Another look at Bayes map iterated filtering

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Abstract

A new proof of convergence of Bayes map iterated filtering algorithm, based on supermartingale inequality, is proposed. Relying on log concavity assumption, this approach with verifiable conditions is simpler than the previous approach and is generalizable to more sophisticated algorithms. The assumption could be relaxed at the cost of more technical details.

Keywords: Iterated filtering, Bayes map, Hidden Markov model, Stochastic approximation, Time series

1. Introduction

Inference methodology for stochastic dynamic systems is often termed “plug-and-play” if only simulated estimation is needed to plug into the inference procedure (Bretó et al., 2009). The plug-and-play methodology has been used to free researchers from demanding closed-form expression requirements for transition probabilities, imposed by previously available statistical methodology, allowing researchers to broaden classes of modeling and considering novel hypotheses. Unlike the mainstream statistical techniques (Expectation-Maximization algorithms and Bayesian Markov Chain Monte Carlo), plug-and-play approaches are a relatively recent and exciting development because of their less restrictive requirements. Examples of plug-and-play methodologies follow the frequentist paradigm (Ionides et al., 2011; Lindström et al., 2012), the Bayesian paradigm (Andrieu et al., 2010; Toni et al., 2009), or work by matching selected summary statistics (Wood, 2010). Amongst these approaches, we are interested in iterated filtering because of its computational efficiency. Iterated filtering was originally proposed by Ionides et al. (2006) and later theoretically developed by Ionides et al. (2011). Recently, Lindström et al. (2012) extended it to improve on numerical performance while Doucet et al. (2013) expanded it to include filtering/smoothing with quite attractive theoretical properties. Ionides et al. (2015) generalized Lindström et al. (2012)’s approach and combine the idea with data cloning (Lele et al., 2007), developed a Bayes map iterated filtering with an entirely different theoretical approach. We revisit the approach of Ionides et al. (2015) using a supermartingale inequality, which relies on easily verifiable conditions, thus of more general interests. The paper is organized as follows. In the next section, we introduce some notations and develop the framework of Bayes map iterated filtering. In Sections 3, we prove the convergence of this approximation filter to the maximum likelihood estimator (MLE) by super-martingale inequality.

2. Problem definition

We consider a general latent variable model $(X, Y, \theta) = (X_{1:n}, Y_{1:n}, \theta)$ for $n \in N$ where $X_{i:j}$ denotes $(X_i, \dots, X_j)$ and (X, Y) is a discrete time Markov chain with X also a Markov chain, taking values in some measurable space $\mathcal{X}$. We suppose that X is unobserved while Y is observed and taking fixed value y in observation space $\mathcal{Y}$. We also suppose that the joint density $f_{XY}(x, y; \theta)$ of a random variables (X, Y) depending on a parameter $\theta \in \Theta$. The marginal densities of X and Y are denoted $f_X(x; \theta)$ and $f_Y(y; \theta)$ respectively while the conditional density of the observed variable Y given the latent variable X , also known as *measurement model*, is denoted as $f_{Y|X}(y|x; \theta)$. Let f be a Markov kernel acting on

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$\mathcal{X}$ to itself, g be a Markov kernel acting from $\mathcal{X}$ to $\mathcal{Y}$. The likelihood function $L(\theta) = f_Y(y; \theta) = \int f_\theta(y|x)g_\theta(x)dx$. It is assumed that the likelihood is intractable. We are interested in computing the MLE. Set $\ell(\theta) = \log L(\theta)$, and

$$\theta_\star = \arg \max_{\theta \in \Theta} \ell(\theta).$$

The data cloning approach to computing $\theta_\star$ is to introduce the set of posterior distributions $\{\pi_n, n \geq 1\}$, where

$$\pi_n(\theta|y) = \frac{e^{n\ell(\theta)}\pi(\theta)}{\int e^{n\ell(\theta)}\pi(\theta)d\theta},$$

for some prior distribution π . Lele et al. (2007) show that if $\Theta_n \sim \pi_n$, then Θ_n converges in probability to $\theta_\star$, and $\sqrt{n}(\Theta_n - \theta_\star)$ converges weakly to a Gaussian distribution. The precise statement of their result is as follows.

