Efficient estimation for the heteroscedastic single-index varying coefficient models

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Abstract

We propose a semiparametric estimator for the heteroscedastic single-index varying coefficient model. The estimator is proved to attain the semiparametric efficiency bound. Some simulation studies and a real data application are conducted to evaluate the proposed methods.

Keywords: Single-index varying coefficient model; efficiency; semiparametric efficiency bound; estimating equation.

1 Introduction

The varying coefficient model has gained a lot of attentions during the past decade. Compared to the parametric model, it is more flexible to capture the true underlying relationships between covariates and response. However, the varying coefficient model

$$Y = \alpha(U)^\top Z + \varepsilon, \quad Z \in R^q, \quad (1)$$

would suffer from the “curse of dimensionality” problem if the dimension of U is high. This difficulty motivates us to study more flexible and more general models. We consider a heteroscedastic single-index varying coefficient model of the form

$$Y = g^\top (X^\top \beta) Z + \varepsilon, \quad (2)$$

where $X \in R^p$ and $Z \in R^q$ are vectors of covariates, Y is the response variable, β is a $p \times 1$ vector of unknown parameters with $\|\beta\| = 1$ for the sake of identifiability ($\|\cdot\|$ denotes the Euclidean metric), $g(\cdot)$ is a $q \times 1$ vector of unknown functions and ε is a random error with $E(\varepsilon|X, Z) = 0$ and $Var(\varepsilon|X, Z) = \sigma^2(X, Z)$, $\sigma^2(X, Z)$ is some unknown function.

Model (2) includes many important statistical models as special cases. If $g(\cdot)$ is a $q \times 1$ constant vector, it turns to the linear regression model. If $p = 1$ and $\beta = 1$, it reduces to the varying coefficient model. If some elements of $g(X)$ are constants, it is the varying coefficient partially linear model. If $q = 1$ and $Z = 1$, it reduces to the single-index model. And if $p = q$ and $Z = X$, model (2) becomes the adaptive varying coefficient linear model. Thus, model (2) is easily utilized in real applications since it is flexible enough to capture many different situations.

Existing literature focused on homoscedastic variance case with $\sigma^2(X, Z)$ being a positive constant. Xue and Wang (2012) developed an adjusted empirical likelihood ratio approach for constructing the confidence regions of the parameter vector β . Huang and Zhang (2013) proposed a maximum empirical likelihood estimator. Feng and Xue (2013, 2015) studied some variable selection problems in the single-index varying coefficient model. To the best of our knowledge, there are no existing studies on the heteroscedastic situation, which is very common in practice.

Moreover, the aforementioned papers did not consider semiparametric efficiency problem. Among a class of regular asymptotically linear estimators, semiparametric efficient estimators, which achieve the so-called semiparametric efficiency bound, have the smallest asymptotic variance. Tsiatis (2006) gave a general introduction to semiparametric efficiency bound and efficient estimators. Semiparametric efficiency problems have been studied extensively. Ma and Zhu (2013) studied the efficient estimators for the heteroscedastic partially linear single-index models allowing high dimensional covariates by constructing a semiparametrically efficient score function. Xue and Pang (2013) studied the statistical properties for the parametric estimator of (2) with homoscedastic variance, and further studied its efficiency when model (2) reduced to the single-index model. Lai and Wang (2014) and Lai et al. (2014) studied the efficient estimators for the partially linear single-index models.

In this paper, our goal is to search for the efficient estimator of β for the heteroscedastic single-index varying coefficient model (2). The rest of this paper is organized as follows. In Section 2, the semiparametric efficient estimator for the parameter vector β is defined and the asymptotic properties of the proposed estimators are also investigated. In Section 3, numerical studies are reported. All technical details are delayed to Appendix.

2 Semiparametric efficient estimator

In order to obtain the efficient estimator for β in model (2), we firstly define the semi-parametrically efficient score function for the parameter of interest β , which can be used to construct the efficient estimating equations.

Suppose that $\{(Y_i, X_i, Z_i), i = 1, \dots, n\}$ is a random sample generated from the single-index varying coefficient model (2). For the sake of identifiability, we assume that $\|\beta\| = 1$ and the first component of β is positive. This assumption means that the true value of β is a boundary point on the unit sphere, and hence $g(X^\top \beta)$ does not have a derivative at the point β (Zhu and Xue (2006)). For the convenience of computing the score vector, we use the delete-one-component method proposed by Yu and Ruppert (2002). Let $\beta = (\beta_1, \dots, \beta_p)^\top$ and $\beta^{(1)} = (\beta_2, \dots, \beta_p)^\top$, then, we may write

$$\beta(\beta^{(1)}) = ((1 - \|\beta^{(1)}\|^2)^{1/2}, \beta_2, \dots, \beta_p)^\top.$$

The true parameter vector $\beta_0^{(1)}$ must satisfy the constraint $\|\beta_0^{(1)}\| < 1$. Thus, β is infinitely differentiable in a neighborhood of $\beta_0^{(1)}$, and the Jacobian matrix is

$$J_{\beta^{(1)}} = \frac{\partial \beta}{\partial \beta^{(1)}} = (b_1, \dots, b_p)^\top,$$

where $b_s (1 < s \leq p)$ is a $(p-1)$ -dimensional unit vector with s th component 1, and $b_1 = -(1 - \|\beta^{(1)}\|^2)^{-1/2} \beta^{(1)}$. Thus, we can compute the efficient score function $S_\beta^{eff}(Y, X, Z)$ for the heteroscedastic single-index varying coefficient model in the following theorem.

Theorem 1 *Assume that the conditional probability density function of ε given (X, Z) , $p_1(\varepsilon|X, Z)$, is differentiable with respect to ε and $0 < E(\varepsilon^2|X, Z) < \infty$ almost everywhere. Then the semiparametrically efficient score of model (2) is given by*

$$\begin{aligned} S_{\beta_0}^{eff}(Y, X, Z) &= \varepsilon w(X, Z) \left\{ g'^\top(X^\top \beta_0) Z J_{\beta_0^{(1)}}^\top X - E[w(X, Z) g'^\top(X^\top \beta_0) Z \right. \\ &\quad \left. \times J_{\beta_0^{(1)}}^\top X Z^\top | X^\top \beta_0] \{ E[w(X, Z) Z Z^\top | X^\top \beta_0] \}^{-1} Z \right\}, \end{aligned} \quad (3)$$

where $\varepsilon = Y - Z^\top g(X^\top \beta_0)$ and $w(X, Z) = \{E(\varepsilon^2|X, Z)\}^{-1}$.

