Resolution of a conjecture on variance functions for one-parameter natural exponential families

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Abstract

We prove a conjecture of Bar-Lev, Bshouty and Enis stating that a polynomial with a simple root at 0 and a complex root with positive imaginary part is the variance function of a one-parameter natural exponential family with mean domain $(0, \infty)$ if and only if the real part of the complex root is not positive.

Keywords: Natural exponential family; polynomial variance functions.

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1 Introduction

The one-parameter natural exponential family (NEF) is an important family of probability measures and has been widely used in statistical modelling ([McCullagh and Nelder, 1989](#); [Letac, 1992](#)). Let β be a positive Radon measure on $\mathbb{R}$ that is not concentrated on one point. Suppose the interior Θ of

$$\tilde{\Theta} = \left\{ \theta \in \mathbb{R} : g(\theta) = \int e^{x\theta} \beta(dx) < \infty \right\}$$

is not empty. Then $\mathcal{F} = \{F_\theta : \theta \in \Theta\}$ with

$$F_\theta(dx) = \exp[\theta x - \log g(\theta)] \beta(dx)$$

forms a NEF with respect to the basis β . Without loss of generality, we can assume $0 \in \Theta$, so that β is a probability measure.

For each NEF $\mathcal{F}$, the mapping $\mu(\theta) = \frac{d}{d\theta} \log g(\theta)$ defines the mean function $\mu : \Theta \rightarrow U$ with $U = \mu(\Theta)$, and U is called the “mean domain”. Let $\theta = \theta(\mu)$ be the inverse function of μ . Define the function v on U as

$$v(\mu) = \int (x - \mu)^2 F_{\theta(\mu)}(dx)$$

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for $\mu \in U$. Then the pair (v, U) is called the variance function (VF) of $\mathcal{F}$. Note that v is a positive, real analytic function on U and that v characterizes the NEF $\mathcal{F}$.

Let $\mathcal{G}_D$ be the set of positive, real analytic functions on a domain $D \subseteq \mathbb{R}$. In order to use an $f \in \mathcal{G}_D$ to model the variance-mean relationship for some data and assume that the associated random errors follow a NEF, it is crucial to ensure that (f, D) is indeed the VF of some NEF. Therefore, identifying which $f \in \mathcal{G}_D$ are VFs of NEFs is of importance. For few of these results, see, e.g., [Bar-Lev et al. \(1991\)](#); [Letac \(1992, 2016\)](#).

Let $\mathbb{R}^+ = (0, \infty)$. The authors of a ground breaking paper [Bar-Lev et al. \(1992\)](#) raised the following:

Conjecture. *Let n be a fixed positive integer and define*

$$v(u) = a_0 u (u - u_1)^n (u - \bar{u}_1)^n, \quad (1)$$

where $a_0 > 0$, $\Im(u_1) > 0$, and $\bar{u}_1$ is the complex conjugate of u_1 . Then $(v, \mathbb{R}^+)$ is a variance function if and only if $\Re(u_1) \leq 0$.

The sufficiency in the conjecture holds as follows. When $\Re(u_1) \leq 0$, the polynomial v has non-negative coefficients and the sum of the coefficients are positive. So, by [Bar-Lev \(1987\)](#), $(v, \mathbb{R}^+)$ is the VF of an infinitely divisible NEF. In particular, when $v'(0) = 1$ Proposition 4.4 of [Letac and Mora \(1990\)](#) implies that the corresponding NEF is concentrated on the set $\mathbb{N}$ of non-negative integers. Thus, it is left to prove the necessity. To this end, [Bar-Lev et al. \(1992\)](#) derived a powerful result, their Theorem 1, that converts proving the necessity into determining the sign of $\Re(\tau)$ of the residue $\tau = -2\pi i \operatorname{Res}(1/v, u_1)$, where $i = \sqrt{-1}$. However, [Bar-Lev et al. \(1992\)](#) only verified the necessity for $n \in \{1, 2, 3\}$ by directly computing τ , since $1/v$ is the reciprocal of the polynomial v with degree $2n + 1$ and computing τ becomes intractable when n is large.

