When are two Markov chains similar?

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Abstract

We present sufficient conditions for two Markov chains to be similar, thus confirming a conjecture of Pollett (J. Appl. Prob 38A (2001), 53-65). Our proof technique also shows one of the sufficient conditions given in Theorem 2 of Lenin et al. (J. Appl. Prob 37 (2000), 835-849) for two birth-death processes to be similar is unnecessary.

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1 Introduction

This note addresses the question of when two continuous-time Markov chains (CTMCs) are similar, as defined in Lenin et al. [3] and Pollett [4].

Suppose $X := \{X(t); t \geq 0\}$ and $\bar{X} := \{\bar{X}(t); t \geq 0\}$ are two CTMCs having state space E , and rate matrices $\mathbf{Q} := [q(x, y)]_{x, y \in E}$ and $\bar{\mathbf{Q}} := [\bar{q}(x, y)]_{x, y \in E}$, respectively. Further associated with X is its collection of transition times $\{T_n\}_{n \geq 0}$, where $T_0 := 0$, as well as its embedded discrete-time

Markov chain (DTMC) $\{X_n\}_{n \geq 0}$, where $X_n := X(T_n)$ for each integer $n \geq 0$. Similarly, we let $\{\bar{T}_n\}_{n \geq 0}$ and $\{\bar{X}_n\}_{n \geq 0}$ represent the transition times and the embedded DTMC associated with $\bar{X}$.

We assume X and $\bar{X}$ share the same transition structure, meaning for any pair of states $x, y \in E$, $x \neq y$, $q(x, y) > 0$ if and only if $\bar{q}(x, y) > 0$. This assumption ensures both chains share the same communicating classes. Furthermore, we do not force either X or $\bar{X}$ to be regular, meaning either chain could explode in a finite amount of time: to model such a phenomenon, we create an additional state Δ outside of E , we let $T := \lim_{n \rightarrow \infty} T_n$ represent the time at which X explodes, and we define

$$X(t) = \begin{cases} \sum_{n=0}^{\infty} X_n \mathbf{1}(T_n \leq t < T_{n+1}), & t < T; \\ \Delta, & t \geq T. \end{cases}$$

with $\bar{X}$ being constructed analogously, so that both processes are well-defined on $[0, \infty)$. Finally, we assume both X and $\bar{X}$ are stable and conservative over E , meaning $\mathbf{Q}$ satisfies $q(x) := -q(x, x) = \sum_{y \neq x} q(x, y) < \infty$ for each $x \in E$, and similarly for $\bar{\mathbf{Q}}$.

It is known—see e.g. Section 2.2 of Anderson [1]—that the transition functions $\{p_{x,y}\}_{x,y \in E}$ associated with X are the minimal transition functions that solve the corresponding Kolmogorov forward and backward equations, where for each $t \geq 0$, $x, y \in E$,

$$p_{x,y}(t) := \mathbb{P}_x(X(t) = y)$$

with $\mathbb{P}_x$ representing a conditional probability, given $X(0) = x$. Throughout we will still use the notation $\mathbb{P}_x$ when conditioning on other Markov chains being in state x at time zero, with $\mathbb{E}_x$ being the corresponding expectation: this usage of notation is fairly standard, and should not cause any additional confusion. For example, for each $x, y \in E$ we follow the convention that the transition function $\bar{p}_{x,y}$ associated with $\bar{X}$ evaluated at time t , $\bar{p}_{x,y}(t)$, can also be expressed as $\mathbb{P}_x(\bar{X}(t) = y)$.

Our goal is to find sufficient conditions for X and $\bar{X}$ to be similar over a common communicating class C , where similarity is defined as follows.

Definition 1.1 *We say X and $\bar{X}$ are similar over a communicating class C if there exists a collec-*

tion of positive numbers $\{c_{x,y}\}_{x,y \in C}$ satisfying, for each $x, y \in C$,

$$\bar{p}_{x,y}(t) = c_{x,y}p_{x,y}(t), \quad t \geq 0.$$

The next theorem is the main result of this note: it will be proven in Section 3.

Theorem 1.1 *Suppose that for some communicating class C of both X and $\bar{X}$, the following conditions are satisfied:*

- (a) *for each $x \in C$, $q(x) = \bar{q}(x)$.*
- (b) *for each $x \in C$, every feasible path $x = x_0, x_1, \dots, x_{n-1}, x_n = x$ under Q (and also $\bar{Q}$) starting and ending at x satisfies*

$$\prod_{\ell=1}^n q(x_{\ell-1}, x_\ell) = \prod_{\ell=1}^n \bar{q}(x_{\ell-1}, x_\ell).$$

Then X and $\bar{X}$ are similar over C .