This theorem provides us the semiparametrically efficient score function for β even for the heteroscedastic model when $w(X, Z) = \{E(\varepsilon^2|X, Z)\}^{-1} = \{\sigma^2(X, Z)\}^{-1}$. By Theorem 4.1 of Tsiatis (2006), it is proved directly that the semiparametric efficiency bound of model (2) is

$$\{E(S_{\beta_0}^{eff} S_{\beta_0}^{eff\top})\}^{-1} = \{E(w(X, Z) S_e(\beta_0) S_e^\top(\beta_0))\}^{-1}, \quad (4)$$

where

$$\begin{aligned} S_e(\beta_0) &= g'^\top(X^\top \beta_0) Z J_{\beta_0(1)}^\top X - D_2(X^\top \beta_0) D_1^{-1}(X^\top \beta_0) Z, \\ D_1(X^\top \beta_0) &= E[w(X, Z) Z Z^\top | X^\top \beta_0], \\ D_2(X^\top \beta_0) &= E[w(X, Z) g'^\top(X^\top \beta_0) Z J_{\beta_0(1)}^\top X Z^\top | X^\top \beta_0]. \end{aligned}$$

Based on Theorem 1, the following efficient estimating equation

$$\sum_{i=1}^n \hat{S}_\beta^{eff}(Y_i, X_i, Z_i) = 0 \quad (5)$$

can be constructed to compute the semiparametrically efficient estimator whose asymptotic variance attains the semiparametric efficiency bound, where

$$\begin{aligned} \hat{S}_\beta^{eff}(Y_i, X_i, Z_i) &= \{Y_i - Z_i^\top \hat{g}(X_i^\top \beta)\} \hat{w}(X_i, Z_i) \left\{ \hat{g}'^\top(X_i^\top \beta) Z_i J_{\beta(1)}^\top X_i \right. \\ &\quad \left. - \hat{E}[\hat{w}(X_i, Z_i) \hat{g}'^\top(X_i^\top \beta) Z_i J_{\beta(1)}^\top X_i Z_i^\top | X_i^\top \beta] \right\} \{ \hat{E}[\hat{w}(X_i, Z_i) Z_i Z_i^\top | X_i^\top \beta] \}^{-1} Z_i \}, \end{aligned}$$

with $\hat{g}(\cdot)$, $\hat{g}'(\cdot)$, $\hat{E}(\cdot | X^\top \beta)$ and $\hat{w}(\cdot)$ denoting the consistent estimators of $g(\cdot)$, $g'(\cdot)$, $E(\cdot | X^\top \beta)$ and $w(\cdot)$ respectively. Such estimators can be obtained as follows.

For a given β , we employ the local linear regression technique (Fan and Gijbels (1996)) to estimate $g(\cdot)$ and $g'(\cdot)$. The local linear estimators for $g(t)$ and $g'(t)$ are defined as $\hat{g}(t) = \hat{a}$ and $\hat{g}'(t) = \hat{b}$, where $\hat{a}$ and $\hat{b}$ minimize the sum of weighted squares

$$\sum_{i=1}^n [Y_i - \{a + hb \frac{X_i^\top \beta - t}{h}\}^\top Z_i]^2 K_h(X_i^\top \beta - t), \quad (6)$$

with respect to a and b , $K_h(\cdot) = \frac{1}{h} K(\cdot/h)$, $K(\cdot)$ is a kernel function and h is a bandwidth.

Therefore, we can obtain

$$\begin{pmatrix} \hat{g}(t) \\ h \hat{g}'(t) \end{pmatrix} = S_n^{-1}(t, \beta) \xi_n(t, \beta), \quad (7)$$

where

$$S_n(t, \beta) = \begin{pmatrix} S_{n,0}(t, \beta) & S_{n,1}(t, \beta) \\ S_{n,1}(t, \beta) & S_{n,2}(t, \beta) \end{pmatrix}, \quad \xi_n(t, \beta) = \begin{pmatrix} \xi_{n,0}(t, \beta) \\ \xi_{n,1}(t, \beta) \end{pmatrix},$$

and

$$\begin{aligned} S_{n,k}(t, \beta) &= \frac{1}{n} \sum_{i=1}^n Z_i Z_i^\top \left(\frac{X_i^\top \beta - t}{h} \right)^k K_h(X_i^\top \beta - t), \quad k = 0, 1, 2; \\ \xi_{n,k}(t, \beta) &= \frac{1}{n} \sum_{i=1}^n Z_i Y_i \left(\frac{X_i^\top \beta - t}{h} \right)^k K_h(X_i^\top \beta - t), \quad k = 0, 1. \end{aligned}$$

To obtain the nonparametric estimators of the conditional expectations in (5), the Nadaraya-Watson kernel method is used here. We define the estimators respectively by

$$\begin{aligned} & \hat{E}[\hat{w}(X, Z) \hat{g}'^\top(X^\top \beta) Z J_{\beta(1)}^\top X Z^\top | X^\top \beta] \\ &= \frac{\sum_{i=1}^n K_h(X_i^\top \beta - X^\top \beta) \hat{w}(X_i, Z_i) \hat{g}'^\top(X_i^\top \beta) Z_i J_{\beta(1)}^\top X_i Z_i^\top}{\sum_{i=1}^n K_h(X_i^\top \beta - X^\top \beta)}, \\ & \hat{E}[\hat{w}(X, Z) Z Z^\top | X^\top \beta] = \frac{\sum_{i=1}^n K_h(X_i^\top \beta - X^\top \beta) \hat{w}(X_i, Z_i) Z_i Z_i^\top}{\sum_{i=1}^n K_h(X_i^\top \beta - X^\top \beta)}. \end{aligned} \quad (8)$$

