On Shape-preserving Capability of Cubic L^1 Spline Fits

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Abstract

Cubic L^1 spline fits have shown some favorable shape-preserving property for geometric data. To quantify the shape-preserving capability, we consider the basic shape of two parallel line segments in a given window. When one line segment is sufficiently longer than the other, the spline fit can preserve its linear shape in at least half of the window. We propose to use the minimum of such length difference as a shape-preserving metric because it represents the extra information that the spline fits need to preserve the shape. We analytically calculate this metric in a 3-node window for second-derivative-based, first-derivative-based and function-value-based spline fits. In a 5-node window, we compute this metric numerically. In both cases, the shape-preserving metric is rather small, which explains the observed strong shape-preserving capability of spline fits. Moreover, the function-value-based spline fits are indicated to preserve shape better than the other two types of spline fits. This study initiates a quantitative research on shape preservation of L^1 spline fits.

Keywords: L^1 spline, shape preservation, approximation, spline fit, univariate

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1. Introduction

Shape preservation is an important objective in the interpolation and approximation of data with abrupt changes in magnitude and/or spacing, such as natural and urban terrain, geophysical features, robot paths, financial data and many other irregular phenomena. Over the past decade and more, L^1 splines have been developed for multiscale univariate [1, 2, 3, 4] and bivariate [5, 6, 7] data and have shown excellent performances in shape preservation, such as L^1 interpolating splines [8, 9, 10, 11, 12], L^1 smoothing splines [13, 3] and L^1 spline fits [14, 15]. Applications on urban terrain reconstruction can be seen, for example, in [16] and [17].

Cubic L^1 spline fits are defined by minimizing an L^1 data fitting functional over a manifold of L^1 interpolating splines. Such L^1 interpolating splines are calculated by minimizing an L^1 interpolating functional. Regarding to the order of derivative that is involved in the L^1 interpolating functional, there are three types of spline fits which are second-derivative based, first-derivative based and function-value based, respectively [18, 19, 20]. Global algorithms and local moving-window approach have been designed and both produce spline fits that preserve shapes well [14, 15, 10, 12].

Nevertheless, to the best of our knowledge, no theoretic analysis on the shape-preserving capability of the L^1 spline fits is available while the demand for it is growing. Fundamentally, such a theory should define shape preservation in a mathematical form so that different data-approximating principles could be fairly compared. In particular, comparison of the three type of spline fits is of interest. Practically, it should enable us to explain or predict the approximating results of cubic L^1 spline fits in various circumstances. It could help users improve the spline nodes placement or acquire more data where needed. This study is the first step towards this direction. We propose a metric to assess the capability of spline fits in preserving a given linear shape. Then we compare second-derivative-based, first-derivative-based and function-value-based spline fits against this metric. Throughout this paper, by “spline fits” we refer to the univariate cubic L^1 spline fits. Bivariate splines, multivariate splines and polynomials of higher orders are beyond the scope of this paper.

One of the basic principles of shape preservation is that, if some portion of the data is aligned along a straight line segment, the spline fit should reconstruct that line segment. We consider two parallel horizontal line segments, which is a simple and often seen pattern in irregular and multiscale

geometric data. For example, a building and its ground around may form such a pattern. This pattern is characterized by a Heaviside step function. Another practical significance of Heaviside step function is in G01 codes in CNC machining, where there is often data lying along a straight line and they need to be fitted by splines [21]. Spline fits, however, are designed to be C^1 -smooth and thus cannot perfectly recover the jump between the two parallel line segments. Additionally, the spline nodes in applied scenarios are usually fixed a priori. Then, the shape-preserving performance of a spline fit is highly dependent on the relative positions of spline nodes and data. In this study, we approximate the Heaviside step function by spline fits in a window of odd-number equally spaced spline nodes. Naturally, the definition of spline fits is generalized to handle not only discrete data but also functions. In this window, the two parallel line segments may have different lengths. Without loss of generality, we assume the line segment on the left is longer than the line segment on the right. When this length difference is large enough, the spline fit will preserve the linear shape of the longer line segment in at least the left half of the window. The shorter line segment is not necessarily preserved. The reason to this phenomenon is that the longer line segment provides sufficient extra information that leads the spline fit to preserve its linear shape. This extra information can be represented by the length difference between the line segments. Hence, we define the “shape-preserving metric” to be the minimum of such length difference that the linear shape of the longer line segment can be preserved. A smaller metric value indicates the stronger shape-preserving capability.

To calculate the shape-preserving metric, we develop an analytic approach in a 3-node window and a numerical procedure in a 5-node window. In the 3-node window, to keep the information contained in the Heaviside step function from being completely lost, we impose boundary conditions that the first and third spline node have the same function value and first derivative as the Heaviside step function. Then, we can solve the spline fit problem analytically and obtain the explicit relation between the length difference and the function value and first derivative at the middle spline node. Thus, the metric is the minimum length difference such that the middle spline node has the same function value and first derivative as the Heaviside step function. In the 5-node window, we keep increasing the length difference by a given precision and solve the spline fit problem numerically. The metric is the corresponding length difference when the linear shape is preserved for the first time. No boundary conditions are fixed in this case.

Results in 3-node and 5-node windows show that all three types of spline fits have relatively small shape-preserving metrics compared to the space between adjacent spline nodes. Furthermore, we find the second-derivative-based spline fits yield the biggest metric value and the function-value-based spline fits have the smallest. This suggests that the function-value-based spline fits may perform better in shape preservation than the other two types of spline fits.

The remaining of this paper is organized as follows. In Section 2, we extend the second-derivative-based cubic L^1 spline fits to function approximation and introduce first-derivative-based spline fits and function-value-based spline fits. In Section 3, we define the shape-preserving metric by approximating the Heaviside step function in a given window. In Section 4, we calculate the shape-preserving metrics for the three types of spline fits in a 3-node window with fixed boundary conditions. In Section 5, we numerically compute the shape-preserving metrics in a 5-node window. Section 6 summarizes the paper with some concluding remarks.

2. Spline Fits for Functions

Cubic L^1 spline fits are originally designed in [14] to be used for approximating discrete data and they are based on second derivatives of the splines. In this section, we extend the use of spline fits to function approximation and introduce first-derivative and function-value-based spline fits. Suppose the function to be approximated is $f(x)$, $-\infty < x < \infty$. Let the monotonically increasing spline nodes be x_i , $i = 0, 1, \dots, I$. The function values and first derivatives of the spline fit at these nodes are denoted by z_i and b_i , respectively. Let $\mathbf{z} = (z_0, z_1, \dots, z_I)$ and $\mathbf{b} = (b_0, b_1, \dots, b_I)$. Also let $h_i = x_{i+1} - x_i$ and $\Delta z_i = \frac{z_{i+1} - z_i}{h_i}$. We consider C^1 -smooth piecewise cubic polynomials $z(x)$ with nodes x_i , $i = 0, 1, \dots, I$, of the form

$$z(x) = z_i + b_i(x - x_i) + \frac{1}{h_i} [-(2b_i + b_{i+1}) + 3\Delta z_i] (x - x_i)^2 + \frac{1}{h_i^2} [b_i + b_{i+1} - 2\Delta z_i] (x - x_i)^3 \quad (1)$$

in each interval $[x_i, x_{i+1}]$, $i = 0, 1, \dots, I-1$. A cubic L^1 spline fit is a function $z(x)$ of form (1) that minimizes the L^1 fitting functional

$$F(\mathbf{z}, \mathbf{b}) := \int_{x_0}^{x_I} |z(x) - f(x)| dx$$

105 over a manifold of cubic L^1 interpolating splines. The free parameters in a
 106 cubic L^1 spline fit are the function values z_i , $i = 0, 1, \dots, I$, at the nodes.
 107 For each $\mathbf{z}$, the first derivatives b_i , $i = 0, 1, \dots, I$ of the corresponding inter-
 108 polating spline are calculated by minimizing an L^1 interpolating functional
 109 as described below.

110 The second-derivative-based interpolating functional is

$$G(\mathbf{b}) := \int_{x_0}^{x_I} \left| \frac{d^2 z}{dx^2} \right| dx$$

111 with $\mathbf{z}$ fixed. Therefore, the second-derivative-based spline fit optimization
 112 problem is

$$\begin{aligned} \min_{\mathbf{z}} \quad & \int_{x_0}^{x_I} |z(x) - f(x)| dx \\ \text{s.t.} \quad & \mathbf{b} = \arg \min_{\mathbf{b}} \int_{x_0}^{x_I} \left| \frac{d^2 z}{dx^2} \right| dx. \end{aligned} \quad (2)$$

113 The lower-level problem is called the interpolating spline problem.

114 For the first-derivative-based interpolating spline, the interpolating func-
 115 tional is

$$G(\mathbf{b}) := \sum_{i=0}^{I-1} \int_{x_i}^{x_{i+1}} \frac{1}{h_i} \left| \frac{dz}{dx} - \frac{d\zeta}{dx} \right| dx$$

116 where $\zeta(x)$ is the piecewise linear function that connects (x_i, z_i) , $i = 0, 1, \dots, I$,
 117 i.e.,

$$\zeta(x) = z_i + \frac{\Delta z_i}{h_i} (x - x_i) \quad (3)$$

118 in each interval $[x_i, x_{i+1}]$. Consequently, the first-derivative-based spline fit
 119 optimization problem is

$$\begin{aligned} \min_{\mathbf{z}} \quad & \int_{x_0}^{x_I} |z(x) - f(x)| dx \\ \text{s.t.} \quad & \mathbf{b} = \arg \min_{\mathbf{b}} \sum_{i=0}^{I-1} \int_{x_i}^{x_{i+1}} \frac{1}{h_i} \left| \frac{dz}{dx} - \frac{d\zeta}{dx} \right| dx. \end{aligned}$$

120 The function-value-based interpolating functional is

$$G(\mathbf{b}) := \sum_{i=0}^{I-1} \int_{x_i}^{x_{i+1}} \frac{1}{h_i^2} |z(x) - \zeta(x)| dx$$

121 where $\zeta(x)$ is the same as (3). The function-value-based spline fit optimiza-
122 tion problem is

$$\begin{aligned} \min_{\mathbf{z}} \quad & \int_{x_0}^{x_I} |z(x) - f(x)| dx \\ \text{s.t.} \quad & \mathbf{b} = \arg \min_{\mathbf{b}} \sum_{i=0}^{I-1} \int_{x_i}^{x_{i+1}} \frac{1}{h_i^2} |z(x) - \zeta(x)| dx. \end{aligned}$$

123 For each given $\mathbf{z}$, we denote the optimal solution to the interpolating
124 spline problem by $\mathbf{b}^*(\mathbf{z})$. The optimal solution to the spline fit problem is
125 denoted by $\mathbf{z}^*$.