The next lemma provides a simple, yet important consequence of Conditions (a) and (b) and will be used in Section 3 to prove Theorem 1.1.

Lemma 1.1 *Suppose Condition (b) of Theorem 1.1 holds, and let x, y be two distinct states in C . Then for any two feasible paths $x = x_0, x_1, \dots, x_{m-1}, x_m = y$ and $x = y_0, y_1, \dots, y_{n-1}, y_n = y$ under Q from x to y , we have*

$$\prod_{\ell=1}^m \frac{\bar{q}(x_{\ell-1}, x_\ell)}{q(x_{\ell-1}, x_\ell)} = \prod_{\ell=1}^n \frac{\bar{q}(y_{\ell-1}, y_\ell)}{q(y_{\ell-1}, y_\ell)}.$$

Proof Observe first that since $x, y \in C$, there exists a feasible path $y = z_0, z_1, \dots, z_{k-1}, z_k = x$ from y to x . Furthermore, Condition (b) also implies

$$\prod_{\ell=1}^k \frac{\bar{q}(z_{\ell-1}, z_\ell)}{q(z_{\ell-1}, z_\ell)} \prod_{\ell=1}^m \frac{\bar{q}(x_{\ell-1}, x_\ell)}{q(x_{\ell-1}, x_\ell)} = 1 = \prod_{\ell=1}^k \frac{\bar{q}(z_{\ell-1}, z_\ell)}{q(z_{\ell-1}, z_\ell)} \prod_{\ell=1}^n \frac{\bar{q}(y_{\ell-1}, y_\ell)}{q(y_{\ell-1}, y_\ell)} \quad (1)$$

since $z_0, z_1, \dots, z_{k-1}, z_k = x_0, x_1, \dots, x_m$ and $z_0, z_1, \dots, z_{k-1}, z_k = y_0, y_1, \dots, y_n$ are both feasible paths starting and ending at state y . After dividing the far-left and far-right terms of (1) by their common multiple, we find

$$\prod_{\ell=1}^m \frac{\bar{q}(x_{\ell-1}, x_{\ell})}{q(x_{\ell-1}, x_{\ell})} = \prod_{\ell=1}^n \frac{\bar{q}(y_{\ell-1}, y_{\ell})}{q(y_{\ell-1}, y_{\ell})}$$

which proves our claim. $\diamond$

Conditions (a) and (b) from Theorem 1.1 were conjectured on pg. 64 of [4] to be sufficient for two Markov chains to be similar over C . Theorem 1.1 also generalizes Theorem 2 of [3] where two sufficient conditions equivalent to (a) and (b) are given for two birth-death processes to be similar. In fact (as also pointed out in [4]) the authors of [3] further assume an associated Stieltjes moment problem has a unique solution, but the arguments we provide here show this extra assumption is not needed to establish similarity, once Conditions (a) and (b) are known to hold.

Interestingly, in [3] the authors also show (see Theorem 1 of [3]) that similarity of two birth-death processes over a communicating class C implies Conditions (a) and (b) of Theorem 1.1 must hold on C , i.e. that these are also necessary conditions for similarity on C . The author has tried showing that—as also conjectured in [4]—Conditions (a) and (b) are also necessary for similarity to hold over C in this more general context, but currently it is not clear this is the case.

2 Preliminaries

Our approach involves working with the Laplace transforms of the transition functions associated with both X and $\bar{X}$, as opposed to working with the transition functions directly. For each $x, y \in E$, we define for $Re(\alpha) > 0$

$$\pi_{x,y}(\alpha) := \int_0^\infty e^{-\alpha t} p_{x,y}(t) dt, \quad \bar{\pi}_{x,y}(\alpha) := \int_0^\infty e^{-\alpha t} \bar{p}_{x,y}(t) dt.$$

The first proposition in this section yields a useful expression for each Laplace transform $\pi_{x,y}$, $x, y \in E$.

Proposition 2.1 *For each $x, y \in E$,*

$$\pi_{x,y}(\alpha) = \frac{\mathbb{E}_x \left[\int_0^{\tau_x} e^{-\alpha t} \mathbf{1}(X(t) = y) dt \right]}{1 - \mathbb{E}[e^{-\alpha \tau_x}]}, \quad \text{Re}(\alpha) > 0 \quad (2)$$

where $\tau_x := \inf\{t \geq 0 : X(t-) \neq X(t) = x\}$, with $X(t-) := \lim_{s \uparrow t} X(s)$ for each $t > 0$.