Note that $w(X, Z) = \{E(\varepsilon^2 | X, Z)\}^{-1} = \frac{1}{\sigma^2(X, Z)}$. Clearly, the nonparametric regression estimator of $\sigma^2(X, Z)$ suffers from the curse of dimensionality when the dimensionality of (X, Z) is high. To overcome this difficulty, Ma and Zhu (2013) assume there is a low dimensional variable $\xi = \xi(X, Z)$ such that $\sigma^2(X, Z) = \text{var}(\varepsilon | \xi)$ and ξ has a known form and ξ is univariate. In this paper, we assume that there exists a link function $\varphi(\cdot)$ and parameter γ_0 such that

$$\sigma^2(X, Z) = \text{var}(\varepsilon | X, Z) = \varphi(\gamma_0^\top U),$$

where $U = (X^\top, Z^\top)^\top$ and $\|\gamma_0\| = 1$. Hence, the single-index model is utilized to model the heterogeneity. Let $\hat{\varepsilon}_i = Y_i - Z_i^\top \hat{g}_I(X_i^\top \hat{\beta}_I)$, where $\hat{\beta}_I$ and $\hat{g}_I(\cdot)$ are the consistent initial estimators, which can be obtained by solving (5) and minimizing (6) with $w(X, Z) \equiv 1$. Thus, considering $E[\varepsilon_i^2 | U] = \varphi(\gamma_0^\top U)$, γ_0 and $\varphi(\cdot)$ can be estimated by the usual kernel method, for example Carroll et al. (1997), and we obtain $\hat{\sigma}^2(X, Z)$. Two bandwidths h_1 and h_2 are additionally involved. The bandwidth h_1 is selected for estimation of γ_0 , while h_2 is for estimation of $\varphi(\cdot)$. These bandwidths h , h_1 and h_2 in the nonparametric estimating process can be selected by the cross-validation technique.

Define the solution of (5) on $\beta^{(1)}$, $\hat{\beta}^{(1)}$ say, to be the estimator of $\beta_0^{(1)}$. For $\beta \in \mathbf{B}_n$ in (5), $\mathbf{B}_n = \{\beta : \|\beta - \beta_0\| \leq Cn^{-1/2}\}$, and a positive constant C , we have the following asymptotic properties.

Theorem 2 *If model (2) is a heteroscedastic model with $E(\varepsilon^2|X, Z) = \sigma^2(X, Z)$, where $\sigma^2(X, Z)$ is twice times continuously differentiable with bounded derivative, then we have that, under the regularity conditions C1-C8 given in the Appendix,*

$$\sqrt{n}(\hat{\beta}^{(1)} - \beta_0^{(1)}) \xrightarrow{\mathcal{D}} N(0, \Sigma_1^{-1}), \quad (9)$$

where

$$\Sigma_1 = E[w(X, Z)S_e(\beta_0)S_e(\beta_0)^\top].$$

Corollary 1 *Under the conditions of Theorem 2, we have*

$$\sqrt{n}(\hat{\beta} - \beta_0) \xrightarrow{\mathcal{D}} N(0, J_{\beta_0^{(1)}} \Sigma_1^{-1} J_{\beta_0^{(1)}}^\top).$$

Remark 1 *From Theorem 2, we can see that the asymptotic variance of the estimator is Σ_1^{-1} , hence attains the semiparametric efficiency bound proposed in (4).*

3 Numerical examples

3.1 Simulations

To illustrate the finite sample performances of the proposed method, we consider the example below. In the simulations, the Epanechnikov kernel $K(t) = 0.75(1 - t^2)_+$ is used. Moreover, since root-n consistency of the proposed estimators does not require undersmoothing of nonparametric functions, the optimal bandwidth can be selected to be the optimal one for estimation of nonparametric functions. We use 5-fold cross validation to choose bandwidths for estimators.

Let $\{Y_i, X_i, Z_i; i = 1, \dots, n\}$ be the i.i.d sample where $X_i = (X_{i1}, \dots, X_{ip})^\top$ is generated from uniform distribution on $[0, 1]^p$ with independent components, $Z_{i1} = 1$ and $(Z_{i2}, \dots, Z_{iq})^\top$ follows a multivariate normal distribution with $\text{cov}(Z_{ij_1}, Z_{ij_2}) = 0.5^{|j_1 - j_2|}$ for $2 \leq j_1, j_2 \leq q$. Consider the model

$$Y_i = \sum_{s=1}^q g_s(X_i^\top \beta) Z_{i(s+1)} + 0.5e_i, \quad (10)$$

where $\beta = (\beta_1, \dots, \beta_p)^\top$ with true value β_0 and e_i is the error term. We set $p = 2, q = 3$ and consider the model (10) with $g_1(u) = 2 \sin(2\pi u), g_2(u) = \exp(2u - 1), g_3(u) = 8u(1 - u)$, and $\beta_0 = (2, 1)^\top / \sqrt{5}$. Two error distributions were considered: (1-1) homoscedastic: $e_i \sim N(0, 1)$ and (1-2) heteroscedastic: $e_i \sim N(0, \exp(U_i^\top \gamma_0 / 2))$ with $U_i = (X_i^\top, Z_{i2}, \dots, Z_{iq})^\top$ and $\gamma_0 = (1, \dots, 1) / \sqrt{p + q - 1}$. Let $n = 100$ and 200 , and a total of 500 simulation replications were conducted.

To examine the performances of the efficient estimator, we consider the following four different estimating equations:

$$\begin{aligned}
(a) \quad & n^{-1} \sum_{i=1}^n \{Y_i - Z_i^\top \hat{g}(X_i^\top \beta)\} \hat{w}(X_i, Z_i) \left\{ \hat{g}'^\top(X_i^\top \beta) Z_i J_{\beta(1)}^\top X_i \right. \\
& \quad \left. - \hat{E}[\hat{w}(X_i, Z_i) \hat{g}'^\top(X_i^\top \beta) Z_i J_{\beta(1)}^\top X_i Z_i^\top | X_i^\top \beta] \{ \hat{E}[\hat{w}(X_i, Z_i) Z_i Z_i^\top | X_i^\top \beta] \}^{-1} Z_i \right\} = 0, \\
(b) \quad & n^{-1} \sum_{i=1}^n \{Y_i - Z_i^\top \hat{g}(X_i^\top \beta)\} \left\{ \hat{g}'^\top(X_i^\top \beta) Z_i J_{\beta(1)}^\top X_i \right. \\
& \quad \left. - \hat{E}[\hat{g}'^\top(X_i^\top \beta) Z_i J_{\beta(1)}^\top X_i Z_i^\top | X_i^\top \beta] \{ \hat{E}[Z_i Z_i^\top | X_i^\top \beta] \}^{-1} Z_i \right\} = 0, \\
(c) \quad & n^{-1} \sum_{i=1}^n \{Y_i - Z_i^\top \hat{g}(X_i^\top \beta)\} \hat{w}(X_i, Z_i) \hat{g}'^\top(X_i^\top \beta) Z_i J_{\beta(1)}^\top X_i = 0, \\
(d) \quad & n^{-1} \sum_{i=1}^n \{Y_i - Z_i^\top \hat{g}(X_i^\top \beta)\} \hat{g}'^\top(X_i^\top \beta) Z_i J_{\beta(1)}^\top X_i = 0.
\end{aligned}$$

The estimator defined by (a) is semiparametric efficient when the model is heteroscedastic, while the estimator defined by (b) is semiparametric efficient when the model is homoscedastic. Estimators defined by (c) and (d) correspond to two normal estimators. Table 1 summarizes the results. Moreover, we computed absolute errors (AE) for each functional component. AE_j is defined as

$$AE_j = \sum_{i=1}^n |\hat{g}_j(X_i^\top \hat{\beta}) - g_j(X_i^\top \beta_0)|, j = 1, 2, 3.$$

Insert Table 1 about here

Several conclusions can be made from Table 1. When the sample size n increases, all four estimators (a)-(b) have smaller errors. Estimator (b) performs well for homoscedastic

case, so does estimator (a) for heteroscedastic case. This finding agrees with semiparametric efficiency theory in Section 2.