Assumption 1. *There exist $\theta_\star$ such that $\theta \mapsto \ell(\theta)$ attains a local maximum at $\theta_\star$.*

Assumption 2. *π is continuous at $\theta_\star$, $\theta \mapsto \ell(\theta)$ is of class C^2 in a neighborhood of $\theta_\star$ and $-H_{\theta_\star}$ is positive definite, where $H_{\theta_\star} \stackrel{\text{def}}{=} \nabla^2 \ell(\theta)|_{\theta=\theta_\star}$.*

Assumption 3. *For any $\delta > 0$, $\gamma(\delta) < \ell(\theta_\star)$, where $\gamma(\delta) \stackrel{\text{def}}{=} \sup\{\ell(\theta), \|\theta - \theta_\star\| > \delta\}$.*

Theorem 4. *Lele et al. (2007) Assume Assumptions 1 - 3. Set $\Sigma \stackrel{\text{def}}{=} \{-H_{\theta_\star}\}^{-1/2}$, and $\Psi_n \stackrel{\text{def}}{=} \sqrt{n}\Sigma(\Theta_n - \theta_\star)$. As $n \rightarrow \infty$, Θ_n converges in probability to $\theta_\star$, and Ψ_n converges weakly to $N(0, I_d)$.*

The main ingredient of the proof is the following Taylor expansion. For any $u \in \mathbb{R}^p$, there exists $t \in (0, 1)$ such that

$$\ell(\theta_\star + \Sigma u) - \ell(\theta_\star) = -\frac{1}{2}\|u\|^2 + \frac{1}{2}u'\Sigma(\nabla^2 \ell(\theta_\star + t\Sigma u) - \nabla^2 \ell(\theta_\star))\Sigma u.$$

As a consequence, we get the following lemma.

Lemma 5. *Assume Assumption 1 - 2. Then for all $\theta \in \Theta$,*

$$\lim_{n \rightarrow \infty} n(\ell(\theta_\star + \frac{1}{\sqrt{n}}\Sigma\theta) - \ell(\theta_\star)) = -\|\theta\|^2/2,$$

and the convergence is uniform on compact sets.

One limitation of the data cloning approach is that it is not clear how the technique can be used in situations where f_θ , the density of the state process is intractable. We extend the data cloning approach by iterating sequentially the filtering of the time series data.

3. Main result

Suppose we wish to solve the maximization problem

$$\theta_\star \stackrel{\text{def}}{=} \arg \max_{\theta \in \mathbb{R}^d} \ell(\theta), \tag{1}$$

for some smooth function ℓ , that we think of as a log-likelihood function. Our motivation comes from partially observed Markov processes, and latent variables models where ℓ can be written as $\ell(\theta) \stackrel{\text{def}}{=} \log \int q_\theta(x)f_\theta(y|x)d_{X_0}$. But for the time being ℓ is arbitrary.

Let us write $\mathcal{P}$ to denote the space of all probability measures on $\mathbb{R}^d$. Let K_n denote the probability kernel on $\mathbb{R}^d \times \mathcal{B}(\mathbb{R}^d)$ given by

$$K_n(\theta, d\vartheta) = \frac{1}{\sigma_n^d} K(\sigma_n^{-1}(\vartheta - \theta))d\vartheta, \tag{2}$$

for positive scale parameter σ_n , and where K is the density of the normal distribution on $\mathbb{R}^d$ with mean zero and covariance Λ . Without any loss of generality in the theory we assume that $\Lambda = I_d$, the $d \times d$ identity matrix.

We consider this Bayesian operator $B_n : \mathcal{P} \rightarrow \mathcal{P}$ which transforms the probability measure μ into $B_n(\mu)$ defined as

$$\begin{aligned} B_n(\mu)(A) &= \frac{\int \mu(d\theta) \int K_n(\theta, d\vartheta) e^{\ell(\vartheta)} 1_A(\vartheta)}{\int \mu(d\theta) \int K_n(\theta, d\vartheta) e^{\ell(\vartheta)}} \\ &= \frac{\int \mu(d\theta) \int K(du) e^{\bar{\ell}(\theta + \sigma_n u)} 1_A(\theta + \sigma_n u)}{\int \mu(d\theta) \int K(du) e^{\bar{\ell}(\theta + \sigma_n u)}}, \end{aligned} \quad (3)$$

by the change of variable $\vartheta = \theta + \sigma_n u$, where $\bar{\ell} = \ell - \ell(\theta_*)$. Given some initial probability measure $\pi_0 \in \mathcal{P}$ we consider the sequence of probability measures $\{\pi_n, n \geq 0\}$, defined recursively by

$$\pi_n = B_n(\pi_{n-1}), \quad n \geq 1.$$

The goal here is to show that as $n \rightarrow \infty$, π_n concentrates around the solution θ_* of the maximization problem (1). We assume that ℓ is strongly concave.

