Short-range dependent processes subordinated to the Gaussian may not be strong mixing

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Abstract

There are all kinds of weak dependence. For example, strong mixing. Short-range dependence (SRD) is also a form of weak dependence. It occurs in the context of processes that are subordinated to the Gaussian. Is a SRD process strong mixing if the underlying Gaussian process is long-range dependent? We show that this is not necessarily the case.

A stationary discrete-time process with covariance $\gamma(n)$, $n \geq 0$ displays *long-range dependence* (LRD) if

$$\gamma(n) = n^{2H-2}L(n),$$

where $H \in (1/2, 1)$ and where $L(x)$, $x > 0$ is a slowly varying function at infinity, for example, $L(x)$ is a constant or $\log x$. The index H is often called the Hurst index.

A typical way to obtain an LRD Gaussian process $\{Z_i\}$ is to start with *fractional Brownian motion* $B_H(t)$, $t \geq 0$ with Hurst index $H \in (1/2, 1)$. Fractional Brownian motion is a Gaussian process with stationary increments, mean zero and covariance

$$\mathbb{E}B_H(t_1)B_H(t_2) = \frac{1}{2}\{t_1^{2H} + t_2^{2H} - |t_1 - t_2|^{2H}\}.$$

The increment process $Z_i = B_H(i+1) - B_H(i)$, $i \geq 0$, called *fractional Gaussian noise*, has then covariance

$$\gamma(n) = \mathbb{E}Z_iZ_{i+n} = \frac{1}{2}\{(n+1)^{2H} - 2n^{2H} + |n-1|^{2H}\} \sim H(2H-1)n^{2H-2} \text{ as } n \rightarrow \infty,$$

and hence it satisfies the LRD conditions with $H \in (1/2, 1)$ and $L(n) \sim H(2H-1)/2$. It is possible to construct many other examples of Gaussian LRD processes $\{Z_i\}$.

Let $\{Z_i\}$ be a standardized Gaussian LRD process. Observe that the LRD property implies

$$\sum_{n=-\infty}^{\infty} \gamma(n) = \infty,$$

namely, that the sum of the covariances diverges. We will now consider instantaneous transformations $X_i = P(Z_i)$, where $\mathbb{E}P(Z_i)^2 < \infty$. The resulting process $\{X_i\}$ is said to be *short-range dependent* (SRD) if the sum of its covariances converges. The process $\{X_i\}$, however, may be either LRD or SRD depending on the choice of the function $P(x)$.

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Suppose now that $P(\cdot)$ is a finite-order polynomial. It can then be expressed as

$$P(x) = c_0 + \sum_{k=m}^n c_k H_k(x), \quad 1 \leq m \leq n,$$

with $c_m \neq 0$, where $H_k(x)$ is the k -th order Hermite polynomial. The bottom index m plays an important role. It is called the *Hermite rank* of $P(x)$ and/or of the process $\{P(X_i)\}$.

It is known from Breuer and Major [2] that when

$$(2H - 2)m + 1 < 0, \tag{1}$$

which can only happen when $m \geq 2$, then $\{X_i\}$ is SRD and as $N \rightarrow \infty$,

$$N^{-1/2} \sum_{i=1}^{\lfloor Nt \rfloor} (P(Z_i) - \mathbb{E}P(Z_i)) \xrightarrow{f.d.d.} \sigma B(t),$$

where $\sigma^2 = \sum_n \gamma(n)$, $B(t)$ is the standard Brownian motion and $\xrightarrow{f.d.d.}$ denotes convergence of finite-dimensional distributions. This seems to suggest that $\{P(Z_i)\}$ has weak dependence.

Strong mixing is also a form of weak dependence. Recall that a stationary process $\{X_i\}$ is said to be *strong mixing* if

$$\lim_{k \rightarrow \infty} \sup \{|P(A)P(B) - P(A \cap B)|, A \in \mathcal{F}_{-\infty}^0, B \in \mathcal{F}_k^\infty\} = 0,$$

where $\mathcal{F}_a^b$ is the σ -field generated by $X_a, \dots, X_b$.

It is natural to ask whether $\{P(Z_i)\}$ is strong mixing. We will show that this may *not* be the case.

Theorem 1. *Suppose that $\{Z_i\}$ has covariance $\gamma(n) = n^{2H-2}L(n)$, where H satisfies (1). The SRD process $\{X_i = P(Z_i)\}$ is not strong mixing if there exists a polynomial $Q(x)$ such that the Hermite rank m' of $Q(P(x))$ satisfies*

$$(2H - 2)m' + 1 > 0. \tag{2}$$

Note that (2) implies, in view of (1), that $m' < m$.