3.2 Real data analysis

As an illustration, we apply the proposed method to a body fat data which is available at [http:// lib.stat.cmu.edu/datasets/bodyfat](http://lib.stat.cmu.edu/datasets/bodyfat). A variety of popular health books suggest that people could assess their health conditions by estimating their percentage of body fat. Unfortunately, it is inconvenient or costly nowadays to accurately measure body fat. The aim for analyzing this data set is to build a predictive model for percentage of body fat. To achieve this goal, we use the data set of 252 men with thirteen baseline predictors: AGE (age), BMI(body mass index) and circumferences of various skinfold measurements [NK(neck), CT(chest), AN(2 abdomen), HIP (hip), RAN (the ratio of 2 abdomen to hip), TN(thigh), KE(knee), AK(ankle), BS(biceps), FA(forearm) and WI(wrist)]. The response (Y) is the logarithm of percentage of body fat. Due to the generating process of the response and possibly outliers, we delete a sample with $Y = 0$, a sample with $Y = 0.7$ and a sample with height=29.5 inches. The sample size is thus 249.

Before applying our method, the predictors are transformed so that their marginal distribution are approximately $N(0, 1)$. Moreover, the first 200 samples are used to fit the model, and then the rest 49 observations are utilized to evaluate the prediction ability of the underlying model. The prediction performance is measured by the absolute prediction error (APE), which is defined as $\{|y_i - \hat{y}_i|\}, i = 1, \dots, 49$. Consider the following homoscedastic model

$$\begin{aligned} Y = & g_0(U) + g_1(U)NK + g_2(U)CT + g_3(U)AN + g_4(U)HIP + g_5(U)TN \\ & + g_6(U)KE + g_7(U)AK + g_8(U)BS + g_9(U)FA + g_{10}(U)WI + \varepsilon, \end{aligned} \quad (11)$$

where $U = \beta_1 AGE + \beta_2 BMI + \beta_3 RAN$ and ε is the error term. Moreover, the variables AGE, BMI and RAN are transformed so that the marginal distributions are all $U[0, 1]$. Let $\beta = (\beta_1, \beta_2, \beta_3)^\top$. Based on the efficient estimating equation in Section 2, we get $\tilde{\beta} = (0.3372, 0.9374, -0.0868)^\top$ and $R^2 = 0.8093$, where

$$R^2 = (SST - SSE)/SST, \quad SST = \sum (Y_i - \text{mean}(Y_i))^2, \quad SSE = \sum (\tilde{Y}_i - Y_i)^2$$

and $\tilde{Y}_i$ is the fitted value based on model (11). It is remarkable that the R^2 value implies (11) is a reasonable model to fit this real data. The median of APEs of the proposed model (11) is 0.3775(0.2713), where the values in the parentheses are MAD of APEs.

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Appendix. Technical Proofs

In order to study the proposed asymptotic properties, the following standard assumptions are imposed.

- C1. The density function $f(t)$ of $W^\top \eta$ is bounded away from zero and infinity on $\mathbf{T}_\eta = \{t : t = W^\top \eta, W \in \mathbf{A}_W, \eta \in \mathbf{B}_n^\eta = \{\eta : \|\eta\| = 1, \|\eta - \eta_0\| \leq Cn^{-1/2}\}\}$, $\mathbf{A}_W$ is the compact support of W . $f(t)$ has continuous derivatives up to order 2 on $\mathbf{T}_\eta$. Here (W, η) can be (X, β) and (U, γ) , respectively.
- C2. The coefficients $g_j(t), j = 1, \dots, q$, both have bounded and continuously derivatives up to order 2 on $\mathbf{T}_\beta$.
- C3. Assume $E(\varepsilon^2 | X_i, Z_i) = \sigma^2(X_i, Z_i) = \varphi(U^\top \gamma_0)$, $U = (X^\top, Z^\top)^\top$ and $\|\gamma_0\| = 1$. $\varphi(\cdot)$ is bounded away from 0 and infinity, and has bounded derivative up to order 2 on T_γ .

- C4. The functions $D_1(t) = E[h(X, Z)ZZ^\top | t]$ and $D_2(t) = E[h(X, Z)g'^\top(t)ZJ_{\beta_0^{(1)}}^\top XZ^\top | t]$, are twice continuously differentiable with bounded derivatives, where $h(X, Z) = w(X, Z)$ or 1. $D_1(t)$ is nonsingular and $E(\|Z\|^4 | t)$ is also bounded.
- C5. The kernel K is a bounded and symmetric probability density function, satisfying the Lipschitz condition and

$$\int_{-\infty}^{\infty} u^2 K(u) du \neq 0, \quad \int_{-\infty}^{\infty} |u|^i K(u) du < \infty, \quad i = 1, 2, \dots$$

- C6. For any $s > 2$, we have $E|Y_i|^{2s} < \infty$, $E|Z_j|^{2s} < \infty$, $j = 1, \dots, q$, $Z \in \mathbf{A}_Z$.
- C7. $n^{2\epsilon-1}h_v \rightarrow \infty$ for some $\epsilon < 1 - s^{-1}$, $h_v \rightarrow 0$, $h_v = h, h_1$ or h_2 . $nh_1^4 \rightarrow 0$, $nh_1^2 h^4 \rightarrow 0$, $\frac{\log^2 n}{nhh_1^3} \rightarrow 0$, $\frac{\log^3 n}{nhh_2} \rightarrow 0$, $nh^5 = O(1)$, $nh_2^5 = O(1)$.
- C8. The matrix $\Sigma_1 = E[g'^\top(X^\top \beta_0)ZJ_{\beta_0^{(1)}}^\top X - D_2(X^\top \beta_0)D_1^{-1}(X^\top \beta_0)Z]^\otimes 2$ is positive definite.

