Bayesian Information in an Experiment and the Fisher Information Distance

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Abstract

There are two forms of Fisher information; for the parameter of a model and for the information in a density model. These two forms are shown to be fundamentally connected through a measure of gain in information from a Bayesian experiment.

Keywords: Shannon entropy; information; Bayesian design.

1. Introduction

For the statistical model $f(x|\theta)$, with $x \in \mathbb{R}$ and $\theta \in \Theta \subset \mathbb{R}$, we assume that $f(x|\theta)$ is differentiable with respect to both x and θ . To start we have θ as a 1 dimensional parameter, which is extended to higher dimensions later in the paper. The amount of information about θ in a single observation experiment is given by the Fisher information;

$$I(\theta) = \int \left(\frac{\partial}{\partial \theta} \log f(x|\theta) \right)^2 f(x|\theta) dx. \quad (1)$$

It has a number of applications, most notably in large sample asymptotic studies; see, for example, [8].

On the other hand, a well known measure of the information carried by a density function is the Shannon entropy, which in the continuous case is often referred to as the differential Shannon entropy. This is defined as

$$H(Y) = - \int f_Y(y) \log f_Y(y) dy, \quad (2)$$

where the random variable Y has density function f_Y . See [14]. This was originally defined in the discrete case; i.e. Y is a discrete random variable. Some of the key properties, such as positivity, are lost in the differential case.

Nevertheless, the Shannon entropy was adopted as a measure of the information in a prior and posterior distribution; see [9]. To elaborate, suppose $p(\theta)$ denotes a prior distribution for model $f(x|\theta)$ and $p(\theta|x)$ is the posterior distribution. The idea is that the gain in information by observing outcome X is given by

$$\Delta(X) = H(\Theta|X) - H(\Theta) \quad (3)$$

where

$$H(\Theta) = - \int p(\theta) \log p(\theta) d\theta \quad \text{and} \quad H(\Theta|X = x) = - \int p(\theta|x) \log p(\theta|x) d\theta.$$

The expected gain in information by the experiment is therefore given by $\Delta = \int \Delta(x) m(x) dx$ where $m(x) = \int f(x|\theta) p(\theta) d\theta$. One can easily find simplified forms for Δ , which can also be represented as

$$\Delta = \int D(p(\cdot|x), p(\cdot)) m(x) dx$$

where D denotes the Kullback-Leibler divergence;

$$D(p_1, p_2) = \int p_1(x) \log \frac{p_1(x)}{p_2(x)} dx.$$

Applications in the Bayesian design of experiments can be found in [13].

If $I(\theta)$ represents the information about θ in a single observation experiment and $p(\theta)$ is the prior, then we ask why the expected information in the experiment is not simply

$$\tilde{\Delta} = \int I(\theta) p(\theta) d\theta, \quad (4)$$

rather than Δ . We will discuss this, showing that (4) arises by using the Fisher information distance, $F(p_1, p_2)$ (to be defined later), i.e.

$$\tilde{\Delta} = \int F(p(\cdot|x), p(\cdot)) m(x) dx,$$

rather than the Kullback-Leibler divergence, in section 3. However, for now, we note that $\int I(\theta) p(\theta) d\theta$ does appear in a Bayesian type Cramér-Rao bound; see [4].

In section 2 we show that an alternative recognized measure of information of a density, $J(Y)$, also known as the Fisher information, is such that if the gain in information given observation X is

$$\Delta^*(X) = J(\Theta|X) - J(\Theta),$$

then the expected gain,

$$\Delta^* = \int \Delta^*(x) m(x) dx,$$

coincides with $\tilde{\Delta}$.

2. Fisher information for a density function

Widely used in the functional analysis community, the Fisher information for a density f_Y is given by

$$J(Y) = \int \frac{f'_Y(y)^2}{f_Y(y)} dy. \quad (5)$$

See, for example, [2]. This can also be represented as

$$J(Y) = \int f_Y(y) \left(\frac{\partial}{\partial y} \log f_Y(y) \right)^2 dy$$

and of course $J(Y)$ will coincide with (1) when $f_Y(y) = f(y - \theta)$; i.e. a location model.