126 3. Shape-preserving Metric

Consider the following Heaviside step function with a jump at $x = \epsilon \geq 0$:

$$H_\epsilon(x) = \begin{cases} 0, & \text{if } x \leq \epsilon, \\ 1, & \text{if } x > \epsilon. \end{cases}$$

127 $H_\epsilon(x)$ characterizes the geometric pattern of two parallel line segments, as
128 shown in Figure 1. W.l.o.g., the gap between two parallel line segments is
129 set to be 1. We approximate $H_\epsilon(x)$ in a window of $I + 1$ equally spaced
130 spline nodes $x_0 = -\frac{I}{2}, \dots, x_{I/2} = 0, \dots, x_I = \frac{I}{2}$ where $I \geq 2$ is an even
131 number. For convenience, the splines nodes are set to be equally spaced and
132 the space between each two adjacent nodes is 1. General spacing settings
133 can be handled without difficulty. If the spline fit $z(x)$ satisfies $z(x) = H_\epsilon(x)$
134 for all $x_0 = -\frac{I}{2} \leq x \leq 0 = x_{I/2}$, we say the spline fit ϵ -preserves the linear
135 shape. (If not specified otherwise, we use “preserve” instead of “ ϵ -preserve”
136 for simplicity.) For example, it is easy to see that when $\epsilon = \frac{I}{2}$, we have
137 $H_\epsilon(x) = 0$. The spline fit satisfies $z_i = b_i = 0$, $i = 0, \dots, I$ and preserves the
138 linear shape.

139 The spline fit can preserve the linear shape when the longer line seg-
140 ment contains sufficient extra information over the shorter one. Such extra
141 information is represented by the length difference of the two parallel line
142 segments in $H_\epsilon(x)$, which is 2ϵ . For simplicity, we use ϵ to denote this length
143 difference. We propose the shape-preserving metric as defined below.

144 **Definition 1.** The ϵ -shape-preserving metric ϵ^* of cubic L^1 spline fits for
 145 approximating $H_\epsilon(x)$ is defined as the minimum of such ϵ that the spline fit
 146 $z(x)$ satisfies $z(x) = H_\epsilon(x)$ for $x_0 \leq x \leq x_{I/2}$.

147 For simplicity, we use “shape-preserving metric” to refer to “ ϵ -shape-
 148 preserving metric” in the rest of this paper.

149 The existence of ϵ^* is guaranteed. Actually, $\epsilon^* \leq \frac{I}{2}$. As we will show in
 150 the later sections, when $\epsilon < \epsilon^*$, the spline fit fails to preserve the linear shape.
 151 When $\epsilon \geq \epsilon^*$, the spline fit preserves the linear shape. Figure 1 illustrates
 152 that the second-derivative-based spline fit fails to preserve the linear shape of
 153 $H_0(x)$ in a 5-node window. Figure 2 illustrates the second-derivative-based
 154 spline fit preserves the linear shape of $H_{0.5}(x)$ in a 5-node window.

Figure 1: Second-derivative-based spline fit fails to preserve linear shape of $H_0(x)$

155 4. Calculate ϵ^* in a 3-node Window

156 In this section, we calculate ϵ^* for the three types of spline fits in a 3-
 157 node window, i.e., $I = 2$. Noting that a 3-node window is relatively small,
 158 we impose boundary conditions so that the information of $H_\epsilon(x)$ can be
 159 contained at the boundary. Specifically, we set

Figure 2: Second-derivative-based spline fit preserves linear shape of $H_{0.5}(x)$

$$z_0 = 0, b_0 = 0, z_2 = 1 \text{ and } b_2 = 0. \quad (4)$$

160 That means that, the boundary nodes coincide with $H_\epsilon(x)$ in terms of func-
 161 tion values and first derivatives. In this case, the linear shape will be pre-
 162 served for $-1 \leq x \leq 0$ if $z_1 = 0$ and $b_1 = 0$.

163 We adopt an analytic approach. For second-derivative-based spline fits,
 164 we can solve the interpolating spline problem in (2) for the first derivative
 165 b_1 , which is a function in z_1 . Then we solve the spline fit problem (2) and
 166 analyze the relation between the optimal solution z_1^* and ϵ . We prove that
 167 there exists $\epsilon^* > 0$ such that $z_1^* = 0$ if and only if $\epsilon \geq \epsilon^*$. In this case,
 168 $b_1^*(z_1^*) = 0$ and thus spline fit preserves linear shape. The value of ϵ^* can
 169 be analytically calculated. Shape-preserving metrics of first-derivative-based
 170 and function-value-based spline fits are calculated by similar procedures.

171 4.1. Second-derivative-based Spline Fits

172 The solution to the interpolating spline problem is given by the following
 173 theorem:

174 **Theorem 1.** *Given $z_0 = 0, b_0 = 0, z_2 = 1$ and $b_2 = 0$, the first deriva-*
 175 *tive b_1^* that minimizes the second-derivative-based L^1 interpolating functional*

176 $\int_{-1}^1 \left| \frac{d^2 z}{dx^2} \right| dx$ is a function of z_1 with the following explicit form:

$$b_1^*(z_1) = \begin{cases} 0, & \text{if } z_1 \leq 0, \\ 4z_1(1 - z_1), & \text{if } 0 < z_1 \leq \frac{1}{4}, \\ 1 - z_1, & \text{if } \frac{1}{4} < z_1 \leq \frac{2}{5}, \\ \text{smaller root of} \\ z_1 = \frac{1}{12}(6 - \sqrt{2}\sqrt{21b_1^2 - \sqrt{b_1^2(81b_1^2 - 156b_1 + 76)} - 38b_1 + 18}), & \text{if } \frac{2}{5} < z_1 \leq \frac{1}{2}, \\ \text{smaller root of} \\ z_1 = \frac{1}{12}(6 + \sqrt{2}\sqrt{21b_1^2 - \sqrt{b_1^2(81b_1^2 - 156b_1 + 76)} - 38b_1 + 18}), & \text{if } \frac{1}{2} < z_1 \leq \frac{3}{5}, \\ z_1, & \text{if } \frac{3}{5} < z_1 < \frac{3}{4}, \\ 4z_1(1 - z_1), & \text{if } \frac{3}{4} < z_1 \leq 1, \\ 0, & \text{if } z_1 > 1. \end{cases} \quad (5)$$

177 *Proof.* By direct calculation, we have

$$\begin{aligned} z''(-1) &= -2b_1 + 6z_1, & z''(0^-) &= 4b_1 - 6z_1, \\ z''(0^+) &= -4b_1 - 6z_1 + 6, & z''(1) &= 2b_1 + 6z_1 - 6. \end{aligned}$$

178 Note that $z''(x)$ is discontinuous at $x = 0$. The expression of $G(b_1)$ depends
 179 on the signs of $z''(-1)$, $z''(0^-)$, $z''(0^+)$ and $z''(1)$. Accordingly, there are 11
 180 cases as shown in Table 1, along with their defining linear inequalities in
 181 terms of b_1 and z_1 . The second column denotes the signs of $(z''(-1), z''(0^-)$,
 182 $z''(0^+), z''(1))$. Figure 3 illustrates the 11 cases in a (z_1, b_1) plane. Within
 183 each case, we minimize $G(b_1)$ for any given z_1 and denote its minimizer by
 184 $\bar{b}_1(z_1)$. The results are listed in Table 1. We use Case 9 and Case 5 as
 185 examples to show how to find $\bar{b}_1(z_1)$.

186 In Case 9, $z''(-1) \leq 0$, $z''(0^-) \geq 0$, $z''(0^+) \leq 0$ and $z''(1) \geq 0$. The roots
 187 of $z''(x) = 0$ in $[-1, 0]$ and $[0, 1]$ are

$$u_1 = \frac{-2b_1 + 3z_1}{3(b_1 - 2z_1)} \text{ and } u_2 = \frac{2b_1 - 3(1 - z_1)}{3(b_1 - 2(1 - z_1))},$$

188 respectively. Hence, for any fixed z_1 ,

$$\begin{aligned}
G(b_1) &= \int_{-1}^{u_1} -\frac{d^2 z}{dx^2} dx + \int_{u_1}^0 \frac{d^2 z}{dx^2} dx + \int_0^{u_2} -\frac{d^2 z}{dx^2} dx + \int_{u_2}^1 \frac{d^2 z}{dx^2} dx \\
&= 2[b_1 - z'(u_1) - z'(u_2)].
\end{aligned}$$

189 The first derivative of $G(b_1)$ is

$$G'(b_1) = 2 \left[\frac{(b_1 + z_1 - 1)(b_1 + 3z_1 - 3)}{3(b_1 + 2z_1 - 2)^2} + \frac{(b_1 - z_1)(b_1 - 3z_1)}{3(b_1 - 2z_1)^2} + 1 \right].$$

190 Note that $b_1 \geq 3z_1$ and $b_1 \geq 3(1 - z_1)$. Then $G'(b_1) > 0$ for all $z_1 \in \mathbb{R}$.
191 Therefore, the minimizer b_1 of $G(b_1)$ in this case is

$$\bar{b}_1(z_1) = \max\{3z_1, 3(1 - z_1)\}.$$

192 $G'(\bar{b}_1) > 0$ in this case.