In this note, we will prove the necessity for all $n \in \mathbb{N}$ using a different strategy that exploits the algebraic and analytic properties of v . The positive answer to the conjecture enlarges the family of polynomials that can be the VFs of NEFs and the variance-mean relationships that can be used for statistical modelling using NEFs.

2 Proof of necessity

Let $(v, \mathbb{R}^+)$ be a VF of the form (1) generated by the probability measure β with finite, positive mean $\mu_0 = \mu(0)$. Define $\theta_0 = \int_{\mu_0}^{\infty} dt/v(t)$. Since $n \geq 1$, then $0 < \theta_0 < \infty$ and $\Theta = (-\infty, \theta_0)$. Using the notations in [Bar-Lev et al. \(1992\)](#), we introduce $\tau = -2\pi i \operatorname{Res}(1/v, u_1)$

Figure 1: Contour Γ in the proof of [Theorem 1](#), where the arrow indicates orientation.

as in the previous section, $d = 2\pi/v'(0)$, $\Theta + id/2 = \{z \in \mathbb{C} : z = \theta + id/2, \theta \in \Theta\}$ and

$$S_0^+ = \{z \in \mathbb{C} : \Re(z) \in \Theta \text{ and } 0 < \Im(z) < d/2\}.$$

A particular case of Theorem 1 of [Bar-Lev et al. \(1992\)](#) is:

Proposition 1. *With the above hypothesis on v and notations, $\theta_0 + \tau \notin S_0^+ \cup (\Theta + id/2)$ must hold.*

Here is our main result:

Theorem 1. *Let v be defined by (1) where n is a positive integer. Then $(v, \mathbb{R}^+)$ is a variance function if and only if $\Re(u_1) \leq 0$.*

Proof. The sufficiency holds as already explained in the previous section. So, it suffices to show that $(v, \mathbb{R}^+)$ is not a VF when $\Re(u_1) > 0$ in order to prove the necessity. By [Proposition 1](#), we only have to show that $\Re(u_1) > 0$ implies $\theta_0 + \tau \in \Theta + id/2$. Without loss of generality, we can assume $a_0 = 1$. Pick ρ and R such that $5^{-1}|u_1| > \rho > 0$ and $R > 2|u_1|$. Define the contour $\Gamma = \bigcup_{i=1}^4 \Gamma_i$ depicted in [Figure 1](#), where $\Gamma_1 = \{Re^{i\omega} : 0 \leq \omega \leq \pi\}$, $\Gamma_2 = \{\rho e^{i\omega} : \pi \geq \omega \geq 0\}$, $\Gamma_3 = \{x : -R \leq x \leq -\rho\}$ and $\Gamma_4 = \{x : \rho \leq x \leq R\}$. Then $\tau = -\int_{\Gamma} d\zeta/v(\zeta)$. However, $\int_{\Gamma_2} d\zeta/v(\zeta) = -i\pi/v'(0)$ by applying the residue theorem to the whole circle that includes Γ_2 as a semi-circle. Further, $\lim_{R \rightarrow \infty} \left| \int_{\Gamma_1} d\zeta/v(\zeta) \right| \leq \lim_{R \rightarrow \infty} \pi/(6R^{n+1}) = 0$, and $L_R = \int_{\Gamma_3 \cup \Gamma_4} d\zeta/v(\zeta)$ is real. So, $\Im(\tau) = \pi/v'(0) = 2^{-1}d$ and

$$\Re(\tau) = -\lim_{R \rightarrow \infty, \rho \rightarrow 0} L_R = -\int_0^\infty \left(\frac{1}{|\zeta - u_1|^{2n}} - \frac{1}{|\zeta + u_1|^{2n}} \right) \frac{d\zeta}{\zeta}. \quad (2)$$

Note that $\theta_0 + \tau = \theta_0 + \Re(\tau) + id/2$ and $\Im(u_1) > 0$. If $\Re(u_1) > 0$, then $\Re(\tau) < 0$ and $\theta_0 + \tau \in \Theta + id/2$, which implies that $(v, \mathbb{R}^+)$ is not a VF. This completes the proof. $\square$

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