On the Law of Large Numbers for Discrete Fourier Transform

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Abstract

We shall establish the law of large numbers for the discrete Fourier transform of random variables with finite first moment under condition $P(|X_n| > x) \leq P(|X_1| > x)$ for all $x \geq 0$; for $1 < p < 2$, we establish the Marcinkiewicz-Zygmund type rate of convergence for the discrete Fourier transform of random variables with finite p th moment under condition $\frac{1}{n} \sum_{k=1}^n P(|X_k| > x) \leq MP(|X_1| > x)$ for all $x \geq 0$ and some positive constant M .

Key words: Law of large numbers; Discrete Fourier transform; Rate of convergence

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1 Introduction

The law of large numbers is valid for pairwise independent identically distributed random variables, result due to Etemadi (1981). This is a surprising result since a sequence of pairwise independent identically distributed random variables may not be ergodic. A way to look into the speed of convergence of this result when the variables have finite moments of order r , $1 < r < 2$, is provided by Baum and Katz (1965) in the i.i.d. case and by Stocia (2011) in the martingale difference case. By carefully examining the proof in Stocia (2011), we notice that the proof can be adapted to centered pairwise independent random variables and we can formulate the following result.

Proposition 1.1. *Let $(X_n)_{n \geq 1}$ be a pairwise independent sequence of random variables with the same distribution.*

(a) *Assume $X_n \in L^1$. Then*

$$\frac{S_n}{n} \rightarrow EX_1, \quad P - \text{a.s. as } n \rightarrow \infty,$$

where $S_n = \sum_{k=1}^n X_k$.

(b) *Assume $X_n \in L^p$, $1 < p < 2$. Then for all $1 \leq r \leq p$ and $\epsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{p/r-2} P(|S_n| > \epsilon n^{1/r}) < \infty.$$

We note that, by Korchevsky (2015), $\frac{S_n - ES_n}{n^{1/p}} \rightarrow 0$, $P - \text{a.s. as } n \rightarrow \infty$, also holds.

The goal of our note is to study the law of large numbers for the discrete Fourier transform of a sequence of random variables with finite p th moment for $p = 1$ and its Marcinkiewicz-Zygmund type rate of convergence for $1 < p < 2$ under weaker conditions than identical distribution as described in the abstract and to show that, from some point of view, the variables have similar properties as pairwise independent random variables with the same distribution.

2 Main Results

Let $(X_n)_{n \geq 1}$ denote a sequence of real-valued random variables on a probability space $(\Omega, \mathcal{F}, P)$. No dependence between the variables is assumed. For $-\pi \leq t < \pi$, define the discrete Fourier transform

$$S_n(t) = \sum_{k=1}^n e^{ikt} X_k. \quad (1)$$

We shall establish an analogue of the result by Baum and Katz (1965) for the discrete Fourier transform.

Theorem 2.1. *Assume*

$$P(|X_n| > x) \leq P(|X_1| > x), \quad \text{for all } x \geq 0, \quad n \geq 1. \quad (\text{UD})$$

If $E|X_1| < \infty$, then for almost all $t \in [-\pi, \pi)$,

$$\lim_{n \rightarrow \infty} \frac{S_n(t)}{n} = 0, \quad P - a.s.$$

The following theorem describes the rate of convergence in the strong law of large numbers.

Theorem 2.2. *Assume*

$$\frac{1}{n} \sum_{k=1}^n P(|X_k| > x) \leq MP(|X_1| > x), \quad \text{for all } x \geq 0, \quad n \geq 1. \quad (\text{MD})$$

Let $1 < p < 2$, $1 \leq r \leq p$. If $E|X_1|^p < \infty$, then for every $\epsilon > 0$ and for almost all $t \in [-\pi, \pi)$,

$$\sum_{n=1}^{\infty} n^{p/r-2} P\left[\max_{1 \leq k \leq n} |S_k(t)| > \epsilon n^{1/r}\right] < \infty. \quad (2)$$

Remark 2.3. In fact, Gut (1992) gave an example to show that condition (MD) is strictly weaker than condition (UD). But in Theorem 2.1, it is an open question whether condition (UD) can be weakened to condition (MD).