193 In Case 5, $z''(-1) \geq 0$, $z''(0^-) \leq 0$, $z''(0^+) \geq 0$ and $z''(1) \leq 0$. We have

$$\begin{aligned}
G(b_1) &= \int_{-1}^{u_1} \frac{d^2 z}{dx^2} dx + \int_{u_1}^0 -\frac{d^2 z}{dx^2} dx + \int_0^{u_2} \frac{d^2 z}{dx^2} dx + \int_{u_2}^1 -\frac{d^2 z}{dx^2} dx \\
&= 2[z'(u_1) + z'(u_2) - b_1]
\end{aligned}$$

194 and

$$G'(b_1) = 2 \left[-\frac{(b_1 + z_1 - 1)(b_1 + 3z_1 - 3)}{3(b_1 + 2z_1 - 2)^2} - \frac{(b_1 - 3z_1)(b_1 - z_1)}{3(b_1 - 2z_1)^2} - 1 \right].$$

195 Within this case, for $z_1 \leq 0$, we have $G'(3z_1) = -\frac{2(3z_1-1)(11z_1-5)}{(5z_1-2)^2} \leq 0$.

196 Consequently, $\bar{b}_1(z_1) = 3z_1$. We also have $\bar{b}_1(z_1) = \frac{3}{2}z_1$ for $0 \leq z_1 \leq \frac{2}{5}$,
197 $\bar{b}_1(z_1) = \frac{3}{2}(1 - z_1)$ for $\frac{3}{5} \leq z_1 \leq 1$, and $\bar{b}_1(z_1) = 3(1 - z_1)$ for $z_1 \geq 1$.

198 For $\frac{2}{5} \leq z_1 \leq \frac{1}{2}$, we have $G'(\frac{3}{2}z_1) \geq 0$ and $G'(0) < 0$. Hence, $\bar{b}_1(z_1)$ solves
199 the equation $G(b_1) = 0$. Figure 4 shows the curve $G'(b_1) = 0$. We can see
200 that in the region corresponding to Case 5, $\bar{b}_1(z_1)$ is the smaller root of the
201 equation $G'(b_1) = 0$. Finding an explicit form of $\bar{b}_1(z_1)$ is a difficult task. (We
202 tried in Mathematica and the obtained expression of $\bar{b}_1(z_1)$ is complicated
203 and inaccurate.) Nevertheless, by simplifying $G'(b_1) = 0$, we have

$$z_1 = \frac{1}{12}(6 - \sqrt{2}\sqrt{21b_1^2 - \sqrt{b_1^2(81b_1^2 - 156b_1 + 76)} - 38b_1 + 18}). \quad (6)$$

204 Therefore, for $\frac{2}{5} \leq z_1 \leq \frac{1}{2}$, $\bar{b}_1(z_1)$ is the smaller root of Equation 6.

205 For $\frac{1}{2} \leq z_1 \leq \frac{3}{5}$, $\bar{b}_1(z_1)$ is the smaller root of the equation

$$z_1 = \frac{1}{12}(6 + \sqrt{2}\sqrt{21b_1^2 - \sqrt{b_1^2(81b_1^2 - 156b_1 + 76)} - 38b_1 + 18}).$$

206 Similarly, we can calculate $\bar{b}_1(z_1)$ and the sign of $G'(\bar{b}_1)$ for all other cases.
 207 Note that $G(b_1)$ is continuous and convex in b_1 . Then $b_1^*(z_1)$ should be the
 208 $\bar{b}_1(z_1)$ that satisfies $G'(\bar{b}_1) = 0$. By observing Table 1, we have the expression
 209 as in Equation (5). Figure 5 shows the graph of $b_1^*(z_1)$. $\square$

210 Now we study the L^1 fitting functional and calculate the shape-preserving
 211 metric ϵ^* .

212 **Lemma 1.** *For two splines $z(x)$ and $\tilde{z}(x)$ in the 3-node window with bound-*
 213 *ary conditions (4), if $b_1^*(z_1) = b_1^*(\tilde{z}_1)$, then $z(x) - \tilde{z}(x) = z(-x) - \tilde{z}(-x)$ for*
 214 *$-1 \leq x \leq 1$.*

215 *Proof.* When $-1 \leq x \leq 0$, we have

$$\begin{aligned} z(x) - \tilde{z}(x) &= \frac{1}{h_0}(x - x_0)^2(\Delta z_0 - \Delta \tilde{z}_0)[3 - \frac{2}{h_0}(x - x_0)] \\ &= (z_1 - \tilde{z}_1)(1 - 3x^2 - 2x^3) \end{aligned}$$

216 and

$$\begin{aligned} z(-x) - \tilde{z}(-x) &= z_1 - \tilde{z}_1 + \frac{1}{h_1}(-x - x_1)^2(\Delta z_1 - \Delta \tilde{z}_1)[3 - \frac{2}{h_1}(-x - x_1)] \\ &= (z_1 - \tilde{z}_1)(1 - 3x^2 - 2x^3). \end{aligned}$$

217 Therefore, $z(x) - \tilde{z}(x) = z(-x) - \tilde{z}(-x)$ for $-1 \leq x \leq 0$. Similar proof
 218 applies to $0 \leq x \leq 1$. $\square$

Table 1: Cases in calculating $b_1^*(z_1)$ for second-derivative-based interpolating splines

Case	$z''(x)$	$\bar{b}_1(z_1)$	$G'(\bar{b}_1)$
1	$(+, +, +, -)$	$\begin{cases} 3z_1, & \text{if } 0 \leq z_1 \leq \frac{1}{4}, \\ 1 - z_1, & \text{if } \frac{1}{4} \leq z_1 \leq \frac{2}{5}, \\ \frac{3}{2}z_1, & \text{if } \frac{2}{5} \leq z_1 \leq \frac{1}{2}. \end{cases}$	$\begin{cases} < 0, & \text{if } 0 \leq z_1 \leq \frac{1}{4}, \\ = 0, & \text{if } \frac{1}{4} \leq z_1 \leq \frac{2}{5}, \\ > 0, & \text{if } \frac{2}{5} \leq z_1 \leq \frac{1}{2}. \end{cases}$
2	$(+, +, -, +)$	$\max\{\frac{3}{2}z_1, 3(1 - z_1)\}$	> 0
3	$(+, +, -, -)$	$\max\{\frac{3}{2}z_1, \frac{3}{2}(1 - z_1)\}$	> 0
4	$(+, -, +, +)$	$\frac{3}{2}(1 - z_1)$	< 0
5	$(+, -, +, -)$	$\begin{cases} 3z_1, & \text{if } z_1 \leq 0, \\ \frac{3}{2}z_1, & \text{if } 0 \leq z_1 \leq \frac{2}{5}, \\ \text{smaller root of} \\ z_1 = \frac{1}{12}(6 - \sqrt{2}\sqrt{21b_1^2 - \sqrt{b_1^2(81b_1^2 - 156b_1 + 76)} - 38b_1 + 18}), & \text{if } \frac{2}{5} \leq z_1 \leq \frac{1}{2}, \\ \text{smaller root of} \\ z_1 = \frac{1}{12}(6 + \sqrt{2}\sqrt{21b_1^2 - \sqrt{b_1^2(81b_1^2 - 156b_1 + 76)} - 38b_1 + 18}), & \text{if } \frac{1}{2} \leq z_1 \leq \frac{3}{5}, \\ \frac{3}{2}(1 - z_1), & \text{if } \frac{3}{5} \leq z_1 \leq 1, \\ 3(1 - z_1), & \text{if } z_1 \geq 1. \end{cases}$	$\begin{cases} \leq 0, & \text{if } z_1 < \frac{2}{5}, \\ = 0, & \text{if } \frac{2}{5} \leq z_1 \leq \frac{3}{5}, \\ \leq 0, & \text{if } z_1 > \frac{3}{5}. \end{cases}$
6	$(+, -, -, +)$	$\begin{cases} 3(1 - z_1), & \text{if } \frac{2}{3} \leq z_1 \leq \frac{3}{4}, \\ 4z_1(1 - z_1), & \text{if } \frac{3}{4} \leq z_1 \leq 1, \\ 0, & \text{if } z_1 \geq 1. \end{cases}$	$\begin{cases} > 0, & \text{if } \frac{2}{3} \leq z_1 \leq \frac{3}{4}, \\ = 0, & \text{if } \frac{3}{4} \leq z_1 \leq 1, \\ = 0, & \text{if } z_1 \geq 1. \end{cases}$
7	$(+, -, -, -)$	$\begin{cases} \frac{3}{2}(1 - z_1), & \text{if } \frac{1}{2} \leq z_1 \leq \frac{3}{5}, \\ z_1, & \text{if } \frac{3}{5} \leq z_1 \leq \frac{3}{4}, \\ 3(1 - z_1), & \text{if } \frac{3}{4} \leq z_1 \leq 1. \end{cases}$	$\begin{cases} > 0, & \text{if } \frac{1}{2} \leq z_1 \leq \frac{3}{5}, \\ = 0, & \text{if } \frac{3}{5} \leq z_1 \leq \frac{3}{4}, \\ < 0, & \text{if } \frac{3}{4} \leq z_1 \leq 1. \end{cases}$
8	$(-, +, +, -)$	$\begin{cases} 0, & \text{if } z_1 \leq 0, \\ 4z_1(1 - z_1), & \text{if } 0 \leq z_1 \leq \frac{1}{4}, \\ 3z_1, & \text{if } \frac{1}{4} \leq z_1 \leq \frac{1}{3}. \end{cases}$	$\begin{cases} = 0, & \text{if } z_1 \leq 0, \\ = 0, & \text{if } 0 \leq z_1 \leq \frac{1}{4}, \\ > 0, & \text{if } \frac{1}{4} \leq z_1 \leq \frac{1}{3}. \end{cases}$
9	$(-, +, -, +)$	$\max\{3z_1, 3(1 - z_1)\}$	> 0
10	$(-, +, -, -)$	$\max\{3z_1, \frac{3}{2}(1 - z_1)\}$	> 0
11	$(-, -, +, -)$	$\frac{3}{2}z_1$	< 0

219 **Lemma 2.** Given any $0 < \epsilon \leq 1$, $0 \leq z_1^* \leq \frac{1}{4}$ holds for the second-derivative-
 220 based spline fit for $H_\epsilon(x)$.