Proof Fix any two states $x, y \in E$, where possibly $x = y$. Then for each $t \geq 0$,

$$\begin{aligned} p_{x,y}(t) &= \mathbb{P}_x(X(t) = y, \tau_x > t) + \mathbb{P}_x(X(t) = y, \tau_x \leq t) \\ &= \mathbb{P}(X(t) = y, \tau_x > t) + \int_0^t p_{x,y}(t-s) d\mathbb{P}_x(\tau_x \leq s). \end{aligned} \quad (3)$$

Integrating both sides of (3) by $e^{-\alpha t} dt$, $\text{Re}(\alpha) > 0$, over $[0, \infty)$ and simplifying gives (2). $\diamond$

Equation (2) simplifies even further when $x = y$, as the next corollary illustrates.

Corollary 2.1 *For each state $x \in E$,*

$$\pi_{x,x}(\alpha) = \frac{1}{(q(x) + \alpha)(1 - \mathbb{E}_x[e^{-\alpha \tau_x}])}, \quad \text{Re}(\alpha) > 0.$$

Proof Starting with the numerator of (2), we find

$$\mathbb{E}_x \left[\int_0^{\tau_x} e^{-\alpha t} \mathbf{1}(X(t) = x) dt \right] = \mathbb{E}_x \left[\int_0^{T_1} e^{-\alpha t} dt \right] = \frac{1}{\alpha} \mathbb{E}_x [1 - e^{-\alpha T_1}] = \frac{1}{q(x) + \alpha} \quad (4)$$

which proves our claim, once we plug (4) into (2). $\diamond$

The final result of this section shows that the Laplace transforms $\pi_{x,y}$, $x, y \in E$, can be expressed in terms of expected values of random products governed by a CTMC $\{\tilde{X}(t); t \geq 0\}$ on E having a

rate matrix $\tilde{\mathbf{Q}}$ that satisfies the following properties: (i) for any two states $x, y \in E$, $x \neq y$, $\tilde{q}(x, y) > 0$ if and only if $q(y, x) > 0$, and (ii) for each $x \in E$, we have $-\tilde{q}(x, x) = \sum_{x \neq y} \tilde{q}(x, y) = -q(x, x)$.

In order to make use of $\tilde{X}$ and $\tilde{\mathbf{Q}}$, we must assume X and $\bar{X}$ do not have any absorbing states, and there may be many situations (such as in [3]) where this is an unreasonable assumption. Fortunately, we can remove absorbing states in a harmless manner by adding new states in the following way: if a state $z_0 \in E$ is an absorbing state, simply create another new state z'_0 , and extend $\mathbf{Q}$ by choosing positive values for $q(z_0, z'_0)$ and $q(z'_0, z_0)$, and setting $q(z_0) := -q(z_0, z'_0)$, $q(z'_0) := -q(z'_0, z_0)$.

Theorem 2.1 *Suppose $x, y \in E$, where $x \neq y$. Then the Laplace transform $\pi_{x,y}$ of $p_{x,y}$ satisfies*

$$\pi_{x,y}(\alpha) = \pi_{x,x}(\alpha) \mathbb{E}_y \left[e^{-\alpha \tilde{\tau}_x} \prod_{\ell=1}^{\tilde{\eta}_x} \frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)} \right], \quad \text{Re}(\alpha) > 0 \quad (5)$$

where $\tilde{\tau}_x := \inf\{t > 0 : \tilde{X}(t-) \neq \tilde{X}(t) = x\}$, and $\tilde{\eta}_x := \inf\{n \geq 1 : \tilde{X}_n = x\}$.

This result is a generalization of Theorem 1.2 from [2], since (i) in [2] we only work with chains that do not explode with probability one, and (ii) in [2] Equation (5) is only shown to hold when $\alpha > 0$, but here we find it is still valid when we only assume $\text{Re}(\alpha) > 0$.