Remark 2.4. Differently from the i.i.d. case, the reciprocal of Theorem 2.2 is no longer true. That is, (2) does not imply X_1 have finite p th moment.

Corollary 2.5. *Under the assumption of Theorem 2.2, for almost all $t \in [-\pi, \pi)$,*

$$\lim_{n \rightarrow \infty} \frac{S_n(t)}{n^{1/p}} = 0, \quad P - a.s.$$

Remark 2.6. Both theorems hold if the variables have the same distribution.

3 Proofs

Throughout this whole paper, $C > 0$ denotes a generic constant which may take different values from line to line.

In order to prove our main theorems, we shall first establish two preparatory lemmas. We begin by a truncation argument.

Lemma 3.1. *Assume $(X_n)_{n \geq 1}$ satisfies condition (UD) and $E|X_1| < \infty$. Let $Y_k = X_k I\{|X_k| \leq k\}$ and $S_n^*(t) = \sum_{k=1}^n e^{ikt} Y_k$. Then for all t in $[-\pi, \pi)$,*

$$\lim_{n \rightarrow \infty} \left| \frac{1}{n} S_n(t) - \frac{1}{n} S_n^*(t) \right| = 0, \quad P - a.s.$$

Proof. As $P(|X_n| > x) \leq P(|X_1| > x)$, we obtain

$$\sum_{n=1}^{\infty} P(X_n \neq Y_n) = \sum_{n=1}^{\infty} P(|X_n| > n) \leq \sum_{n=1}^{\infty} P(|X_1| > n) \leq \int_0^{\infty} P(|X_1| > x) dx = E|X_1| < \infty.$$

By the Borel-Cantelli Lemma, we know $P(X_n \neq Y_n \text{ i.o.}) = 0$. That is, for almost all $\omega \in \Omega$, $X_n(\omega) = Y_n(\omega)$, for all n sufficiently large, say for all $n \geq m(\omega) := m$. Thus for all $t \in [-\pi, \pi)$,

$$\left| \frac{1}{n} S_n(t) - \frac{1}{n} S_n^*(t) \right| = \frac{1}{n} \left| \sum_{k=1}^n e^{ikt} (X_k - Y_k) \right| \leq \frac{1}{n} \sum_{k=1}^m |e^{ikt}| \cdot |X_k - Y_k| = \frac{1}{n} \sum_{k=1}^m |X_k - Y_k| \rightarrow 0, \quad P - a.s.$$

That is,

$$\lim_{n \rightarrow \infty} \left| \frac{1}{n} S_n(t) - \frac{1}{n} S_n^*(t) \right| = 0, \quad P - a.s.$$

□

Lemma 3.2. *Assume $(X_n)_{n \geq 1}$ satisfies condition (MD) and $E|X_1|^p < \infty$. Then*

$$\sum_{n=1}^{\infty} n^{p/r-1/r-2} \sum_{k=1}^n E \left[|X_k| I(|X_k| > n^{1/r}) \right] \leq C E|X_1|^p < \infty; \quad (3)$$

$$\sum_{n=1}^{\infty} n^{p/r-2/r-2} \sum_{k=1}^n E \left[|X_k|^2 I(|X_k| \leq n^{1/r}) \right] \leq C E|X_1|^p < \infty. \quad (4)$$

Proof. We shall use the following facts: For $X \geq 0$ a.s. and $A > 0$,

$$E(X) = \int_0^{\infty} P(X > x) dx; \quad E(XI(X > A)) = AP(X > A) + \int_A^{\infty} P(X > x) dx \quad (5)$$

and

$$E(XI(X \leq A)) = -AP(X > A) + \int_0^A P(X > x) dx \leq \int_0^A P(X > x) dx. \quad (6)$$

Therefore

$$E(X^2 I(X^2 \leq A)) \leq \int_0^A P(X^2 > x) dx. \quad (7)$$

As $(X_n)_{n \geq 1}$ satisfies condition (MD), by fact (5), we get

$$\sum_{k=1}^n E \left[|X_k| I(|X_k| > n^{1/r}) \right] \leq M n E \left[|X_1| I(|X_1| > n^{1/r}) \right].$$