221 *Proof.* We prove this lemma by considering different ranges of z_1 according
 222 to the expression of $b_1^*(z_1)$.

223 First, we prove that $z_1^* \leq \frac{1}{2}$. Consider two splines $z(x)$ and $\tilde{z}(x)$ with
 224 $z_1 > \frac{1}{2}$ and $\tilde{z}_1 = 1 - z_1$. By symmetry, the first derivatives at node x_1 for

Figure 3: The 11 cases in calculating $b_1^*(z_1)$ for second-derivative-based interpolating splines

both splines are equal, i.e., $b_1^*(z_1) = b_1^*(\tilde{z}_1)$. We have $z(x) = 1 - \tilde{z}(-x)$ and

$$z(x) - \tilde{z}(x) = \frac{1}{h_0}(x - x_0)^2(\Delta z_0 - \Delta \tilde{z}_0)[3 - \frac{2}{h_0}(x - x_0)] \geq 0$$

for all $-1 \leq x \leq 0$. By Lemma 1 we have $z(x) - \tilde{z}(x) \geq 0$ for $-1 \leq x \leq 1$.
 When $\epsilon = 0$, we know $F_0(z_1, b_1^*(z_1)) = F_0(\tilde{z}_1, b_1^*(z_1))$. When $0 < \epsilon \leq 1$, we
 consider

$$F_\epsilon(z_1, b_1^*(z_1)) - F_0(z_1, b_1^*(z_1)) = \int_0^\epsilon [|z(x) - 0| - |z(x) - 1|] dx = \int_0^\epsilon [z(x) - |z(x) - 1|] dx$$

Figure 4: Curve $G'(b_1) = 0$ in Case 5

229 and

$$F_\epsilon(\tilde{z}_1, b_1^*(\tilde{z}_1)) - F_0(\tilde{z}_1, b_1^*(\tilde{z}_1)) = \int_0^\epsilon [|\tilde{z}(x) - 0| - |\tilde{z}(x) - 1|] dx = \int_0^\epsilon [|\tilde{z}(x)| - 1 + \tilde{z}(x)] dx.$$

230 (1) If $\frac{1}{2} < z_1 \leq \frac{3}{4}$, then $\frac{1}{2} < z(x) < 1$ and $0 < \tilde{z}(x) < z(x)$ for $0 \leq x \leq \epsilon$.

231 Hence,

$$z(x) - |z(x) - 1| = 2z(x) - 1 > 2\tilde{z}(x) - 1 = |\tilde{z}(x)| - 1 + \tilde{z}(x).$$

232 (2) If $\frac{3}{4} < z_1 \leq 1$, then $0 < \tilde{z}(x) < \min\{z(x), 1\}$ for $0 \leq x \leq \epsilon$. Hence,

$$z(x) - |z(x) - 1| = \min\{2z(x) - 1, 1\} > 2\tilde{z}(x) - 1 = |\tilde{z}(x)| - 1 + \tilde{z}(x).$$

Figure 5: $b_1^*(z_1)$ of second-derivative-based interpolating splines in a 3-node window with boundary conditions

233 (3) If $z_1 > 1$, then $z(x) > 1$ and $\tilde{z}(x) < 1$ for $0 \leq x \leq \epsilon$. Hence,

$$z(x) - |z(x) - 1| = 1 > \max\{2\tilde{z}(x) - 1, -1\} = |\tilde{z}(x)| - 1 + \tilde{z}(x).$$

234 Therefore, for $0 < \epsilon \leq 1$, $F_\epsilon(z_1, b_1^*(z_1)) > F_\epsilon(\tilde{z}_1, b_1^*(\tilde{z}_1))$ with $z_1 > \frac{1}{2}$ and
 235 $\tilde{z}_1 = 1 - z_1$. The second-derivative-based spline fit has to satisfy $z_1^* \leq \frac{1}{2}$.

236 Second, we prove that $z_1^* \leq \frac{2}{5}$. Suppose that $\frac{2}{5} < z_1 \leq \frac{1}{2}$. It is easy to
 237 see there exists $\frac{1}{4} \leq \tilde{z}_1 \leq \frac{2}{5}$ such that $b_1^*(\tilde{z}_1) = b_1^*(z_1)$. Notice that $b_1^*(z_1) \leq$
 238 $3z_1$, $b_1^*(z_1) \leq 3(1 - z_1)$, $b_1^*(\tilde{z}_1) \leq 3z_1$ and $b_1^*(\tilde{z}_1) \leq 3(1 - z_1)$. Thus, we have
 239 $0 \leq \tilde{z}(x) \leq z(x) \leq 1$ for all $-1 \leq x \leq 1$ and the equality holds only at

240 $x = -1$ and 1 . By Lemma 1, we know $\int_{-1}^{-\epsilon} |z(x) - 0|dx - \int_{-1}^{-\epsilon} |\tilde{z}(x) - 0|dx =$
 241 $\int_{\epsilon}^1 |\tilde{z}(x) - 1|dx - \int_{\epsilon}^1 |z(x) - 1|dx$. Note $\int_{-\epsilon}^{\epsilon} |z(x) - 0|dx > \int_{-\epsilon}^{\epsilon} |\tilde{z}(x) - 0|dx$.
 242 Hence, $F_{\epsilon}(z_1, b_1^*(z_1)) > F_{\epsilon}(\tilde{z}_1, b_1^*(\tilde{z}_1))$. Consequently, $z_1^* \leq \frac{2}{5}$.
 243 Third, we show that $z_1^* \leq \frac{1}{4}$. When $\frac{1}{4} \leq z_1 \leq \frac{2}{5}$, $b_1^*(z_1) = 1 - z_1$. Then,
 244 the fitting functional becomes

$$F_{\epsilon}(z_1, b_1^*(z_1)) = \frac{\epsilon^4 z_1}{2} - \frac{\epsilon^4}{2} - \frac{2\epsilon^3 z}{3} + \frac{2\epsilon^3}{3} - \epsilon^2 z_1 + \epsilon^2 + 2\epsilon z_1 - \epsilon + \frac{z_1}{6} + \frac{1}{3}.$$

245 The first derivative in z_1 is

$$\begin{aligned} \frac{dF_{\epsilon}(z_1, b_1^*(z_1))}{dz_1} &= \frac{1}{6}(3\epsilon^4 - 4\epsilon^3 - 6\epsilon^2 + 12\epsilon + 1) \\ &= \frac{1}{6}[3\epsilon^4 + 4\epsilon(1 - \epsilon^2) + 6\epsilon(1 - \epsilon) + 2\epsilon + 1] > 0. \end{aligned}$$

246 Therefore, the minimum z_1^* of $F_{\epsilon}(z_1, b_1^*(z_1))$ in $z_1 \leq \frac{2}{5}$ satisfies $z_1^* \leq \frac{1}{4}$.

247 Finally, we prove that $z_1^* \geq 0$. Suppose that $z_1 < 0$. Then we know
 248 $b_1^*(z_1) = 0$. For $-1 \leq x \leq 0$, $z(x) = z_1(1+x)^2(1-2x) \leq 0$. For $0 \leq x \leq 1$,
 249 $z(x) = 3x^2(1-z_1) - 2x^3(1-z_1) + z_1$. It is easy to verify that $z'(x) \geq 0$.
 250 Thus, $z(x) = 0$ has one root in $0 \leq x \leq 1$. Denote this root by $r(z_1)$. Using
 251 the implicit function theorem, we have

$$\frac{dr(z_1)}{dz_1} = -\frac{\frac{\partial z(x)}{\partial z_1}}{\frac{\partial z(x)}{\partial x}} = -\frac{-3x^2 + 2x^3 + 1}{6x(1-z_1)(1-x)} \leq 0.$$

252 Hence, $r(z_1)$ decreases as z_1 increases. For any $0 < \epsilon \leq 1$, there exists some
 253 node function value $z_1(\epsilon)$ such that $r(z_1(\epsilon)) = \epsilon$. When $z_1 < \tilde{z}_1 \leq z_1(\epsilon)$, we
 254 can easily see $z(x) \leq \tilde{z}(x)$ and, consequently, $F_{\epsilon}(z_1, b_1^*(z_1)) > F_{\epsilon}(\tilde{z}_1, b_1^*(\tilde{z}_1))$.
 255 When $z_1(\epsilon) < z_1 \leq 0$,

$$F_{\epsilon}(z_1, b_1^*(z_1)) = \int_{-1}^0 -z(x)dx + \int_0^{r(z_1)} -z(x)dx + \int_{r(z_1)}^{\epsilon} z(x)dx + \int_{\epsilon}^1 1-z(x)dx.$$

256 Taking the derivative with respect to z_1 , we have

$$\frac{dF_{\epsilon}(z_1, b_1^*(z_1))}{dz_1} = \int_{-1}^0 -\frac{dz(x)}{dz_1}dx + \int_0^{r(z_1)} -\frac{dz(x)}{dz_1}dx + \int_{r(z_1)}^{\epsilon} \frac{dz(x)}{dz_1}dx + \int_{\epsilon}^1 -\frac{dz(x)}{dz_1}dx.$$

257 Notice that $\frac{dz(x)}{dz_1} \geq 0$ for $-1 \leq x \leq 1$. By direct calculation, we have
 258 $\int_{-1}^0 -\frac{dz(x)}{dz_1} dx + \int_0^1 \frac{dz(x)}{dz_1} dx = 0$. Hence, $\frac{dF_\epsilon(z_1, b_1^*(z_1))}{dz_1} \leq 0$ and $z_1^* \geq 0$.
 259 Combining all arguments above, we have $0 \leq z_1^* \leq \frac{1}{4}$. $\square$

260 **Theorem 2.** Fix $z_0 = 0, b_0 = 0, z_2 = 1$ and $b_2 = 0$ in the 3-node window.
 261 The second-derivative-based spline fit for $H_\epsilon(x)$ preserves its linear shape in
 262 $-1 \leq x \leq 0$, i.e., $z_1^* = 0$ and $b_1^*(z_1^*) = 0$, if and only if $\epsilon \geq 0.250$. This
 263 means that the shape-preserving metric of second-derivative-based spline fits
 264 is 0.250.