Proof Fix two states $x, y \in E$, $x \neq y$, and note that for $\text{Re}(\alpha) > 0$,

$$\begin{aligned} \pi_{x,y}(\alpha) &= \frac{1}{(q(x) + \alpha)(1 - \mathbb{E}_x[e^{-\alpha \tau_x}])} (q(x) + \alpha) \mathbb{E}_x \left[\int_0^{\tau_x} e^{-\alpha t} \mathbf{1}(X(t) = y) dt \right] \\ &= \pi_{x,x}(\alpha) (q(x) + \alpha) \mathbb{E}_x \left[\int_0^{\tau_x} e^{-\alpha t} \mathbf{1}(X(t) = y) dt \right] \end{aligned}$$

so to complete the proof, it remains to show that

$$(q(x) + \alpha) \mathbb{E}_x \left[\int_0^{\tau_x} e^{-\alpha t} \mathbf{1}(X(t) = y) dt \right] = \mathbb{E}_y \left[e^{-\alpha \tilde{\tau}_x} \prod_{\ell=1}^{\tilde{\eta}_x} \frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)} \right]. \quad (6)$$

The argument used to establish (6) is just slightly different from that used to prove Theorem 1.2 of [2], in that we do not make use of a clearing model. Starting with the right-hand-side of (6), we

observe after summing over all finite feasible paths from y to x under $\tilde{\mathbf{Q}}$ that

$$\begin{aligned}
& \mathbb{E}_y \left[e^{-\alpha \tilde{\tau}_x} \prod_{\ell=1}^{\tilde{\eta}_x} \frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)} \right] \\
&= \sum_{n=1}^{\infty} \sum_{\substack{x_0=y, x_n=x, x_1, \dots, x_{n-1} \neq x}} \left[\prod_{\ell=1}^n \frac{q(x_\ell, x_{\ell-1})}{\tilde{q}(x_{\ell-1}, x_\ell)} \right] \left[\prod_{\ell=1}^n \frac{\tilde{q}(x_{\ell-1})}{\tilde{q}(x_{\ell-1}) + \alpha} \right] \prod_{\ell=1}^n \frac{\tilde{q}(x_{\ell-1}, x_\ell)}{\tilde{q}(x_{\ell-1})} \\
&= \frac{q(x) + \alpha}{q(y) + \alpha} \sum_{n=1}^{\infty} \sum_{\substack{x_0=y, x_n=x, x_1, \dots, x_{n-1} \neq x}} \prod_{\ell=1}^n \frac{q(x_\ell, x_{\ell-1})}{q(x_\ell) + \alpha} \\
&= \frac{q(x) + \alpha}{q(y) + \alpha} \sum_{n=1}^{\infty} \mathbb{E}_x [e^{-\alpha T_n} \mathbf{1}(X_n = y, \eta_x > n)] \\
&= (q(x) + \alpha) \sum_{n=1}^{\infty} \mathbb{E}_x \left[e^{-\alpha T_n} \mathbf{1}(X_n = y, \eta_x > n) \frac{1}{\alpha} \frac{\alpha}{q(y) + \alpha} \right] \\
&= (q(x) + \alpha) \sum_{n=1}^{\infty} \mathbb{E}_x \left[e^{-\alpha T_n} \mathbf{1}(X_n = y, \eta_x > n) \frac{1}{\alpha} \mathbb{E} \left[1 - e^{-\alpha(T_{n+1} - T_n)} \mid X_0, \dots, X_n, T_1, \dots, T_n \right] \right] \\
&= (q(x) + \alpha) \mathbb{E}_x \left[\mathbb{E} \left[\sum_{n=1}^{\infty} \mathbf{1}(\eta_x > n) \int_{T_n}^{T_{n+1}} e^{-\alpha t} \mathbf{1}(X(t) = y) dt \mid X_0, \dots, X_n, T_1, \dots, T_n \right] \right] \\
&= (q(x) + \alpha) \mathbb{E}_x \left[\int_0^{\tau_x} e^{-\alpha t} \mathbf{1}(X(t) = y) dt \right]
\end{aligned}$$

which establishes (6). Here the fifth equality follows from observing

$$\mathbb{E}[1 - e^{-\alpha(T_{n+1} - T_n)} \mid X_0, \dots, X_n, T_1, \dots, T_n] \mathbf{1}(X_n = y) = \frac{\alpha}{\alpha + q(y)} \mathbf{1}(X_n = y)$$

with probability one, and the sixth equality is a consequence of both Fubini's theorem, as well as $e^{-\alpha T_n} \mathbf{1}(X_n = y, \eta_x > n)$ being a measurable function of $T_0, \dots, T_n, X_1, \dots, X_n$. $\diamond$

Remark In the proof of Theorem 2.1, we are implicitly making use of the fact that at each return to a state $x \in E$, the total number of transitions made by X before this return time must be finite: if the chain explodes before we revisit x , then $\tau_x = \infty$, meaning x is not revisited.

