A Self-Improvement to the Cauchy-Schwarz Inequality

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Abstract

We present a self improvement to the Cauchy-Schwarz inequality, which in the probability case yields

$$[E(XY)]^2 \leq E(X^2) E(Y^2) - \left(|E(X)|\sqrt{\text{Var}(Y)} - |E(Y)|\sqrt{\text{Var}(X)} \right)^2.$$

It is to be noted that the additional term to the inequality only involves the marginal first two moments for X and Y , and not any joint property. We also provide the discrete improvement to the inequality.

Keywords: Cauchy-Schwarz; Cramer-Rao Inequality, Wasserstein distance

1. Introduction

The Cauchy-Schwarz inequality is arguably one of the most widely used inequalities in mathematics; see [6]. Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be two real sequences; then

$$\left(\sum_{i=1}^n x_i y_i \right)^2 \leq \sum_{i=1}^n x_i^2 \sum_{i=1}^n y_i^2 \quad (1)$$

with equality if and only if there exists a constant c such that $x_i = c y_i$ for all $i = 1, \dots, n$. There are a number of proofs available in the literature and textbooks.

There are refinements of (1), such as [2], but are quite specific in the details on the x and y . A survey of the inequality is given in [4] and a recent generalization is presented in [7].

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The aim in this paper is to provide a new and elementary (self) improvement of the Cauchy-Schwarz inequality which only uses the terms

$$\sum_{i=1}^n x_i, \quad \sum_{i=1}^n x_i^2, \quad \sum_{i=1}^n y_i, \quad \text{and} \quad \sum_{i=1}^n y_i^2.$$

These are the first two marginal moments for x and y . The self improvement is similar in spirit to one appearing in [8], which worked on the Jensen inequality, and also in [3], which worked on the logarithmic Sobolev inequality.

Other improvements of the Cauchy-Schwarz inequality use more than these first two moments; see for example [1], which use joint properties of x and y .

In the following we will use $\bar{x}$ and $\bar{y}$ to denote $n^{-1} \sum_{i=1}^n x_i$ and $n^{-1} \sum_{i=1}^n y_i$, respectively, and V_x and V_y to denote

$$\sum_{i=1}^n x_i^2 - n\bar{x}^2 \quad \text{and} \quad \sum_{i=1}^n y_i^2 - n\bar{y}^2,$$

respectively. It is clear that V_x and V_y are nonnegative. The new result is that

$$\left(\sum_{i=1}^n x_i y_i \right)^2 \leq \sum_{i=1}^n x_i^2 \sum_{i=1}^n y_i^2 - n \left(|\bar{x}| \sqrt{V_y} - |\bar{y}| \sqrt{V_x} \right)^2. \quad (2)$$

We prove this in section 2 where we also present the improved inequality for the probability setting. Section 3 contains some applications.

2. Proof of (2)

Now (2) is a self-improvement of the Cauchy-Schwarz inequality. To see this, for some as yet unspecified h_1 and h_2 , we have

$$\left(\sum_{i=1}^n (x_i - h_1)(y_i - h_2) \right)^2 \leq \sum_{i=1}^n (x_i - h_1)^2 \sum_{i=1}^n (y_i - h_2)^2,$$

so

$$\left(\sum_{i=1}^n x_i y_i - h_1 n \bar{y} - h_2 n \bar{x} + n h_1 h_2 \right)^2 \leq \left(\sum_{i=1}^n x_i^2 - 2 h_1 n \bar{x} + n h_1^2 \right) \left(\sum_{i=1}^n y_i^2 - 2 h_2 n \bar{y} + n h_2^2 \right).$$

Setting $h_1 h_2 = h_1 \bar{y} + h_2 \bar{x}$, to isolate the $\sum_{i=1}^n x_i y_i$ term from the left side, and after some algebra on the right side, we arrive at

$$\left(\sum_{i=1}^n x_i y_i \right)^2 \leq \sum_{i=1}^n x_i^2 \sum_{i=1}^n y_i^2 - n h_1 V_y [2\bar{x} - h_1] - n h_2 V_x [2\bar{y} - h_2].$$

Now setting $h_1 = h$ and $h_2 = h\bar{y}/(h - \bar{x})$ for some h , we have

$$h_1 V_y [2\bar{x} - h_1] + h_2 V_x [2\bar{y} - h_2] = h(2\bar{x} - h) \left[V_y - V_x \frac{\bar{y}^2}{(h - \bar{x})^2} \right]. \quad (3)$$

Putting $h = \bar{x} + t$ the aim now would be to maximize,

$$F(t) = (\bar{x}^2 - t^2)(V_y - V_x \bar{y}^2/t^2).$$

Noting that $F(\infty) = -\infty$ and $F(0+) = -\infty$, we see the maximizing t^2 is given by $\hat{t}^2 = |\bar{x}\bar{y}| \sqrt{V_x/V_y}$, and so (3) becomes

$$\left(|\bar{x}| \sqrt{V_y} - |\bar{y}| \sqrt{V_x} \right)^2.$$

This completes the proof.

In the special case when $x_i = 1$ for all i , we obtain an equality in (2).