265 *Proof.* To solve the spline fit problem, we only need to consider $0 \leq z_1 \leq \frac{1}{4}$.
 266 In this range of z_1 , $b_1 = 4z_1(1 - z_1) > 3z_1$. This means $z''(-1) \leq 0$ and
 267 the cubic L^1 spline fit intersects with $H_\epsilon(x)$ in the interval $-1 < x < 0$ at
 268 another point besides x_0 . Solve the equation $z(x) = 0$, we have this point
 269 $\bar{x} = \frac{1}{4z_1 - 2}$. The cubic L^1 spline fit has no other intersection with $H_\epsilon(x)$ than
 270 x_2 in the interval $0 \leq x \leq 1$ because $b_1^*(z_1) < 3(1 - z_1)$ when $0 \leq z_1 \leq \frac{1}{4}$.

271 The L^1 fitting functional then becomes

$$\begin{aligned} & F_\epsilon(z_1, b_1^*(z_1)) \\ &= \int_{-1}^{\bar{x}} [H_\epsilon(x) - z(x)] dx + \int_{\bar{x}}^0 [z(x) - H_\epsilon(x)] dx + \int_0^\epsilon [z(x) - H_\epsilon(x)] dx + \int_\epsilon^1 [H_\epsilon(x) - z(x)] dx \\ &= -2\epsilon^4 z_1^2 + 3\epsilon^4 z_1 - \epsilon^4 + \frac{16\epsilon^3 z_1^2}{3} - \frac{22\epsilon^3 z_1}{3} + 2\epsilon^3 - 4\epsilon^2 z_1^2 + 4\epsilon^2 z_1 + 2\epsilon z_1 - \epsilon \\ &\quad - \frac{8z_1^5}{3(2z_1 - 1)^3} + \frac{8z_1^4}{3(2z_1 - 1)^3} - \frac{2z_1^3}{(2z_1 - 1)^3} + \frac{z_1^2}{3} + \frac{z_1^2}{(2z_1 - 1)^3} - \frac{5z_1}{6} - \frac{3z_1}{16(2z_1 - 1)^3} + \frac{1}{2}. \end{aligned}$$

272 The first derivative in z_1 is

$$\begin{aligned} & \frac{\partial F_\epsilon(z_1, b_1^*(z_1))}{\partial z_1} \\ &= \epsilon^4(3 - 4z_1) + \frac{2}{3}\epsilon^3(16z_1 - 11) + \epsilon^2(4 - 8z_1) + 2\epsilon + \frac{4z_1(73 - 64z_1(3(z_1 - 2)z_1 + 4)) - 31}{48(1 - 2z_1)^4}. \end{aligned}$$

273 Then we have

$$\frac{\partial^2 F_\epsilon(z_1, b_1^*(z_1))}{\partial \epsilon \partial z_1} = -2(\epsilon - 1)^2(\epsilon(8z_1 - 6) - 1)$$

274 and

$$\frac{\partial^2 F_\epsilon(z_1, b_1^*(z_1))}{\partial z_1^2} = \frac{2(37 - 64z_1)z_1 - 11}{12(2z_1 - 1)^5} - \frac{4}{3}\epsilon^2(\epsilon(3\epsilon - 8) + 6).$$

275 It is easy to verify that $\frac{\partial^2 F_\epsilon(z_1, b_1^*(z_1))}{\partial z_1^2} \geq \frac{\partial^2 F_\epsilon(z_1, b_1^*(z_1))}{\partial z_1^2} \Big|_{z_1=0}$ for $0 \leq z_1 \leq \frac{1}{4}$ and
 276 $0 \leq \epsilon \leq 1$. Notice that $\frac{\partial^2 F_{0.5}(z_1, b_1^*(z_1))}{\partial z_1^2} \Big|_{z_1=0} = 0$. Therefore, $\frac{\partial^2 F_{0.5}(z_1, b_1^*(z_1))}{\partial z_1^2} \geq 0$
 277 for $0 \leq z_1 \leq \frac{1}{4}$. Then $\frac{\partial F_{0.5}(z_1, b_1^*(z_1))}{\partial z_1} \geq \frac{\partial F_{0.5}(z_1, b_1^*(z_1))}{\partial z_1} \Big|_{z_1=0} = \frac{15}{24} > 0$. On
 278 the other hand, $\frac{\partial^2 F_\epsilon(z_1, b_1^*(z_1))}{\partial \epsilon \partial z_1} \geq 0$ for $0 \leq z_1 \leq \frac{1}{4}$ and $0 \leq \epsilon \leq 1$. Hence,
 279 $\frac{\partial F_\epsilon(z_1, b_1^*(z_1))}{\partial z_1} \geq 0$ for $0 \leq z_1 \leq \frac{1}{4}$ and $0.5 \leq \epsilon \leq 1$. This implies that $z_1^* = 0$ for
 280 $0.5 \leq \epsilon \leq 1$ and $\epsilon^* \leq 0.5$.

281 When $0 \leq z_1 \leq \frac{1}{4}$ and $0 \leq \epsilon \leq 0.5$, by noticing $\frac{\partial^2 F_\epsilon(z_1, b_1^*(z_1))}{\partial z_1^2}$ is de-
 282 creasing in ϵ , we have $\frac{\partial^2 F_\epsilon(z_1, b_1^*(z_1))}{\partial z_1^2} \geq \frac{\partial^2 F_{0.5}(z_1, b_1^*(z_1))}{\partial z_1^2} \Big|_{z_1=0} = 0$. Consequently,
 283 $F_\epsilon(z_1, b_1^*(z_1))$ is convex in $0 \leq z_1 \leq \frac{1}{4}$ for $0 \leq \epsilon \leq 0.5$. Then $z_1 = 0$ minimizes
 284 $F_\epsilon(z_1, b_1^*(z_1))$ if $\frac{\partial F_\epsilon(z_1, b_1^*(z_1))}{\partial z_1} \Big|_{z_1=0} \geq 0$. By direct calculation we have

$$\frac{\partial F_\epsilon(z_1, b_1^*(z_1))}{\partial z_1} \Big|_{z_1=0} = 3\epsilon^4 - \frac{22\epsilon^3}{3} + 4\epsilon^2 + 2\epsilon - \frac{31}{48}.$$

285 Recall that $\frac{\partial^2 F_\epsilon(z_1, b_1^*(z_1))}{\partial \epsilon \partial z_1} \geq 0$. Solve the equation $\frac{\partial F_\epsilon(z_1, b_1^*(z_1))}{\partial z_1} \Big|_{z_1=0} = 0$ and we
 286 have its unique solution

$$\begin{aligned} \epsilon^* &= \frac{11}{18} - \frac{1}{18} \sqrt{\frac{7}{2} \left(9\sqrt[3]{7 - \sqrt{22}} + 9\sqrt[3]{7 + \sqrt{22}} + 14 \right)} \\ &\quad + \frac{1}{18} \sqrt{\frac{7}{2} \left(-9\sqrt[3]{7 - \sqrt{22}} - 9\sqrt[3]{7 + \sqrt{22}} + 28 \sqrt{\frac{14}{9\sqrt[3]{7 - \sqrt{22}} + 9\sqrt[3]{7 + \sqrt{22}} + 14}} + 28 \right)} \\ &\simeq 0.250. \end{aligned}$$

287 Therefore, when $\epsilon \geq 0.250$, the second-derivative-based spline fit for $H_\epsilon(x)$
 288 satisfies $z_1^* = 0$ and $b_1^*(z_1^*) = 0$. $\square$

289 4.2. First-derivative-based Spline Fits

290 **Theorem 3.** Given $z_0 = 0, b_0 = 0, z_2 = 1$ and $b_2 = 0$, the first deriva-
 291 tive b_1^* that minimizes the first-derivative-based L^1 interpolating functional

292 $\int_{-1}^1 \left| \frac{dz}{dx} - \frac{d\zeta}{dx} \right| dx$ satisfies the following properties:

293 (1) $b_1^*(z_1) = 0$, if $z_1 \leq 0$,

294 (2) $z_1 \leq b_1^*(z_1) \leq 2z_1$, if $0 < z_1 \leq 0.255443$,

295 (3) $0 < b_1^*(z_1) \leq z_1$, if $0.255443 < z_1 \leq \frac{1}{2}$,

296 (4) $0 < b_1^*(z_1) \leq 1 - z_1$, if $\frac{1}{2} < z_1 \leq 0.755443$,

297 (5) $1 - z_1 \leq b_1^*(z_1) \leq 2(1 - z_1)$, if $0.755443 < z_1 \leq 1$,

298 (6) $b_1^*(z_1) = 0$, if $z_1 > 1$.

299 *Proof.* Note that the interpolating functional $\int_{-1}^1 \left| \frac{dz}{dx} - \frac{d\zeta}{dx} \right| dx$ is convex in z_1
 300 and b_1 . Consequently, $G(b_1)$ is convex in b_1 for any given z_1 .

301 When $-1 \leq x \leq 0$, we have

$$z'(x) = 3(b_1 - 2z_1)(x + 1)^2 - 2(b_1 - 3z_1)(x + 1)$$

302 and $\zeta'(x) = z_1$.