Remark 2 Condition C1 ensures that the denominators of the proposed nonparametric estimators are bounded away from 0 and infinity on $t \in \mathbf{T}_\eta$ for η in a neighbour of η_0 . Condition C2-C4 are some derivative conditions, they are standard smoothness conditions. Condition C5 is the usual assumptions for the kernel function. C6 and C7 are the necessary condition for the asymptotic normality or the uniform consistency of the estimators. Condition C8 ensures that the asymptotic variance for the estimator β_0 exists.

Proof of Theorem 1. Let $p(\varepsilon, X, Z)$, $p_1(\varepsilon | X, Z)$, $p_2(X, Z)$ represent the density function of (ε, X, Z) , the conditional density function of ε on (X, Z) and the marginal density function of (X, Z) , respectively.

Note that

$$p(\varepsilon, X, Z) = p_1(\varepsilon | X, Z)p_2(X, Z).$$

Following Tsiatis (2006), we first consider the parametric submodels to derive the efficient score. We consider submodels $p_1(\varepsilon | X, Z, \theta_1)$, $p_2(X, Z, \theta_2)$, $g_i(X_i^\top \beta, \theta_{3i})$, $i = 1, 2, \dots, q$. Then the parameter of interest is $\beta^{(1)}$, the nuisance parameters are $\{\theta_1, \theta_2, \theta_{3i}, i = 1, 2, \dots, q\}$.

Denote the score function for $\beta^{(1)}$ by S_β . From

$$\begin{aligned} \frac{\partial}{\partial \beta^{(1)}} \log p(\varepsilon, X, Z) &= \frac{\partial}{\partial \beta^{(1)}} \log p_1(\varepsilon|X, Z) = \frac{p'_{1,\varepsilon}(\varepsilon|X, Z)}{p_1(\varepsilon|X, Z)} \frac{\partial \varepsilon}{\partial \beta^{(1)}} \\ \text{and} \quad \frac{\partial \varepsilon}{\partial \beta^{(1)}} &= \frac{\partial [Y - g^\top(X^\top \beta)Z]}{\partial \beta^{(1)}} = -g'^\top(X^\top \beta)Z J_{\beta^{(1)}}^\top X, \end{aligned}$$

we have

$$S_\beta = -\frac{p'_{1,\varepsilon}(\varepsilon|X, Z)}{p_1(\varepsilon|X, Z)} g'^\top(X^\top \beta)Z J_{\beta^{(1)}}^\top X. \quad (12)$$

Denote the nuisance tangent space corresponding to p_1, p_2 and g respectively Λ_1, Λ_2 and Λ_3 . Thus we have

$$\begin{aligned} \Lambda_1 &= \{S(\varepsilon, X, Z) : E[\varepsilon S(\varepsilon, X, Z)|X, Z] = 0, E[S(\varepsilon, X, Z)|X, Z] = 0\}, \\ \Lambda_2 &= \{S(X, Z) : E[S(X, Z)] = 0\}, \\ \Lambda_3 &= \{S(\varepsilon, X, Z) : E[S(\varepsilon, X, Z)|X, Z] = 0, E[\varepsilon S(\varepsilon, X, Z)|X, Z] = h^\top(X^\top \beta)Z\}, \end{aligned}$$

$h(\cdot)$ is a function dependent on $X^\top \beta$. Obviously, we find that Λ_1, Λ_2 and Λ_3 are mutually orthogonality. Hence the nuisance tangent space $\Lambda = \Lambda_1 \oplus \Lambda_2 \oplus \Lambda_3$, that is,

$$\begin{aligned} \Lambda &= \{f(Y, X, Z) : E[f] = 0, E[\varepsilon f|X, Z] = h^\top(X^\top \beta)Z, \\ &\quad h(X^\top \beta) \text{ is a function of } X^\top \beta\}. \end{aligned}$$

Note that $\Lambda^\perp$ is defined to be the orthogonal complement of Λ . In next lemma we derive the elements of $\Lambda^\perp$.

Lemma 1 For model (2), $\Lambda^\perp = \{\varepsilon f(X, Z) : E[f(X, Z)Z^\top | X^\top \beta] = 0\}$.

Proof: For any $h(Y, X, Z) \in \mathcal{H}$, $\mathcal{H}$ denotes the Hilbert space of q -dimensional mean-zero finite-variance measurable functions (equipped with the covariance inner product), it suffices to show that $\varepsilon h_1(X, Z) \in \Lambda^\perp, h(Y, X, Z) - \varepsilon h_1(X, Z) \in \Lambda$ and this separation for $h(Y, X, Z)$ is unique. To ensure $h(Y, X, Z) - \varepsilon h_1(X, Z) \in \Lambda$, it must satisfy that $E[h(Y, X, Z) - \varepsilon h_1(X, Z)] = 0$ and

$$E[\varepsilon h(Y, X, Z) - \varepsilon^2 h_1(X, Z)|X, Z] = h_2^\top(X^\top \beta)Z \text{ for some } h_2.$$

Thus

$$h_1(X, Z) = \frac{1}{E[\varepsilon^2|X, Z]} \{E[\varepsilon h(Y, X, Z)|X, Z] - h_2^\top(X^\top \beta)Z\}.$$

Since $\varepsilon h_1(X, Z) \in \Lambda^\perp$, $E[h_1(X, Z)Z^\top | X^\top \beta] = 0$. Together with the above two equalities, we have

$$\begin{aligned} & E\left\{\frac{1}{E[\varepsilon^2|X, Z]} \left[E[\varepsilon h(Y, X, Z)|X, Z]Z^\top - h_2^\top(X^\top \beta)ZZ^\top\right] \middle| X^\top \beta\right\} = 0 \\ \Rightarrow & h_2^\top(X^\top \beta) = E\left\{\frac{E[\varepsilon h(Y, X, Z)|X, Z]Z^\top}{E[\varepsilon^2|X, Z]} \middle| X^\top \beta\right\} \left\{E\left[\frac{ZZ^\top}{E[\varepsilon^2|X, Z]} \middle| X^\top \beta\right]\right\}^{-1}. \end{aligned}$$

Therefore any $h(Y, X, Z) \in \mathcal{H}$ can be separated into these two spaces uniquely. That is, $\Lambda^\perp = \{\varepsilon f(X, Z) : E[f(X, Z)Z^\top | X^\top \beta] = 0\}$. $\triangle$

Hereafter we focus to derive the efficient score function. Note that one score function of $\beta^{(1)}$ is $S_\beta = -\frac{p'_{1,\varepsilon}(\varepsilon|X,Z)}{p_1(\varepsilon|X,Z)}g'^\top(X^\top \beta)ZJ_{\beta^{(1)}}^\top X$, then the efficient score can be defined as $S_\beta^{eff} = S_\beta - \Pi(S_\beta|\Lambda) = \Pi(S_\beta|\Lambda^\perp)$.

