations would be used in the resulting estimator. We therefore prefer to compute the smaller dimensional estimates.

	Min.	Median	Mean	Max.	Std.
$RV(\text{Google}) \times 10^5$	2.81	21.17	38.35	929.70	59.52
$RV(\text{IBM}) \times 10^5$	1.48	10.07	21.37	1144.00	47.77
$RV(\text{Oracle}) \times 10^5$	3.96	23.19	35.39	996.60	49.82
$RCov(\text{Google}, \text{IBM}) \times 10^5$	-0.13	4.61	11.13	490.40	25.31
$RCov(\text{Google}, \text{Oracle}) \times 10^5$	-1.97	6.09	13.25	488.20	27.18
$RCov(\text{IBM}, \text{Oracle}) \times 10^5$	-1.18	4.66	11.21	511.70	25.86
$\hat{\theta}_t^{\text{MM}}(\text{rGumbel})$	1.01	1.27	1.30	1.99	0.17
$\hat{\theta}_t^{\text{MM}}(\text{Clayton})$	0.02	0.54	0.59	2.11	0.35

Table 2: Descriptive statistics of the realized kernels (variances and covariances) and realized copulae (method of moments for Clayton and rotated Gumbel copulae) of the Google-IBM-Oracle portfolio. Sample period: March 1, 2006 to May 31, 2013.

	Min.	Median	Mean	Max.	Std.
$RV(\text{IBM}) \times 10^5$	1.40	9.94	21.40	1141.00	48.70
$RV(\text{Pfizer}) \times 10^5$	1.91	13.99	23.65	894.90	37.42
$RV(\text{Exxon}) \times 10^5$	1.61	14.19	27.86	2233.00	72.62
$RCov(\text{IBM}, \text{Pfizer}) \times 10^5$	-2.29	2.82	7.67	449.10	20.61
$RCov(\text{IBM}, \text{Exxon}) \times 10^5$	-3.54	3.76	10.21	648.40	27.77
$RCov(\text{Pfizer}, \text{Exxon}) \times 10^5$	-1.96	3.18	9.03	552.60	24.74
$\hat{\theta}_t^{\text{MM}}(\text{rGumbel})$	1.00	1.22	1.25	1.96	0.16
$\hat{\theta}_t^{\text{MM}}(\text{Clayton})$	0.01	0.43	0.51	2.06	0.33

Table 3: Descriptive statistics of the realized kernels (variances and covariances) and realized copulae (method of moments for Clayton and rotated Gumbel copulae) of the IBM-Pfizer-Exxon portfolio. Sample period: March 1, 2006 to May 31, 2013.

497 In Figure 2, we display the time series of realized correlations of the
498 two portfolios. The top panel shows the homogeneous portfolio containing
499 Google-IBM-Oracle. As is to be expected, all realized correlations track each
500 other very closely. The second portfolio with IBM-Pfizer-Exxon, which is seen
501 in the lower panel, exhibits more heterogeneous paths of realized correlation.
502 Since these intra-day high-frequency estimates capture only an
503 open-to-close (co)variance, they need to be adjusted to be usable for

	Min.	Median	Mean	Max.	Std.	Skewn.	Kurt.
Google	-0.103	0.000	0.000	0.182	0.02	0.48	8.88
IBM	-0.087	0.001	0.000	0.111	0.02	0.02	5.08
Oracle	-0.124	0.000	0.000	0.121	0.02	0.07	5.06
Pfizer	-0.108	0.000	0.000	0.098	0.02	0.08	4.98
Exxon	-0.159	0.000	0.000	0.167	0.02	0.18	14.66

Table 4: Descriptive statistics of the daily log return data of the stocks used for the out-of-sample VaR prediction. Sample period is December 15, 2004 to May 31, 2013.

estimating a VaR for a close-to-close P&L distribution; see Andersen et al. (2013). To this end, we follow Halbleib and Voev (2011) and adjust each (co)variance estimate by an average scaling factor that incorporates the overnight information. This factor is obtained by the “simple scaling method” suggested by Hansen and Lunde (2005). As a drawback of this method, one could be concerned about mixing high-frequency information from during the trading period and overnight low frequency information. While this is true, the scaling method relies on a single constant; consequently, this “mix of information” appears to be very mild.

For the estimation of the competitor models which are based on daily data and for the out-of-sample VaR predictions, we use daily closing price data on these five stocks. The actual estimation periods differ between the models. The RCop and the RCov models, the rolling window approach, the Patton, the GAS and the GRAS model are estimated on a 200 days window first starting March 1, 2006. The LCP method requires an initialization phase which starts September 7, 2009. The actual intervals of homogeneity found by this procedure and then used for estimation and prediction can be seen in the lower panels of Figures 3 and 4. The DCC models showed very inadequate VaR predictions on small samples, and we decided to base the estimation on 1,000 observations. For the Google-IBM-Oracle portfolio, the estimation of the DCC models starts August 20, 2004, since Google Inc. was first traded in public on August 19, 2004. We therefore cannot test the DCC models on the entire data set for the Google-IBM-Oracle portfolio. For the other portfolio, estimation starts March 12, 2002. The descriptive statistics of the subsample of daily returns, on which we evaluate the quantile estimates (this is from December 15, 2006 to May 31, 2013) can be found in Table 4.

Figure 2: Realized correlation estimated by means of the realized kernel estimator between March 1, 2006 and May 31, 2013. Top panel: Google-IBM-Oracle portfolio. Lower panel: IBM-Pfizer(PFE)-Exxon(XOM) portfolio.

530 4.2. In-sample results

531 For the empirical application, we consider an Archimedean copula, the
 532 Clayton copula, and the rotated Gumbel (rGumbel) copula, which is not
 533 Archimedean. Both copulae exhibit lower tail-dependence, which is likely to
 534 be crucial for modeling risk measures for stock portfolios. In-sample results

for the estimated copula parameters are displayed in the lower panels of Tables 2 and 3.