303 When $z_1 \leq 0$ and $b_1 \geq z_1$, the equation $z'(x) - \zeta'(x) = 0$ has two roots in
 304 $-1 \leq x \leq 0$. Moreover,

$$w_1 = \frac{-\sqrt{b_1^2 - 3b_1z_1 + 3z_1^2} - 2b_1 + 3z_1}{3(b_1 - 2z_1)}, w_2 = \frac{\sqrt{b_1^2 - 3b_1z_1 + 3z_1^2} - 2b_1 + 3z_1}{3(b_1 - 2z_1)},$$

$$G_0(b_1) = \int_{-1}^{w_1} z'(x) - \zeta'(x) dx + \int_{w_1}^{w_2} \zeta'(x) - z'(x) dx + \int_{w_2}^0 z'(x) - \zeta'(x) dx,$$

305 and

$$\frac{dG_0}{db_1} = \int_{-1}^{w_1} \frac{\partial z'(x)}{\partial b_1} dx + \int_{w_1}^{w_2} -\frac{\partial z'(x)}{\partial b_1} dx + \int_{w_2}^0 \frac{\partial z'(x)}{\partial b_1} dx.$$

When $b_0 = 0$, we have

$$\left. \frac{dG_0}{db_1} \right|_{b_1=0} = -\frac{\sqrt{z_1^2}}{3\sqrt{3}z_1} = \frac{1}{3\sqrt{3}}.$$

306 Similarly, when $z_1 \leq 0$ and $b_1 \leq 1 - z_1$, we can calculate

$$\frac{dG_1}{db_1}\bigg|_{b_1=0} = -\frac{\sqrt{(1-z_1)^2}}{3\sqrt{3}(1-z_1)} = -\frac{1}{3\sqrt{3}}.$$

Hence,

$$\frac{dG}{db_1}\bigg|_{b_1=0} = \frac{dG_0}{db_1}\bigg|_{b_1=0} + \frac{dG_1}{db_1}\bigg|_{b_1=0} = 0.$$

307 Therefore, $b_1^*(z_1) = 0$ for $z_1 \leq 0$.

308 When $z_1 \geq 0$, $z_1 \leq b_1 \leq 2z_1$ and $b_1 \leq 1 - z_1$, the equation $z'(x) - \zeta'(x) = 0$
 309 has one root w_2 in $-1 \leq x \leq 0$ and the following two roots w_3 and w_4 in
 310 $0 \leq x \leq 1$:

$$w_3 = \frac{-\sqrt{b_1^2 - 3b_1(1-z_1) + 3(1-z_1)^2} - 2b_1 + 3(1-z_1)}{3(b_1 - 2(1-z_1))},$$

$$w_4 = \frac{\sqrt{b_1^2 - 3b_1(1-z_1) + 3(1-z_1)^2} - 2b_1 + 3(1-z_1)}{3(b_1 - 2(1-z_1))}.$$

311 Then

$$G(b_1) = \int_{-1}^{w_2} \zeta'(x) - z'(x)dx + \int_{w_2}^0 z'(x) - \zeta'(x)dx + \int_0^{w_4} \zeta'(x) - z'(x)dx$$

$$+ \int_{w_4}^{w_3} z'(x) - \zeta'(x)dx + \int_{w_3}^1 \zeta'(x) - z'(x)dx.$$

312 We can verify that $G(b_1)|_{b_1 \rightarrow 2z_1} > 0$ for $0 \leq z_1 \leq \frac{1}{3}$ and $G(b_1)|_{b_1=1-z_1} > 0$
 313 for $\frac{1}{3} \leq z_1 \leq \frac{1}{2}$. Furthermore, $G(b_1)|_{b_1=z_1}$ is increasing in $0 \leq z_1 \leq \frac{1}{2}$,
 314 $G(b_1)|_{b_1=z_1=0} < 0$ and $G(b_1)|_{b_1=z_1=\frac{1}{2}} > 0$. Denote the root of $G(b_1)|_{b_1=z_1} = 0$
 315 by θ . We can compute $\theta \approx 0.255443$. Therefore, $z_1 \leq b_1^*(z_1) \leq 2z_1$, if
 316 $0 \leq z_1 \leq 0.255443$. Moreover, $b_1^*(z_1) \leq z_1$, if $0.255443 \leq z_1 \leq \frac{1}{2}$.

317 When $z_1 \geq 0$, $b_1 \leq z_1$ and $b_1 \leq 1 - z_1$, we can calculate

$$\frac{dG}{db_1}\bigg|_{b_1=0} = -\frac{2}{3\sqrt{3}} < 0.$$

318 Then $0 < b_1^*(z_1) \leq z_1$, if $0.255443 \leq z_1 \leq \frac{1}{2}$. The remaining part of the
 319 theorem holds by symmetry. $\square$

Figure 6: $b_1^*(z_1)$ of first-derivative-based interpolating spline in a 3-node window with boundary conditions

Figure 6 shows the graph of $b_1^*(z_1)$.

Lemma 3. *Given any $0 < \epsilon \leq 1$, $0 \leq z_1^* \leq \frac{1}{2}$ holds for the first-derivative-based spline fit for $H_\epsilon(x)$.*

Proof. Note that $0 \leq b_1^*(z_1) \leq 3(1 - z_1)$ for $\frac{1}{2} \leq z_1 \leq 1$, $b_1^*(z_1) = 0$ for $z_1 \geq 1$, and $b_1^*(z_1) = 0$ for $z_1 \leq 0$. The proof of Lemma 2 can be applied. $\square$

Theorem 4. *Fix $z_0 = 0, b_0 = 0, z_2 = 1$ and $b_2 = 0$ in the 3-node window. The first-derivative-based spline fit for $H_\epsilon(x)$ preserves its linear shape in $-1 \leq x \leq 0$, i.e., $z_1^* = 0$ and $b_1^*(z_1^*) = 0$, if and only if $\epsilon \geq 0.109$. This means that the shape-preserving metric of first-derivative-based spline fits is 0.109.*

Proof. We only need to consider $0 \leq z_1 \leq \frac{1}{2}$. First, we prove that $z_1^* = 0$ minimizes $F_\epsilon(z_1, b_1^*(z_1))$ locally in $0 \leq z_1 \leq 0.255443$ only if $\epsilon \geq 0.109$. Second, that we prove when $\epsilon \geq 0.109$, $z_1^* = 0$ minimizes $F_\epsilon(z_1, b_1^*(z_1))$ globally in $0 \leq z_1 \leq \frac{1}{2}$.

We notice that $0 \leq z(x) \leq 1$ for $-1 \leq x \leq 1$ because $b_1^*(z_1) < \min\{3z_1, 3(1 - z_1)\}$. Therefore,

$$\begin{aligned} F_\epsilon(z_1, b_1^*(z_1)) &= \int_{-1}^0 [z(x) - H_\epsilon(x)] dx + \int_0^\epsilon [z(x) - H_\epsilon(x)] dx + \int_\epsilon^1 [H_\epsilon(x) - z(x)] dx \\ &= \left(\frac{\epsilon^4}{2} - \frac{4\epsilon^3}{3} + \epsilon^2 - \frac{1}{6} \right) b_1 + (\epsilon^4 - 2\epsilon^3 + 2\epsilon) z_1 - \epsilon^4 + 2\epsilon^3 - \epsilon + \frac{1}{2}. \end{aligned}$$

336 The first derivative in z_1 is

$$\frac{\partial F_\epsilon(z_1, b_1^*(z_1))}{\partial z_1} = \left(\frac{\epsilon^4}{2} - \frac{4\epsilon^3}{3} + \epsilon^2 - \frac{1}{6} \right) \frac{db_1}{dz_1} + (\epsilon^4 - 2\epsilon^3 + 2\epsilon).$$

337 The second derivative in z_1 is

$$\frac{\partial^2 F_\epsilon(z_1, b_1^*(z_1))}{\partial z_1^2} = \left(\frac{\epsilon^4}{2} - \frac{4\epsilon^3}{3} + \epsilon^2 - \frac{1}{6} \right) \frac{d^2 b_1}{dz_1^2}.$$

338 We observe from Figure 6 that $b_1^*(z_1)$ is concave in $0 \leq z_1 \leq 0.255443$.
 339 Thus $\frac{d^2 b_1}{dz_1^2} \leq 0$. On the other hand, $\left(\frac{\epsilon^4}{2} - \frac{4\epsilon^3}{3} + \epsilon^2 - \frac{1}{6} \right) = \frac{1}{6}(\epsilon-1)^3(3\epsilon+1) \leq 0$
 340 for $0 \leq \epsilon \leq 1$. Consequently, we have $\frac{\partial^2 F_\epsilon(z_1, b_1^*(z_1))}{\partial z_1^2} \geq 0$ and then $F_\epsilon(z_1, b_1^*(z_1))$
 341 is convex in z_1 . $z_1^* = 0$ minimizes $F_\epsilon(z_1, b_1^*(z_1))$ in $0 \leq z_1 \leq 0.255443$ if and
 342 only if $\frac{\partial F_\epsilon(z_1, b_1^*(z_1))}{\partial z_1} \Big|_{z_1=0} \geq 0$. Notice that

$$\frac{\partial^2 F_\epsilon(z_1, b_1^*(z_1))}{\partial \epsilon \partial z_1} = 2(\epsilon-1)^2 \epsilon \frac{db_1}{dz_1} + 2(\epsilon-1)^2(2\epsilon+1) \geq 0$$

343 for $0 \leq \epsilon \leq 1$. We need to solve the following equation for ϵ^* :

$$-\frac{\epsilon^4 - 2\epsilon^3 + 2\epsilon}{\frac{\epsilon^4}{2} - \frac{4\epsilon^3}{3} + \epsilon^2 - \frac{1}{6}} = \frac{db_1}{dz_1} \Big|_{z_1=0}. \quad (7)$$

344 Now we calculate $\frac{db_1}{dz_1} \Big|_{z_1=0}$. Suppose $\frac{db_1}{dz_1} \Big|_{z_1=0} = k$. Solve the equation

$$\lim_{z_1 \rightarrow 0} \frac{dG(b_1)}{db_1} \Big|_{b_1=kz_1} = 0$$

345 and we have $k = 1.37885$. Substituting $\frac{db_1}{dz_1} \Big|_{z_1=0} = 1.37885$ to Equation (7),
 346 we can obtain $\epsilon^* = 0.109$.