So then

$$\sum_{n=1}^{\infty} n^{p/r-1/r-2} \sum_{k=1}^n E \left[|X_k| I(|X_k| > n^{1/r}) \right] \leq M \sum_{n=1}^{\infty} n^{p/r-1/r-1} E \left[|X_1| I(|X_1| > n^{1/r}) \right].$$

By Stoica (2011) (page 912), we have

$$\sum_{n=1}^{\infty} n^{p/r-1/r-1} E \left[|X_1| I(|X_1| > n^{1/r}) \right] \leq C E |X_1|^p < \infty.$$

Combining this result with our computation, we obtain (3).

By condition (MD) and (7), we get

$$\begin{aligned} \sum_{k=1}^n E X_k^2 I(|X_k| \leq n^{1/r}) &\leq \sum_{k=1}^n \int_0^{n^{2/r}} P(X_k^2 > x) dx \\ &\leq n M \int_0^{n^{2/r}} P(X_1^2 > x) dx \\ &\leq C n \sum_{k=1}^{n^{2/r}} P(X_1^2 > k). \end{aligned}$$

Now

$$\begin{aligned} \sum_{n=1}^{\infty} n^{p/r-2/r-2} n \sum_{k=1}^{n^{2/r}} P(X_1^2 > k) &\leq \sum_{k=1}^{\infty} P(X_1^2 > k) \sum_{n \geq k^{r/2}} n^{p/r-2/r-1} \\ &\leq C \sum_{k=1}^{\infty} P(X_1^2 > k) k^{\frac{p-2}{2}} \leq C E |X_1|^p, \end{aligned}$$

because

$$E |X_1|^p = E[(X_1^2)^{p/2}] = \frac{p}{2} \int_0^{\infty} x^{\frac{p-2}{2}} P(X_1^2 > x) dx.$$

Therefore we have (4). □

3.1 Proof of Theorem 2.1

Proof. First, let us show that $\sum_{k=1}^n \frac{e^{ikt} Y_k}{k}$ converges a.s. for almost all $t \in [-\pi, \pi)$.

From Durrett (2013) (page 64), we know that $\sum_{k=1}^{\infty} \frac{EY_k^2}{k^2} \leq 4E|X_1| < \infty$. So, for almost all $\omega \in \Omega$, $\sum_{k=1}^{\infty} \frac{Y_k^2(\omega)}{k^2} < \infty$.

Then, by Carleson's Theorem (1966), for almost all $\omega \in \Omega$, $\sum_{k=1}^n \frac{e^{ikt}Y_k(\omega)}{k}$ converges almost everywhere in t . That is, for almost all $\omega \in \Omega$, there exists $I_\omega \subset [-\pi, \pi)$ with $\lambda(I_\omega) = 2\pi$, such that for all $t \in I_\omega$, $\sum_{k=1}^n \frac{e^{ikt}Y_k(\omega)}{k}$ converges, where λ is the Lebesgue measure on $[-\pi, \pi)$.

Let $\Omega_0 = \{\omega : \sum_{k=1}^{\infty} \frac{Y_k^2(\omega)}{k^2} < \infty\}$, then $P(\Omega_0) = 1$.

It is convenient to work on the product space $[-\pi, \pi) \times \Omega$ with product measure $\tilde{P} := \lambda \times P$.

Define $A = \{(t, \omega) : \sum_{k=1}^{\infty} \frac{e^{ikt}Y_k(\omega)}{k} \text{ is convergent}\} \subset [-\pi, \pi) \times \Omega$.

Using Fubini Theorem,

$$\begin{aligned} \tilde{P}(A) &= \int_{[-\pi, \pi) \times \Omega} I_A(t, \omega) d\tilde{P} = \int_{\Omega} \int_{-\pi}^{\pi} I_A(t, \omega) \lambda(dt) dP \\ &= \int_{\Omega_0} \int_{-\pi}^{\pi} I_A(t, \omega) \lambda(dt) dP = \int_{\Omega_0} 2\pi dP \\ &= 2\pi \\ &= \int_{-\pi}^{\pi} \int_{\Omega} I_A(t, \omega) dP \lambda(dt). \end{aligned}$$

Thus, for almost all $t \in [-\pi, \pi)$, $\sum_{k=1}^n \frac{e^{ikt}Y_k}{k}$ converges a.s.