In the first portfolio (Google-IBM-Oracle) in Table 2, the estimated parameters of the rotated Gumbel fluctuate between one (independence case) and quite substantial dependence of around two (Kendall's tau is about 0.5), with the mean estimate being $\theta_{\text{rGum}}^{\text{MM}} \approx 1.30$. The estimated Clayton copula parameters are between zero (independence case) and about two. The mean is 0.59. For the second portfolio (IBM-Pfizer-Exxon) in Table 3, the estimates of the copula shape parameters are lower. Here, estimates are on average around 1.25 for the rotated Gumbel copula and 0.51 for the Clayton copula.

In the top panels of Figures 3 and 4, we plot the time series of the realized Clayton shape parameters (red line). The time series of the copula parameter exhibits substantial variation, but follows pronounced trends reflecting different levels of dependence in the market. We can identify three periods of increased dependence: the second half of 2008 to the beginning of 2009, mid-2010, and end of 2011. These three periods correspond to the market unrest caused by the default of Lehman Brothers in 2008, the European sovereign debt crisis, which worsened mid-2010 in the face of a potential default of Greece, and the U.S. debt ceiling crisis concentrating around August 2, 2011, the presumable date on which the U.S. government might have exhausted its borrowing authority. These time series patterns of the realized Clayton copula parameter suggest very slowly decaying autocorrelation functions, as we confirm in Figure 5.

Apart from the Rcop parameter, the top panels of Figures 3 and 4 also feature the time series of the Clayton copula parameters of the rolling window (black line, window size 200 days), the adaptive LCP method (blue line), and the Patton and the GRAS model (green and lighter green lines, respectively). Since the copula parameters of the GAS model are very close to those of the GRAS model, they are omitted. The different time series reveal that the copula structure estimated by the GRAS (GAS) model and to a lesser extent the LCP method are close. These models also feature the most variable parameter tracks. On the other hand, the rolling window and the Patton model are similar and have smooth parameter tracks.

Figure 3: Estimated copula parameters for the case of the Clayton copula on the Google-IBM-Oracle portfolio. Top panel: rolling window (black line), LCP method (blue line), and RCop (red line), Patton model (green), GRAS model (lighter green). Lower panel: estimated interval length of the LCP procedure (blue line) and (constant) interval length of rolling window approach (200 days, black line). LCP (blue) is run with $m_0 = 40$, which is the smallest possible interval of homogeneity.

Figure 4: Estimated copula parameters for the case of the Clayton copula on the IBM-Pfizer-Exxon portfolio. Top panel: rolling window (black line), LCP method (blue line), and RCop (red line), Patton model (green), GRAS model (lighter green). Lower panel: estimated interval length of the LCP procedure (blue line) and (constant) interval length of rolling window approach (200 days, black line). LCP (blue) is run with $m_0 = 40$, which is the smallest possible interval of homogeneity.

Figure 5: Sample autocorrelations and robust standard errors of the Clayton realized copula parameter. Top panel: Google-IBM-Orcacle portfolio. Lower panel: IBM-Pfizer-Exxon portfolio.

568 Since the LCP method estimates and formally tests for intervals of
569 homogeneity, this method allows for deeper insights into the dynamics of the
570 dependence structure. In the lower panels of Figures 3 and 4, we compare
571 the estimated interval lengths of the LCP method with the rolling window
572 approach. Up to 2009, for both portfolios, the LCP method tends to identify
573 rather short intervals of homogeneity with lengths of less than 100 days. It
574 is first between mid-2009 to early 2010 that the LCP method recovers much
575 larger interval widths for both portfolios. Toward the end of the sample,
576 we again find shorter intervals for the technology sector portfolio, whereas
577 the interval widths for the mixed sector portfolio fluctuates more strongly. It
578 remains to be checked in the out-of-sample analysis whether the adaptiveness
579 of dependence structure is critical for the precision of daily VaR estimates.

580 In the lower panels of Tables 5 and 6, we provide average in-sample
581 estimates for the RCop forecasting models. Estimation is accomplished by
582 ordinary least squares. The size of the estimates for the log-RV models are
583 as reported in the literature; see Corsi (2009), among others. The dynamics
584 of RV are mainly driven by yesterday's realization. The influence of the
585 weekly component is only half as large (or less). The monthly RV aggregate
586 is smallest in magnitude. This contrasts with the covariance dynamics as
587 implied by the copula parameters. The shape parameters of the RCop models
588 appear to be mainly driven by the daily and weekly aggregates, with the latter
589 even outweighing the first one in the IBM-Pfizer-Exxon portfolio; see Table 6.
590 As before, the monthly component is of minor importance. These findings
591 suggest that the dynamics of the spillovers and the (lower) tail-dependence
592 as reflected by the time-varying copula parameter are more sluggish than
593 those of RV: after some initial shock, the cross-sectional dependence tends to
594 subside more slowly than RV, still allowing the system to maintain high
595 coefficients of tail-dependence and, as a consequence, a high probability
596 of incurring simultaneous price deteriorations in all stocks, even when the
597 individual variances might have moderated already.

598 *4.3. Out-of-sample results*

599 Since the RCop model has a time-dependent copula structure, for which
600 only a single cross-sectional vector of observations at a daily level is available,
601 it appears difficult to subject the copula assumption to a specification test.
602 As a means of model validation, we therefore backtest the model VaR
603 out-of-sample.

604 *4.3.1. Portfolio construction and backtesting procedure*

605 Computing risk measures for portfolios of stocks followed by a backtesting
 606 analysis are standard procedures in applied risk management; see, e.g.,
 607 Berkowitz and O'Brien (2002), Audrino and Barone-Adesi (2005), Giacomini
 608 et al. (2009), Jin (2010), Berkowitz et al. (2011), among others. Closest to
 609 our research is Giot and Laurent (2004), who appear to be among the first
 610 to include both low and high-frequency data for such an analysis.