By the proof of Lemma 1, we have $S_\beta^{eff}(Y, X, Z) = \varepsilon h_1(X, Z)$, where

$$\begin{aligned} h_1(X, Z) &= \frac{1}{E[\varepsilon^2|X, Z]} [E[\varepsilon S_\beta|X, Z] - h_2^\top(X^\top \beta)Z], \\ h_2^\top(X^\top \beta) &= E\left\{\frac{E[\varepsilon S_\beta|X, Z]Z^\top}{E[\varepsilon^2|X, Z]} \middle| X^\top \beta\right\} \left\{E\left[\frac{ZZ^\top}{E[\varepsilon^2|X, Z]} \middle| X^\top \beta\right]\right\}^{-1}. \end{aligned}$$

Set $w(X, Z) = \frac{1}{E[\varepsilon^2|X, Z]}$. Note that $E_{p_1}\{-\frac{p'_{1,\varepsilon}(\varepsilon|X,Z)}{p_1(\varepsilon|X,Z)}\varepsilon|X, Z\} = 1$. For $h_1(X, Z)$, we have

$$\begin{aligned} h_1(X, Z) &= w(X, Z) \left\{g'^\top(X^\top \beta)ZJ_{\beta^{(1)}}^\top X \right. \\ &\quad \left. - E[w(X, Z)g'^\top(X^\top \beta)ZJ_{\beta^{(1)}}^\top XZ^\top | X^\top \beta] \{E[w(X, Z)ZZ^\top | X^\top \beta]\}^{-1}Z\right\}. \end{aligned}$$

Therefore

$$\begin{aligned} S_{\beta_0}^{eff}(Y, X, Z) &= \varepsilon w(X, Z) \left\{g'^\top(X^\top \beta_0)ZJ_{\beta_0^{(1)}}^\top X \right. \\ &\quad \left. - E[w(X, Z)g'^\top(X^\top \beta_0)ZJ_{\beta_0^{(1)}}^\top XZ^\top | X^\top \beta_0] \{E[w(X, Z)ZZ^\top | X^\top \beta_0]\}^{-1}Z\right\}. \end{aligned}$$

Recall that $D_1(X_i^\top \beta) = E[w(X_i, Z_i)Z_iZ_i^\top | X_i^\top \beta]$ and

$$D_2(X_i^\top \beta) = E[w(X_i, Z_i)g'^\top(X_i^\top \beta)Z_iJ_{\beta^{(1)}}^\top X_iZ_i^\top | X_i^\top \beta].$$

Denote $U(X_i, Z_i, \beta) = g'^\top(X_i^\top \beta)Z_iJ_{\beta^{(1)}}^\top X_i - D_2(X_i^\top \beta)D_1^{-1}(X_i^\top \beta)Z_i$. Thus the efficient score function can be written as $S_{eff} = \varepsilon w(X, Z)U(X_i, Z_i, \beta)$. $\square$

To prove Theorem 2, we first give some lemmas.

Lemma 2 *Let $(T_1, Y_1), \dots, (T_n, Y_n)$ be i.i.d. random vectors, where Y_i s are scalar random variables. Assume further that $E|Y_1|^s < \infty$ and $\sup_t \int |y|^s f(t, y) dy < \infty$, where f denotes the joint density of (T_1, Y_1) . Let $K(\cdot)$ be a bounded positive function with a bounded support, satisfying the Lipschitz condition. Then*

$$\sup_{t \in \mathbf{T}} \left| \frac{1}{n} \sum_{i=1}^n [K_h(T_i - t)Y_i - E(K_h(T_i - t)Y_i)] \right| = O_p\left\{\left(\frac{\log n}{nh}\right)^{1/2}\right\},$$

provided $0 < h \rightarrow 0$ and $n^{2\epsilon-1}h \rightarrow \infty$ for some $\epsilon < 1 - s^{-1}$, $s > 2$, where h is a bandwidth and $\mathbf{T}$ is some closed set.

The proof of the Lemma can be found in Mack and Silverman (1982).

Lemma 3 *Suppose that conditions in Theorem 2 hold. Then*

$$\begin{aligned} \sup_{t \in \mathbf{T}, \beta \in \mathbf{B}_n} \|\hat{g}(t; \beta) - g(t)\| &= O_p\left(\left\{\frac{\log n}{nh}\right\}^{1/2} + h^2\right), \\ \sup_{t \in \mathbf{T}, \beta \in \mathbf{B}_n} \|\hat{g}'(t; \beta) - g'(t)\| &= O_p\left(\left\{\frac{\log n}{nh^3}\right\}^{1/2} + h\right). \end{aligned}$$

This lemma is proved in Theorem 1 of Xue and Wang (2012).

Lemma 4 *Suppose that conditions in Theorem 2 hold. Then for any $\beta \in \mathbf{B}_n$,*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n (g(X_i^\top \beta) - \hat{g}(X_i^\top \beta))^T Z_i w(X_i, Z_i) S_{ei}(\beta) = o_p(1),$$

where $S_{ei}(\beta) = g'^\top(X_i^\top \beta) Z_i J_{\beta(1)}^\top X_i - D_2(X_i^\top \beta) D_1^{-1}(X_i^\top \beta) Z_i$.

Proof. From (7), and by Lemma 2, we have $\frac{1}{n} \sum_{i=1}^n Z_i Z_i^\top K_h(t_i - t) - f(t) E(ZZ^\top | t) = O_p(h^2 + [\frac{\log n}{nh}]^{1/2})$, let $\Sigma_{11}(t) = f(t) E(ZZ^\top | t)$, $t = X^\top \beta$, thus,

$$\begin{aligned} \hat{g}(t) - g(t) &= \Sigma_{11}^{-1}(t) \frac{1}{n} \sum_{i=1}^n Z_i \varepsilon_i K_h(t_i - t) + \Sigma_{11}^{-1}(t) \frac{1}{n} \sum_{i=1}^n Z_i Z_i^\top K_h(t_i - t) h g'(t) \left(\frac{t_i - t}{h}\right) \\ &\quad + \Sigma_{11}^{-1}(t) \frac{1}{n} \sum_{i=1}^n Z_i Z_i^\top K_h(t_i - t) \frac{h^2}{2} g''(t) \left(\frac{t_i - t}{h}\right)^2 + O_p(h^4). \end{aligned}$$