347 Now we prove that when $\epsilon \geq \epsilon^*$, $z_1^* = 0$ is the global minimum of
 348 $F_\epsilon(z_1, b_1^*(z_1))$ in $0 \leq z_1 \leq \frac{1}{2}$. We observe that $b_1^*(z_1)$ is concave in $0.255443 \leq$
 349 $z_1 \leq \frac{1}{2}$. Moreover, $\frac{db_1}{dz_1} \Big|_{z_1=0.255443} \leq 1$ because $b_1^*(0.255443) = 0.255443$ and
 350 $b_1 \leq z_1$ for $0.255443 \leq z_1 \leq \frac{1}{2}$. Hence, $\frac{\partial F_\epsilon(z_1, b_1^*(z_1))}{\partial z_1} \geq 0$ in $0.255443 \leq z_1 \leq \frac{1}{2}$.
 351 Therefore, when $\epsilon \geq \epsilon^*$, $\frac{\partial F_\epsilon(z_1, b_1^*(z_1))}{\partial z_1}$ is monotonically increasing in $0 \leq z_1 \leq \frac{1}{2}$.
 352 The minimum is $z_1^* = 0$. $\square$

353 **Remark 1.** In the proof of Theorem 4, we used the concavity of function
 354 $b_1^*(z_1)$ in intervals $0 \leq z_1 \leq 0.255443$ and $0.255443 \leq z_1 \leq \frac{1}{2}$. This prop-
 355 erty of $b_1^*(z_1)$ is assumed without a mathematical proof because no analytic
 356 expression of $b_1^*(z_1)$ is available. Nevertheless, we can observe it in Figure 6.

357 4.3. Function-value-based Spline Fits

358 **Theorem 5.** Given $z_0 = 0, b_0 = 0, z_2 = 1$ and $b_2 = 0$, the first deriva-
 359 tive b_1^* that minimizes the function-value-based L^1 interpolating functional
 360 $\int_{-1}^1 |z(x) - \zeta(x)| dx$ satisfies the following properties:

- 361 (1) $b_1^*(z_1) = 0$, if $z_1 \leq 0$,
- 362 (2) $0 \leq b_1^*(z_1) \leq z_1$, if $0 \leq z_1 \leq \frac{1}{2}$,
- 363 (3) $0 \leq b_1^*(z_1) \leq 1 - z_1$, if $\frac{1}{2} \leq z_1 \leq 1$,
- 364 (4) $b_1^*(z_1) = 0$, if $z_1 \geq 1$.

365 *Proof.* Note that $G(b_1)$ is convex in b_1 for any given z_1 . When $-1 \leq x \leq 0$,
 366 we have

$$z(x) = (b_1 - 2z_1)(x + 1)^3 - (b_1 - 3z_1)(x + 1)^2$$

367 and $\zeta(x) = z_1(1 + x)$.

368 When $z_1 \leq 0$ and $b_1 \geq z_1$, the equation $z(x) - \zeta(x) = 0$ has a root in
 369 $-1 < x < 0$ as $r_1 = \frac{z_1 - b_1}{b_1 - 2z_1}$. We have

$$G_0(b_1) = \int_{-1}^{r_1} z(x) - \zeta(x) dx + \int_{r_1}^0 \zeta(x) - z(x) dx$$

370 and

$$\frac{dG_0}{db_1} \Big|_{b_1=0} = \frac{1}{32}.$$

371 When $z_1 \leq 0$ and $b_1 \leq 1 - z_1$, the equation $z(x) - \zeta(x) = 0$ has a root in
 372 $0 < x < 1$ as $r_2 = \frac{b_1 + z_1 - 1}{b_1 + 2z_1 - 2}$. We have

$$G_1(b_1) = \int_0^{r_2} \zeta(x) - z(x) dx + \int_{r_2}^1 z(x) - \zeta(x) dx$$

373 and

$$\frac{dG_1}{db_1}\big|_{b_1=0} = -\frac{1}{32}.$$

374 Hence,

$$\frac{dG(b_1)}{db_1}\big|_{b_1=0} = 0$$

375 when $z_1 \leq 0$. Consequently, $b_1^*(z_1) = 0$ for $z_1 \leq 0$.

376 When $z_1 \geq 0$, $b_1 \leq z_1$ and $b_1 \leq 1 - z_1$, we have

$$G(b_1) = \int_{-1}^{r_1} \zeta(x) - z(x)dx + \int_{r_1}^0 z(x) - \zeta(x)dx + \int_0^{r_2} \zeta(x) - z(x)dx + \int_{r_2}^1 z(x) - \zeta(x)dx$$

377 and

$$\frac{dG}{db_1}\big|_{b_1=0} = -\frac{1}{16} < 0.$$

378 Hence $b_1^*(z_1) > 0$ for $0 \leq z_1 \leq 1$.

379 When $0 \leq z_1 \leq \frac{1}{2}$, we have

$$\frac{dG}{db_1}\big|_{b_1=z_1} = \frac{(z_1 - 1)^3(9z_1 - 5)}{6(2 - 3z_1)^4} > 0.$$

380 Hence $b_1^*(z_1) \leq z_1$ for $0 \leq z_1 \leq 1$. By symmetry we can prove the remaining
381 of the theorem. $\square$

382 Figure 7 shows the graph of $b_1^*(z_1)$.

383 **Lemma 4.** *Given any $0 < \epsilon \leq 1$, $0 \leq z_1^* \leq \frac{1}{2}$ holds for the function-value-*
384 *based spline fit for $H_\epsilon(x)$.*

385 *Proof.* Note that $0 \leq b_1^*(z_1) \leq 3(1 - z_1)$ for $\frac{1}{2} \leq z_1 \leq 1$, $b_1^*(z_1) = 0$ for $z_1 \geq 1$,
386 and $b_1^*(z_1) = 0$ for $z_1 \leq 0$. The proof of Lemma 2 can be applied. $\square$

387 **Theorem 6.** *Fix $z_0 = 0, b_0 = 0, z_2 = 1$ and $b_2 = 0$ in the 3-node window.*
388 *The function-value-based spline fit for $H_\epsilon(x)$ preserves its linear shape in*
389 *$-1 \leq x \leq 0$, i.e., $z_1^* = 0$ and $b_1^*(z_1^*) = 0$, if and only if $\epsilon \geq 0.050$. This*
390 *means that the shape-preserving metric of function-value-based spline fits is*
391 *0.050.*

Figure 7: $b_1^*(z_1)$ for function-value-based interpolating spline in a 3-node window with boundary conditions

392 *Proof.* Proof of Theorem 4 applies. For function-value-based spline fits,
 393 $\frac{db_1}{dz_1}|_{z_1=0} = 0.61191$ and $\epsilon^* = 0.050$. We also need the assumption of con-
 394 cavity of $b_1^*(z_1)$ here. This property can be observed in Figure 7. $\square$

395 Table 2 shows the shape-preserving metrics of the three types of spline
 396 fits. The shape-preserving metrics are no more than 25% of the space between
 397 adjacent spline nodes. This implies the strong shape-preserving capability
 398 of spline fits. We can see that the shape-preserving metrics of the second-
 399 derivative-based, first-derivative-based and function-value-based spline fits
 400 decrease in sequence. This indicates that the function-value-based spline fits
 401 have the strongest shape-preserving capability among these three types of
 402 spline fits.

Table 2: Shape-preserving metrics calculated in a 3-node window

	Second-derivative-based	First-derivative-based	Function-value-based
ϵ^*	0.250	0.109	0.050

403 5. Compute ϵ^* in a 5-node Window

404 In the local-calculation approaches for interpolating spline and spline fits,
 405 a window of 5 node is of the appropriate size for both accuracy and efficiency
 406 [10, 18, 15, 12]. To evaluate the shape-preserving performance of the local-
 407 calculation approach and provide further comparison of the three types of

spline fits, we compute ϵ^* in a 5-node window. No boundary conditions are imposed in this case. Considering that a similar analysis as in Section 4 for the 5-node window case is extremely cumbersome, we compute ϵ^* numerically in this section. The numerically computational procedure is as follows.

Step 1 Set a precision increment of $0 < \Delta\epsilon < 1$. Let $\epsilon = 0$.

Step 2 Solve the spline fit problem numerically. If $z_0 = z_1 = z_2 = 0$ and $b_0 = b_1 = b_2 = 0$, go to Step 3. Otherwise, let $\epsilon = \epsilon + \Delta\epsilon$ and repeat Step 2.

Step 3 Output $\epsilon^* = \epsilon$. Stop.

In Step 2, we need to solve the interpolating spline problem for $\mathbf{b}^*(\mathbf{z})$. While analytic methods have been developed for second-derivative-based and first-derivative-based splines, no such method is available for function-value-based splines. To conduct a fair comparison of the three types of spline fits, we use the Lagrangian multiplier primal affine method as in [14] and [20]. This algorithm is satisfyingly accurate for our purpose of comparing the metrics. We use the MATLAB function `fmincon` to numerically minimize the L^1 fitting functional, which is in general not analytically expressed. The starting point for `fmincon` is set as $\mathbf{z} = (0.5, 0.5, 0.5, 0.5, 0.5)$. The criterion of determining $z_0 = z_1 = z_2 = 0$ is that $|z_0| + |z_1| + |z_2| < 10^{-3}$. The criterion of determining $b_0 = b_1 = b_2 = 0$ is that $|b_0| + |b_1| + |b_2| < 10^{-3}$. We set $\Delta\epsilon = 0.001$.

The computational results are shown in Table 3. The strong capability of spline fits and superior performance of function-value-based spline fits are verified. Moreover, by comparing to Table 2, we note that the metrics in the 5-node window are smaller than those in the 3-node window. This suggests that a 5-node window contains more information about the geometric pattern to be approximated. Hence, in the local-calculation approach for spline fits, the 5-node algorithm should be preferred to the 3-node algorithm.

Table 3: Shape-preserving metrics computed in a 5-node window

	Second-derivative-based	First-derivative-based	Function-value-based
ϵ^*	0.169	0.054	0.034

Analysis and numerical study for windows of 7 or more nodes may provide additional comparison of the three types of splines. In [12], numerical experiments for interpolating splines in 7-node windows and global windows are

conducted respectively, and are compared with 5-node windows. The authors of [12] claim that these experiments provide “a piece of evidence supporting the use of second-derivative-based 5-point-window L_1 splines.” For spline fits, we may also conduct experiments of different sizes. Since the numerical method would be very similar to that for 5-node windows, and the primary purpose of this paper is not calculating shape-preserving metrics of all kinds of windows, we leave the study of other window sizes to future research.