611 For portfolio risk-management, the aggregate portfolio and loss (P&L)
 612 distribution must be determined. Consider a portfolio where we denote the
 613 number of shares in the portfolio by $a_t = (a_{1,t}, \dots, a_{d,t})^\top$ with $a_{i,t} \in \mathbb{R}$. The
 614 market value V_t of this portfolio is given by

$$V_t = \sum_{j=1}^d a_{j,t} S_{j,t} , \quad (31)$$

615 where $S_{j,t}$ is the asset price. In this study, we consider only portfolios which
 616 are equally weighted in terms of wealth allocation. This implies that $a_{j,t} =$
 617 $w_j V_t / S_{j,t}$ where $w_j = 1/d$, $j = 1, \dots, d$. Hence, absolute portfolio weights are
 618 adjusted on a daily basis in order to keep the relative contributions constant.

619 The daily trading P&L on this portfolio is given by

$$L_{t+1} = (V_{t+1} - V_t) = \sum_{j=1}^d a_{j,t} S_{j,t} \{ \exp(r_{j,t+1}) - 1 \} , \quad (32)$$

620 where $r_{j,t}$ denotes the log-return on asset j . Denote the conditional
 621 distribution function of L by

$$F_{L_{t+1}|\mathcal{F}_t}(x) = P(L_{t+1} \leq x | \mathcal{F}_t) . \quad (33)$$

622 For our empirical analysis, we employ the VaR at level α , which is the
 623 most widely applied risk measure in practice. The $\text{VaR}(\alpha)$ is defined as the
 624 α -quantile of $F_{L_{t+1}|\mathcal{F}_t}$:

$$\text{VaR}_{t+1|\mathcal{F}_t}(\alpha) = F_{L_{t+1}|\mathcal{F}_t}^{-1}(\alpha) . \quad (34)$$

625 It follows that $F_{L_{t+1}|\mathcal{F}_t}$ is determined by the d -dimensional distribution of
 626 log-returns. The accuracy of the VaR estimates therefore depends on how
 627 well the RCop model and the alternative approaches presented in Section 3

capture the unknown multi-dimensional conditional distribution of daily returns.

To approximate the quantile in (34), we proceed as follows. As described in Section 4.1, we estimate each model on a trailing time window $[t-w, t]$, for every t . The estimated parameters determine the multivariate conditional return distributions for the rolling window, the LCP method, the copula DCC model, the Patton and the G(R)AS models. For the RCop and the RCov models, we estimate the dynamic equations in (13) and (29) and compute their one-day ahead predictions. As described in Sections 2.2 and 3.3, these predictions together with the distributional assumptions determine the conditional return distributions of these models. We then draw 100,000 random variates from the return distributions of all models; the algorithms to simulate from the copulae used in this work (Clayton, rotated Gumbel, t -copula) can be found, for instance, in McNeil et al. (2005). We then compute the $(t+1)$ -portfolio P&L according to (32). Finally, we calculate $\text{VaR}_{t+1|\mathcal{F}_t}(\alpha)$ as the empirical α -quantile from the simulated P&L distribution.

In the backtesting analysis, one compares the predicted VaR with the realized P&L. A variety of statistical criteria have been suggested in the literature to do so; see Campbell (2006) and Christoffersen (2009) for overviews. Let $\{l_t\}$ be the realizations of the respective P&L distribution. Unconditional coverage testing focuses on the exceedances ratio of the respective VaR. The exceedances ratio $\hat{\alpha}$ is defined by

$$\hat{\alpha} = \frac{N}{T}$$

and $N = \sum_{t=1}^T \mathbf{I}\{l_t < \widehat{\text{VaR}}_t(\alpha)\},$

where N denotes the number of observed exceedances and $\mathbf{I}\{\cdot\}$ is an indicator function. A likelihood ratio test based on binomial theory for $H_0 : \hat{\alpha} = \alpha$ is

$$LR_{uc} = 2 \log \frac{\hat{\alpha}^N (1 - \hat{\alpha})^{T-N}}{\alpha^N (1 - \alpha)^{T-N}},$$

which has asymptotically a $\chi^2(1)$ distribution under H_0 . This test is due to Kupiec (1995). We also consider a simple t -test based on the normal approximation for the binomial distribution and independence testing based on Ljung-Box statistics as suggested by Berkowitz et al. (2011).

651 4.3.2. Backtesting results

652 The out-of-sample evaluation period is from December 15, 2006 to May
653 31, 2013, i.e., the quantile forecasts are evaluated on about six and a half
654 years of data on 1588 sample days altogether.

655 Tables 7 summarizes the results for the one-day ahead quantiles of 1%,
656 5%, and 10% for the Google-IBM-Oracle portfolio. We find that the RCop
657 models perform very well at any quantile. Among all models, their hitting
658 ratios are close to the target levels, and the p -values of the Kupiec test,
659 the t -test, and the Berkowitz et al. (2011) test are among the highest.
660 Moreover, we cannot distinguish between the two copulae (Clayton versus
661 rotated Gumbel). The margins (normal versus GED) are also of minor
662 importance. This is because daily returns that are standardized with RV
663 often appear to be close to normally distributed; see Andersen et al. (2001a).
664 This also holds for our RCop models. It is also irrelevant whether we use
665 the HAR model to predict the copula parameter (models no. 3, 6, 14, 17)
666 or whether we first forecast a covariance matrix based on one of the RCov
667 models and infer the copula parameter in a second step (models no. 4, 5, 7,
668 8, 15, 16, 18, 19). Thus, for small portfolios, the loss in information when
669 modeling only the copula parameter rather than the entire covariance matrix
670 is negligible. Models of similar predictive power for the VaR are the Patton
671 model, the GAS, and the GRAS model. Statistically, one cannot reject these
672 models, however, their hitting ratios are less good. For instance, the GAS
673 and the GRAS model do not match well the low 1% quantile. The t -copula
674 DCC model is disappointing. While the 1% quantiles are matched by both
675 specifications, the 10% quantiles are far from satisfactory. We recall, however,
676 that we can evaluate this model only on a subsample of our evaluation period.