Now (5) is not so well known in the statistics literature; though there are some references. For example, [3] used it to prove the central limit theorem via a Cramér-Rao inequality, and elaborating on this paper, [7], also use (5), as well as the Fisher information distance, to be defined later in this paper. As well, [11] discuss properties of (5) as a direct comparison with the properties of (1).

Theorem 1. *It is that $\Delta^* \equiv \tilde{\Delta}$.*

PROOF. Now

$$J(\Theta|X = x) = \int \frac{\left(\frac{\partial}{\partial \theta} p(\theta|x) \right)^2}{p(\theta|x)} d\theta$$

which is given by

$$\frac{1}{m(x)} \int \frac{(p'(\theta) f(x|\theta) + p(\theta) \frac{\partial}{\partial \theta} f(x|\theta))^2}{p(\theta) f(x|\theta)} d\theta.$$

Taking expectations with respect to $m(x)$, and noting that

$$\int \int p'(\theta) \frac{\partial}{\partial \theta} f(x|\theta) dx d\theta = 0,$$

allowing the usual interchanging of integration and differentiation ([1] and [6], p. 518)¹ we see that

$$\int [J(\Theta|x) - J(\Theta)] m(x) dx = \int \int \frac{(\frac{\partial}{\partial \theta} f(x|\theta))^2}{f(x|\theta)} dx p(\theta) d\theta.$$

The right-hand-side is then seen to be

$$\int \int \left(\frac{\partial}{\partial \theta} \log f(x|\theta) \right)^2 f(x|\theta) dx p(\theta) d\theta$$

as required. □

Now $\tilde{\Delta}$ can also be represented as the expected relative Fisher information distance between the prior and posterior. This distance between densities p_1 and p_2 is given by

$$F(p_1, p_2) = \int p_1(y) \left(\frac{p'_1(y)}{p_1(y)} - \frac{p'_2(y)}{p_2(y)} \right)^2 dy, \quad (6)$$

and is well known in transport theory.

Lemma 2. *It is that*

$$\tilde{\Delta} = \int m(x) F(p(\cdot|x), p(\cdot)) dx.$$

¹Sufficient conditions are given as $\partial/\partial\theta f(x|\theta)$ exists for all x , and there exists a Lebesgue integrable function $g(x)$ such that $|(f(x|\theta_1) - f(x|\theta_2))/(\theta_1 - \theta_2)| \leq g(x)$ for all x and $\theta_1 \neq \theta_2$

PROOF. This is straightforward to prove and relies on the fact that

$$\int \int m(x) \frac{\partial / \partial \theta p(\theta|x) p'(\theta)}{p(\theta)} d\theta dx = \int \frac{p'(\theta)^2}{p(\theta)} d\theta.$$

We can consider both the expected Kullback-Leibler divergence or the expected relative Fisher information distance as a measure of the difference between prior and posterior; interpreted as the expected gain in information in a Bayesian experiment. The attractive feature of using the relative Fisher information distance is the result of the expected distance being the expected Fisher information.

The multivariate versions of Δ^* and $\tilde{\Delta}$, when $\theta \in \Theta \subset \mathbb{R}^d$, also match when the (Fisher) information in θ is given by $I(\theta) = \text{trace } \mathcal{I}(\theta)$, where $\mathcal{I}(\theta)$ is the $d \times d$ Fisher information matrix; i.e.

$$\mathcal{I}_{ij}(\theta) = \int \left[\left(\frac{\partial}{\partial \theta_i} \log f(x|\theta) \right) \left(\frac{\partial}{\partial \theta_j} \log f(x|\theta) \right) \right] f(x|\theta) dx.$$

To derive this, we start with

$$J(\Theta|X) = \int \frac{|\nabla f(x|\theta)|^2}{f(x|\theta)} d\theta$$

which is equal to

$$\frac{1}{m(x)} \int \frac{\sum_{j=1}^d \left(p(\theta) \frac{\partial}{\partial \theta_j} f(x|\theta) + f(x|\theta) \frac{\partial}{\partial \theta_j} p(\theta) \right)^2}{p(\theta) f(x|\theta)} d\theta.$$