Corollary. It is that

$$\frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (x_i y_j - x_j y_i)^2 \geq n \left(|\bar{x}| \sqrt{V_y} - |\bar{y}| \sqrt{V_x} \right)^2.$$

The Lagrange identity (see [6]) has

$$\sum_{i=1}^n x_i^2 \sum_{i=1}^n y_i^2 - \left(\sum_{i=1}^n x_i y_i \right)^2 = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (x_i y_j - x_j y_i)^2$$

and hence the statement of the Corollary follows.

The probability version of the Cauchy-Schwarz inequality is given by

$$[E(XY)]^2 \leq E(X^2) E(Y^2) \quad (4)$$

and is an equality if and only if $Y = cX$ for some constant c .

The self improvement for this inequality follows the same outline as in the proof of (2) and we omit the details. The new inequality is given by

$$[E(XY)]^2 \leq E(X^2) E(Y^2) - \left(|E(X)|\sqrt{\text{Var}(Y)} - |E(Y)|\sqrt{\text{Var}(X)} \right)^2. \quad (5)$$

3. Applications

3.1. Standard normal

We obtain an interesting result when we take Y to be standard normal; then (5) becomes

$$[E(XY)]^2 \leq \text{Var}(X). \quad (6)$$

Of course, this would also hold if Y had mean 0 and variance 1.

To understand this inequality better, let us consider the Wasserstein distance with quadratic cost function; i.e.

$$d(X, Y) = \inf_{\pi \in C(p)} \int \int (x - y)^2 \pi(\mathrm{d}x, \mathrm{d}y)$$

where $C(p)$ is the set of probability measures $\pi(x, y)$ which have the standard normal as marginal for Y and the marginal $p(x)$ for X . See, for example, [5]. It is well known that the minimum is obtained when $\pi(x, y)$ has X from $p(x)$ with

$$Y = \Phi^{-1}(P(X)),$$

where $P(x)$ is the distribution function corresponding to the density $p(x)$ and Φ is the standard normal distribution function. Hence, since the marginals of X and Y are fixed,

$$E(XY) \leq \int x \Phi^{-1}(P(x)) p(x) \mathrm{d}x.$$

If we now take $P(x) = \Phi((x - \mu)/\sigma)$ then

$$\int x \Phi^{-1}(P(x)) p(x) \mathrm{d}x = E \left[X \left(\frac{X - \mu}{\sigma} \right) \right] = \sigma.$$

Hence, (6) is a sharp upper bound with equality when X is also normal.

To see how extensive the improved inequality is, let us consider a simple example with Y as standard normal and X has mean μ and variance σ^2 . Then the Cauchy-Schwarz inequality gives $[E(XY)]^2 \leq \mu^2 + \sigma^2$, whereas the improved inequality yields $[E(XY)]^2 \leq \sigma^2$, which clearly can amount to a large difference.

3.2. General Cramer-Rao inequality

Consider a regular parametric model $p_\theta(\cdot)$ and an observation x from the model. Then for an unbiased estimator $\hat{\theta}$ for θ we have the well known Cramer-Rao lower bound for the variance of $\hat{\theta}$; namely $\text{Var}(\hat{\theta}) \geq 1/I(\theta)$ where $I(\theta)$ is the Fisher information, given by

$$I(\theta) = \int \left(\frac{\partial}{\partial \theta} \log p_\theta(x) \right)^2 p_\theta(x) dx.$$

Here we provide a lower bound for mean square error when the estimator is biased.

Let T be a biased estimator for θ ; so $E(T) = \theta + b(\theta)$. Now using the extended Cauchy-Schwarz inequality (5), we have

$$\left[\int (T - \theta) Y p_\theta(x) dx \right]^2 \leq E[(T - \theta)^2] E(Y^2) - [E(T - \theta)]^2 \text{Var}(Y), \quad (7)$$

where

$$Y = \frac{\partial}{\partial \theta} \log p_\theta(X)$$

and so $E(Y) = 0$ and $E(Y^2) = \text{Var}(Y) = I(\theta)$. The term on the left hand side of (7) is, allowing the switch $\int \partial/\partial \theta \equiv \partial/\partial \theta \int$, equal to $(1 + b'(\theta))^2$ and hence

$$\text{MSE} = E[(T - \theta)^2] \geq \frac{(1 + b'(\theta))^2}{I(\theta)} + b^2(\theta).$$

This is a sharp lower bound for estimators of the form $T = w\hat{\theta} + (1 - w)\mu$, where $\hat{\theta}$ is unbiased and has variance $1/I(\theta)$. For with this form, a Bayes estimator for example, we have

$$(T - \theta)^2 = [w(\hat{\theta} - \theta) + (1 - w)(\mu - \theta)]^2.$$

Hence,

$$E[(T - \theta)^2] = \frac{w^2}{I(\theta)} + (1 - w)^2(\mu - \theta)^2.$$

Now note that $b(\theta) = (1 - w)(\mu - \theta)$ and $b'(\theta) = w - 1$. Clearly, we also have that for biased estimator T , $\text{Var}(T) \geq (1 + b'(\theta))^2 / I(\theta)$, where $\text{bias}(T) = b(\theta)$.

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