677 The rolling window and the LCP method are among the inferior models.
678 They apparently have difficulty matching the 1% quantile. Both models,
679 however, work reasonably well at the 5% and the 10% quantile. Similar
680 observations apply to the RCov models, which assume normal margins and
681 no tail-dependent multivariate dependence structure. They do not match
682 the 1% and the 5% quantile, but are acceptable at the 10% quantiles. It is
683 also worthwhile to compare the RCov models with the RCop models which
684 use the same marginal (normal) assumption (models no. 3, 4, 5, 14, 15,
685 16). Note that the models are estimated either from the same informational
686 basis, i.e., the estimates of RCov (models no. 3, 14), or even from the
687 same forecasts (models no. 4, 5, 15, 16). This comparison evidences that

	β_0		β_D		β_W		β	
	mean	std	mean	std	mean	std	mean	std
Google	-1.713	1.095	0.455	0.113	0.289	0.137	0.058	0.110
IBM	-1.865	1.382	0.463	0.109	0.319	0.117	0.018	0.127
Oracle	-1.602	0.734	0.479	0.109	0.216	0.117	0.116	0.132
$\hat{\theta}_t^{\text{MM}}$ (rGumbel)	0.060	0.034	0.383	0.077	0.302	0.187	0.068	0.102
$\hat{\theta}_t^{\text{MM}}$ (Clayton)	-0.185	0.175	0.385	0.079	0.303	0.197	0.072	0.107

Table 5: Average parameter estimates and standard deviations of the HAR models for the Google-IBM-Oracle portfolio. Parameters are averaged over all estimates obtained for the 200 days' rolling window used for the daily VaR predictions. Sample period: December 15, 2006 to May 31, 2013.

688 non-trivial forms of tail-dependence make the difference for the strikingly
689 different VaR predictions of these models.

690 For the second portfolio (IBM-Pfizer-Exxon) the findings are similar;
691 see Table 8. In general, all RCop models match the 1% and the 10%
692 quantiles very well. The results on the 5% quantile are mixed. While the
693 RCop models, which are based on the Chiriac and Voev (2011) forecast,
694 do well (models no. 5, 8, 16, 19), the other RCop models are weak. This
695 difference in performance between the two portfolios could be due to the more
696 heterogeneous composition of the second portfolio. The Patton model, the
697 GAS and the GRAS model do not cover well the 1% quantile. They all are
698 too aggressive. The copula DCC models perform very well on all quantiles.
699 For the RCov models, we again find that 10% quantiles are well matched,
700 but the 5% and the 1% quantiles are failed.

701 In Figures 6 and 7, we present the exceedances plots for the 1%-VaR
702 risk for both portfolios. The figures elucidate the findings of the previous
703 tables. As is visible in the top panels of both figures, the rolling window
704 and the LCP method exhibit a smoother quantile history than other models.
705 Yet the plots also reveal that the rolling window approach and – albeit to a
706 lesser extent – the adaptive LCP method detect market outbursts too late.
707 Therefore, it is unlikely that another selection of the margins will remedy this
708 deficiency. In contrast, the other models, i.e., the copula DCC, the GRAS
709 (GAS), the Patton, the RCop models, and the RCov model owed to Chiriac
710 and Voev (2011) respond more quickly to shocks in the economy and their

	β_0		β_D		β_W		β	
	mean	std	mean	std	mean	std	mean	std
IBM	-1.866	1.371	0.460	0.112	0.322	0.133	0.019	0.125
Pfizer	-1.982	1.405	0.467	0.109	0.227	0.125	0.086	0.162
Exxon	-1.587	0.863	0.478	0.097	0.338	0.096	0.007	0.116
$\hat{\theta}_t^{\text{MM}}$ (rGumbel)	0.042	0.025	0.321	0.111	0.412	0.122	0.070	0.090
$\hat{\theta}_t^{\text{MM}}$ (Clayton)	-0.181	0.104	0.287	0.093	0.463	0.158	0.081	0.091

Table 6: Average parameter estimates and standard deviations of the HAR models for the IBM-Pfizer-Exxon portfolio. Parameters are averaged over all estimates obtained for the 200 days' rolling window used for the daily VaR predictions. Sample period: December 15, 2006 to May 31, 2013.

quantiles widen accordingly (as do those of the Bauer and Vorkink (2011) model which are not shown for the sake of space). In Figure 6, this is nicely visible for the second half of 2011, where many exceedances occur. While the rolling window and the LCP method do not detect these outbursts, the other models do and only a few exceedances are recorded. Like the RCop models, the RCov models also respond to these events, but since they have zero tail-dependence, they are not sufficiently fat-tailed. This leads to an overly large number of exceedances. The same patterns can be observed for the second portfolio in Figure 7.

4.3.3. Assessing the overall model fit

The results in the previous section show that RCop models are among the most preferred models for VaR predictions. Although it is not the main purpose of this paper, one may ask whether the accuracy in the tails comes at the price of a lower fit in the overall distribution. We assess this question by evaluating the predictive log-likelihood of the forecast portfolio density using the model confidence set approach as articulated in Hansen et al. (2003) and Hansen et al. (2011a).

We run the model confidence tests on significance levels of $\alpha = 1\%$ and $\alpha = 0.1\%$. This implies the confidence sets $\mathcal{M}_\alpha$ are guaranteed to contain any given model with confidence levels of 99% and 99.9%. The implementation strictly follows the recommendations in Hansen et al. (2003), i.e., we block-bootstrap the standard errors with a block length size of 12, the bootstrap sample contains 5,000 runs, and we display results for

734 the range statistic and the semi-quadratic statistic for model evaluation.
735 The results are robust to smaller block length sizes such as 6 and 9. We
736 evaluate the predictive density at the realized portfolio return on our 100,000
737 random draws by using the Rosenblatt-Parzen kernel density estimator with
738 a Gaussian kernel function. Silverman’s rule of thumb is used for the
739 bandwidth selection.