Taking expectations with respect to $m(x)$, noting that the cross terms are 0; i.e.

$$\int \int \frac{\partial}{\partial \theta_j} p(\theta) \frac{\partial}{\partial \theta_j} f(x|\theta) dx d\theta = 0,$$

we see that

$$\int [J(\Theta|x) - J(\Theta)] m(x) dx = \int \int \sum_{j=1}^d \left(\frac{\partial}{\partial \theta_j} \log f(x|\theta) \right)^2 f(x|\theta) p(\theta) dx d\theta,$$

as required.

3. Bayesian design

The connection between the two forms of Fisher information suggests that

$$\tilde{\Delta} = \text{trace} \int \mathcal{I}(\theta) p(\theta) d\theta$$

represents the gain in information in a Bayesian experiment with model $f(x|\theta)$ and prior $p(\theta)$.

Such results are associated with the design of a Bayesian experiment where the model is $f(x|\theta, \xi)$ and the idea is to select the ξ which maximizes the gain in information about θ . For us, based on $\tilde{\Delta}$, we would choose the ξ which maximizes

$$\Delta_T = \text{trace} \int \mathcal{I}_\xi(\theta) p(\theta) d\theta = \int m(x) F(p(\cdot|x, \xi), p(\cdot)) dx, \quad (7)$$

where $\mathcal{I}_\xi(\theta)$ is obtained from the model $f(x|\theta, \xi)$, and in the multivariate case

$$F(p_1, p_2) = \int p_1 \left| \frac{\nabla p_1}{p_1} - \frac{\nabla p_2}{p_2} \right|^2.$$

Now the most popular approach to Bayesian design is based on the expected Kullback-Leibler divergence between prior and posterior; i.e. maximize

$$\Delta_B = \int m(x) D(p(\cdot|x, \xi), p(\cdot)) dx. \quad (8)$$

For a recent application, see [12].

On the other hand, selecting ξ via expected gains in information using $\mathcal{I}_\xi(\theta)$ are regarded as pseudo-Bayesian. Two popular ideas are the pseudo-Bayesian D-design;

$$\Delta_D = \int \log \det \mathcal{I}_\xi(\theta) p(\theta) d\theta,$$

see [5], for example, and the pseudo-Bayesian A-design;

$$\Delta_A = \int \text{trace} \mathcal{I}_\xi(\theta)^{-1} p(\theta) d\theta,$$

where here one minimizes Δ_A . See [10], for example.

However, there is nothing pseudo-Bayesian about (7), which would be known as a pseudo-Bayesian T-design, given the derivation involves simply

replacing the Kullback-Leibler divergence with the relative Fisher information distance.

Here we present a simple illustration; though there are also numerous cases in the literature. Suppose Y is Poisson with parameter $e^{\xi\theta}$ where $\xi > 0$ is a design point, to be chosen. It is easy to obtain the Fisher information for θ as

$$I_{\xi}(\theta) = \xi^2 e^{-\xi\theta}.$$

Hence, we would select ξ by maximizing

$$\Delta_T = \xi^2 \int e^{-\xi\theta} p(\theta) d\theta.$$

So if $p(\theta)$ were gamma with parameters (a, b) , with $a > 2$, then

$$\Delta_T = \frac{\xi^2}{(b + \xi)^a}$$

which is maximized for $a > 2$ by $\xi = 2b/(a - 2)$.

4. Discussion

There are two notions of Fisher information; i.e. (1), which provides information about the parameter of a model, and (5), which provides information about the density itself. When associated with the gain in information in a Bayesian experiment, these two complement each other perfectly, via theorem 2.1. In fact, whereas [9] uses the expected Kullback-Leibler divergence between prior and posterior, we propose the expected relative Fisher information distance between prior and posterior, due to its connection with the expected Fisher information about θ .

If (8) is regarded as fully Bayes, then so must (7), rather than as a pseudo-Bayesian approach, due to its derivation as a measure of distance between prior and posterior.

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