Therefore,

$$\begin{aligned}
& \frac{1}{\sqrt{n}} \sum_{i=1}^n (g(X_i^\top \beta) - \hat{g}(X_i^\top \beta))^T Z_i w(X_i, Z_i) S_{ei}(\beta) \\
&= o_p(1) - \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_i \left\{ \frac{1}{n} \sum_{j=1}^n w(X_j, Z_j) S_{ej}(\beta) Z_j^\top K_h(t_j - t_i) \right\} \Sigma_{11}^{-1}(t) Z_i \\
&\quad + \frac{1}{\sqrt{n}} \sum_{i=1}^n \left\{ \frac{1}{n} \sum_{j=1}^n w(X_j, Z_j) S_{ej}(\beta) Z_j^\top K_h(t_j - t_i) \left[h g'(t_j) \frac{t_j - t_i}{h} - \frac{h^2}{2} g''(t_j) \left(\frac{t_j - t_i}{h} \right)^2 \right] \right\} Z_i Z_i^\top \\
&:= o_p(1) + E_{a1} + E_{a2}.
\end{aligned}$$

Since $E(w(X_j, Z_j) S_{ej}(\beta) Z_j^\top K_h(t_j - t_i)) = 0$, we can get $E(E_{a1}^{\otimes 2}) = O(\frac{1}{nh})$, thus $E_{a1} = O_p(\frac{1}{\sqrt{nh}}) = o_p(1)$. Similarly, $E_{a2} = O_p(\sqrt{h^2}) = o_p(1)$. Lemma 4 is then proved.

□

Proof of Theorem 2. Denote $S_{ei}(\beta) = g'^\top(X_i^\top \beta) Z_i J_{\beta(1)}^\top X_i - D_2(X_i^\top \beta) D_1^{-1}(X_i^\top \beta) Z_i$.

Now we focus on the following estimating equation

$$Q_n(\hat{\beta}) \doteq \frac{1}{\sqrt{n}} \sum_{i=1}^n (Y_i - \hat{g}^\top(X_i^\top \hat{\beta}) Z_i) \hat{w}(X_i, Z_i) \hat{S}_{ei}(\hat{\beta}) = 0,$$

where $\hat{S}_{ei}(\hat{\beta}) = \hat{g}'^\top(X_i^\top \hat{\beta}) Z_i J_{\hat{\beta}(1)}^\top X_i - \hat{D}_2(X_i^\top \hat{\beta}) \hat{D}_1^{-1}(X_i^\top \hat{\beta}) Z_i$. It is remarkable that $Q_n(\beta) = 0$ is the sample version of $n^{-1/2} \sum_{i=1}^n S_{\beta}^{eff}(Y_i, X_i, Z_i) = 0$ with some ‘plug-in’ estimators.

Firstly, it is easy to see that

$$\begin{aligned}
Q_n(\hat{\beta}) &= \frac{1}{\sqrt{n}} \sum_{i=1}^n (g(X_i^\top \beta_0) - \hat{g}(X_i^\top \hat{\beta}))^\top Z_i \hat{w}(X_i, Z_i) \hat{S}_{ei}(\hat{\beta}) + \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_i \hat{w}(X_i, Z_i) \hat{S}_{ei}(\hat{\beta}) \\
&:= E_1 + E_2.
\end{aligned} \tag{13}$$

For E_1 , we have

$$\begin{aligned}
E_1 &= \frac{1}{\sqrt{n}} \sum_{i=1}^n (g(X_i^\top \beta_0) - \hat{g}(X_i^\top \hat{\beta}))^\top Z_i w(X_i, Z_i) S_{ei}(\beta_0) \\
&\quad + \frac{1}{\sqrt{n}} \sum_{i=1}^n (g(X_i^\top \beta_0) - \hat{g}(X_i^\top \hat{\beta}))^\top Z_i [\hat{w}(X_i, Z_i) - w(X_i, Z_i)] S_{ei}(\beta_0) \\
&\quad + \frac{1}{\sqrt{n}} \sum_{i=1}^n (g(X_i^\top \beta_0) - \hat{g}(X_i^\top \hat{\beta}))^\top Z_i w(X_i, Z_i) [\hat{S}_{ei}(\hat{\beta}) - S_{ei}(\beta_0)] \\
&\quad + \frac{1}{\sqrt{n}} \sum_{i=1}^n (g(X_i^\top \beta_0) - \hat{g}(X_i^\top \hat{\beta}))^\top Z_i [\hat{w}(X_i, Z_i) - w(X_i, Z_i)] [\hat{S}_{ei}(\hat{\beta}) - S_{ei}(\beta_0)] \\
&:= E_{11} + E_{12} + E_{13} + E_{14}.
\end{aligned} \tag{14}$$

For E_{11} , we have

$$\begin{aligned}
E_{11} &= \frac{1}{\sqrt{n}} \sum_{i=1}^n (g(X_i^\top \beta_0) - g(X_i^\top \hat{\beta}))^\top Z_i w(X_i, Z_i) S_{ei}(\beta_0) \\
&\quad + \frac{1}{\sqrt{n}} \sum_{i=1}^n (g(X_i^\top \hat{\beta}) - \hat{g}(X_i^\top \hat{\beta}))^\top Z_i w(X_i, Z_i) S_{ei}(\beta_0) \\
&= E_{111} + E_{112}.
\end{aligned} \tag{15}$$

By Lemma 4, $E_{112} = o_p(1)$, and we have

$$E_{11} = -\frac{1}{\sqrt{n}} \sum_{i=1}^n S_{ei}(\beta_0) w(X_i, Z_i) Z_i^\top g'(X_i^\top \beta_0) X_i^\top J_{\beta_0^{(1)}} (\hat{\beta}^{(1)} - \beta_0^{(1)}) + o_p(1). \tag{16}$$

For E_{12} , note the definitions of the initial estimators of $\hat{g}_I(\cdot)$ and $\hat{\beta}_I$, by Lemma 2-3, and the assumption for $\sigma^2(\cdot)$, similar to the proof of Lemma 4 in Ma and Zhu (2013), it follows

$$\sup_{X \in \mathbf{A}_X, Z \in \mathbf{A}_Z} \|\hat{\sigma}^2(X, Z) - \sigma^2(X, Z)\| = O_p(h_2^2 + \frac{\log n}{\sqrt{nh_2}}). \tag{17}$$