740 Tables 9 and 10 present the results. The technology portfolio in Table 9
741 does not take the copula DCC model into account as we could only
742 assess this model on a smaller sample size. In both tables, two findings
743 are remarkable. First, predominantly models estimated on high-frequency
744 data enter the model confidence sets for both tests (range versus the
745 semi-quadratic statistics). This includes the RCop models, the RCov models
746 by Bauer and Vorkink (2011) and Chiriac and Voev (2011), and the GRAS
747 model (Google-IBM-Oracle only). Based on the less robust semi-quadratic
748 statistics, the GAS and the Patton model are also among the 99.9% superior
749 models for the Google-IBM-Oracle portfolio. Thus, whereas for the RCop
750 models, the Patton, and the GRAS model the tail predictions and the
751 overall distributional fit are good, there is indeed a trade-off for the RCov
752 models which are both the most preferred models according to the predictive
753 likelihood. Second, while the Clayton and rotated Gumbel copula are not
754 distinguishable in the VaR predictions, there is a preference for the rotated
755 Gumbel copula when evaluating the predictive likelihood; using similar
756 high-frequency data, De Lira Salvatierra and Patton (2013) make a similar
757 observation.

758 In summary, the results from assessing the predictive likelihood support
759 the importance of high-frequency data for modeling events at a lower
760 frequency. This corroborates findings of Engle and Gallo (2006), Ghysels
761 et al. (2006), and De Lira Salvatierra and Patton (2013), among others.

762 5. Conclusion

763 We introduce a new time-varying conditional copula framework called
764 the realized copula (RCop) model. Based on assumptions on the marginal
765 distributions and the copula family, the copula parameter is estimated from
766 a realized covariance matrix inferred from within-day high-frequency data in
767 a method-of-moments fashion using Hoeffding’s lemma. The resulting time
768 series of copula parameters is captured using a heterogeneous autoregressive

769 model which is well established in the realized variance literature. In our
770 empirical analysis, we test several variants of this modeling approach.

771 RCop models allow us to move beyond the usual Gaussian structure
772 typically adopted by realized variance models. They are highly responsive to
773 shocks in the economic system as are standard realized variance models,
774 but at the same time allow for tail-dependence. In an out-of-sample
775 VaR backtesting analysis, we demonstrate the relevance of this feature.
776 Comparing our approach with a rolling window, an adaptive change point
777 algorithm, three dynamic conditional copula models, and two realized
778 covariance models, we find the RCop models among the most preferred
779 models. Specifically, they capture the important 1% quantile of the profit
780 and loss distributions to an excellent degree.

781 Our empirical results demonstrate the usefulness of intra-day data for
782 modeling events at a lower frequency as pioneered by Engle and Gallo (2006)
783 and Ghysels et al. (2006). Our test portfolios, however, are small. It would
784 therefore be desirable to carry the RCop approach to larger dimensions. As
785 discussed, possible avenues in this direction are mixtures of copulae (Cai
786 and Wang, 2014), hierarchical Archimedean copulae (Whelan, 2004; Härdle
787 et al., 2013; Savu and Trede, 2010), or vine copulae (Joe, 1996; Bedford and
788 Cooke, 2002). Factor copulae, as recently investigated by Patton and Oh
789 (2013), could also be considered. We leave these topics for future research.

No.	Model	0.01				0.05				0.10			
		$\hat{\alpha}$	K	T	BCP	$\hat{\alpha}$	K	T	BCP	$\hat{\alpha}$	K	T	BCP
1	roll. window, Cl.	0.0151	0.057	0.095	0.604	0.0466	0.530	0.520	0.995	0.0957	0.567	0.562	0.997
2	LCP, Cl.	0.0164	0.019	0.045	0.362	0.0453	0.387	0.372	0.980	0.0913	0.242	0.229	0.928
3	RCop, HAR, Cl.	0.0139	0.145	0.189	0.831	0.0504	0.945	0.945	1.000	0.1045	0.550	0.555	0.996
4	RCop, BaVo, Cl.	0.0132	0.219	0.261	0.911	0.0497	0.963	0.963	1.000	0.1058	0.445	0.453	0.989
5	RCop, ChVo, Cl.	0.0088	0.628	0.614	0.999	0.0485	0.781	0.779	1.000	0.1001	0.987	0.987	1.000
6	RCop, HAR, GED, Cl.	0.0132	0.219	0.261	0.911	0.0497	0.963	0.963	1.000	0.1058	0.445	0.453	0.989
7	RCop, BaVo, GED, Cl.	0.0132	0.219	0.261	0.911	0.0497	0.963	0.963	1.000	0.1058	0.445	0.453	0.989
8	RCop, ChVo, GED, Cl.	0.0088	0.628	0.614	0.999	0.0485	0.781	0.779	1.000	0.0995	0.947	0.947	1.000
9	Patton, GED, Cl.	0.0120	0.445	0.471	0.989	0.0460	0.455	0.443	0.990	0.0976	0.750	0.748	1.000
10	GAS, GED, Cl.	0.0145	0.092	0.135	0.726	0.0466	0.530	0.520	0.995	0.1058	0.445	0.453	0.989
11	GRAS, GED, Cl.	0.0139	0.145	0.189	0.831	0.0497	0.963	0.963	1.000	0.1026	0.726	0.728	1.000
12	roll. window, rG.	0.0151	0.057	0.095	0.604	0.0416	0.112	0.092	0.774	0.0850	0.042	0.032	0.527
13	LCP, rG.	0.0164	0.019	0.045	0.362	0.0409	0.087	0.068	0.712	0.0869	0.076	0.064	0.676
14	RCop, HAR, rG.	0.0120	0.445	0.471	0.989	0.0510	0.854	0.855	1.000	0.1102	0.182	0.194	0.878
15	RCop, BaVo, rG.	0.0132	0.219	0.261	0.911	0.0523	0.681	0.685	0.999	0.1108	0.157	0.169	0.848
16	RCop, ChVo, rG.	0.0101	0.976	0.976	1.000	0.0497	0.963	0.963	1.000	0.1058	0.445	0.453	0.989
17	RCop, HAR, GED, rG.	0.0126	0.318	0.354	0.963	0.0504	0.945	0.945	1.000	0.1102	0.182	0.194	0.878
18	RCop, BaVo, GED, rG.	0.0132	0.219	0.261	0.911	0.0516	0.766	0.768	1.000	0.1096	0.210	0.222	0.905
19	RCop, ChVo, GED, rG.	0.0094	0.823	0.819	1.000	0.0504	0.945	0.945	1.000	0.1058	0.445	0.453	0.989
20	Patton, GED, rG.	0.0120	0.445	0.471	0.989	0.0435	0.221	0.200	0.913	0.0945	0.458	0.450	0.990
21	GAS, GED, rG.	0.0132	0.219	0.261	0.911	0.0447	0.325	0.308	0.965	0.0970	0.687	0.684	0.999
22	GRAS, GED, rG.	0.0132	0.219	0.261	0.911	0.0472	0.609	0.603	0.998	0.0932	0.361	0.351	0.975
23	DCC, <i>t</i> -copula	0.0120	0.447	0.473	0.989	0.0418	0.125	0.105	0.799	0.0751	0.001	0.000	0.037
24	DCC, GED, <i>t</i> -copula	0.0102	0.921	0.922	1.000	0.0393	0.042	0.028	0.531	0.0743	0.000	0.000	0.027
25	RCov, BaVo	0.0239	0.000	0.000	0.000	0.0655	0.007	0.013	0.196	0.1108	0.157	0.169	0.848
26	RCov, ChVo	0.0170	0.011	0.031	0.260	0.0586	0.127	0.146	0.802	0.1077	0.313	0.323	0.961