We close the section by giving the general form of minimum length difference needed for preserving linear shape of two parallel line segments. The next theorem shows that the minimum length difference is proportional to the node space and is independent of the gap between the parallel line segments.

Theorem 7. *Let $I \geq 2$ be an even number, $\hat{\epsilon} > 0$, $D_1 > 0$ and $D_2 > 0$. The spline fit $\hat{z}(x)$ with nodes $\hat{x}_0 = -\frac{I}{2}D_1, \dots, \hat{x}_{I/2} = 0, \dots, \hat{x}_I = \frac{I}{2}D_1$, preserves the linear shape of the function*

$$\hat{H}_{\hat{\epsilon}}(x) = \begin{cases} 0, & \text{if } x \leq \hat{\epsilon} \\ D_2, & \text{if } x > \hat{\epsilon} \end{cases}$$

in $\hat{x}_0 \leq x \leq \hat{x}_{I/2}$ if and only if $\hat{\epsilon} \geq D_1\epsilon^$, where ϵ^* is the corresponding shape-preserving metric to the type of the spline fit.*

Proof. Note that for any cubic spline $z(x)$ with first derivatives b_i at node x_i , $i = 0, \dots, I$ and $\hat{z}(x)$ with first derivatives $\hat{b}_i$ at node $\hat{x}_i$, $i = 0, \dots, I$, if $\hat{b}_i = \frac{D_2}{D_1}b_i$, $i = 0, \dots, I$, then $\hat{z}(x) = D_2z(\frac{x}{D_1})$ for all $\hat{x}_0 \leq x \leq \hat{x}_I$. Consequently, if $\mathbf{b}^*(\mathbf{z})$ minimizes the interpolating functional $G(\mathbf{b})$, then $\hat{\mathbf{b}}^*(\mathbf{z}) = \frac{D_2}{D_1}\mathbf{b}^*(\mathbf{z})$ will minimize $G(\hat{\mathbf{b}})$. For the fitting functional, we have $F(\mathbf{z}, \hat{\mathbf{b}}^*(\mathbf{z})) = D_1D_2F(\mathbf{z}, \mathbf{b}^*(\mathbf{z}))$. The spline fits for $\hat{H}_{\hat{\epsilon}}(x)$ and $H_{\epsilon}(x)$ will then have the same function values at nodes $\hat{x}_i$ and x_i , $i = 0, \dots, I$. Therefore, spline fit $\hat{z}(x)$ preserves linear shape of $\hat{H}_{\hat{\epsilon}}(x)$ if and only if $\frac{\hat{\epsilon}}{D_1} \geq \epsilon^*$, i.e., $\hat{\epsilon} \geq D_1\epsilon^*$. $\square$

6. Concluding Remarks

We have defined a metric to quantitatively characterize the shape-preserving capability of three types of cubic L^1 spline fits. To calculate this metric, we have developed an analytic approach in a 3-node window and a numerical procedure in a 5-node window. In both cases, the spline fits are shown to

469 have convincingly strong capability in preserving linear shape. The function-
 470 value-based spline fit yields the smallest metric, which suggests it is better
 471 in shape preservation than the other two types of spline fits. This paper is
 472 of the first theoretic study on shape-preserving property of cubic L^1 spline
 473 fits. Such a study may help explain phenomena in L^1 splines applications. It
 474 may also be used as a reference to improve spline node positions and collect
 475 data of better quality for shape preservation purposes.

476 Shape preservation deserves a more general and precise definition. We
 477 have discussed the principle of preserving linear shapes in given data. An-
 478 other principle is to preserve convex shapes, such as corners and circles. To
 479 fully evaluate the shape-preserving capability, we need to investigate more
 480 shapes such as a triangular function or a circle. The shape-preserving met-
 481 rics in these cases should be defined accordingly. Furthermore, an analytic
 482 approach for calculating the metric in a 5-node window is worth designing.
 483 The analysis in the 3-node window, as well as the analytic solution to the
 484 L^1 interpolating spline problem in a 5-node window [10, 12, 18], shows the
 485 potential in these regards.

486 As indicated, function-value-based spline fits may perform superiorly in
 487 shape preservation. Analytic methods for solving the function-value-based
 488 interpolating spline problem and spline fit problem become more necessary
 489 and appealing. Such computational development may also be used to better
 490 evaluate the shape-preserving capability.

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500 References

- 501 [1] P. Auquier, O. Gibaru, E. Nyiri, C^1 and C^2 -continuous polynomial
 502 parametric L_p splines ($p \geq 1$), Computer Aided Geometric Design (7)
 503 373–394. doi:10.1016/j.cagd.2007.04.007.

- 504 [2] H. Cheng, S.-C. Fang, J. E. Lavery, Univariate cubic L_1 splines - A
505 geometric programming approach, Mathematical Methods of Operations
506 Research (2) 197–229. doi:10.1007/s001860200216.
- 507 [3] J. E. Lavery, Univariate cubic L_p splines and shape-preserving, mul-
508 tiscale interpolation by univariate cubic L_1 splines, Computer Aided
509 Geometric Design (4) 319–336. doi:10.1016/S0167-8396(00)00003-0.
- 510 [4] J. E. Lavery, Shape-preserving, multiscale interpolation by univariate
511 curvature-based cubic L_1 splines in Cartesian and polar coordinates,
512 Computer Aided Geometric Design (4) 257–273. doi:10.1016/S0167-
513 8396(02)00087-0.
- 514 [5] J. E. Lavery, Shape-preserving, multiscale interpolation by bi- and mul-
515 tivariate cubic splines, Computer Aided Geometric Design (4) 321–343.
516 doi:10.1016/S0167-8396(01)00034-6.
- 517 [6] Y. Wang, S.-C. Fang, J. E. Lavery, H. Cheng, A geometric programming
518 approach for bivariate cubic L_1 splines, Computers & Mathematics with
519 Applications (4) 481–514. doi:10.1016/j.camwa.2004.11.003.
- 520 [7] W. Zhang, Y. Wang, S.-C. Fang, J. E. Lavery, Cubic L_1 splines on
521 triangulated irregular networks, Pacific Journal of Optimization 2 (2006)
522 289–317.
- 523 [8] P. Auquier, O. Gibaru, E. Nyiri, On the cubic L_1 spline inter-
524 polant to the Heaviside function, Numerical Algorithms (4) 321–332.
525 doi:10.1007/s11075-007-9140-0.
- 526 [9] H. Cheng, S.-C. Fang, J. E. Lavery, Shape-preserving properties of uni-
527 variate cubic splines, Journal of Computational and Applied Mathemat-
528 ics (2) 361–382. doi:10.1016/j.cam.2004.05.003.
- 529 [10] Q. Jin, J. E. Lavery, S.-C. Fang, Univariate cubic L_1 interpolating
530 splines: analytical results for linearity, convexity and oscillation on 5-
531 point windows, Algorithms (3) 276–293. doi:10.3390/a3030276.
- 532 [11] E. Nyiri, O. Gibaru, P. Auquier, Fast $L_1^k C^k$ polynomial spline inter-
533 polation algorithm with shape-preserving properties, Computer Aided
534 Geometric Design (1) 65–74. doi:10.1016/j.cagd.2010.10.002.

- 535 [12] L. Yu, Q. Jin, J. E. Lavery, S.-C. Fang, Univariate cubic L_1 interpolating
536 splines: spline functional, window Size and analysis-based algorithm,
537 Algorithms (3) 311–328. doi:10.3390/a3030311.
- 538 [13] H. Cheng, S.-C. Fang, J. E. Lavery, A geometric programming frame-
539 work for univariate cubic L_1 smoothing splines, Annals of Operations
540 Research 133 (2005) 229–248.
- 541 [14] J. E. Lavery, Shape-preserving approximation of multiscale univariate
542 data by cubic L_1 spline fits, Computer Aided Geometric Design (1) 43–
543 64. doi:10.1016/j.cagd.2003.07.002.
- 544 [15] Z. Wang, J. E. Lavery, S.-C. Fang, Approximation of irregular geometric
545 data by locally calculated univariate cubic L^1 spline fits, Annals of Data
546 Science (1) 5–14. doi:10.1007/s40745-014-0002-z.
- 547 [16] D. Bulatov, J. E. Lavery, Reconstruction and texturing of 3D urban
548 terrain from uncalibrated monocular images using L_1 splines, Pho-
549 togrammetric Engineering & Remote Sensing 76 (4) (2010) 439–449.
550 doi:10.14358/PERS.76.4.439.
- 551 [17] Y.-M. Lin, W. Zhang, S.-C. Fang, Y. Wang, J. E. Lavery, Computa-
552 tionally efficient models of urban and natural terrain by non-iterative
553 domain decomposition for L_1 smoothing splines, in: Proceedings of the
554 25th army science conference, Department of the Army, no. 1, Washing-
555 ton, DC, USA, 2006.
- 556 [18] Q. Jin, L. Yu, J. E. Lavery, S.-C. Fang, Univariate cubic L_1 interpolating
557 splines based on the first derivative and on 5-point windows: analysis, al-
558 gorithm and shape-preserving properties, Computational Optimization
559 and Applications (2) 575–600. doi:10.1007/s10589-011-9426-y.
- 560 [19] J. E. Lavery, Shape-preserving, first-derivative-based parametric and
561 nonparametric cubic spline curves, Computer Aided Geometric De-
562 sign (3) 276–296. doi:10.1016/j.cagd.2005.11.002.
- 563 [20] J. E. Lavery, Shape-preserving univariate cubic and higher-degree splines
564 with function-value-based and multistep minimization principles, Com-
565 puter Aided Geometric Design (1) 1–16. doi:10.1016/j.cagd.2008.01.004.

- 566 [21] R. T. Farouki, S. Shah, Real-time CNC interpolators for Pythagorean-
567 hodograph curves, *Computer Aided Geometric Design* (7) 583–600.
568 doi:10.1016/0167-8396(95)00047-X.