Now by applying Kronecker Lemma (If $a_n \uparrow \infty$ and $\sum_{n=1}^{\infty} (x_n/a_n)$ converges, then $a_n^{-1} \sum_{m=1}^n x_m \rightarrow 0$), for almost all $t \in [-\pi, \pi)$, we obtain

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n e^{ikt}Y_k = \lim_{n \rightarrow \infty} \frac{1}{n} S_n^*(t) = 0, \quad P - a.s.$$

By Lemma 3.1, it follows that for almost all $t \in [-\pi, \pi)$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} S_n(t) = \lim_{n \rightarrow \infty} \frac{1}{n} S_n^*(t) = 0, \quad P - a.s.$$

□

3.2 Proof of Theorem 2.2

Proof. Define the following random variables for $k = 1, 2, \dots, n$:

$$X'_k = e^{itk} X_k I\{|X_k| \leq n^{1/r}\}, \quad X''_k = e^{itk} X_k I\{|X_k| > n^{1/r}\}.$$

Clearly, $e^{itk} X_k = X'_k + X''_k$ and $S_n(t) = S'_n(t) + S''_n(t)$ where $S'_n(t) = \sum_{k=1}^n X'_k$ and $S''_n(t) = \sum_{k=1}^n X''_k$.

By Markov's Inequality,

$$P \left[\max_{1 \leq k \leq n} |S'_k(t)| > \epsilon n^{1/r} \right] \leq \frac{1}{\epsilon^2} n^{-2/r} E \left[\left(\max_{1 \leq k \leq n} |S'_k(t)| \right)^2 \right], \quad (8)$$

and by the maximal inequality in Hunt and Young (1974),

$$\int_{-\pi}^{\pi} \max_{1 \leq k \leq n} |S'_k(t)|^2 \lambda(dt) \leq C \sum_{k=1}^n |X'_k|^2. \quad (9)$$

Using Fubini's Theorem and properties (8) and (9), we obtain

$$\begin{aligned} \tilde{P} \left[\max_{1 \leq k \leq n} |S'_k(t)| > \epsilon n^{1/r} \right] &= \int_{-\pi}^{\pi} P \left[\max_{1 \leq k \leq n} |S'_k(t)| > \epsilon n^{1/r} \right] \lambda(dt) \\ &\leq \frac{1}{\epsilon^2} n^{-2/r} \int_{-\pi}^{\pi} E \left[\left(\max_{1 \leq k \leq n} |S'_k(t)| \right)^2 \right] \lambda(dt) \\ &\leq \frac{C}{\epsilon^2} n^{-2/r} E \left[\sum_{k=1}^n |X'_k|^2 \right] \\ &= \frac{C}{\epsilon^2} n^{-2/r} \sum_{k=1}^n E \left[(X_k)^2 I(|X_k| \leq n^{1/r}) \right]. \end{aligned}$$

By Lemma 3.2, we get

$$\sum_{n=1}^{\infty} n^{p/r-2} \tilde{P} \left[\max_{1 \leq k \leq n} |S'_k(t)| > \epsilon n^{1/r} \right] < \infty. \quad (10)$$

By Markov's inequality,

$$\begin{aligned} P \left[\max_{1 \leq k \leq n} |S''_k(t)| > \epsilon n^{1/r} \right] &\leq \frac{1}{\epsilon} n^{-1/r} E \left[\max_{1 \leq k \leq n} |S''_k(t)| \right] \\ &\leq \frac{1}{\epsilon} n^{-1/r} E \left[\sum_{k=1}^n |X''_k| \right] \\ &= \frac{1}{\epsilon} n^{-1/r} \sum_{k=1}^n E \left[|X_k| I\{|X_k| > n^{1/r}\} \right]. \end{aligned}$$

Again, by Lemma 3.2, we obtain

$$\sum_{n=1}^{\infty} n^{p/r-2} P \left[\max_{1 \leq k \leq n} |S''_k(t)| > \epsilon n^{1/r} \right] \leq CE |X_1|^p < \infty. \quad (11)$$