Assumption 6. *The function $\ell : \mathbb{R}^d \rightarrow \mathbb{R}$ is concave, admits a unique maximum $\theta_* \in \mathbb{R}^d$, and is twice differentiable. Furthermore, there exists $0 < \kappa \leq L < \infty$ such that for all $\theta \in \mathbb{R}^d$, the spectrum of $\nabla^2 \ell(\theta)$ is contained in $[-L, -\kappa]$*

Assumption 7. *Assume $\sum \sigma_n^2 < \infty$.*

By standard first-order optimality conditions, $\nabla \ell(\theta_*) = 0$. Assumption 6 implies that for all $\theta, u \in \mathbb{R}^d$

$$-\kappa \|u\|^2 \geq \nabla^2 \ell(\theta) \cdot (u, u) \geq -L \|u\|^2. \quad (4)$$

We obtain the following lemma.

Lemma 8. *Suppose Assumption 6, the following holds,*

$$\pi_n(h) = \frac{\int \pi_{n-1}(d\theta) e^{\ell(\theta)} h(\theta)}{\int \pi_{n-1}(d\theta) e^{\ell(\theta)}} + \sigma_n^2 \epsilon_{n,3},$$

for some $\epsilon_{n,3} < \infty$.

Proof. See Supplement S1. □

Lemma 9. *Denote*

$$\pi_N^1(h) = \frac{\int \pi_N(d\theta) e^{\ell(\theta)} h(\theta)}{\int \pi_N(d\theta) e^{\ell(\theta)}},$$

and for an integer $m > 1$,

$$\pi_N^m(h) = \frac{\int \pi_N^{m-1}(d\theta) e^{\ell(\theta)} h(\theta)}{\int \pi_N^{m-1}(d\theta) e^{\ell(\theta)}},$$

then

$$\pi_N^k(h) = \pi_{N+k}(h) - \sum_{i=N+1}^{N+k} \epsilon_{i,3} \sigma_i^2,$$

for some bounded sequence $\{\epsilon_{i,3}, i \geq 1\}$.

Proof. See Supplement S2. □

Theorem 10. *Theorem 1 of (Robbins and Siegmund, 1985). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots$, a sequence of sub- σ -algebras of $\mathcal{F}$. For each $n = 1, 2, \dots$, let z_n, β_n, ξ_n and ζ_n be non-negative $\mathcal{F}_n$ -measurable random variable such that*

$$E(z_{n+1} | \mathcal{F}_n) \leq z_n(1 + \beta_n) + \xi_n - \zeta_n, \quad (5)$$

$\lim_{n \rightarrow \infty} z_n$ exists and is finite and $\sum_1^\infty \zeta_n < \infty$ a.s. on

$$\left\{ \sum_1^\infty \beta_n < \infty, \sum_1^\infty \xi_n < \infty \right\}.$$

Theorem 11. *Suppose that Assumptions 6 and 7 hold. Suppose also that π_0 is absolutely continuous w.r.t. the Lebesgue measure and $\check{\Theta}_n \sim \pi_n$. Then $\check{\Theta}_n$ converges a.s to θ_* as $n \rightarrow \infty$.*

Proof. By Lemma 8,

$$\begin{aligned} \pi_n(h) &= \frac{\int \pi_{n-1}(d\theta) e^{\ell(\theta)} h(\theta)}{\int \pi_{n-1}(d\theta) e^{\ell(\theta)}} + \sigma_n^2 \epsilon_n \\ &= \pi_{n-1}(h) + \left\{ \frac{\int \pi_{n-1}(d\theta) e^{\bar{\ell}(\theta)} h(\theta)}{\int \pi_{n-1}(d\theta) e^{\bar{\ell}(\theta)}} - \pi_{n-1}(h) \right\} + \sigma_n^2 \epsilon_n, \end{aligned}$$

where $\bar{\ell}(\theta) = \ell(\theta) - \ell(\theta_*)$, and $\{\epsilon_n\}$ is a bounded sequence. For $t \in [0, 1]$, we define $\pi_{n,t}(d\theta) = \frac{1}{c_{n,t}} \pi_n(d\theta) e^{t\bar{\ell}(\theta)}$, where $c_{n,t} = \int \pi_n(d\theta) e^{t\bar{\ell}(\theta)}$. We have

$$\begin{aligned} \frac{\int \pi_{n-1}(d\theta) e^{\bar{\ell}(\theta)} h(\theta)}{\int \pi_{n-1}(d\theta) e^{\bar{\ell}(\theta)}} - \pi_{n-1}(h) &= \int \pi_{n-1,1}(d\theta) h(\theta) - \int \pi_{n-1,0}(d\theta) h(\theta) \\ &= \int_0^1 \left\{ \frac{d}{dt} \int \pi_{n-1,t}(d\theta) h(\theta) \right\} dt \end{aligned}$$

where $h(\theta) = \ell(\theta) - \ell(\theta_*)$

$$\begin{aligned} &= \int_0^1 \left\{ \int \frac{d}{dt} \left[\frac{1}{c_{n-1,t}} \pi_{n-1}(d\theta) e^{t\bar{\ell}(\theta)} \right] h(\theta) \right\} dt \\ &= \underbrace{\int_0^1 \left\{ \int \left[\frac{d}{dt} \frac{1}{c_{n-1,t}} \right] \pi_{n-1}(d\theta) e^{t\bar{\ell}(\theta)} h(\theta) \right\} dt}_A + \underbrace{\int_0^1 \left\{ \int \frac{1}{c_{n-1,t}} \pi_{n-1}(d\theta) \bar{\ell}(\theta) e^{t\bar{\ell}(\theta)} h(\theta) \right\} dt}_B. \end{aligned}$$