Therefore, note that $\hat{w}(\cdot) = \frac{1}{\hat{\sigma}^2(\cdot)}$, by Lemma 3, (17) and condition C7, we get

$$E_{12} = \frac{1}{\sqrt{n}} \sum_{i=1}^n (g(X_i^\top \beta_0) - \hat{g}(X_i^\top \hat{\beta}))^\top Z_i [\hat{w}(X_i, Z_i) - w(X_i, Z_i)] S_{ei}(\beta_0) = o_p(1).$$

Similarly, we can prove $E_{13} = o_p(1)$ and $E_{14} = o_p(1)$. Thus

$$E_1 = -\frac{1}{\sqrt{n}} \sum_{i=1}^n S_{ei}(\beta_0) w(X_i, Z_i) Z_i^\top g'(X_i^\top \beta_0) J_{\beta_0^{(1)}}^\top X_i (\hat{\beta}^{(1)} - \beta_0^{(1)}) + o_p(1). \tag{18}$$

For the term E_2 , we have

$$\begin{aligned}
E_2 &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_i w(X_i, Z_i) S_{ei}(\beta_0) + \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_i [\hat{w}(X_i, Z_i) - w(X_i, Z_i)] S_{ei}(\beta_0) \\
&\quad + \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_i [\hat{w}(X_i, Z_i) - w(X_i, Z_i)] [\hat{S}_{ei}(\hat{\beta}) - S_{ei}(\beta_0)] \\
&\quad + \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_i w(X_i, Z_i) [\hat{S}_{ei}(\hat{\beta}) - S_{ei}(\beta_0)] \\
&:= E_{21} + E_{22} + E_{23} + E_{24}.
\end{aligned} \tag{19}$$

Similarly, it can be obtained

$$E_2 = \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_i w(X_i, Z_i) S_{ei}(\beta_0) + o_p(1). \tag{20}$$

By (13),(18) and (20), it follows

$$\begin{aligned} & \frac{1}{\sqrt{n}} \sum_{i=1}^n S_{ei}(\beta_0) w(X_i, Z_i) Z_i^\top g'(X_i^\top \beta_0) X_i^\top J_{\beta_0^{(1)}} (\hat{\beta}^{(1)} - \beta_0^{(1)}) \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_i w(X_i, Z_i) S_{ei}(\beta_0) + o_p(1). \end{aligned} \quad (21)$$

Therefore,

$$\begin{aligned} \sqrt{n}(\hat{\beta}^{(1)} - \beta_0^{(1)}) &= \left[\frac{1}{n} \sum_{i=1}^n w(X_i, Z_i) S_{ei}(\beta_0) Z_i^\top g'(X_i^\top \beta_0) X_i^\top J_{\beta_0^{(1)}} \right]^{-1} \\ &\quad \times \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_i w(X_i, Z_i) S_{ei}(\beta_0) + o_p(1). \end{aligned} \quad (22)$$

Denote the items of the right side of (22) as $A_n^{-1}W_n + o_p(1)$. Clearly, $A_n \rightarrow_p \Sigma_1^*$ and $W_n \rightarrow_d N(0, \Sigma_1)$, where

$$\begin{aligned} \Sigma_1^* &= E \left[w(X, Z) S_e(\beta_0) Z^\top (g'(X^\top \beta_0)) X^\top J_{\beta_0^{(1)}} \right] \\ \text{and } \Sigma_1 &= E \left[w(X, Z) S_e(\beta_0) S_e^\top(\beta_0) \right]. \end{aligned}$$

It is noted that $\Sigma_1^* = \Sigma_1 = E[S_\beta^{eff} S_\beta^{eff\top}]$. Thus we have

$$\sqrt{n}(\hat{\beta}^{(1)} - \beta_0^{(1)}) \xrightarrow{\mathcal{D}} N(0, \Sigma_1^{-1}).$$

□

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Table 1: Summary of the simulation

n	estimator	$(\hat{\beta}_1 - \beta_1)^2$	$(\hat{\beta}_2 - \beta_2)^2$	AE_1	AE_2	AE_3
		$Mean \times 10^4 (SD \times 10^4)$	$Mean \times 10^4 (SD \times 10^4)$	Median(MAD)	Median(MAD)	Median(MAD)
$e_i \sim N(0, 1)$						
100	(a)	0.9593(1.4096)	3.7750(5.4123)	0.1745(0.0314)	0.1327(0.0303)	0.1440(0.0341)
	(b)	0.9141(1.3398)	3.5936(5.1763)	0.1740(0.0313)	0.1320(0.0303)	0.1442(0.0339)
	(c)	1.0473(1.6003)	4.2421(6.5839)	0.1759(0.0317)	0.1326(0.0305)	0.1449(0.0341)
	(d)	0.9938(1.5342)	4.0086(6.2901)	0.1759(0.0315)	0.1321(0.0304)	0.1444(0.0340)
200	(a)	0.4179(0.6400)	1.6527(2.4636)	0.1387(0.0222)	0.0873(0.0194)	0.0952(0.0207)
	(b)	0.3963(0.5852)	1.5691(2.2745)	0.1378(0.0221)	0.0872(0.0194)	0.0952(0.0207)
	(c)	0.4574(0.6692)	1.1847(2.7189)	0.1386(0.0222)	0.0873(0.0194)	0.0951(0.0207)
	(d)	0.4326(0.6112)	1.7488(2.5006)	0.1390(0.0221)	0.0872(0.0194)	0.0954(0.0206)
$e_i \sim N(0, \exp(U_i^\top \gamma_0 / 2))$						
100	(a)	1.1502(1.7886)	4.5352(6.7866)	0.2104(0.0363)	0.1436(0.0340)	0.1605(0.0415)
	(b)	1.1982(1.8526)	4.6741(6.9417)	0.2112(0.0362)	0.1439(0.0338)	0.1597(0.0415)
	(c)	1.3483(2.2517)	5.4827(9.3104)	0.2135(0.0365)	0.1444(0.0342)	0.1599(0.0417)
	(d)	1.3302(2.1566)	5.3800(8.7749)	0.2134(0.0362)	0.1447(0.0339)	0.1600(0.0416)
200	(a)	0.4616(0.7070)	1.8465(2.7567)	0.1466(0.0252)	0.1032(0.0235)	0.1118(0.0237)
	(b)	0.5211(0.7759)	2.0614(3.0027)	0.1467(0.0250)	0.1035(0.0235)	0.1125(0.0238)
	(c)	0.5509(0.8374)	2.2366(3.4693)	0.1464(0.0253)	0.1037(0.0235)	0.1119(0.0237)
	(d)	0.5564(0.7901)	2.2499(3.2306)	0.1466(0.0252)	0.1040(0.0235)	0.1125(0.0237)