3 Proof of Theorem 1.1

We first show, for each $x \in C$, that $\bar{\pi}_{x,x}(\alpha) = \pi_{x,x}(\alpha)$ on $Re(\alpha) > 0$. This of course proves $\bar{p}_{x,x}(t) = p_{x,x}(t)$ for $t \geq 0$, since two continuous functions defined on $[0, \infty)$ that share the same Laplace transform must be equal. Starting with Corollary 2.1, we find

$$\pi_{x,x}(\alpha) = \frac{1}{(q(x) + \alpha)(1 - \mathbb{E}_x[e^{-\alpha\tau_x}])} = \frac{1}{(\bar{q}(x) + \alpha)(1 - \mathbb{E}_x[e^{-\alpha\bar{\tau}_x}])} = \bar{\pi}_{x,x}(\alpha) \quad (7)$$

where $\bar{\tau}_x := \inf\{t > 0 : \bar{X}(t-) \neq \bar{X}(t) = x\}$, since from Conditions (a) and (b) of Theorem 1.1, we get

$$\begin{aligned} \mathbb{E}_x[e^{-\alpha\tau_x}] &= \sum_{n=2}^{\infty} \sum_{\substack{x_0, \dots, x_n: \\ x_0=x, x_n=x, x_1, \dots, x_{n-1} \neq x}} \prod_{\ell=1}^n \frac{q(x_{\ell-1}, x_{\ell})}{q(x_{\ell-1}) + \alpha} \\ &= \sum_{n=2}^{\infty} \sum_{\substack{x_0, \dots, x_n: \\ x_0=x, x_n=x, x_1, \dots, x_{n-1} \neq x}} \prod_{\ell=1}^n \frac{\bar{q}(x_{\ell-1}, x_{\ell})}{\bar{q}(x_{\ell-1}) + \alpha} = \mathbb{E}_x[e^{-\alpha\bar{\tau}_x}]. \end{aligned}$$

It remains to show $\bar{\pi}_{x,y}(\alpha) = c_{x,y}\pi_{x,y}(\alpha)$, for $x, y \in C$, $x \neq y$, and $Re(\alpha) > 0$, where

$$c_{x,y} = \prod_{\ell=1}^n \frac{\bar{q}(x_{\ell}, x_{\ell-1})}{q(x_{\ell}, x_{\ell-1})}$$

is the product formed by any chosen feasible path $x = x_n, x_{n-1}, \dots, x_1, x_0 = y$ from x to y under

Q. Starting with (5), we find after applying both (7) and Lemma 1.1 that

$$\begin{aligned} \bar{\pi}_{x,y}(\alpha) &= \bar{\pi}_{x,x}(\alpha) \mathbb{E}_y \left[e^{-\alpha\bar{\tau}_x} \prod_{\ell=1}^{\bar{\eta}_x} \frac{\bar{q}(\tilde{X}_{\ell}, \tilde{X}_{\ell-1})}{\bar{q}(\tilde{X}_{\ell-1}, \tilde{X}_{\ell})} \right] \\ &= \pi_{x,x}(\alpha) \mathbb{E}_y \left[e^{-\alpha\bar{\tau}_x} \left(\prod_{\ell=1}^{\bar{\eta}_x} \frac{\bar{q}(\tilde{X}_{\ell}, \tilde{X}_{\ell-1})}{q(\tilde{X}_{\ell}, \tilde{X}_{\ell-1})} \right) \left(\prod_{\ell=1}^{\bar{\eta}_x} \frac{q(\tilde{X}_{\ell}, \tilde{X}_{\ell-1})}{\bar{q}(\tilde{X}_{\ell-1}, \tilde{X}_{\ell})} \right) \right] \\ &= c_{x,y}\pi_{x,x}(\alpha) \mathbb{E}_y \left[e^{-\alpha\bar{\tau}_x} \prod_{\ell=1}^{\bar{\eta}_x} \frac{q(\tilde{X}_{\ell}, \tilde{X}_{\ell-1})}{\bar{q}(\tilde{X}_{\ell-1}, \tilde{X}_{\ell})} \right] \\ &= c_{x,y}\pi_{x,y}(\alpha) \end{aligned}$$

which proves Theorem 1.1.

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