Table 7: VaR performance for the Google-IBM-Oracle portfolio. Given is the hitting ratio $\hat{\alpha}$ and the p -values of the Kupiec test (K), the t -test (T), and the Berkowitz et al. (2011) test (BCP) in brackets; p -values indicating rejection at the 10% level are bold-faced. Cl. = Clayton copula, rG = rotated Gumbel copula, BaVo = Bauer and Vorkink (2011), ChVo = Chiriac and Voev (2011), GED indicates GED margins – otherwise margins are normal. Evaluation period: December 15, 2006 to May 31, 2013, except for the DCC models which is August 13, 2008 to May 31, 2013.

No.	Model	0.01			0.05			0.10					
		$\hat{\alpha}$	K	T	BCP	$\hat{\alpha}$	K	T	BCP	$\hat{\alpha}$	K	T	BCP
1	roll. window, Cl.	0.0252	0.000	0.000	0.000	0.0554	0.330	0.346	0.967	0.0945	0.458	0.450	0.990
2	LCP, Cl.	0.0202	0.000	0.004	0.026	0.0586	0.127	0.146	0.802	0.0938	0.408	0.399	0.984
3	RCop, HAR, Cl.	0.0094	0.823	0.819	1.000	0.0605	0.064	0.080	0.633	0.1089	0.241	0.253	0.927
4	RCop, BaVo, Cl.	0.0094	0.823	0.819	1.000	0.0605	0.064	0.080	0.633	0.1058	0.445	0.453	0.989
5	RCop, ChVo, Cl.	0.0082	0.453	0.423	0.990	0.0529	0.600	0.606	0.998	0.0970	0.687	0.684	0.999
6	RCop, HAR, GED, Cl.	0.0107	0.780	0.785	1.000	0.0592	0.102	0.121	0.750	0.1071	0.354	0.363	0.973
7	RCop, BaVo, GED, Cl.	0.0088	0.628	0.614	0.999	0.0598	0.081	0.099	0.693	0.1058	0.445	0.453	0.989
8	RCop, ChVo, GED, Cl.	0.0088	0.628	0.614	0.999	0.0535	0.524	0.532	0.995	0.0976	0.750	0.748	1.000
9	Patton, GED, Cl.	0.0183	0.003	0.014	0.117	0.0586	0.127	0.146	0.802	0.0995	0.947	0.947	1.000
10	GAS, GED, Cl.	0.0151	0.057	0.095	0.604	0.0592	0.102	0.121	0.750	0.1052	0.496	0.502	0.993
11	GRAS, GED, Cl.	0.0145	0.092	0.135	0.726	0.0611	0.050	0.065	0.572	0.1058	0.445	0.453	0.989
12	roll. window, rG.	0.0202	0.000	0.004	0.026	0.0516	0.766	0.768	1.000	0.0806	0.008	0.005	0.216
13	LCP, rG.	0.0170	0.011	0.031	0.260	0.0516	0.766	0.768	1.000	0.0856	0.051	0.041	0.578
14	RCop, HAR, rG.	0.0088	0.628	0.614	0.999	0.0611	0.050	0.065	0.572	0.1096	0.210	0.222	0.905
15	RCop, BaVo, rG.	0.0088	0.628	0.614	0.999	0.0642	0.013	0.021	0.284	0.1102	0.182	0.194	0.878
16	RCop, ChVo, rG.	0.0069	0.193	0.140	0.889	0.0567	0.232	0.250	0.921	0.1039	0.606	0.610	0.998
17	RCop, HAR, GED, rG.	0.0082	0.453	0.423	0.990	0.0617	0.039	0.052	0.510	0.1096	0.210	0.222	0.905
18	RCop, BaVo, GED, rG.	0.0094	0.823	0.819	1.000	0.0642	0.013	0.021	0.284	0.1102	0.182	0.194	0.878
19	RCop, ChVo, GED, rG.	0.0069	0.193	0.140	0.889	0.0579	0.157	0.176	0.848	0.1039	0.606	0.610	0.998
20	Patton, GED, rG.	0.0151	0.057	0.095	0.604	0.0573	0.191	0.210	0.888	0.0957	0.567	0.562	0.997
21	GAS, GED, rG.	0.0151	0.057	0.095	0.604	0.0611	0.050	0.065	0.572	0.1045	0.550	0.555	0.996
22	GRAS, GED, rG.	0.0170	0.011	0.031	0.260	0.0598	0.081	0.099	0.693	0.1026	0.726	0.728	1.000
23	DCC, t -copula	0.0195	0.001	0.006	0.044	0.0542	0.453	0.464	0.990	0.0932	0.361	0.351	0.975
24	DCC, GED, t -copula	0.0088	0.628	0.614	0.999	0.0491	0.872	0.871	1.000	0.0913	0.242	0.229	0.928
25	RV, BaVo	0.0239	0.000	0.000	0.000	0.0712	0.000	0.001	0.021	0.1108	0.157	0.169	0.848
26	RV, ChVo	0.0189	0.002	0.009	0.074	0.0668	0.003	0.007	0.129	0.1033	0.665	0.668	0.999