Using Fubini's Theorem and relation (11),

$$\begin{aligned} \sum_{n=1}^{\infty} n^{p/r-2} \tilde{P} \left[\max_{1 \leq k \leq n} |S''_k(t)| > \epsilon n^{1/r} \right] &= \sum_{n=1}^{\infty} n^{p/r-2} \int_{-\pi}^{\pi} P \left[\max_{1 \leq k \leq n} |S''_k(t)| > \epsilon n^{1/r} \right] \lambda(dt) \\ &\leq CE |X_1|^p < \infty. \end{aligned} \quad (12)$$

Combining (10) and (12), we get

$$\begin{aligned} & \sum_{n=1}^{\infty} n^{p/r-2} \tilde{P}[\max_{1 \leq k \leq n} |S_k(t)| > \epsilon n^{1/r}] \\ & \leq \sum_{n=1}^{\infty} n^{p/r-2} \tilde{P}[\max_{1 \leq k \leq n} |S'_k(t)| > \frac{\epsilon}{2} n^{1/r}] + \sum_{n=1}^{\infty} n^{p/r-2} \tilde{P}[\max_{1 \leq k \leq n} |S''_k(t)| > \frac{\epsilon}{2} n^{1/r}] < \infty. \end{aligned}$$

By Fubini's Theorem, we have

$$\begin{aligned} & \sum_{n=1}^{\infty} n^{p/r-2} \tilde{P}[\max_{1 \leq k \leq n} |S_k(t)| > \epsilon n^{1/r}] \\ & = \sum_{n=1}^{\infty} n^{p/r-2} \int_{-\pi}^{\pi} \int_{\Omega} I\{\max_{1 \leq k \leq n} |S_k(t)| > \epsilon n^{1/r}\} dP \lambda(dt) \\ & = \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} n^{p/r-2} P\left(\max_{1 \leq k \leq n} |S_k(t)| > \epsilon n^{1/r}\right) \lambda(dt) < \infty. \end{aligned}$$

Thus, for almost all $t \in [-\pi, \pi)$,

$$\sum_{n=1}^{\infty} n^{p/r-2} P\left(\max_{1 \leq k \leq n} |S_k(t)| > \epsilon n^{1/r}\right) < \infty.$$

□

3.3 Proof of Remark 2.4

Proof. It is well-known that $E|X_1|^p < \infty$ is equivalent to (2) in the i.i.d. case, but this is no longer true in our setting.

For instance, let

$$X_1 = X_2 = \cdots = X_n = \cdots.$$

Then

$$\max_{1 \leq k \leq n} |S_k(t)| = \max_{1 \leq k \leq n} \left| \sum_{j=1}^k e^{ijt} X_k \right| = \max_{1 \leq k \leq n} \left| \frac{1 - e^{ikt}}{1 - e^{it}} \right| \cdot |X_1|.$$

Because

$$1 \leq \max_{1 \leq k \leq n} \left| \frac{1 - e^{ikt}}{1 - e^{it}} \right| \leq \frac{2}{|1 - e^{it}|},$$

(2) is equivalent to

$$\sum_{n=1}^{\infty} n^{p/r-2} P(|X_1| > C\epsilon n^{1/r}) < \infty,$$

which is certainly not equivalent to $E|X_1|^p < \infty$.

□

3.4 Proof of Corollary 2.5

Proof. By Theorem 2.2, when $1 < r = p < 2$, we have: for almost all $t \in [-\pi, \pi)$,

$$\sum_{n=1}^{\infty} n^{-1} P \left(\max_{1 \leq k \leq n} |S_k(t)| > \epsilon n^{1/p} \right) < \infty,$$

which is equivalent to

$$\sum_{N=1}^{\infty} P \left(\max_{1 \leq k \leq 2^N} |S_k(t)| > \epsilon 2^{N/p} \right) < \infty.$$

Then, by the Borel-Cantelli Lemma, for almost all $t \in [-\pi, \pi)$,

$$S_n(t)/n^{1/p} \rightarrow 0, \quad P - \text{a.s.}$$

□

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