Proof. We argue by contradiction. Suppose that $\{X_i\}$ is strong mixing. Then by the definition of strong mixing, $\{Q(X_i)\}$ is also strong mixing. But (2) implies that (Taqqu [4])

$$s_N^2 := \text{Var} \left[\sum_{i=1}^N Q(X_i) \right] \sim c_H L(N)^{m'} N^{(2H-2)m'+2}, \quad (2H - 2)m' + 2 > 1. \tag{3}$$

On the other hand, $S_N := \sum_{i=1}^N (Q(X_i) - \mathbb{E}Q(X_i))$ is an element living on Wiener chaos of a finite order (see Janson [3], Chapter 2). By Janson [3], Theorem 5.10, for any $p > 2$, there exists a constant $c_p > 0$ depending only on p , such that

$$\mathbb{E} |s_N^{-1} S_N|^p \leq c_p \left(\mathbb{E} |s_N^{-1} S_N|^2 \right)^{p/2} = c_p.$$

Therefore $\{s_N^{-2} S_N^2, N \geq 2\}$ is uniformly integrable. By Theorem 10.2 of Bradley [1], Volume 1, strong mixing and uniform integrability imply that

$$s_N^2 = l(N)N$$

for some slowly varying function $l(N)$. This contradicts (3). $\square$

In some cases, no polynomial $Q(x)$ satisfies the requirement of Theorem 1. For example, when $P(x) = x^2$, then the Hermite rank $m = 2$, and one always has

$$\mathbb{E}Q(Z^2)H_1(Z) = \mathbb{E}Q(Z^2)Z = 0$$

for arbitrary polynomials $Q(x)$ (in fact for arbitrary $L^2(\Omega)$ functions). This is because $Q(Z^2)$ is an even function of Z . So the Hermite rank of $Q(P(x))$ is at least 2, and hence we don't have $m' < m$.

In the simple case where $P(x)$ is a Hermite polynomial, we have the following result:

Theorem 2. Suppose $P(x) = H_m(x)$, $m \in \mathbb{Z}_+$. The polynomial $Q(x)$ required in Theorem 1 exists in either of the following cases:

(a) $m \geq 4$ is even and $H > 3/4$.

(b) $m \geq 3$ is odd.

Proof. Using the product formula ((3.13) of Janson [3]) for Hermite polynomials, one has

$$H_m(x)^2 = \sum_{k=0}^m k! \binom{m}{k}^2 H_{2m-2k}(x), \quad (4)$$

$$H_m(x)^3 = \sum_{k_1=0}^m \sum_{k_2=0}^{(2m-2k_1) \wedge m} k_1! k_2! \binom{m}{k_1}^2 \binom{2m-2k_1}{k_2} \binom{m}{k_2} H_{3m-2k_1-2k_2}(x). \quad (5)$$

For case (a), choose $3/4 < H < 1$, but not too big such that $\{P(X_i) = H_m(X_i)\}$ is SRD. This will happen by constraining H to satisfy (1). Now choose $Q(x) = x^2$. Then by (4),

$$Q(P(x)) = H_m(x)^2 = m! + (m-1)!m^2 H_2(x) + \dots,$$

so $\{Q(P(Z_i))\}$ has Hermite rank $m' = 2$, which is less than $m \geq 4$. Since $m' = 2$, and $H > 3/4$, we conclude that $\{Q(P(Z_i))\}$ is LRD and satisfies (2).

For case (b), choose $Q(x) = x^3$. Then

$$Q(P(x)) = H_m(x)^3 = a_1 H_1(x) + \dots$$

for some $a_1 > 0$. The term $H_1(x)$ appears when $3m - 2k_1 - 2k_2 = 1$, e.g., when $k_1 = (m-1)/2$, $k_2 = m$. The coefficient $a_1 > 0$ because all the coefficients before the Hermite polynomials in (5) are positive. It is then clear that the Hermite rank of $H_m(x)^3$ is $m' = 1$. Hence the polynomial $Q(x)$ satisfies (2). $\square$

We conclude with some remarks.

Remark 3. The process $\{X_i = P(Z_i)\}$ in Theorem 1 is SRD. The theorem states that this process is not strong mixing if there a polynomial $Q(x)$ such that the new process $\{Q(P(Z_i))\}$ is LRD.

Remark 4. In Proposition 2 case (b), we do not need a restriction on H . We require $m \geq 3$ since $m = 1$ is incompatible with (1).

Remark 5. What about the converse? Can a strong mixing process not be subordinated to a Gaussian LRD process? The answer is clearly “yes”. Suppose for example $\{X_i\}$ i.i.d. Gaussian. Then there is no $\{X'_i\} \stackrel{f.d.d.}{=} \{X_i\}$ so that $X'_i = G(Z'_i)$, where $\{Z'_i\}$ is LRD Gaussian, because the covariance $\text{Cov}[X'_i, X'_0] \neq 0$ for large i .

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