Since

$$\frac{d}{dt} \frac{1}{c_{n-1,t}} = \frac{-\frac{d}{dt} c_{n-1,t}}{c_{n-1,t}^2} = \frac{-\int \bar{\ell}(\tilde{\theta}) \pi_{n-1}(d\tilde{\theta}) e^{t\bar{\ell}(\tilde{\theta})}}{c_{n-1,t}^2},$$

we have

$$\begin{aligned} A &= \int_0^1 - \left\{ \int \frac{\bar{\ell}(\tilde{\theta}) \pi_{n-1}(d\tilde{\theta}) e^{t\bar{\ell}(\tilde{\theta})}}{c_{n-1,t}} \int \frac{\pi_{n-1}(d\theta) e^{t\bar{\ell}(\theta)} h(\theta)}{c_{n-1,t}} \right\} dt \\ &= \int_0^1 -E[\bar{\ell}(\check{\Theta})] E[h(\check{\Theta})] dt, \end{aligned}$$

and

$$B = \int_0^1 E[\bar{\ell}(\check{\Theta}) h(\check{\Theta})] dt,$$

where $\tilde{\Theta}$ is a random variable follows the distribution $\pi_{n-1,t}$ as defined above. So

$$\begin{aligned} & \int_0^1 \left\{ \frac{d}{dt} \int \pi_{n-1,t}(d\theta) h(\theta) \right\} dt \\ &= \int_0^1 \text{Cov}_{n-1,t}(\bar{\ell}(\tilde{\Theta}), h(\tilde{\Theta})) dt, \end{aligned}$$

where the covariance and variance are w.r.t to the distribution $\pi_{n-1,t}$. Since we choose $h(u) = \ell(\theta_*) - \ell(u)$, $u \in \Theta$, we have

$$\pi_n(h) = \pi_{n-1}(h) - \int_0^1 \text{Var}_{n-1,t}(\bar{\ell}(\tilde{\Theta})) dt + \sigma_n^2 \epsilon_n.$$

We then make use of the super-martingale theorem (Theorem 10) to conclude that $\pi_n(h)$ converges to a finite limit $\pi_u(h)$ and $\sum_n \int_0^1 \text{Var}_{n-1,t}(\bar{\ell}(\tilde{\Theta})) dt < \infty$. Given ϵ , there exists an N such that $|\pi_N(h) - \pi_u(h)| < \epsilon$. By Lemma 8,

$$\pi_{N+1}(h) = \frac{\int \pi_N(d\theta) e^{\ell(\theta)} h(\theta)}{\int \pi_N(d\theta) e^{\ell(\theta)}} + \sigma_{N+1}^2 \epsilon_{N+1},$$

so

$$\pi_N^m(h) = \frac{\int \pi_N(d\theta) e^{n\ell(\theta)} h(\theta)}{\int \pi_N(d\theta) e^{n\ell(\theta)}} = \pi_{N+m}(h) + \sum_{i=N+1}^{N+m} \epsilon_{i,4} \sigma_i^2.$$

Since $\pi_N^m(h)$ is m iteration of data cloning of $\pi_N(h)$, by Theorem 4, $\pi_N^m(h)$ converges to δ_{θ_*} as $m \rightarrow \infty$. As above $\sum_{i=N+1}^{N+m} \epsilon_{i,4} \sigma_i^2$ and ϵ can be made arbitrarily small, hence $\pi_u(h)$ converges to $h(\theta_*) = 0$. Thus, we can conclude that $E(\bar{\ell}(\tilde{\Theta}_n)) \rightarrow 0$ or $E(\ell(\theta_*) - \ell(\tilde{\Theta}_n)) \rightarrow 0$. Since $\ell(\theta_*) - \ell(\tilde{\Theta}_n) \geq 0$, we have $\ell(\theta_*) - \ell(\tilde{\Theta}_n) \rightarrow 0$. From Assumption 6, ℓ admits a unique maximum θ_* and is twice differentiable, which implies that $\tilde{\Theta}_n \rightarrow \theta_*$. $\square$

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