Table 8: VaR performance for the IBM-Pfizer-Exxon portfolio. Given is the hitting ratio $\hat{\alpha}$ and the p -values of the Kupiec test (K), the t -test (T), and the Berkowitz et al. (2011) test (BCP) in brackets; p -values indicating rejection at the 10% level are bold-faced. Cl. = Clayton copula, rG = rotated Gumbel copula, BaVo = Bauer and Vorkink (2011), ChVo = Chiriac and Voev (2011), GED indicates GED margins – otherwise margins are normal. Evaluation period: December 15, 2006 to May 31, 2013.

No.	Model	avg. pred. logLik	p_R	p_{SQ}
1	roll. window, Cl.	-3.4415	0.0000	0.0002
2	LCP, Cl.	-3.3819	0.0000	0.0002
3	RCop, Cl.	-3.3442	0.0000	0.0006
4	RCop, BaVo, Cl.	-3.3390	0.0000	0.0012*
5	RCop, ChVo, Cl.	-3.3361	0.0006	0.0020*
6	RCop, GED, Cl.	-3.3421	0.0000	0.0008
7	RCop, BaVo, GED, Cl.	-3.3419	0.0000	0.0004
8	RCop, ChVo, GED, Cl.	-3.3357	0.0006	0.0052*
9	Patton, GED, Cl.	-3.3521	0.0000	0.0004
10	GAS, GED, Cl.	-3.3554	0.0000	0.0004
11	GRAS, GED, Cl.	-3.3492	0.0000	0.0008
12	roll. window, rG.	-3.4252	0.0000	0.0002
13	LCP, rG.	-3.3854	0.0000	0.0002
14	RCop, rG.	-3.3202	0.0486**	0.0534**
15	RCop, BaVo, rG.	-3.3199	0.0486**	0.0558**
16	RCop, ChVo, rG.	-3.3173	0.0692**	0.0916**
17	RCop, GED, rG.	-3.3196	0.0486**	0.0558**
18	RCop, BaVo, GED, rG.	-3.3197	0.0486**	0.0640**
19	RCop, ChVo, GED, rG.	-3.3168	0.0832**	0.1060**
20	Patton, GED, rG.	-3.3468	0.0006	0.0016*
21	GAS, GED, rG.	-3.3464	0.0006	0.0018*
22	GRAS, GED, rG.	-3.3392	0.0130**	0.0210**
23	DCC, t -copula	—	—	—
24	DCC, GED, t -copula	—	—	—
24	RV, BaVo	-3.3115	0.1844**	0.1844**
25	RV, ChVo	-3.3073	1.0000**	1.0000**

Table 9: The table gives the model confidence sets based on the predictive log-likelihood for the Google-IBM-Oracle portfolio. Models with predictive log-likelihoods in $\mathcal{M}_{0.001}$ and in $\hat{\mathcal{M}}_{0.01}$ are marked with one star and two stars, respectively. Note that $\hat{\mathcal{M}}_{0.01} \subset \mathcal{M}_{0.001}$. p_R and p_{SQ} denote the p -values based on the range statistic and the semi-quadratic statistic, respectively; see Hansen et al. (2003) for details. Cl. = Clayton copula, rG = rotated Gumbel copula, BaVo = Bauer and Vorkink (2011), ChVo = Chiriac and Voev (2011), GED indicates GED distribution for marginals, otherwise normal. Evaluation period: December 15, 2006 to May 31, 2013.

No.	Model	avg. pred.	logLik	p_R	p_{sq}
1	roll. window, Cl.	-1.9163	0.0000	0.0000	0.0000
2	LCP, Cl.	-1.8519	0.0000	0.0000	0.0000
3	RCop, Cl.	-1.7747	0.0002	0.0002	0.0002
4	RCop, BaVo, Cl.	-1.7703	0.0000	0.0000	0.0002
5	RCop, ChVo, Cl.	-1.7673	0.0002	0.0002	0.0002
6	RCop, GED, Cl.	-1.7737	0.0000	0.0000	0.0002
7	RCop, BaVo, GED, Cl.	-1.7723	0.0000	0.0000	0.0002
8	RCop, ChVo, GED, Cl.	-1.7678	0.0002	0.0002	0.0002
9	Patton, GED, Cl.	-1.8188	0.0000	0.0000	0.0000
10	GAS, GED, Cl.	-1.8069	0.0000	0.0000	0.0000
11	GRAS, GED, Cl.	-1.8073	0.0000	0.0000	0.0002
12	roll. window, rG.	-1.8916	0.0000	0.0000	0.0000
13	LCP, rG.	-1.8555	0.0000	0.0000	0.0000
14	RCop, rG.	-1.7565	0.0010*	0.0008	0.0008
15	RCop, BaVo, rG.	-1.7570	0.0010*	0.0006	0.0006
16	RCop, ChVo, rG.	-1.7531	0.0012*	0.0014*	0.0014*
17	RCop, GED, rG.	-1.7562	0.0010*	0.0008	0.0008
18	RCop, BaVo, GED, rG.	-1.7569	0.0010*	0.0006	0.0006
19	RCop, ChVo, GED, rG.	-1.7528	0.0012*	0.0016*	0.0016*
20	Patton, GED, rG.	-1.8078	0.0000	0.0002	0.0002
21	GAS, GED, rG.	-1.8097	0.0000	0.0002	0.0002
22	GRAS, GED, rG.	-1.8031	0.0000	0.0002	0.0002
23	DCC, t -copula	-1.7934	0.0002	0.0002	0.0002
24	DCC, GED, t -copula	-1.7922	0.0000	0.0002	0.0002
25	RV, BaVo	-1.7437	0.0632**	0.0632**	0.0632**
26	RV, ChVo	-1.7395	1.0000**	1.0000**	1.0000**

Table 10: The table gives the model confidence sets based on the predictive log-likelihood for the IBM-Pfizer-Exxon portfolio. Models with predictive log-likelihoods in $\mathcal{M}_{0.001}$ and in $\hat{\mathcal{M}}_{0.01}$ are marked with one star and two stars, respectively. Note that $\hat{\mathcal{M}}_{0.01} \subset \mathcal{M}_{0.001}$. p_R and p_{SQ} denote the p -values based on the range statistic and the semi-quadratic statistic, respectively; see Hansen et al. (2003) for details. Cl. = Clayton copula, rG = rotated Gumbel copula, BaVo = Bauer and Vorkink (2011), ChVo = Chiriac and Voev (2011), GED indicates GED distribution for marginals, otherwise normal. Evaluation period: December 15, 2006 to May 31, 2013.

Figure 6: Exceedances plots for the VaR(0.01) on the Google-IBM-Oracle portfolio. From top left to lower right: Rolling window, LCP method, copula DCC, GRAS model, Patton model, RCop model with GED margins, RCop model based on Bauer and Vorkink (2011), Chiriac and Voev (2011) model. Profit & loss (blue dots), the lower VaR(0.01) (green solid line), exceedances (red crosses).

Figure 7: Exceedances plots for the VaR(0.01) on the IBM-Pfizer-Exxon portfolio. From top left to lower right: Rolling window, LCP method, copula DCC, GRAS model, Patton model, RCop model with GED margins, RCop model based on Bauer and Vorkink (2011), Chiriac and Voev (2011) model. Profit & loss (blue dots), the lower VaR(0.01) (green solid line), exceedances (red crosses).

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797 Appendix A. Estimation of realized variance

On the intra-day level, we adopt the model framework used in the realized variance literature described in Barndorff-Nielsen et al. (2011). We suppose that during the trading day $[t - 1; t]$ the log-price process follows a Brownian semimartingale which is superimposed by market micro structure noise. More precisely, denote the d -dimensional efficient equilibrium price process by

$$Y_t = Y_{t-1} + \int_{t-1}^t \sigma_u dW_u ,$$

798 where σ_t càdlàg is a volatility matrix process and W_t is a d -dimensional
 799 vector of independent Brownian motions. At observation times $t - 1 = \tau_0 <$
 800 $\tau_1 < \dots < \tau_N = t$, we record the efficient price plus a covariance stationary
 801 additive component

$$P_{\tau_i} = Y_{\tau_i} + U_{\tau_i} ,$$

802 where $E(U_{\tau_i}) = 0$ and $\sum_h |h\Omega_h| < \infty$ with $\Omega_h = \text{Cov}(U_{\tau_i}, U_{\tau_i-h})$ for
 803 $h > 0$. The purpose is to estimate the quadratic variation of Y , i.e.,
 804 $[Y]_{t,t-1} = \int_{t-1}^t \Sigma_u du$ with $\Sigma = \sigma\sigma^\top$. An ex-post estimate of integrated
 805 covariance $[Y]_{t,t-1} = \int_{t-1}^t \Sigma_u du$ will be called *realized covariance* and was
 806 denoted by H_t in Section 2.1. For the estimation of $[Y]$ from discrete,
 807 non-synchronous, and noisy price observations, we use the realized kernel
 808 estimator, which is described in the following.

809 The multivariate realized kernel method introduced by Barndorff-Nielsen
 810 et al. (2011) yields consistent, positive semi-definite estimates of the
 811 covariation of equity prices in the presence of noise and non-synchronous
 812 trading. Data synchronization is accomplished via refresh time sampling
 813 (RTS); see also Harris et al. (1995) and Hautsch et al. (2012). The refresh

814 times are defined as the times needed for all the assets in the portfolio to trade
815 or to refresh posted prices. After all assets are traded, the most recent price
816 is used to form the RTS time scale. Formally, the first refresh time is defined
817 as $\tau_1^* = \max\{\tau_{1,1}, \dots, \tau_{d,1}\}$. All subsequent refresh times are defined as $\tau_{i+1}^* =$
818 $\arg \min\{\tau_{j,k_j} | \tau_{j,k_j} > \tau_i^*, \forall k_j = 1, \dots, N_j; j \in 1 \dots d\}$, where N_j denotes the
819 number of price observations made for asset j . This synchronization leads to
820 a new high-frequency vector of returns $p_i = P_{\tau_i^*} - P_{\tau_{i-1}^*}$, where $i = 1, \dots, n$,
821 and n is the number of RTS observations. With RTS, the sample size n of
822 retained data depends on the degree of non-synchronicity between assets.

The multivariate realized kernel estimator is defined as

$$K(P) = \sum_{h=-H}^H k\left(\frac{|h|}{H+1}\right) \Gamma_h ,$$

823 with Γ_h being a matrix of autocovariances given by

$$\Gamma_h = \begin{cases} \sum_{j=|h|+1}^n p_j p_{j-h}^\top, & h \geq 0 \\ \sum_{j=|h|+1}^n p_{j-h} p_j^\top, & h < 0 \end{cases} ,$$

824 and $k(x)$ being the Parzen kernel

$$k(x) = \begin{cases} 1 - 6x^2 + 6x^3 & 0 \leq x \leq 1/2 \\ 2(1-x)^3 & 1/2 \leq x \leq 1 \\ 0 & x > 1 \end{cases} .$$

825 In the estimation of the multivariate bandwidth parameter H , we strictly
826 follow the suggestions outlined in Barndorff-Nielsen et al. (2009).

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